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Derivation of Fokker–Planck equations for stochastic systems under excitation of multiplicative non-Gaussian white noise

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A B S T R A C T

Fokker–Planck equations describe time evolution of probability densities of stochastic dynamical systems and play an important role in quantifying propagation and evolution of uncertainty. Although Fokker–Planck equations can be written explicitly for systems excited by Gaussian white noise, they have remained unknown in general for systems excited by multiplicative non-Gaussian white noise. In this paper, we derive explicit forms of Fokker–Planck equations for one dimensional systems modeled by Marcus stochastic differential equations under multiplicative non-Gaussian white noise. As examples to illustrate the theoretical results, the derived formula is used to obtain Fokker–Planck equations for nonlinear dynamical systems under excitation of (i) α-stable white noise; (ii) combined Gaussian and Poisson white noise, respectively.

1. Introduction and statement of the problem

Stochastic differential equations (SDEs) are ubiquitous in a vast variety of fields, ranging from biology and physical sciences to finance and social sciences [15,10]. The probability density function associated with the stochastic process governed by an SDE is one of the qualities fully characterizing the statistical behavior of the solution to the SDE. Fokker–Planck equation provides the evolution of probability density functions and is an important tool to study how uncertainties propagate and evolve in physical and engineering dynamical systems [12,15,10,7].

Dynamical systems excited by Gaussian white noise are often modeled by SDEs driven by Brownian motions (or Wiener processes). For SDEs driven by Brownian motions, there are explicit formulas to obtain the associated Fokker–Planck equations, regardless the SDEs are in sense of Itô or Stratonovich [15,10]. However, it is *not* the case for SDEs with non-Gaussian noises.

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Dynamical systems excited by non-Gaussian white noise are usually modeled by SDEs driven by non-Gaussian Lévy processes. The connection between the Lévy driven SDE and the related partial integro-differential equations has been extensively studied since [21] and has found wide applications in many areas including finance [4]. This connection has been extended into nonlinear cases recently, see [8] among others. For SDEs driven by Lévy processes, there are two popular definitions, i.e., they are defined in sense of Itô or in sense of Marcus [13,14,11,1]. Marcus SDEs are often appropriate models in engineering and scientific practice, because they preserve certain physical quantities such as energy [13,14,11,19,20]. It is recently shown [19] that Marcus SDE is equivalent to the well known Di Paola–Falsone SDE [5,6] which is widely used in engineering and physics [9,22,20]. Comparison of Marcus integral and Stratonovich integral is recently discussed in [3] for systems with jump noise.

Solutions of both Itô and Marcus SDEs often are Markov processes [1]. It is well known that Fokker–Planck equations for Markov processes require the adjoint operators of the infinitesimal generators of the Markov processes [1]. Unlike the Gaussian cases, Fokker–Planck equations for SDEs driven by non-Gaussian Lévy processes remain *unknown* due to the difficulty in obtaining the explicit expressions for the adjoint of the infinitesimal generators associated with such SDEs [1]. While Fokker–Planck equations for Itô SDEs driven by some particular Lévy processes have been discussed by many authors ([18,17] among others), the research on Fokker–Planck equations for Marcus SDEs has received much less attention. A recent result about Fokker–Planck equations for Marcus SDEs is presented in [18], where an explicit form of Fokker–Planck equations is derived for Marcus SDEs under the condition that coefficients of the noise terms are strictly nonzero (i.e., the coefficient does not change sign). However, it is an *open* problem what the Fokker–Planck equations are like for Marcus stochastic differential equations under general conditions.

Lévy processes are stochastic processes with properties of independent and stationary increments, as well as stochastically continuous sample paths [1,16]. Examples of Lévy processes include Brownian motions, compound Poisson processes, α-stable processes and others. A one-dimensional Lévy process $L(t)$, taking values in $\mathbb{R}$, is characterized by a drift $b \in \mathbb{R}$, a positive real number A and a Borel measure ν defined on $\mathbb{R}$ and concentrated on $\mathbb{R}\backslash\{0\}$. In fact, this measure ν satisfies the following condition [1]

$$\int_{\mathbb{R}\backslash\{0\}} (y^2 \wedge 1)\nu(\mathrm{d}y) < \infty, \tag{1}$$

where $y^2 \wedge 1$ represents the minimum of y^2 and 1. This measure ν is called a Lévy jump measure for the Lévy process $L(t)$. A Lévy process with the generating triplet (b, A, ν) has the Lévy–Itô decomposition

$$\mathrm{d}L(t) = b\mathrm{d}t + \mathrm{d}B(t) + \int_{|y|<1} y\tilde{N}(\mathrm{d}t, \mathrm{d}y) + \int_{|y|\geq 1} yN(\mathrm{d}t, \mathrm{d}y), \tag{2}$$

where $N(\mathrm{d}t, \mathrm{d}x)$ is the Poisson random measure, $\tilde{N}(\mathrm{d}t, \mathrm{d}x) = N(\mathrm{d}t, \mathrm{d}x) - \nu(\mathrm{d}x)\mathrm{d}t$ is the compensated Poisson random measure, and $B(t)$ is the Brownian motion (i.e., Wiener process) with variance A.

Different kinds of Lévy processes can be obtained by taking different triplet (b, A, ν). Just as a Gaussian white noise can be regarded as the formal derivative of a Brownian motion process, a non-Gaussian white noise can be regarded as the formal derivative of some non-Gaussian Lévy process.

We shall consider stochastic dynamical systems described by the following SDE in sense of Marcus,

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + \sigma(X(t)) \diamond \mathrm{d}L(t), \tag{3}$$

where $X(t)$ is a scalar process, and $L(t)$ is the one-dimensional Lévy process with the generating triplet (b, A, ν). Note that we only consider one-dimensional case in this paper. The solution of equation (3) is interpreted as

$$X(t) = X(t) + \int_0^t f(X(s))\mathrm{d}s + \int_0^t \sigma(X(s-)) \diamond \mathrm{d}L(s), \tag{4}$$

where $X(s-) = \lim_{u<s, u\to s} X(u)$, and "$\diamond$" indicates Marcus integral [13,14,1] defined by

$$\int_0^t \sigma(X_{s-}) \diamond \mathrm{d}L(s) = \int_0^t \sigma(X_{s-})\mathrm{d}L(s) + \frac{A}{2}\int_0^t \sigma(X(s-))\sigma'(X(s-))\mathrm{d}s$$
$$+ \sum_{0\le s\le t} \left[\xi(\Delta L(s), X(s-)) - X(s-) - \sigma(X(s-))\Delta L(s)\right], \tag{5}$$

with $\xi(r,x)$ being the value at $z=1$ of the solution of the following ordinary differential equation (ODE):

$$\frac{\mathrm{d}}{\mathrm{d}z} y(z) = r\sigma(y(z)), \qquad y(0) = x. \tag{6}$$

The first term at the right hand side of (5) is the Itô integral, and the second term is the correction term due to the continuous component of $L(t)$, as also appears in Stratonovich integral, and the last term is the correction term due to jumps. Note that the correction term due to jumps is expressed as the sum of some recursively infinite series in Di Paola–Falsone SDEs [5,6], which have been extensively used in engineering and physics. Since it has been shown that Marcus SDEs and Di Paola–Falsone SDEs are essentially equivalent [19], the result for Marcus SDEs in this paper is also true for Di Paola–Falsone SDEs. For more discussion on the relationship between Marcus and Di Paola–Falsone SDEs, readers are referred to [19,20].

In this paper, we shall derive explicit forms of Fokker–Planck equations which govern the probability density functions for the solution of the SDE (3). The result presented here is applicable under much more general conditions than that in [18]. While the coefficient of the noise term is required to be strictly nonzero in [18], the result here allows the coefficient σ to have finite zeros.

Let $p(x,t\big|X(0)=x_0)$ represent the probability density function for the solution $X(t)$ of the SDE (3), and for convenience, we drop off the initial condition and simply use $p(x,t)$ instead of $p(x,t\big|X(0)=x_0)$. Throughout this paper, we assume the following.

Assumption *(H1).* The probability density function $p(x,t)$ for the solution $X(t)$ of (3) exists, and $p(x,t)$ is continuously differentiable with respect to t and twice continuously differentiable with respect to x.

We are not going to present conditions for existence and regularity of the solution and the probability density associated with the SDE (3), which is out of the scope of this paper. Note that the existence and regularity of probability density for solutions of SDEs driven by Lévy processes are active research topics itself.

This paper is organized as follows. In subsection 2.1, we derive Fokker–Planck equations for systems modeled by (3) with $\sigma(x)\neq 0$. The condition that $\sigma(x)\neq 0$ is relaxed in section subsection 2.2 by assuming σ has finite zeros. In section 3, we apply the theoretical result presented in section 2 to obtain the Fokker–Planck equations for some nonlinear dynamical systems under excitation of α-stable white noise, or combined Gaussian and Poisson white noise. Section 4 is the conclusion.

2. Derivation of Fokker–Planck equation

We derive Fokker–Planck equations for the SDE (3) in two cases: (i) σ has no zero, and (ii) σ has zeros.

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2.1. Cases where σ has no zero

In this subsection, we derive Fokker–Planck equations for the SDE (3) with a different approach from that in [18]. The advantage of the approach here lies in that it can be modified to be applicable in cases where σ has zeros.

Definition 1. The transform H associated with the coefficient σ in the SDE (3) is defined as

$$H(x) = \int_a^x \frac{\mathrm{d}t}{\sigma(t)}, \tag{7}$$

where a is any constant.

Lemma 1. *Assume that σ is Lipschitz and has no zero. Then the transform H has the following properties:*

(i) *H is well defined and monotone on $(-\infty, \infty)$;*
(ii) *H is bijective (i.e., one-to-one and onto) and maps from $(-\infty, \infty)$ to $(-\infty, \infty)$;*
(iii) *H has the inverse transform H^{-1}, and H^{-1} is bijective and maps from $(-\infty, \infty)$ to $(-\infty, \infty)$;*
(iv) *$\forall y \in \mathbb{R}$, $H^{-1}(H(\cdot) + y)$ is bijective and maps from $(-\infty, \infty)$ to $(-\infty, \infty)$.*

Proof of Lemma 1. Conclusion (i) follow from the fact that σ is Lipschitz continuous and has no zero.

To show (ii), since H is monotone and defined on $(-\infty, \infty)$, it is sufficient to show that H goes to infinity as x goes to infinity. Without loss of generality, let us assume $\sigma > 0$. It follows from (7) that for $x > a$,

$$\begin{aligned} H(x) &= \int_a^x \frac{\mathrm{d}t}{|\sigma(t)|} = \int_a^x \frac{\mathrm{d}t}{|\sigma(t) - \sigma(a) + \sigma(a)|} \geq \int_a^x \frac{\mathrm{d}t}{|\sigma(t) - \sigma(a)| + |\sigma(a)|} \\ &\geq \int_a^x \frac{\mathrm{d}t}{L|t-a| + |\sigma(a)|}, \end{aligned} \tag{8}$$

where L is the Lipschitz constant satisfying $|\sigma(t) - \sigma(a)| \leq L|t - a|$. It follows from (8) that $H(x) \to +\infty$ as $x \to +\infty$. Similarly, $H(x) \to -\infty$ as $x \to -\infty$ for $\sigma > 0$.

(ii) implies (iii) and (iv). $\square$

Now for the case σ is nonzero, we present the Fokker–Planck equation for the SDE (3) in the following theorem.

Theorem 1. *Suppose the assumption H1 holds, f is differentiable, σ is Lipschitz continuous and twice differentiable, and $\sigma(x) \neq 0$ for all $x \in \mathbb{R}$, then the probability density function $p(x, t)$ for the solution $X(t)$ of the SDE (3) satisfies the following Fokker–Planck equation*

$$\begin{aligned} \frac{\partial p(x,t)}{\partial t} &= -\frac{\partial}{\partial x}\left[\left(f(x) + b\sigma(x) + \frac{A}{2}\sigma(x)\sigma'(x)\right)p(x,t)\right] + \frac{A}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x)p(x,t)\right) \\ &+ \int_{\mathbf{R}\setminus\{0\}} \left[\frac{\sigma(H^{-1}(H(x)-y))}{\sigma(x)} p(H^{-1}(H(x)-y),t)\right. \\ &\qquad \left. - p(x,t) + y\,\mathbf{I}_{(-1,\,1)}(y)\frac{\partial}{\partial x}(\sigma(x)p(x,t))\right]\nu(\mathrm{d}y), \end{aligned} \tag{9}$$

where H is the transform defined as in (7), and H^{-1} is the inverse of H.

Remark 1. The Fokker–Planck equation (9) is the same as the one presented in [18]. However, it is derived here in a different approach from that in [18]. The advantage of the approach here lies in that it can be modified to be applicable in cases where the coefficient σ of the multiplicative noise in the SDE (3) has zeros.

To prove Theorem 1, we need the following Lemma, which can be found in many books on theory of distribution (e.g. [2]).

Lemma 2. *Assume that $\gamma_1 \in C(\mathbb{R})$ and $\gamma_2 \in C(\mathbb{R})$. If $\int_{\mathbb{R}} \phi(x)\gamma_1(x)\,\mathrm{d}x = \int_{\mathbb{R}} \phi(x)\gamma_2(x)\,\mathrm{d}x$ for all $\phi \in C_0^\infty(\mathbb{R})$, then $\gamma_1(x) = \gamma_2(x)$ for all $x \in \mathbb{R}$.*

Proof of Theorem 1. It follows from (2), (3) and (5) that [1]

$$
\begin{aligned}
\mathrm{d}X(t) = {} & f(X(t))\mathrm{d}t + b\sigma(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}B(t) + \frac{A}{2}\sigma(X(t))\sigma'(X(t))\mathrm{d}t \\
& + \int\limits_{|y|<1} [\xi(y, X(t-)) - X(t-)]\,\tilde{N}(\mathrm{d}t, \mathrm{d}y) \\
& + \int\limits_{|y|\geq 1} [\xi(y, X(t-)) - X(t-)]\,N(\mathrm{d}t, \mathrm{d}y) \\
& + \int\limits_{|y|<1} [\xi(y, X(t-)) - X(t-) - \sigma(X(t-)\,y\,]\,\nu(\mathrm{d}y)\mathrm{d}t.
\end{aligned} \tag{10}
$$

It follows from the definition for ξ from the ODE (6) and the definition of the transform H in (7) that

$$
H(\xi(\Delta L(t), X(t-))) - H(X(t-)) = \Delta L(t). \tag{11}
$$

It follows from Lemma 1 that

$$
\xi(\Delta L(t), X(t-)) = H^{-1}(H(X(t-)) + \Delta L(t)). \tag{12}
$$

Substituting (12) into (10), we get

$$
\begin{aligned}
dX(t) = {} & f(X(t))\mathrm{d}t + b\sigma(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}B(t) + \frac{A}{2}\sigma(X(t))\sigma'(X(t))\mathrm{d}t \\
& + \int\limits_{|y|<1} [H^{-1}(H(X(t-)) + y) - X(t-)]\,\tilde{N}(\mathrm{d}t, \mathrm{d}y) \\
& + \int\limits_{|y|\geq 1} [H^{-1}(H(X(t-)) + y) - X(t-)]\,N(\mathrm{d}t, \mathrm{d}y) \\
& + \int\limits_{|y|<1} [H^{-1}(H(X(t-)) + y) - X(t-) - \sigma(X(t-)\,y\,]\,\nu(\mathrm{d}y)\mathrm{d}t.
\end{aligned} \tag{13}
$$

By Itô's formula [1], for $\phi(x) \in C_0^\infty(\mathbb{R})$, it follows from (13) that

$$\begin{aligned}
\phi(X(t+\Delta t)) - \phi(X(t)) &= \int_t^{t+\Delta t} \phi'(X(s-))f(X(s-))\,\mathrm{d}s + \int_t^{t+\Delta t} b\phi'(X(s-))\sigma(X(s))\,\mathrm{d}s \\
&+ \int_t^{t+\Delta t} \phi'(X(s-))\sigma(X(s))\,\mathrm{d}B(s) + \frac{A}{2}\int_t^{t+\Delta t} \phi''(X(s-))\sigma^2(X(s))\,\mathrm{d}s \\
&+ \frac{A}{2}\int_t^{t+\Delta t} \phi'(X(s-))\sigma(X(s-))\sigma'(X(s-))\,\mathrm{d}s \\
&+ \int_t^{t+\Delta t}\int_{|y|\geq 1} \left[\phi\left(H^{-1}(H(X(s-))+y)\right) - \phi(X(s-))\right] N(\mathrm{d}s,\mathrm{d}y) \\
&+ \int_t^{t+\Delta t}\int_{|y|<1} \left[\phi\left(H^{-1}(H(X(s-))+y)\right) - \phi(X(s-))\right] \widetilde{N}(\mathrm{d}s,\mathrm{d}y) \\
&+ \int_t^{t+\Delta t}\int_{|y|<1} \left[\phi\left(H^{-1}(H(X(s-))+y)\right) - \phi(X(s-)) - \phi'(X(s-))\,\sigma(X(s-))\,y\right] \nu(\mathrm{d}y)\mathrm{d}s\,. \qquad (14)
\end{aligned}$$

Taking expectation at both sides of (14), we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \phi(x)p(x,t+\Delta t)\mathrm{d}x - \int_{-\infty}^{\infty} \phi(x)p(x,t)\mathrm{d}x \\
&= \int_t^{t+\Delta t}\int_{-\infty}^{\infty} \phi'(x)f(x)p(x,s)\,\mathrm{d}x\,\mathrm{d}s + b\int_t^{t+\Delta t}\int_{-\infty}^{\infty} \phi'(x)\sigma(x)p(x,s)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{A}{2}\int_t^{t+\Delta t}\int_{-\infty}^{\infty} \phi'(x)\sigma(x)\sigma'(x)p(x,s)\,\mathrm{d}x\,\mathrm{d}s + \frac{A}{2}\int_t^{t+\Delta t}\int_{-\infty}^{\infty} \phi''(x)\sigma^2(x)p(x,s)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_t^{t+\Delta t}\int_{-\infty}^{\infty}\int_{|y|\geq 1} \left[\phi\left(H^{-1}(H(x)+y)\right) - \phi(x)\right] p(x,s)\,\nu(\mathrm{d}y)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_t^{t+\Delta t}\int_{-\infty}^{\infty}\int_{|y|<1} \left[\phi\left(H^{-1}(H(x)+y)\right) - \phi(x) - \phi'(x)\sigma(x)\,y\right] p(x,s)\,\nu(\mathrm{d}y)\,\mathrm{d}x\,\mathrm{d}s\,. \qquad (15)
\end{aligned}$$

To derive the above equation, we have used the following facts [1]:

$$\mathbf{E}\left\{\int_t^{t+\Delta t} \phi'(X(s-)\sigma(X(s))\,\mathrm{d}B(s)\right\} = 0, \qquad (16)$$

$$\mathbf{E}\left\{\int_t^{t+\Delta t}\int_{|y|\geq 1} \left[\phi\left(H^{-1}(H(X(s-))+y)\right) - \phi(X(s-))\right] \tilde{N}(\mathrm{d}s,\mathrm{d}y)\right\} = 0, \qquad (17)$$

$$\mathbf{E}\left\{\int_t^{t+\Delta t}\int_{|y|<1}\left[\phi\left(H^{-1}(H(X(s-))+y)\right)-\phi(X(s-))\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}y)\right\}=0, \tag{18}$$

and

$$\begin{aligned}&\mathbf{E}\left\{\int_t^{t+\Delta t}\int_{|y|\geq 1}\left[\phi\left(H^{-1}(H(X(s-))+y)\right)-\phi(X(s-))\right]N(\mathrm{d}s,\mathrm{d}y)\right\}\\&=\mathbf{E}\left\{\int_t^{t+\Delta t}\int_{|y|\geq 1}\left[\phi\left(H^{-1}(H(X(s-))+y)\right)-\phi(X(s-))\right]\nu(\mathrm{d}y)\,\mathrm{d}s\right\}\\&=\int_t^{t+\Delta t}\int_{-\infty}^{\infty}\int_{|y|\geq 1}\left[\phi\left(H^{-1}(H(x)+y)\right)-\phi(x)\right]p(x,s)\,\nu(\mathrm{d}y)\,\mathrm{d}x\,\mathrm{d}s\,.\end{aligned} \tag{19}$$

Note that the first identity in (19) follows from (17).

Equation (15) can be rewritten as

$$\begin{aligned}&\int_t^{t+\Delta t}\int_{-\infty}^{\infty}\phi(x)\frac{\mathrm{d}p(x,s)}{\mathrm{d}s}\,\mathrm{d}x\,\mathrm{d}s\\&=\int_t^{t+\Delta t}\int_{-\infty}^{\infty}\phi'(x)\left[f(x)p(x,s)+b\sigma(x)p(x,s)+\frac{A}{2}\sigma(x)\sigma'(x)p(x,s)\right]\mathrm{d}x\,\mathrm{d}s\\&+\frac{A}{2}\int_t^{t+\Delta t}\int_{-\infty}^{\infty}\phi''(x)\sigma^2(x)p(x,s)\,\mathrm{d}x\,\mathrm{d}s\\&+\int_t^{t+\Delta t}\int_{-\infty}^{\infty}\int_{\mathbf{R}\backslash\{0\}}\left\{\phi\left(H^{-1}(H(x)+y)\right)-\phi(x)-\phi'(x)\,\sigma(x)\,y\,\mathbf{I}_{(-1,\,1)}(y)\right\}p(x,s)\,\nu(\mathrm{d}y)\,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{20}$$

where $\mathbf{I}_{(-1,\,1)}(y)$ is the indicator function.

Since (20) is true for any t and Δt, it follows that

$$\begin{aligned}&\int_{-\infty}^{\infty}\phi(x)\frac{\partial p(x,t)}{\partial t}\mathrm{d}x\\&=\int_{-\infty}^{\infty}\phi'(x)\left[f(x)p(x,t)+b\sigma(x)p(x,t)+\frac{A}{2}\sigma(x)\sigma'(x)p(x,t)\right]\mathrm{d}x\\&+\frac{A}{2}\int_{-\infty}^{\infty}\phi''(x)\sigma^2(x)p(x,t)\,\mathrm{d}x\\&+\int_{-\infty}^{\infty}\mathrm{d}x\int_{\mathbf{R}\backslash\{0\}}\left\{\phi\left(H^{-1}(H(x)+y)\right)-\phi(x)-\phi'(x)\sigma(x)\,y\,\mathbf{I}_{(-1,\,1)}(y)\right\}p(x,t)\,\nu(\mathrm{d}y).\end{aligned} \tag{21}$$

By integration by parts, the first two integrals in the right hand side of (21) become

$$\begin{aligned}&\int_{-\infty}^{\infty} \phi'(x)\left(f(x)+b\sigma(x)+\frac{A}{2}\sigma(x)\sigma'(x)\right)p(x,t)\mathrm{d}x\\&=-\int_{-\infty}^{\infty}\phi(x)\frac{\partial}{\partial x}\left[\left(f(x)+b\sigma(x)+\frac{A}{2}\sigma(x)\sigma'(x)\right)p(x,t)\right]\mathrm{d}x,\end{aligned}\tag{22}$$

and

$$\int_{-\infty}^{\infty}\phi''(x)\sigma^2(x)p(x,t)\,\mathrm{d}x=\int_{-\infty}^{\infty}\phi(x)\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)p(x,t)\right]\,\mathrm{d}x\,,\tag{23}$$

respectively.

By interchanging order of integrals, the last term of (21) becomes

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}x\int_{\mathbf{R}\backslash\{0\}}\left[\phi\left(H^{-1}(H(x)+y)\right)-\phi(x)-\phi'(x)\sigma(x)\,y\,\mathbf{I}_{(-1,\ 1)}(y)\right]p(x,t)\,\nu(\mathrm{d}y)\\&=\int_{\mathbf{R}\backslash\{0\}}\nu(\mathrm{d}y)\int_{-\infty}^{\infty}\left[\phi\left(H^{-1}(H(x)+y)\right)-\phi(x)-\phi'(x)\,\sigma(x)\,y\,\mathbf{I}_{(-1,\ 1)}(y)\right]p(x,t)\,\mathrm{d}x\end{aligned}\tag{24}$$

The interchanging order of integrals above is justified by

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{\mathbf{R}\backslash\{0\}}\left|\left[\phi\left(H^{-1}(H(x)+y)\right)-\phi(x)-\phi'(x)\,\sigma(x)\,y\,\mathbf{I}_{(-1,\ 1)}(y)\right]p(x,t)\right|\nu(\mathrm{d}y)\,\mathrm{d}x\\&\leq\int_{-\infty}^{\infty}\int_{|y|<1}\left|\left[\phi\left(H^{-1}(H(x)+y)\right)-\phi(x)-\phi'(x)\,\sigma(x)\,y\,\right]p(x,t)\right|\nu(\mathrm{d}y)\,\mathrm{d}x\\&\quad+\int_{-\infty}^{\infty}\int_{|y|\geq1}\left|\left[\phi\left(H^{-1}(H(x)+y)\right)-\phi(x)\right]p(x,t)\right|\nu(\mathrm{d}y)\,\mathrm{d}x\\&<+\infty.\end{aligned}\tag{25}$$

To prove the last inequality in (25), we have used (1) and the fact that $\phi(x)\in C_0^\infty(\mathbb{R})$.

Next, let us examine the integral inside the last term of (24), which can be written as

$$\begin{aligned}&\int_{-\infty}^{\infty}\left[\phi\left(H^{-1}(H(x)+y)\right)-\phi(x)-\phi'(x)\sigma(x)\,y\,\mathbf{I}_{(-1,\ 1)}(y)\right]p(x,t)\,\mathrm{d}x\\&=\int_{-\infty}^{\infty}\phi\left(H^{-1}(H(x)+y)\right)p(x,t)\,\mathrm{d}x-\int_{-\infty}^{\infty}\phi(x)p(x,t)\,\mathrm{d}x\\&\quad-\int_{-\infty}^{\infty}\phi'(x)\,\sigma(x)\,y\,\mathbf{I}_{(-1,\ 1)}(y)p(x,t)\,\mathrm{d}x.\end{aligned}\tag{26}$$

Denote

$$z = H^{-1}(H(x) + y), \tag{27}$$

it follows from (27) and (7) that

$$x = H^{-1}(H(z) - y), \quad \frac{\mathrm{d}x}{\mathrm{d}z} = \frac{\sigma(H^{-1}(H(z) - y)}{\sigma(z)}. \tag{28}$$

For the first integral in the right hand side of (26), by the change of variable and using (28) and (iii) in Lemma 1, we can get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \phi\left(H^{-1}(H(x) + y)\right) p(x,t)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \phi(z) \frac{\sigma(H^{-1}(H(z) - y))}{\sigma(z)} p(H^{-1}(H(z) - y), t)\,\mathrm{d}z \\
&= \int_{-\infty}^{\infty} \phi(x) \frac{\sigma(H^{-1}(H(x) - y))}{\sigma(x)} p(H^{-1}(H(x) - y), t)\,\mathrm{d}x.
\end{aligned} \tag{29}$$

For the last integral at the right hand side of (26), we have

$$\int_{-\infty}^{\infty} \phi'(x)\,\sigma(x)\,y\mathbf{I}_{(-1,\,1)}(y)p(x,t)\,\mathrm{d}x = -\int_{-\infty}^{\infty} \phi(x)\,y\,\mathbf{I}_{(-1,\,1)}(y)\frac{\partial}{\partial x}(\sigma(x)\,p(x,t))\,\mathrm{d}x. \tag{30}$$

Substituting (29) and (30) into (26), we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left[\phi\left(H^{-1}(H(x) + y)\right) - \phi(x) - \phi'(x)\,\sigma(x)\,y\,\mathbf{I}_{(-1,\,1)}(y)\right] p(x,t)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \phi(x) \left[\frac{\sigma(H^{-1}(H(x) - y))}{\sigma(x)} p(H^{-1}(H(x) - y), t)\right. \\
&\qquad \left. - p(x,t) + y\,\mathbf{I}_{(-1,\,1)}(y)\frac{\partial}{\partial x}\left(\sigma(x)p(x,t)\right)\right] \mathrm{d}x.
\end{aligned} \tag{31}$$

Substituting (31) into (24), we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}x \int_{\mathbf{R}\setminus\{0\}} \left[\phi\left(H^{-1}(H(x) + y)\right) - \phi(x) - \phi'(x)\sigma(x)\,y\,\mathbf{I}_{(-1,\,1)}(y)\right] p(x,t)\,\nu(\mathrm{d}y) \\
&= \int_{\mathbf{R}\setminus\{0\}} \nu(\mathrm{d}y) \int_{-\infty}^{\infty} \phi(x) \left[\frac{\sigma(H^{-1}(H(x) - y))}{\sigma(x)} p(H^{-1}(H(x) - y), t)\right. \\
&\qquad \left. - p(x,t) + y\,\mathbf{I}_{(-1,\,1)}(y)\frac{\partial}{\partial x}(\sigma(x)p(x,t))\right] \mathrm{d}x.
\end{aligned} \tag{32}$$

By interchanging order of integrals, (32) becomes

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_{\mathbf{R}\backslash\{0\}} \left[\phi\left(H^{-1}(H(x)+y)\right) - \phi(x) - \phi'(x)\,\sigma(x)\,y\,\mathbf{I}_{(-1,\,1)}(y)\right] p(x,t)\,\nu(\mathrm{d}y)$$
$$= \int_{-\infty}^{\infty} \mathrm{d}x \int_{\mathbf{R}\backslash\{0\}} \phi(x) \left[\frac{\sigma(H^{-1}(H(x)-y))}{\sigma(x)} p(H^{-1}(H(x)-y),t) \right.$$
$$\left. - p(x,t) + y\,\mathbf{I}_{(-1,\,1)}(y) \frac{\partial}{\partial x}(\sigma(x)p(x,t))\right] \nu(\mathrm{d}y). \tag{33}$$

Substituting (22), (23), and (33) into (21), we get

$$\int_{-\infty}^{\infty} \phi(x) \frac{\partial p(x,t)}{\partial t} \mathrm{d}x$$
$$= - \int_{-\infty}^{\infty} \phi(x) \frac{\partial}{\partial x} \left[\left(f(x) + b\sigma(x) + \frac{A}{2}\sigma(x)\sigma'(x)\right) p(x,t)\right] \mathrm{d}x$$
$$+ \frac{A}{2} \int_{-\infty}^{\infty} \phi(x) \frac{\partial^2}{\partial x^2} \left[\sigma^2(x)p(x,t)\right] \mathrm{d}x$$
$$+ \int_{-\infty}^{\infty} \mathrm{d}x \int_{\mathbf{R}\backslash\{0\}} \phi(x) \left[\frac{\sigma(H^{-1}(H(x)-y))}{\sigma(x)} p(H^{-1}(H(x)-y),t) \right.$$
$$\left. - p(x,t) + y\,\mathbf{I}_{(-1,\,1)}(y) \frac{\partial}{\partial x}(\sigma(x)p(x,t))\right] \nu(\mathrm{d}y) \tag{34}$$

Since the above equation is true for any $\phi(x) \in C_0^\infty(\mathbb{R})$, it follows from Assumption H1 and Lemma 2 that the probability density function p satisfies (9). □

2.2. *Cases where σ has zeros*

When σ has zeros, the transform H given in (7) is not well defined. Suppose σ has n zeros represented by x_i $(i = 1, 2, \cdots, n)$. We denote $x_0 = -\infty$ and $x_{n+1} = +\infty$ for convenience. Without loss of generality, we suppose

$$-\infty = x_0 < x_1 < \cdots < x_n < x_{n+1} = +\infty. \tag{35}$$

Definition 2. The transforms $H_i (i = 0, 1, 2, \cdots, n)$ are defined as

$$H_i(x) = \int_{a_i}^{x} \frac{\mathrm{d}y}{\sigma(y)} \qquad \text{for} \quad x \in (x_i, x_{i+1}), \tag{36}$$

where a_i is any constant in (x_i, x_{i+1}).

The transforms $\{H_i\}$ defined in (36) have the following properties.

Lemma 3. *Assume that σ is Lipschitz and has n zeros. Then each transform $H_i (i = 1, 2, \cdots, n)$ defined by (36) has the following properties:*

(i) *each H_i is well defined and monotone on (x_i, x_{i+1});*
(ii) *each H_i is bijective and maps from (x_i, x_{i+1}) to $(-\infty, \infty)$;*
(iii) *each H_i has the inverse H_i^{-1}, and H_i^{-1} is bijective and maps from $(-\infty, \infty)$ to (x_i, x_{i+1});*
(iv) *$\forall y \in \mathbb{R}$, each $H_i^{-1}(H_i(\cdot) + y)$ is bijective from (x_i, x_{i+1}) to (x_i, x_{i+1}), and has the following property*

$$\begin{cases} \lim_{x \to x_i+} H_i^{-1}(H_i(x) + y) = x_i, \\ \lim_{x \to x_{i+1}-} H_i^{-1}(H_i(x) + y) = x_{i+1}, \end{cases} \tag{37}$$

where $\lim_{x \to x_i+}$ represents the right limit at $x = x_i$, and $\lim_{x \to x_{i+1}-}$ the left limit at $x = x_{i+1}$.

Proof of Lemma 3. (i) follows from the fact that $\sigma(x) \neq 0$, $\forall x \in (x_i, x_{i+1})$.

To show (ii), since H_i is monotone and defined on (x_i, x_{i+1}), it is sufficient to show that H_i goes to infinity as x approaches x_i or x_{i+1}. Without loss of generality, we suppose $\sigma(x) > 0$ for $x \in (x_i, x_{i+1})$.

For H_0 and H_n, we have

$$H_0(x) = \int_{a_0}^{x} \frac{\mathrm{d}t}{|\sigma(t)|} = \int_{a_0}^{x} \frac{\mathrm{d}t}{|\sigma(t) - \sigma(x_1)|} \geq \int_{a_0}^{x} \frac{\mathrm{d}t}{L|t - x_1|}, \qquad \forall x \in (x_0, x_1), \tag{38}$$

and

$$H_n(x) = \int_{a_n}^{x} \frac{\mathrm{d}t}{|\sigma(t)|} = \int_{a_n}^{x} \frac{\mathrm{d}t}{|\sigma(t) - \sigma(x_n)|} \geq \int_{a_n}^{x} \frac{\mathrm{d}t}{L|t - x_n|}, \qquad \forall x \in (x_n, x_{n+1}), \tag{39}$$

respectively.

For H_i $(i = 1, 2, \cdots, n-1)$, we have

$$H_i(x) = \int_{a_i}^{x} \frac{\mathrm{d}t}{|\sigma(t)|} = \int_{a_i}^{x} \frac{\mathrm{d}t}{|\sigma(t) - \sigma(x_i)|} \geq \int_{a_i}^{x} \frac{\mathrm{d}t}{L|t - x_i|}, \qquad \forall x \in (x_i, x_{i+1}), \tag{40}$$

and

$$H_i(x) = \int_{a_i}^{x} \frac{\mathrm{d}t}{|\sigma(t)|} = \int_{a_i}^{x} \frac{\mathrm{d}t}{|\sigma(t) - \sigma(x_{i+1})|} \geq \int_{a_i}^{x} \frac{\mathrm{d}t}{L|t - x_{i+1}|}, \qquad \forall x \in (x_i, x_{i+1}). \tag{41}$$

It follows from (38) to (41) that $H_i(x)(i = 0, 1, 2, \cdots, n)$ goes to infinity as x approaches x_i or x_{i+1}. Now, the proof of (ii) is finished.

(iii) and (iv) are implied by (ii). □

Definition 3. The transform $\tilde{H} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ associated with SDE (3) is defined by

$$\tilde{H}(x, y) = \begin{cases} H_i^{-1}\big(H_i(x) + y\big), & \text{for } x \in (x_i, x_{i+1}) \text{ and } y \in \mathbb{R}, \\ x, & \text{for } x = x_1, x_2, \cdots, x_n \text{ and } y \in \mathbb{R}. \end{cases} \tag{42}$$

The transform $\tilde{H}$ has the following properties.

Lemma 4. *Suppose σ is a real analytic function (i.e., it possesses derivatives of all orders and its function value agrees with its Taylor series in a neighborhood of every point) and has n zeros denoted as in (35), then for any given $y \in \mathbb{R}$, $\tilde{H}(x, y)$ is continuously differentiable with respect to x for all $x \in \mathbb{R}$. Moreover,*

$$\frac{\partial \tilde{H}(x,y)}{\partial x} = \begin{cases} \frac{\sigma(\tilde{H}(x,y))}{\sigma(x)} & \text{for } x \in \bigcup_{i=0}^{n}(x_i, x_{i+1}), \\ \sum_{k=0}^{\infty} \frac{1}{k!}\Phi_k(x)y^k & \text{for } x = x_1, x_2, \cdots, x_n, \end{cases} \tag{43}$$

where $\Phi_k(x_i)$ is defined as

$$\begin{cases} \Phi_0(x_i) = 1, \\ \Phi_k(x_i) = \lim_{x\to x_i} \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma(x)\cdots\left(\frac{\mathrm{d}}{\mathrm{d}x}\sigma(x)\right)\right)\right)}_{k-fold} & (k = 1, 2, \cdots). \end{cases} \tag{44}$$

Proof of Lemma 4. For $x \neq x_i (i = 1, 2, \cdots, n)$, since σ is analytic, it is straightforward to check that $\tilde{H}(x,y)$ is continuously differentiable with respect to x, and by direct computation we can get

$$\frac{\partial \tilde{H}(x,y)}{\partial x} = \frac{\sigma(\tilde{H}(x,y))}{\sigma(x)} \qquad \text{for } x \in \bigcup_{i=0}^{n}(x_i, x_{i+1}). \tag{45}$$

In the following, we show that $\tilde{H}(x,y)$ is continuously differentiable with respect to x for $x = x_1, x_2, \cdots, x_n$.

First, we see from (37) that $\tilde{H}(x,y)$ is continuous at $x = x_1, x_2, \cdots, x_n$. Since σ is analytic and $\frac{\partial \tilde{H}(x,y)}{\partial y} = \sigma(\tilde{H}(x,y))$, by Taylor expansion with respect to y at $y = 0$ and $x \neq x_i$ $(i = 1, 2, \cdots, n)$, we have

$$\tilde{H}(x,y) = x + \sigma(x)y + \frac{1}{2!}\sigma(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\sigma(x)\right)y^2 + \frac{1}{3!}\sigma(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma(x)\frac{\mathrm{d}}{\mathrm{d}x}\sigma(x)\right)\right)y^3 + \cdots \tag{46}$$

and

$$\sigma(\tilde{H}(x,y)) = \sigma(x) + \sigma(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\sigma(x)\right)y + \frac{1}{2!}\sigma(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma(x)\frac{\mathrm{d}}{\mathrm{d}x}\sigma(x)\right)\right)y^2 + \cdots . \tag{47}$$

By using (46), we get

$$\begin{aligned} \left.\frac{\partial \tilde{H}(x,y)}{\partial x}\right|_{x=x_i} &= \lim_{x\to x_i}\frac{\tilde{H}(x,y) - \tilde{H}(x_i,y)}{x - x_i} \\ &= \lim_{x\to x_i}\frac{\tilde{H}(x,y) - x_i}{x - x_i} = \sum_{k=0}^{\infty}\frac{1}{k!}\Phi_k(x_i)y^k, \end{aligned} \tag{48}$$

where $\Phi_k(x_i)$ is defined in (44), and the convergence of the infinite series can be checked straightforwardly by using the fact that σ is Lipschitz continuous. (48) indicates that $\tilde{H}(x,y)$ is differentiable at $x = x_i$ $(i = 1, 2, \cdots, n)$.

It follows from (47) that

$$\lim_{x\to x_i}\frac{\sigma(\tilde{H}(x,y))}{\sigma(x)} = \sum_{k=0}^{\infty}\frac{1}{k!}\Phi_k(x_i)y^k. \tag{49}$$

It follows from (45), (48) and (49) that $\tilde{H}(x,y)$ is continuously differentiable with respect to x. □

Theorem 2. *Suppose the assumption H1 holds, f is differentiable, σ is Lipschitz continuous, analytic, and has n zeros $\{x_i\}_{i=1,2,\cdots,n}$ as defined in (35), then the probability density function $p(x,t)$ for the solution $X(t)$ of the SDE (3) satisfies the following equation*

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\left(f(x)+b\sigma(x)+\frac{A}{2}\sigma(x)\sigma'(x)\right)p(x,t)\right] + \frac{A}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)p(x,t)\right]$$
$$+ \int\limits_{\mathbf{R}\backslash\{0\}} \left[\frac{\partial \tilde{H}(x,-y)}{\partial x} p(\tilde{H}(x,-y),t) \right.$$
$$\left. - p(x,t) + y\,\mathbf{I}_{(-1,\,1)}(y)\frac{\partial}{\partial x}(\sigma(x)p(x,t))\right]\nu(\mathrm{d}y), \tag{50}$$

where $\tilde{H}(x,y)$ is defined in (42) and (36), and $\frac{\partial \tilde{H}(x,y)}{\partial x}$ is given in (43).

Proof of Theorem 2. The proof follows the same steps as the proof of Theorem 1 presented in subsection 2.1. Here, we only state the difference.

When σ has n zeros as defined in (35), equation (12) in subsection 2.1 now changes to

$$\begin{cases} \xi(\Delta L(t), X(t-)) = H_i^{-1}\big(H_i(X(t-)) + \Delta L(t)\big), & \text{for } X(t-) \in (x_i, x_{i+1}), \\ \xi(\Delta L(t), X(t-)) = X(t-). & \text{for } X(t-) = x_1, x_2, \cdots, x_n. \end{cases} \tag{51}$$

With the help of $\tilde{H}$ defined in (42), (51) can be written as

$$\xi(\Delta L(t), X(t-)) = \tilde{H}(X(t-), \Delta L(t)). \tag{52}$$

Comparing $H^{-1}(H(x)+y)$ used in the case σ being nonzero with $\tilde{H}(x,y)$, one can see that they are both continuously differential with respect to x, and the role of $H^{-1}(H(x)+y)$ in the proof of Theorem 1 can now be replaced completely by $\tilde{H}(x,y)$. Replacing $H^{-1}(H(x)+y)$ in the proof of Theorem 1 with $\tilde{H}(x,y)$, we arrive at the equation (50). □

3. Examples

In this section, we present a couple of simple examples to illustrate the results we have obtained in Theorem 2.

3.1. Example 1

Let $\sigma(x) = x$, and $L(t)$ be the α-stable Lévy motion with the triplet $b = 1$, $A = 0$ and $\nu(\mathrm{d}x) = \frac{\mathrm{d}x}{|x|^{1+\alpha}}$. For more details about α-stable Lévy motion, see [16] among others. Then the SDE (3) becomes

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + X(t) \diamond \mathrm{d}L(t). \tag{53}$$

According to (36),

$$H_0(x) = \ln\frac{x}{a_0} \qquad \text{for } x \in (-\infty, 0), \tag{54}$$

and

$$H_1(x) = \ln\frac{x}{a_1} \qquad \text{for } x \in (0, +\infty), \tag{55}$$

where a_0 and a_1 are any given constants satisfying $a_0 \in (-\infty, 0)$ and $a_1 \in (0, +\infty)$. It follows from (54) and (55) that

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$$H_0^{-1}(x) = a_0 e^x \qquad \text{for } x \in (-\infty, 0), \tag{56}$$

and

$$H_1^{-1}(x) = a_1 e^x \qquad \text{for } x \in (0, +\infty). \tag{57}$$

Therefore,

$$H_0^{-1}(H_0(x) + y) = xe^y \qquad \text{for } x \in (-\infty, 0), \tag{58}$$

and

$$H_1^{-1}(H_1(x) + y) = xe^y \qquad \text{for } x \in (0, +\infty). \tag{59}$$

With (58) and (59), $\tilde{H}$ defined by (42) becomes

$$\tilde{H}(x, y) = \begin{cases} xe^y & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{60}$$

Namely,

$$\tilde{H}(x, y) = xe^y. \tag{61}$$

Therefore, according to (50) in Theorem 2, Fokker–Planck equation for (53) can be expressed as

$$\begin{aligned} \frac{\partial p(x,t)}{\partial t} &= -\frac{\partial}{\partial x}\left[f(x)p(x,t) + xp(x,t)\right] \\ &\quad + \int\limits_{\mathbf{R}\backslash\{0\}} \left[e^{-y}p(xe^{-y}, t) - p(x,t) + y\,\mathbf{I}_{(-1,\,1)}(y)\frac{\partial}{\partial x}(xp(x,t))\right]\frac{\mathrm{d}y}{|y|^{1+\alpha}}. \end{aligned} \tag{62}$$

3.2. Example 2

All the parameters in this example are the same as those in Example 1 in the previous subsection except that a combined Gaussian and white noise is used in (53) instead of α-stable white noise. A combined Gaussian and Poisson white noise is corresponding to a Lévy process which consists of two components: (i) Brownian motion; (ii) compound Poisson process, and can be expressed as

$$L(t) = B(t) + \sum_{i=1}^{N(t)} r_i, \tag{63}$$

where $B(t)$ is a standard scalar Brownian motion with variance matrix $\tilde{A}$, $N(t)$ $(t > 0)$ is a Poisson process with intensity parameter λ, r_i $(i = 1, 2, \cdots)$ are i.i.d random numbers, with probability density function $\mu(x)$, which are also independent of $N(t)$. The Lévy process expressed in (63) has a triplet as $b = \lambda \int_{|y|<1} y\mu(\mathrm{d}y)$, $A = \tilde{A}$, and $\nu(\mathrm{d}x) = \lambda\mu(\mathrm{d}y)$ [1].

Same as Example 1, we get the following Fokker–Planck equation for SDE (53)

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\left(f(x) + \left(\lambda \int\limits_{|y|<1} y\mu(\mathrm{d}y)\right)x + \frac{\tilde{A}}{2}x\right)p(x,t)\right] + \frac{\tilde{A}}{2}\frac{\partial^2}{\partial x^2}\left(x^2 p(x,t)\right)$$

$$+\lambda \int\limits_{\mathbf{R}\backslash\{0\}} \left[e^{-y} p(xe^{-y}, t) - p(x,t) + y\, \mathbf{I}_{(-1,\,1)}(y) \frac{\partial}{\partial x}(xp(x,t)) \right] \mu(\mathrm{d}y). \tag{64}$$

4. Conclusion

In this work, we have derived the Fokker–Planck equations for Marcus SDEs with multiplicative non-Gaussian white noises. The main results are summarized in Theorem 1 in subsection 2.1 and Theorem 2 in subsection 2.2. The Fokker–Planck equations are essentially non-local partial differential equations and may involve singular integrals depending on the specific Lévy processes.

It is a challenging but important task to develop efficient numerical methods for these Fokker–Planck equations, which is left for our future research.

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