J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Well-posedness of Korteweg–de Vries–Benjamin Bona Mahony equation on a finite domain ☆

Jie Li *, Kangsheng Liu

Department of Mathematics, Zhejiang University, Hangzhou, China

A R T I C L E   I N F O

Article history:
Received 10 April 2016
Available online xxxx
Submitted by J. Lenells

Keywords:
Korteweg–de Vries–Benjamin Bona Mahony equation
Finite domain
Well-posedness
Nonhomogeneous boundary
Semigroup
Nonlinear interpolation

A B S T R A C T

In this paper, we consider the Korteweg–de Vries–Benjamin Bona Mahony equation on a finite domain with initial value and nonhomogeneous boundary conditions. This particular problem arises from the phenomenon of long wave with small amplitude in fluid. We get the global well-posedness of this system.

1. Introduction

The Korteweg–de Vries (KdV) equation and the Benjamin Bona Mahony equation (BBM) are two typical examples associated with the effects of dissipation, dispersion, nonlinearity and also provide a description of the propagation of waves with small amplitude in water or soliton in other liquid medium. The KdV equation is described as follows:

$$u_t + u_{xxx} + u_x + uu_x = 0.$$

The BBM equation is an alternative to the KdV equation [1] which is described as follows:

$$u_t - u_{txx} + u_x + uu_x = 0.$$

The well-posedness of KdV equation has been tremendously researched, including the whole line, quarter plane, periodic domain or finite domain [11,7,3,4,15,12]. There are also some results of the well-posedness

☆ This project was supported by the Natural Science Foundation of China (Grant No. 11231007).

* Corresponding author.
E-mail addresses: yuhang_lijie@163.com (J. Li), ksliu@zju.edu.cn (K. Liu).

http://dx.doi.org/10.1016/j.jmaa.2017.02.038

of the BBM equation [8,13,16,5,18]. Since the term u_{txx} plays an very important role in BBM equation, this leads some difficulties to get the well-posedness of it. L. Rosier and B.Y. Zhang proved the unique continuation property for small data in $H^1(\mathbb{T})$ of BBM equation in periodic domain [16]. J.L. Bona and N. Tzvetkov gave some sharp well-posedness results for the BBM equation in the quarter plane [5], they showed that the initial-value problem is globally well posed in H^s if $s \geq 0$. M. Francius, E. Pelinovsky and A. Slunyaev introduced the wave dynamics of the following equation [10]:

$$u_t - u_{txx} - C_1 u_{xxx} + C_2 u_x + u u_x = 0, \tag{1.1}$$

where $C_1, C_2 \in \mathbb{R}$. Let $C_1 = 0$, $C_2 = 1$, then Eq. (1.1) comes to be BBM equation. In the limit of weakly nonlinear long wave, Eq. (1.1) is asymptotically closed to KdV equation (see, for instance [9]). Let $C_1 = -1$, $C_2 = 1$, then Eq. (1.1) comes to be the so-called KdV–BBM equation:

$$u_t - u_{txx} + u_{xxx} + u_x + u u_x = 0.$$

In this paper, we consider the KdV–BBM equation with nonhomogeneous boundary-value conditions described as follows:

$$\begin{cases} u_t - u_{txx} + u_{xxx} + u_x + u u_x = 0, \quad u(x,0) = \phi(x), \quad x \in [0,L], \quad t \geq 0, \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t). \end{cases} \tag{1.2}$$

Our aim is to show the well-posedness of the system (1.2) in the space $H^s(0,L)$ when the initial value and boundary value are drawn from the product space $H^s(0,L) \times H^{s_1}(0,T) \times H^{s_2}(0,T) \times H^{s_3}(0,T)$, where T is any positive constant and $s_1 = s_2 = s_3 = s - 1$ for any $s \geq 1$. Throughout this paper, for any $s \geq 0$, $H^s(0,L)$ denotes the Sobolev space

$$H^s(0,L) = \left\{ f : (0,L) \to \mathbb{R}; \quad \|f\|_{H^s(0,L)} := \left\|(1-\partial_x^2)^{s/2} f\right\|_{L^2(0,L)} < \infty \right\}.$$

Its dual space is denoted by $H^{-s}(0,L)$. With the definition of Sobolev space, we define the following two product spaces:

$$X_{s,T} = H^{s+1}(0,L) \times H^s(0,T) \times H^s(0,T) \times H^s(0,T),$$

$$E_{s,T} = H^{s+1}(0,L) \times H^{s+1/2+\epsilon}(0,T) \times H^{s+1/2+\epsilon}(0,T) \times H^{s+1/2+\epsilon}(0,T),$$

where $s \geq 0$ and ϵ is any positive constant number. With these definitions, we can give the main results of this paper.

Theorem 1.1 *(Local well-posedness). For any $T > 0$, $L > 0$, $s \geq 0$ and $(\phi, \overrightarrow{h}) = (\phi, h_1, h_2, h_3) \in X_{s,T}$ be given, there exists a $T^* \in (0,T]$ depending only on $\|(\phi, \overrightarrow{h})\|_{X_{s,T}}$ such that there exists a unique solution u of Eq. (1.2) with $u \in C([0,T^*], H^{s+1}(0,L))$.*

Theorem 1.2 *(Global well-posedness). For any $T > 0$, $L > 0$, $s \geq 0$ and $(\phi, \overrightarrow{h}) = (\phi, h_1, h_2, h_3) \in E_{s,T}$ be given, there exists a unique solution u of Eq. (1.2) with $u \in C([0,T], H^{s+1}(0,L))$.*

We first present various linear estimates of the linearized KdV–BBM equation associated to Eq. (1.2) in Section 2. With the help of these estimates, we discuss the nonlinear problem of KdV–BBM equation (1.2) and give its local well-posedness in section 3. The global well-posedness is presented in section 4.

Please cite this article in press as: J. Li, K. Liu, Well-posedness of Korteweg–de Vries–Benjamin Bona Mahony equation on a finite domain, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.02.038

2. Linear estimates

In this section, we will discuss the following linearized system associated to Eq. (1.2),

$$\begin{cases} u_t - u_{txx} + u_{xxx} + u_x = f, \quad u(x,0) = \phi(x), \quad x \in [0,L], \quad t \geq 0, \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t). \end{cases} \tag{2.1}$$

We first consider the linear problem only with initial value $\phi(x)$ as follows:

$$\begin{cases} u_t - u_{txx} + u_{xxx} + u_x = 0, \quad u(x,0) = \phi(x), \quad x \in [0,L], \quad t \geq 0, \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0. \end{cases} \tag{2.2}$$

The linear operator of Eq. (2.2) in the space $L^2(0,L)$ is described as follows

$$A = -(1-\partial_x^2)^{-1}(\partial_x^3 + \partial_x).$$

Multiply both sides of Eq. (2.2) by u and integrate over $(0,L)$ with respect to x. This leads to the following estimate:

$$\frac{d}{dt}\Big[\int_0^L (u^2 + u_x^2)dx\Big] = -u_x^2(0,t) \leq 0. \tag{2.3}$$

Since the solution u of Eq. (2.2) can be written in the form $u(x,t) = W_0(t)\phi(x)$, where $W_0(t)$ is the semigroup generated by the infinitesimal generator A [14]. According to (2.3), we know that the semigroup $W_0(t)$ is a C_0-semigroup in the space $H^1(0,L)$ and

$$\|W_0(t)\|_{H^1(0,L)} \leq 1, \quad \|u(\cdot,t)\|_{H^1(0,L)} \leq \|\phi\|_{H^1(0,L)}.$$

So, we have the following proposition.

Proposition 2.1. *For any $\phi \in H^1(0,L)$, the solution $u(x,t) = W_0(t)\phi(x)$ of Eq. (2.2) satisfies*

$$\|u(\cdot,t)\|^2_{H^1(0,L)} + \int_0^t u_x^2(0,\tau)d\tau = \|\phi\|^2_{H^1(0,L)},$$

and $\|W_0(t)\|_{H^1(0,L)} \leq 1$.

In order to make it convenient to get the estimate of f, we write Eq. (2.1) in the following form and let $\phi = 0$ and $\overrightarrow{h} = (0,0,0)$.

$$\begin{cases} u_t = Au + \widetilde{f}, \quad u(x,0) = 0, \quad x \in [0,L], \quad t \geq 0, \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0, \end{cases} \tag{2.4}$$

where $\widetilde{f} = (1-\partial_x^2)^{-1} f$.

Proposition 2.2. *For any $f \in L^1(0,t;H^{-1}(0,L))$, the solution u of Eq. (2.4) satisfies*

$$\|u(\cdot,t)\|_{H^1(0,L)} \leq C\|f\|_{L^1(0,t;H^{-1}(0,L))}.$$

Please cite this article in press as: J. Li, K. Liu, Well-posedness of Korteweg–de Vries–Benjamin Bona Mahony equation on a finite domain, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.02.038

Proof. Since the mild solution u of Eq. (2.4) has the form

$$u(x,t) = \int_0^t W_0(t-\tau)\widetilde{f}(x,\tau)d\tau.$$

Using the generalized Minkowski's inequality, we have

$$\begin{aligned}
\|u(\cdot,t)\|_{H^1(0,L)} &= \|\int_0^t W_0(t-\tau)\widetilde{f}(x,\tau)d\tau\|_{H^1(0,L)} \\
&\le \Big(\int_0^L \Big|\int_0^t W_0(t-\tau)\widetilde{f}(x,\tau)d\tau\Big|^2 dx\Big)^{1/2} + \Big(\int_0^L \Big|\int_0^t W_0(t-\tau)\widetilde{f}_x(x,\tau)d\tau\Big|^2 dx\Big)^{1/2} \\
&\le \int_0^t \Big(\int_0^L |W_0(t-\tau)\widetilde{f}(x,\tau)|^2 dx\Big)^{1/2} d\tau + \int_0^t \Big(\int_0^L |W_0(t-\tau)\widetilde{f}_x(x,\tau)|^2 dx\Big)^{1/2} d\tau \\
&\le 2\int_0^t \big\|W_0(t-\tau)\widetilde{f}(\cdot,\tau)\big\|_{H^1(0,L)} d\tau \\
&\le 2\int_0^t \|\widetilde{f}\|_{H^1(0,L)} d\tau \\
&= 2\|\widetilde{f}\|_{L^1(0,t;H^1(0,L))}.
\end{aligned}$$

Since $\widetilde{f} = (1-\partial_x^2)^{-1} f$ and according to the definition of Sobolev space, we have

$$\|u(\cdot,t)\|_{H^1(0,L)} \le C\|f\|_{L^1(0,t;H^{-1}(0,L))}. \qquad \square$$

Next, we consider the nonhomogeneous boundary-value linear problem

$$\begin{cases} u_t - u_{txx} + u_{xxx} + u_x = 0, \quad u(x,0) = 0, \quad x \in [0,L], \quad t \ge 0, \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t). \end{cases} \tag{2.5}$$

We use the method of Laplace transform on the time t and get the explicit solution of Eq. (2.5). After taking Lapace transform on Eq. (2.5), we have

$$\begin{cases} s\widehat{u}(x,s) - s\widehat{u}_{xx}(x,s) + \widehat{u}_{xxx}(x,s) + \widehat{u}_x(x,s) = 0, \quad u(x,0) = \phi(x), \quad x \in [0,L], \\ \widehat{u}(0,s) = \widehat{h}_1(s), \quad u(L,t) = \widehat{h}_2(s), \quad u_x(L,t) = \widehat{h}_3(s), \end{cases} \tag{2.6}$$

where

$$\hat{u}(x,s) = \int_0^{+\infty} e^{-st} u(x,t)dt$$

and

$$\hat{h}_j(s) = \int_0^{+\infty} e^{-st} h_j(t)dt, \quad j = 1,2,3.$$

The solution $\hat{u}(x,s)$ of Eq. (2.13) can be written in the following form:

$$\hat{u}(x,s) = \sum_{j=1}^{3} C_j(s) e^{\lambda_j(s)x},$$

where $\lambda_j(s)$, $j = 1,2,3$, are the three solutions of the following characteristic equation

$$s - s\lambda^2 + \lambda^3 + \lambda = 0, \tag{2.7}$$

and $C_j(s)$, $j = 1,2,3$, solve the following equations,

$$\begin{cases} C_1(s) + C_2(s) + C_1(s) = \hat{h}_1(s), \\ C_1(s)e^{\lambda_1(s)} + C_2(s)e^{\lambda_2(s)} + C_3(s)e^{\lambda_3(s)} = \hat{h}_2(s), \\ C_1(s)\lambda_1(s)e^{\lambda_1(s)} + C_2(s)\lambda_2(s)e^{\lambda_2(s)} + C_3(s)\lambda_3(s)e^{\lambda_3(s)} = \hat{h}_3(s). \end{cases} \tag{2.8}$$

Let $\lambda_1 = i\rho$ be one of the three roots of λ in Eq. (2.7). Then we have

$$s = \frac{i(\rho^3 - \rho)}{\rho^2 + 1},$$

and the other two roots of λ:

$$\lambda_2 = \frac{\sqrt{\rho^4 - \rho^2 - 1} - i\rho}{\rho^2 + 1}, \quad \lambda_3 = \frac{-\sqrt{\rho^4 - \rho^2 - 1} - i\rho}{\rho^2 + 1}.$$

Solving Eq. (2.8) with Cramer's rule, we can get $C_j(s)$, $j = 1,2,3$ as follows:

$$C_j = \frac{\Delta_j(s)}{\Delta(s)}, \quad j = 1,2,3,$$

where $\Delta(s)$ is the determinant of the coefficient matrix of Eq. (2.8) and $\Delta_j(s)$ the determinants of the matrices that are obtained by replacing the ith-column of $\Delta(s)$ by the column vector $(\hat{h}_1(s), \hat{h}_2(s), \hat{h}_3(s))$, $j = 1,2,3$. Taking inverse Laplace transform of $\hat{u}$, we have:

$$u(x,t) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{st} \hat{u}(x,s) ds = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{st} \frac{\Delta_j(s)}{\Delta(s)} e^{\lambda_j(s)x} ds$$

for any $\delta > 0$. We can also write the solution u_1 of Eq. (2.5) with $h_2 = h_3 = 0$ as follows:

$$u_1(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{st} \frac{\Delta_{j,1}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_1(s) ds \equiv W_1(t)h_1, \tag{2.9}$$

where $\Delta_{j,1}(s)$ is obtained from $\Delta_j(s)$ with $\hat{h}_1(s) = 1$, $\hat{h}_2(s) = \hat{h}_2(s) = 0$. With the same method we can get $u_2(x,t)$ and $u_3(x,t)$ with $h_1(t) \equiv h_3(t) \equiv 0$ and $h_1(t) \equiv h_2(t) \equiv 0$ respectively. So we can write $u_k(x,t)$ in the general form:

$$u_k(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{st} \frac{\Delta_{j,k}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_k(s) ds \equiv W_k(t)h_k, \tag{2.10}$$

for $k = 1, 2, 3$. Because the right-hand side of Eq. (2.10) is continuous with δ for any $\delta \geq 0$ and the left hand-side of Eq. (2.10) does not depend on δ, we can take $\delta = 0$ and change Eq. (2.10) to the following form:

$$u_k(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{0}^{+i\infty} e^{st} \frac{\Delta_{j,k}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_k(s) ds + \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{-i\infty}^{0} e^{st} \frac{\Delta_{j,k}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_k(s) ds$$
$$\equiv I_k(x,t) + II_k(x,t),$$

for $k = 1, 2, 3$. Since we have let $s = s(\rho) = i(\rho^3 - \rho)/(\rho^2 + 1)$, we can write $I_k(x,t)$ and $II_k(x,t)$ in the following equivalent forms:

$$I_k(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{1}^{+\infty} e^{s(\rho)t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,k}^+(\rho)}{\Delta^+(\rho)} \frac{\partial s(\rho)}{\partial \rho} \hat{h}_k^+(\rho) d\rho$$

and

$$II_k(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{1}^{+\infty} e^{-s(\rho)t} e^{\lambda_j^-(\rho)x} \frac{\Delta_{j,k}^-(\rho)}{\Delta^-(\rho)} \frac{\partial s(\rho)}{\partial \rho} \hat{h}_k^-(\rho) d\rho$$

where $\hat{h}_k^+(\rho) = \hat{h}_k(s(\rho))$, $\Delta^+(\rho)$, $\Delta_{j,k}^+(\rho)$, $\lambda_j^+(\rho)$ are obtained from $\Delta(s)$, $\Delta_{j,k}(s)$, $\lambda_j(s)$ by replacing s with $s(\rho)$, for $j, k = 1, 2, 3$, respectively. We also point out that $\Delta^-(\rho) = \overline{\Delta^+}(\rho)$ and $\Delta_{j,k}^-(\rho) = \overline{\Delta_{j,k}^+}(\rho)$ for $j, k = 1, 2, 3$, and $\hat{h}_k^-(\rho) = \overline{\hat{h}_k^+}(\rho)$.

The following two lemmas will play a very important role in proving the estimate about the boundary-value h_j, $j = 1, 2, 3$.

Lemma 2.1. *Let Kf be the function defined by*

$$Kf(x) = \int_{0}^{+\infty} e^{\gamma(\nu)x} f(\nu) d\nu, \quad \text{for} \quad \text{any} \quad f \in L^2(0, +\infty),$$

where $\gamma(\nu)$ is a continuous complex-valued function defined on $(0, +\infty)$ and satisfies the following conditions:

(i) There exist $\delta > 0$ and $b > 0$ such that

$$\mathop{Inf}_{0<\nu<\delta} \frac{\|Re\gamma(\nu)\|}{\nu} \geq b \quad , \quad and \quad Re\gamma(\nu) > 0 \quad ;$$

(ii) There exists a complex number $\alpha + i\beta$ such that

$$\lim_{\nu \to +\infty} \frac{\gamma(\nu)}{\nu} = \alpha + i\beta.$$

Then there exists a constant C such that for all $f \in L^2(0, +\infty)$,

$$\|Kf\|_{L^2(0,1)} \leq C(\|e^{Re\gamma(\cdot)} f(\cdot)\|_{L^2(0,+\infty)} + \|f(\cdot)\|_{L^2(0,+\infty)}).$$

Proof. Using *Hölder* inequality, we have

$$
\begin{aligned}
\|Kf\|^2_{L^2(0,1)} &\leq \int_0^1 \Big(\int_0^{+\infty} e^{Re(\gamma(\nu))x} \mid f(\nu) \mid d\nu \Big)^2 dx \\
&= \int_0^1 \int_0^{+\infty} e^{Re(\gamma(\nu))x} \mid f(\nu) \mid d\nu \int_0^{+\infty} e^{Re(\gamma(\xi))x} \mid f(\xi) \mid d\xi dx \\
&= \int_0^{+\infty} \int_0^{+\infty} \Big(\int_0^1 e^{Re(\gamma(\nu)+\gamma(\xi))x} dx \Big) \mid f(\nu) \mid\mid f(\xi) \mid d\nu d\xi \\
&\leq \int_0^{+\infty} \int_0^{+\infty} \frac{(e^{Re(\gamma(\nu)+\gamma(\xi))}+1)}{Re(\gamma(\nu)+\gamma(\xi))} \mid f(\nu) \mid\mid f(\xi) \mid d\nu d\xi \\
&\leq \Big\| \int_0^{+\infty} \frac{e^{Re\gamma(\nu)} \mid f(\nu) \mid}{\mid Re(\gamma(\nu)+\gamma(\xi)) \mid} d\nu \Big\|_{L^2(0,+\infty)} \|e^{Re\gamma(\cdot)} f(\cdot)\|_{L^2(0,+\infty)} \\
&\quad + \Big\| \int_0^{+\infty} \frac{\mid f(\nu) \mid}{\mid Re(\gamma(\nu)+\gamma(\xi)) \mid} d\nu \Big\|_{L^2(0,+\infty)} \|f(\cdot)\|_{L^2(0,+\infty)}.
\end{aligned}
$$

Observe that

$$
\|e^{Re\gamma(\mu\xi)} f(\mu\xi)\|_{L^2(0,\infty)} \leq \frac{1}{\sqrt{\mu}} \|e^{Re\gamma(\cdot)} f(\cdot)\|_{L^2(0,+\infty)}, \qquad \textit{for} \quad \textit{any} \quad \mu \in (0,+\infty)
$$

and under the condition (i), we have

$$
\frac{\xi}{\mid Re(\gamma(\mu\xi)+\gamma(\xi)) \mid} \leq \frac{C}{\xi+1}, \qquad \textit{for} \quad \textit{any} \quad \xi \in (0,+\infty).
$$

Using the generalized Minkowski's inequality, we have

$$
\begin{aligned}
\Big\| \int_0^{+\infty} \frac{e^{Re\gamma(\nu)} \mid f(\nu) \mid}{Re(\gamma(\nu)+\gamma(\xi))} d\nu \Big\|_{L^2(0,+\infty)} &= \Big\| \int_0^{+\infty} \frac{e^{Re\gamma(\mu\xi)} \mid f(\mu\xi) \mid \xi d\mu}{Re(\gamma(\mu\xi)+\gamma(\xi))} \Big\|_{L^2(0,+\infty)} \\
&\leq \int_0^{+\infty} \Big\| \frac{e^{Re\gamma(\mu\xi)} \mid f(\mu\xi) \mid \xi d\mu}{Re(\gamma(\mu\xi)+\gamma(\xi))} \Big\|_{L^2(0,+\infty)} d\mu \\
&\leq C \int_0^{+\infty} \frac{1}{\sqrt{\mu}(1+\mu)} d\mu \|e^{Re\gamma(\cdot)} f(\cdot)\|_{L^2(0,+\infty)} \\
&\leq C \|e^{Re\gamma(\cdot)} f(\cdot)\|_{L^2(0,+\infty)}
\end{aligned}
$$

for some constant $C > 0$. Using the same argument, we can also have the following inequality

$$
\Big\| \int_0^{+\infty} \frac{\mid f(\nu) \mid}{Re(\gamma(\nu)+\gamma(\xi))} d\nu \Big\|_{L^2(0,+\infty)} \leq C \|f(\cdot)\|_{L^2(0,+\infty)}.
$$

So, we complete the proof. □

Please cite this article in press as: J. Li, K. Liu, Well-posedness of Korteweg–de Vries–Benjamin Bona Mahony equation on a finite domain, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.02.038

Using Lemma 2.1, we can directly have the following Lemma 2.2.

Lemma 2.2. *Fixed $a > 0$. Define function Gf as follows*

$$Gf(x) = \int_0^a e^{i\gamma(\mu)x} f(\mu)d\mu, \quad \textit{for} \quad \textit{any} \quad f \in L^2(0,a),$$

where $\gamma(\cdot)$ is a continuous and real-valued function defined on the interval $[0,a]$ which is also C^1 on $(0,a)$. $\gamma(\mu)$ also satisfies the condition that: there exists a constant C_1 such that

$$\frac{1}{|\gamma'(\mu)|} \leq C_1, \quad \textit{for} \quad \textit{any} \quad 0 < \mu < a.$$

Then there exists a constant C_2 such that the following inequality holds

$$\|Gf\|_{L^2(0,a)} \leq C_2 \|f\|_{L^2(0,a)}$$

With the help of Lemma 2.1 and Lemma 2.2, we can discuss the regularity of the solution u_k of Eq. (2.5) with $h_i \equiv 0$, $i \in \{1,2,3\}\backslash\{k\}$, for $k = 1,2,3$.

Proposition 2.3. *There exists a constant C such that*

$$\|u_1\|_{C(\mathbb{R}^+;H^1(0,L))} \leq C\|h_1\|_{L^2(\mathbb{R}^+)}, \qquad \sup_{x\in[0,L]} \|\partial_x u_1(x,\cdot)\|_{L^2(\mathbb{R}^+)} \leq C\|h_1\|_{L^2(\mathbb{R}^+)}, \tag{2.11}$$

for all $h_1(t) \in L^2(\mathbb{R}^+)$; and also

$$\sup_{x\in[0,L]} \|\partial_{xx} u_1(x,\cdot)\|_{L^2(\mathbb{R}^+)} \leq \|h_1\|_{H^1(\mathbb{R}^+)}, \tag{2.12}$$

for all $h_1(t) \in H^1(\mathbb{R}^+)$.

Proof. We first give the explicit estimate of $\lambda_j^+(\rho)$, $\Delta^+(\rho)$, and $\frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)}$, $j = 1,2,3$. We write $\Delta^+(\rho)$ in another form

$$\Delta^+(\rho) = A_1 + A_2 + A_3,$$

where

$$\begin{aligned} A_1 &= \Delta_{1,1}^+(\rho) = (\lambda_3^+(\rho) - \lambda_2^+(\rho))e^{\lambda_2^+(\rho)+\lambda_3^+(\rho)}, \\ A_2 &= \Delta_{2,1}^+(\rho) = (\lambda_1^+(\rho) - \lambda_3^+(\rho))e^{\lambda_1^+(\rho)+\lambda_3^+(\rho)}, \\ A_3 &= \Delta_{3,1}^+(\rho) = (\lambda_2^+(\rho) - \lambda_1^+(\rho))e^{\lambda_1^+(\rho)+\lambda_2^+(\rho)}. \end{aligned}$$

With easy calculation, we have the following estimates

$$\frac{\Delta_{1,1}^+(\rho)}{\Delta^+(\rho)} \sim \frac{1}{\rho}, \qquad \frac{\Delta_{2,1}^+(\rho)}{\Delta^+(\rho)} \sim 1, \qquad \frac{\Delta_{3,1}^+(\rho)}{\Delta^+(\rho)} \sim 1,$$

as $\rho \to +\infty$. According to Lemma 2.1, Lemma 2.2, we know that there exists a constant C such that

$$\|I_1(\cdot,t)\|^2_{L_2(0,1)} \le \sum_{j=1}^{3}\int_1^{+\infty}\left|\frac{\Delta^+_{j,1}(\rho)}{\Delta^+(\rho)}\right|^2\left(e^{Re\lambda_j^+(\rho)}+1\right)^2\left|\hat{h_1}^+(\rho)\frac{\partial s(\rho)}{\partial\rho}\right|^2 d\rho.$$

Since $\frac{\partial s(\rho)}{\partial\rho}=\frac{\rho^4+4\rho^2+\rho-1}{(\rho^2+1)^2}\sim 1$ as $\rho\to+\infty$, we have

$$\|I_1(\cdot,t)\|^2_{L_2(0,L)} \le C\int_1^{+\infty}|\hat{h_1}^+(\rho)|^2 d\rho.$$

Let $\xi i = s(\rho)i$, we have $\rho\sim\xi$ and $d\xi\sim d\rho$ as $\rho\to+\infty$. So, we have the following estimate:

$$\begin{aligned}\|I_1(\cdot,t)\|^2_{L_2(0,L)} &\le C\int_0^{\infty}|\hat{h_1}^+(\rho)|^2 d\xi\\ &\le C\int_0^{\infty}|\hat{h_1}(\xi)|^2 d\xi\\ &\le C\|h_1\|^2_{L^2(\mathbb{R}^+)}.\end{aligned}$$

With the same method, we also have

$$\begin{aligned}\|\partial_x I_1(\cdot,t)\|_{L_2(0,L)} &\le C\|h_1\|_{L^2(\mathbb{R}^+)},\\ \|II_1(\cdot,t)\|_{L_2(0,L)} &\le C\|h_1\|_{L^2(\mathbb{R}^+)},\\ \|\partial_x II_1(\cdot,t)\|_{L_2(0,L)} &\le C\|h_1\|_{L^2(\mathbb{R}^+)}.\end{aligned}$$

So we get

$$\|u_1\|_{C(\mathbb{R}^+;H^1(0,L))} \le C\|h_1\|_{L^2(\mathbb{R}^+)}.$$

Now, we shall discuss the regularity of $I_1(x,t)$ on the interval $[0,L]$.

$$\begin{aligned}\partial_x I_1(x,t) &= \sum_{j=1}^{3}\frac{1}{2\pi}\int_1^{+\infty} e^{is(\rho)t}\lambda_j^+(\rho)e^{\lambda_j^+(\rho)x}\frac{\Delta^+_{j,1}(\rho)}{\Delta^+(\rho)}s'(\rho)\hat{h_1}^+(\rho)d\rho\\ &= \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty} e^{i\omega t}\lambda_j^+(\theta(\omega))e^{\lambda_j^+(\theta(\omega))x}\frac{\Delta^+_{j,1}(\theta(\omega))}{\Delta^+(\theta(\omega))}\hat{h_1}(i\omega)d\omega,\end{aligned}$$

where $\theta(\omega)$ is the solution of the equation $\omega=s(\rho)$, for $\rho\ge 1$. Using Plancherel theorem on t, we have

$$\begin{aligned}\|\partial_x I_1(x,\cdot)\|^2_{L^2(0,+\infty)} &\le C\sum_{j=1}^{3}\int_0^{+\infty}\left|\lambda_j^+(\theta(\omega))e^{\lambda_j^+(\theta(\omega))x}\frac{\Delta^+_{j,1}(\theta(\omega))}{\Delta^+(\theta(\omega))}\right|^2|\hat{h_1}(i\omega)|^2 d\omega\\ &\le C\sum_{j=1}^{3}\int_0^{+\infty}|\lambda_j^+(\theta(\omega))|^2\left(\sup_{0\le x\le 1}|e^{\lambda_j^+(\theta(\omega))x}|^2\right)\left|\frac{\Delta^+_{j,1}(\theta(\omega))}{\Delta^+(\theta(\omega))}\right|^2|\hat{h_1}(i\omega)|^2 d\omega.\end{aligned}$$

Using the estimates of $\lambda_j^+(\rho)$, $\frac{\Delta^+_{j,1}(\rho)}{\Delta^+(\rho)}$, $j=1,2,3$, we have the following inequality

$$\|\partial_x I_1(x,\cdot)\|^2_{L^2(0,+\infty)} \le C\sum_{j=1}^{3}\int_0^{+\infty} |\hat{h_1}(i\omega)|^2 d\omega$$
$$\le C\|h_1\|^2_{L^2(0,+\infty)}$$

Using the same strategy, we can also get the following estimate

$$\|I_1(x,\cdot)\|_{L^2(0,+\infty)} \le C\|h_1\|_{L^2(0,+\infty)},$$

and

$$\|\partial_x II_1(x,\cdot)\|_{L^2(0,+\infty)} \le C\|h_1\|_{L^2(0,+\infty)},$$
$$\|II_1(x,\cdot)\|_{L^2(0,+\infty)} \le C\|h_1\|_{L^2(0,+\infty)}.$$

Clearly,

$$\begin{aligned}&\partial_x I_1(x,t) - \partial_x I_1(x_0,t)\\ &\quad= \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty} e^{i\omega t}\lambda_j^+(\theta(\omega))(e^{\lambda_j^+(\theta(\omega))x} - e^{\lambda_j^+(\theta(\omega))x_0})\frac{\Delta^+_{j,1}(\theta(\omega))}{\Delta^+(\theta(\omega))}\hat{h_1}(i\omega)d\omega.\end{aligned}$$

Observe that

$$\begin{aligned}&\|\partial_x I_1(x,t) - \partial_x I_1(x_0,t)\|^2_{L^2(0.+\infty)}\\ &\le \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}\left|\lambda_j^+(\theta(\omega))(e^{\lambda_j^+(\theta(\omega))x} - e^{\lambda_j^+(\theta(\omega))x_0})\right|^2\left|\frac{\Delta^+_{j,1}(\theta(\omega))}{\Delta^+(\theta(\omega))}\right|^2 d\omega\\ &\le C||h_1||^2_{L^2(0,+\infty)}.\end{aligned}$$

Using Fatou's lemma, we have

$$\begin{aligned}&\lim_{x\to 0}\|\partial_x I_1(x,t) - \partial_x I_1(x_0,t)\|^2_{L^2(0.+\infty)}\\ &\le \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}\left|\lambda_j^+(\theta(\omega))\lim_{x\to 0}(e^{\lambda_j^+(\theta(\omega))x} - e^{\lambda_j^+(\theta(\omega))x_0})\right|^2\left|\frac{\Delta^+_{j,1}(\theta(\omega))}{\Delta^+(\theta(\omega))}\right|^2 d\omega\\ &= 0.\end{aligned}$$

So we have proved $\partial_x I_1(x,\cdot) \in C_b([0,1];L^2(0,+\infty))$ with

$$\sup_{x\in[0,1]}\|\partial_x I_1(x,\cdot)\|_{L^2(0,+\infty)} \le C\|h_1\|_{L^2(0,+\infty)}$$

for all $h_1 \in L^2(0,+\infty)$. Using the same method, we can also get the following estimate

$$I_1(x,\cdot) \in C_b([0,1];L^2(0,+\infty)),$$

with

$$\sup_{x\in[0,1]}\|I_1(x,\cdot)\|_{L^2(0,+\infty)} \le C\|h_1\|_{L^2(0,+\infty)}.$$

It's obvious that II_1 has the same property as I_1. So we have proved (2.11). Similarly, we can also get (2.12). The proof is complete. □

Since the processes of getting the regularities of u_2 and u_3 are similar to that presented in the proof of Proposition 2.3, we only give the results in the following proposition and omit the proof.

Proposition 2.4. *There exists a constant C such that*

$$\|u_2\|_{C(\mathbb{R}^+;H^1(0,L))} \le C\|h_2\|_{L^2(\mathbb{R}^+)}, \quad \sup_{x\in[0,L]} \|\partial_x u_2(x,\cdot)\|_{L^2(\mathbb{R}^+)} \le C\|h_2\|_{L^2(\mathbb{R}^+)},$$

$$\|u_3\|_{C(\mathbb{R}^+;H^1(0,L))} \le C\|h_3\|_{L^2(\mathbb{R}^+)}, \quad \sup_{x\in[0,L]} \|\partial_x u_3(x,\cdot)\|_{L^2(\mathbb{R}^+)} \le C\|h_3\|_{L^2(\mathbb{R}^+)},$$

for all $h_2(t) \in L^2(\mathbb{R}^+)$, $h_3(t) \in L^2(\mathbb{R}^+)$; and also

$$\sup_{x\in[0,L]} \|\partial_{xx} u_2(x,\cdot)\|_{L^2(\mathbb{R}^+)} \le C\|h_2\|_{H^1(\mathbb{R}^+)}, \quad \sup_{x\in[0,L]} \|\partial_{xx} u_3(x,\cdot)\|_{L^2(\mathbb{R}^+)} \le C\|h_3\|_{H^1(\mathbb{R}^+)},$$

for all $h_2(t) \in H^1(\mathbb{R}^+)$, $h_3(t) \in H^1(\mathbb{R}^+)$.

We can write the solution of Eq. (2.5) in the following abstract form

$$u(t) = \sum_{j=1}^{3} W_j(t)h_j, \tag{2.13}$$

where W_j is defined in (2.10). Let's define a new product space

$$\mathcal{H}_{s,T} = H^s(0,T) \times H^s(0,T) \times H^s(0,T),$$

for $s \ge 0$ and $T > 0$, and the norm of $\vec{h}(t) = (h_1, h_2, h_3)$ in the space $\mathcal{H}_{s,T}$ as follows

$$\|\vec{h}\|_{\mathcal{H}_{s,T}} \equiv (\|h_1\|^2_{H^s(0,T)} + \|h_2\|^2_{H^s(0,T)} + \|h_3\|^2_{H^s(0,T)})^{\frac{1}{2}}.$$

Using the estimates we have got in Proposition 2.3, Proposition 2.4, we have the following theorem directly.

Theorem 2.1. *There exists a unique solution $u(x,t)$ of Eq. (2.5) with*

$$u(x,t) \in C_b(0,+\infty; H^1(0,L)), \quad u_x \in C_b([0,L]; L^2(0,+\infty)) \quad \textit{and} \quad u_{xx} \in C_b([0,L]; H^1(0,+\infty)).$$

Moreover there exists a constant C, such that

$$\|u\|_{C(0,+\infty;H^1(0,L))} \le C\|\vec{h}\|_{\mathcal{H}_{0,+\infty}}, \ \textit{for any } \vec{h} \in \mathcal{H}_{0,+\infty};$$

$$\sup_{x\in[0,L]} \|u_x(x,\cdot)\|_{L^2(0,+\infty)} \le C\|\vec{h}\|_{\mathcal{H}_{0,+\infty}}, \ \textit{for any } \vec{h} \in \mathcal{H}_{0,+\infty};$$

$$\sup_{x\in[0,L]} \|u_{xx}(x,\cdot)\|_{H^1(0,+\infty)} \le C\|\vec{h}\|_{\mathcal{H}_{1,+\infty}}, \ \textit{for any } \vec{h} \in \mathcal{H}_{1,+\infty}.$$

Now, let's discuss the Kato-smoothing property of Eq. (2.2). Let ϕ^* be the zero extension of ϕ, namely $\phi^* = \phi$ on $(0,L)$ and $\phi^* = 0$ on $\mathbb{R} \setminus (0,L)$. Then, we have

$$\tilde{v}(x,t) = \tilde{W}(t)\phi^*,$$

where the semigroup $\tilde{W}(t)$ is generated by the linear operator $\tilde{A}$, which is defined by

$$\tilde{A}f = -(1-\partial_x^2)^{-1}(\partial_x^3 + \partial_x)f, \quad \textit{for} \quad f \in \mathbb{D}(\tilde{A}) = H^1(\mathbb{R}).$$

We also assume that v_b is the solution of the following problem

$$\begin{cases} v_t - v_{txx} + v_{xxx} + v_x = 0, \quad v(x,0) = 0, \quad \textit{for} \quad x \in [0,L], \quad t \geq 0, \\ v(0,t) = \tilde{v}(0,t) = \xi_1(t), \\ v(L,t) = \tilde{v}(L,t) = \xi_2(t), \\ v_x(L,t) = \tilde{v}_x(L,t) = \xi_3(t). \end{cases} \tag{2.14}$$

As what we did in (2.10), we have $v_b = W_b(t)\vec{\xi}$ and the following result.

Proposition 2.5. *We can write the solution of Eq. (2.1) in another form:*

$$u(x,t) = W_0(t)\phi = \tilde{W}(t)\phi^* - W_b(t)\vec{\xi},$$

for any $\phi \in L^2(0,L)$*, where* ϕ^* *and* $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ *are defined as above.*

In order to have more precise estimates of the solution $u(t)$ of Eq. (2.1), the following Lemma will be used.

Lemma 2.3. *There exists a constant* C*, such that for any* $\phi^* \in H^1(\mathbb{R})$*, the solution* $\tilde{v}(x,t) = \tilde{W}(t)\phi^*$ *satisfies the following estimate*

$$\sup_{x\in\mathbb{R}} \|\partial_x \tilde{v}(x,\cdot)\|_{L^2(0,+\infty)} \leq C\|\phi^*\|_{H^1(\mathbb{R})}, \tag{2.15}$$

and also

$$\sup_{x\in\mathbb{R}} \|\partial_{xx} \tilde{v}(x,\cdot)\|_{L^2(0,+\infty)} \leq C\|\phi^*\|_{H^2(\mathbb{R})}, \ \textit{for any } \phi^* \in H^2(\mathbb{R}). \tag{2.16}$$

Proof. We first solute the following linear equation

$$\tilde{v}_t - \tilde{v}_{txx} + \tilde{v}_x + \tilde{v}_{xxx} = 0, \quad \tilde{v}(x,0) = \phi^*(x), \tag{2.17}$$

where $x \in (-\infty, +\infty)$, $t \geq 0$. Using Fourier transform on x, we can have the solution of the Eq. (2.17) as follows

$$\tilde{v}(x,t) = K(x,t) * \phi^*(x), \quad \text{where} \quad \mathbb{F}[K(x,t)] = \hat{K}(\xi,t) = e^{\frac{i(\xi^3-\xi)}{\xi^2+1}t},$$

and

$$\begin{aligned} \mathbb{F}[\partial_x \tilde{v}(x,t)] &= i\xi\hat{\tilde{v}}(\xi,t) \\ &= i\xi\hat{\phi^*}(\xi) \cdot e^{\frac{i(\xi^3-\xi)}{\xi^2+1}t}. \end{aligned}$$

Then, we have the following estimate

$$\int_{-\infty}^{+\infty} |\partial_x \tilde{v}(x,t)|^2 dt = \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} e^{ix\xi} i\xi \hat{\tilde{v}}(\xi,t) d\xi \right|^2 dt$$
$$= \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} i\xi e^{\frac{i(\xi^3-\xi)}{\xi^2+1}t} e^{ix\xi} \hat{\phi}^*(\xi) d\xi \right|^2 dt.$$

Let $\xi = \psi(\eta)$, where ξ is the solution of the equation $\frac{i(\xi^3-\xi)}{\xi^2+1} = \eta$. We have the following estimate,

$$\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} i\xi e^{\frac{i(\xi^3-\xi)}{\xi^2+1}t} e^{ix\xi} \hat{\phi}^*(\xi) d\xi \right|^2 dt = \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} i\psi(\eta) e^{i\eta t} e^{ix\psi(\eta)} \hat{\phi}(\psi(\eta))\psi'(\eta) d\eta \right|^2 dt$$
$$= \int_{-\infty}^{+\infty} \left| \mathbb{F}^{-1}[i\psi(\eta) e^{ix\psi(\eta)} \hat{\phi}(\psi(\eta))\psi'(\eta)](t) \right|^2 dt$$
$$= \int_{-\infty}^{+\infty} \left| i\psi(\eta) e^{ix\psi(\eta)} \hat{\phi}(\psi(\eta))\psi'(\eta) \right|^2 dt$$
$$\leq \int_{-\infty}^{+\infty} |\psi(\eta)|^2 |\hat{\phi}(\psi(\eta))|^2 |\psi'(\eta)|^2 d\eta$$
$$\leq C \int_{-\infty}^{+\infty} (1+\eta^2) |\hat{\phi}(\eta)|^2 d\eta$$
$$= C\|\phi\|^2_{H^1(\mathbb{R})}.$$

Using Fatou's lemma and the similar process in the proof of Proposition 2.3, we can have the continuity of $\partial_x \tilde{v}(x,t)$ on x. So we proved (2.15). With the same method, we can also prove (2.16) and we omit the details. □

The following estimate follows from Theorem 2.1, Proposition 2.5 and Lemma 2.3.

Proposition 2.6. *The solution $u(x,t)$ of Eq. (2.2) satisfies the following properties*

$$u_x \in C_{b,x}([0,L]; H_t^1(0,+\infty)), \ \textit{for any } \phi(x) \in H^1(0,L);$$
$$u_{xx} \in C_{b,x}([0,L]; H_t^2(0,+\infty)), \ \textit{for any } \phi(x) \in H^2(0,L);$$

moreover, there exists a constant C such that

$$\sup_{x\in[0,L]} \|u_x(x,\cdot)\|_{L^2(0,+\infty)} \leq C\|\phi\|_{H^1(0,L)},$$
$$\sup_{x\in[0,L]} \|u_{xx}(x,\cdot)\|_{L^2(0,+\infty)} \leq C\|\phi\|_{H^2(0,L)}.$$

The following results reveal the regularity of the solution u of Eq. (2.1) concerned with the non-homogeneous term f.

Proposition 2.7. *Let $T > 0$ be given and $\phi = h_1 = h_2 = h_3 = 0$ in Eq. (2.1), the solution u of Eq. (2.1) has the following regularities:*

$$\sup_{x\in[0,L]} \|u_x(x,\cdot)\|_{L^2(0,T)} \leq C\|f\|_{L^1(0,T;H^{-1}(0,L))}, \textit{ for any } f \in L^1(0,T;H^{-1}(0,L));$$

$$\sup_{x\in[0,L]} \|u_{xx}(x,\cdot)\|_{L^2(0,T)} \leq C\|f\|_{L^1(0,T;L^2(0,L))}, \textit{ for any } f \in L^1(0,T;L^2(0,L)).$$

Proof. It's equivalent to discuss the property of solution u of Eq. (2.4) with $\widetilde{f} = (1-\partial_x^2)^{-1}f$. With semigroup theory, we can write the solution u of Eq. (2.4) in the following form:

$$u(x,t) = \int_0^t W_0(t-\tau)\widetilde{f}(x,\tau)d\tau.$$

We have

$$u_x(x,t) = \int_0^t \partial_x[W_0(t-\tau)\widetilde{f}(x,\tau)]d\tau.$$

Using the generalized Minkowski's inequality, we have

$$\begin{aligned}\|u_x(\cdot,t)\|_{L^2(0,T)} &= \Big[\int_0^T \Big|\int_0^T \partial_x[W_0(t-\tau)\widetilde{f}](\tau)d\tau\Big|^2 dt\Big]^{1/2} \\ &\leq \int_0^T \Big(\int_0^T |\partial_x[W_0(t-\tau)\widetilde{f}]|^2 dt\Big)^{1/2} d\tau.\end{aligned}$$

For

$$\sup_{x\in[0,L]} \|u(x,t)\|_{H^1(\mathbb{R}^+)} \leq C\|\phi\|_{H^1(0,L)},$$

we have

$$\begin{aligned}\sup_{x\in[0,L]} \|u_x(x,t)\|_{L^2(0,T)} &\leq \int_0^T \sup_{x\in[0,L]} \Big(\int_0^T |\partial_x[W_0(t-\tau)\widetilde{f}]|^2 dt\Big)^{1/2} d\tau \\ &\leq C\int_0^T \|\widetilde{f}(\cdot,t)\|_{H^1(0,L)} d\tau \\ &= C\|\widetilde{f}\|_{L^1(0,T;H^1(0,L))},\end{aligned}$$

for any $\widetilde{f} \in L^1(0,T;H^1(0,L))$. Since $\widetilde{f} = (1-\partial_x^2)^{-1}f$ and according to the definition of Sobolev space, we have

$$\sup_{x\in[0,L]} \|u(x,t)\|_{L^2(0,T)} \leq C\|f\|_{L^1(0,T;H^{-1}(0,L))},$$

for any $f \in L^1(0,T;H^{-1}(0,L))$. With the same strategy, we also have

$$\sup_{x\in[0,L]} \|u_{xx}(x,\cdot)\|_{L_t^2(0,T)} \leq C\|f\|_{L^1(0,T;L^2(0,L))},$$

for any $f \in L^1(0,T;L^2(0,L))$. The proof is complete. □

Please cite this article in press as: J. Li, K. Liu, Well-posedness of Korteweg–de Vries–Benjamin Bona Mahony equation on a finite domain, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.02.038

3. Local well-posedness

In this section, we will discuss the following full nonlinear IBVP problem

$$\begin{cases} u_t - u_{txx} + u_{xxx} + u_x + uu_x = 0, \quad u(x,0) = \phi(x), \quad x \in [0,L], \quad t \geq 0, \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t). \end{cases} \tag{3.1}$$

For convenience, we give the following declaration. For any given $T > 0$, $L > 0$ and $s \geq 0$, we denote

$$K_{s,T} := C(0,T; H^{s+1}(0,L)), \text{ with } \|u\|_{K_{s,T}} = \|u\|_{C(0,T;H^{s+1}(0,L))}, \text{ for any } u \in K_{s,T}.$$

The space $K_{s,T}$ has the following very useful property.

Lemma 3.1. *Let $s \geq 0$ be given. There exists a positive constant C, such that for any $u, v \in K_{s,T}$,*

$$\int_0^T \|(1-\partial_x^2)^{-1}\partial_x(uv)\|_{H^{s+1}(0,L)} dt \leq CT^2 \|u\|_{K_{s,T}} \|v\|_{K_{s,T}}. \tag{3.2}$$

Proof. For the definition of Sobolev space, we have

$$\|(1-\partial_x^2)^{-1}\partial_x(u(\cdot,t)v(\cdot,t))\|_{H^{s+1}(0,L)} \leq \|u(\cdot,t)v(\cdot,t)\|_{H^s(0,L)},$$

for any $s \geq 0$. When $s = 0$, using Gagliardo–Nirenberg interpolation inequality [6], we have

$$\begin{aligned} \|u(\cdot,t)v(\cdot,t)\|_{L^2(0,L)} &\leq \|u(\cdot,t)\|_{L^\infty(0,L)} \|v(\cdot,t)\|_{L^2(0,L)} \\ &\leq C(\|u(\cdot,t)\|_{L^2(0,L)} + \|u(\cdot,t)\|_{L^2(0,L)}^{1/2} \|u_x(\cdot,t)\|_{L^2(0,L)}^{1/2}) \|v(\cdot,t)\|_{L^2(0,L)} \\ &\leq C\|u(\cdot,t)\|_{H^1(0,L)} \|v(\cdot,t)\|_{H^1(0,L)}. \end{aligned}$$

So we have

$$\int_0^T \|(1-\partial_x^2)^{-1}\partial_x(u(\cdot,t)v(\cdot,t))\|_{H^1(0,L)} dt \leq CT^2 \|u\|_{K_{0,T}} \|u\|_{K_{0,T}}.$$

When $s = 1$, because

$$\|u(\cdot,t)v(\cdot,t)\|_{H^1(0,L)} \leq 2\big(\|u(\cdot,t)v(\cdot,t)\|_{L^2(0,L)} + \|\partial_x(u(\cdot,t)v(\cdot,t))\|_{L^2(0,L)}\big),$$

and

$$\begin{aligned} \|u(\cdot,t)v_x(\cdot,t)\|_{L^2(0,L)} &\leq C(\|u(\cdot,t)\|_{L^2(0,L)} + \|u(\cdot,t)\|_{L^2(0,L)}^{1/2} \|u_x(\cdot,t)\|_{L^2(0,L)}^{1/2}) \|v_x\|_{L^2(0,L)} \\ &\leq C(\|u(\cdot,t)\|_{L^2(0,L)} + \|u_x(\cdot,t)\|_{L^2(0,L)}) \|v_x\|_{L^2(0,L)} \\ &\leq C\|u(\cdot,t)\|_{H^1(0,L)} \|v(\cdot,t)\|_{H^1(0,L)}, \end{aligned}$$

and also

$$\|u_x(\cdot,t)v(\cdot,t)\|_{L^2(0,L)} \leq C\|u(\cdot,t)\|_{H^1(0,L)} \|v(\cdot,t)\|_{H^1(0,L)}.$$

So, we have

Please cite this article in press as: J. Li, K. Liu, Well-posedness of Korteweg–de Vries–Benjamin Bona Mahony equation on a finite domain, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.02.038

$$\int_0^T \|(1-\partial_x^2)^{-1}\partial_x(u(\cdot,t)v(\cdot,t))\|_{H^2(0,L)}dt \le CT^2\|u\|_{K_{0,T}}\|u\|_{K_{0,T}}$$
$$\le CT^2\|u\|_{K_{1,T}}\|u\|_{K_{1,T}}.$$

Estimate (3.2) with case $0 < s < 1$ follows form the nonlinear interpolation theory developed by Bona and Scott [2]. The proof for the case $s > 1$ is similar, so we omit the details. The proof is complete. □

For the following forced linear system

$$\begin{cases} u_t - u_{txx} + u_{xxx} + u_x = f, \quad u(x,0) = \phi(x), \quad x \in [0,L], \quad t \ge 0, \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t), \end{cases} \tag{3.3}$$

using the linear estimates we have got in Section 2, we know that for any $(\phi, \vec{h}) \in X_{0,T}$ and $f \in L^1(0,T;H^{-1}(0,L))$, we have the following estimate of the solution u of Eq. (3.3)

$$\|u\|_{K_{0,T}} \le C(\|(\phi,\vec{h})\|_{X_{0,T}} + \|f\|_{L^1(0,T;H^{-1}(0,L))}), \tag{3.4}$$

where C is a positive constant independent of $(\phi, \vec{h}) \in X_{0,T}$ and f. We can also have the estimate of the solution u of Eq. (3.3) in the space $K_{s,T}$ for the any $s \ge 0$.

Lemma 3.2. *Let $T > 0$, $L > 0$, $s \ge 0$, $f \in L^1(0,T;H^{s-1}(0,L))$ and $(\phi,\vec{h}) \in X_{s,T}$ be given, Eq. (3.3) admits a unique solution $u \in K_{s,T}$ with*

$$\|u\|_{K_{s,T}} \le C(\|(\phi,\vec{h})\|_{X_{s,T}} + \|f\|_{L^1(0,T;H^{s-1}(0,L))}), \tag{3.5}$$

where C is a positive constant independent of $(\phi,\vec{h}) \in X_{s,T}$ and f.

Proof. We first consider the case $s = 1$. According to Eq. (3.3), let $v = u_x$ then we have

$$\begin{cases} v_t - v_{txx} + v_{xxx} + v_x = f_x, \quad v(x,0) = \phi_x(x), \quad x \in [0,L], \quad t \ge 0, \\ v(0,t) = g_1(t), \quad v(L,t) = g_2(t), \quad v_x(L,t) = g_3(t), \end{cases} \tag{3.6}$$

where

$$g_1(t) = u_x(0,t), \quad g_2(t) = u_x(L,t) \quad \textit{and} \quad g_3(t) = u_{xx}(L,t),$$

and we denote $\overrightarrow{g(t)} = (g_1(t), g_2(t), g_3(t))$. Using (3.4), we directly have

$$\|v\|_{K_{0,T}} \le C(\|(\phi_x,\vec{g})\|_{X_{0,T}} + \|f_x\|_{L^1(0,T;H^{-1}(0,L))}). \tag{3.7}$$

According to Prop. 2.3, Prop. 2.4, Prop. 2.6 and Prop. 2.7, we have

$$\|g_1(t)\|_{L^2(0,L)} \le C(\|\phi\|_{H^1(0,L)} + \|h_1\|_{L^2(0,T)} + \|h_2\|_{L^2(0,T)} + \|h_3\|_{L^2(0,T)} + \|f\|_{L^1(0,T;H^{-1}(0,L))}),$$
$$\|g_2(t)\|_{L^2(0,L)} \le C(\|\phi\|_{H^1(0,L)} + \|h_1\|_{L^2(0,T)} + \|h_2\|_{L^2(0,T)} + \|h_3\|_{L^2(0,T)} + \|f\|_{L^1(0,T;H^{-1}(0,L))}),$$
$$\|g_3(t)\|_{L^2(0,L)} \le C(\|\phi\|_{H^2(0,L)} + \|h_1\|_{H^1(0,T)} + \|h_2\|_{H^1(0,T)} + \|h_3\|_{H^1(0,T)} + \|f\|_{L^1(0,T;L^2(0,L))}).$$

With (3.7) and $\|u\|_{K_{1,T}} \le 2(\|u\|_{K_{0,T}} + \|u_x\|_{K_{0,T}})$, we get

$$\|u\|_{K_{1,T}} \leq C(\|(\phi, \vec{h})\|_{X_{1,T}} + \|f\|_{L^1(0,T;L^2(0,L))}).$$

Since (3.3) has shown the result of the case $s = 0$, we can get the results of case of $0 < s < 1$ with interpolation. We can also use the iteration method to prove (3.5) for the case $s > 1$. The proof is complete. □

The following theorem shows the local well-posedness result for Eq. (3.1) in $K_{s,T}$ for any $s \geq 0$.

Theorem 3.1. *Let $T > 0$, $L > 0$ and $s \geq 0$ be given, for any $(\phi, \vec{h}) \in X_{s,T}$, there exists a $T^* \in (0, T]$ depending only on $\|(\phi, \vec{h})\|_{X_{s,T}}$ and a unique solution $u \in K_{s,T^*}$ of Eq. (3.1). Moreover, for any $T' < T^*$, there exists a neighbourhood $B(0, r)$ in $X_{s,T}$ such that there exists a unique solution u of Eq. (3.1) in the space $K_{s,T'}$ for any $(\phi, \vec{h}) \in B(0, r)$.*

Proof. We can write the solution u of Eq. (3.1) in the following form

$$u_t = W_0(t)\phi + W_b(t)\vec{h} - \int_0^t W_0(t-\tau)(1-\partial_x^2)^{-1}\partial_x(u^2/2)d\tau, \tag{3.8}$$

where $W_b(t) = (W_1(t), W_2(t), W_3(t))$ is defined as in (2.10). Let

$$G_{s,\theta,r} = \{v \in K_{s,\theta}, \|v\|_{K_{s,\theta}} \leq r\},$$

where θ and r are positive numbers and will be determined later. It's obvious that the set $G_{s,\theta,r}$ is a closed, convex, bounded and complete metric space induced from $K_{s,\theta}$. For any $(\phi, \vec{h}) \in X_{s,T}$ and $v \in G_{s,\theta,r}$, we define a map Γ as follows

$$\Gamma(v) = W_0(t)\phi + W_b(t)\vec{h} - \int_0^t W_0(t-\tau)(1-\partial_x^2)^{-1}\partial_x(v^2/2)d\tau.$$

Using Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}\|\Gamma(v)\|_{K_{s,\theta}} &\leq C_0\|(\phi, \vec{h})\|_{X_{s,T}} + C_1\int_0^\theta \|(1-\partial_x^2)^{-1}\partial_x(v^2/2)\|_{H^{s+1}(0,L)}(\tau)d\tau \\ &\leq C_0\|(\phi, \vec{h})\|_{X_{s,T}} + C_1\theta^2\|v\|_{K_{s,\theta}}^2,\end{aligned}$$

where $0 < \theta \leq T$. Since

$$\begin{aligned}\Gamma(v_1) - \Gamma(v_2) &= -\int_0^\theta W_0(t-\tau)(1-\partial_x^2)^{-1}\partial_x(v_1^2/2 - v_2^2/2)(\tau)d\tau \\ &= -\frac{1}{2}\int_0^\theta W_0(t-\tau)(1-\partial_x^2)^{-1}\partial_x[(v_1+v_2)(v_1-v_2)](\tau)d\tau,\end{aligned}$$

so we have

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{K_{s,\theta}} \leq C\theta^2\|v_1 + v_2\|_{K_{s,\theta}}\|v_1 - v_2\|_{K_{s,\theta}}.$$

We choose proper $0 < \theta \leq T$ and r such that

$$\begin{cases} r = 2C_0\|(\phi, \vec{h})\|_{X_{s,T}}, \\ 0 < C_1\theta^2 r \leq \frac{1}{2}, \end{cases}$$

then we have

$$\|\Gamma(v)\|_{K_{s,\theta}} \leq r,$$

for any $v \in G_{s,\theta,r}$, and we also have

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{K_{s,\theta}} \leq \frac{1}{2}\|v_1 - v_2\|_{K_{s,\theta}},$$

for any $v_1, v_2 \in G_{s,\theta,r}$. It means that the map Γ is a contraction map in $G_{s,\theta,r}$ and the fixed point $u = \Gamma(v)$ is the unique solution of Eq. (3.1) in $G_{s,\theta,r}$. The proof is complete. □

4. Global well-posedness

Theorem 3.1 shows the result of local well-posedness of Eq. (3.1) in the time interval $(0, T^*)$, where T^* locate in $(0, T]$ and only depends on $\|(\phi, \vec{h})\|_{X_{0,T}}$. If Eq. (3.1) has the well-posedness in $[0, T]$, that means T^* may be equal to T, then we say that Eq. (3.1) is global well-posed. In this section we will discuss the global well-posedness of Eq. (3.1). The space $E_{s,T}$ has been defined in Section 1, we restate it in the following

$$E_{s,T} = H^{s+1}(0, L) \times H^{s+1/2+\epsilon}(0, T) \times H^{s+1/2+\epsilon}(0, T) \times H^{s+1/2+\epsilon}(0, T),$$

where $s \geq 0$ and ϵ is any positive constant.

The following result gives a global priori H^s-*estimate* for smooth solutions of Eq. (3.1).

Proposition 4.1. *For any given $T > 0$, $L > 0$ and $s \geq 0$, there exists a continuous nondecreasing function $\Lambda_s : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{H^{s+1}(0,L)} \leq \Lambda_s(\|(\phi, \vec{h})\|_{E_{s,T}}), \tag{4.1}$$

for any smooth solution u of Eq. (3.1).

Proof. Let u be a smooth solution of Eq. (3.1) and $u = v + w$, where v be the solution of the following Eq. (4.2)

$$\begin{cases} v_t - v_{txx} + v_{xxx} + v_x = 0, \quad v(x, 0) = \psi(x), \quad x \in [0, L], \quad t \geq 0, \\ v(0, t) = h_1(t), \quad v(L, t) = h_2(t), \quad v_x(L, t) = h_3(t), \end{cases} \tag{4.2}$$

with $\psi(x) = (1 - x)h_1(0) + xh_2(0) - x(1 - x)(h_3(0) - h_2(0) + h_1(0))$ and w be the solution of the following Eq. (4.3)

$$\begin{cases} w_t - w_{txx} + w_{xxx} + w_x + ww_x = -(vw)_x - vv_x, \quad x \in [0, L], \quad t \geq 0, \\ w(x, 0) = \phi(x) - \psi(x), \quad w(0, t) = 0, \quad w(L, t) = 0, \quad w_x(L, t) = 0. \end{cases} \tag{4.3}$$

Using Sobolev embedding theorem [17], we have

$$\sup_{0 \leq t \leq T} |h_i(t)| \leq C_\epsilon \|h_i(t)\|_{H^{1/2+\epsilon}[0,T]}, \quad i = 1, 2, 3,$$

where ϵ is any positive constant. Since $\psi(x) = (1-x)h_1(0) + xh_2(0) - x(1-x)(h_3(0) - h_2(0) + h_1(0))$, we have

$$\begin{aligned}\|\psi\|_{H^{3/2+\epsilon}[0,L]} &\le 3C \sup_{0\le t\le T; i=1,2,3} |h_i(t)| \\ &\le 3CC_\epsilon(\|h_1(t)\|_{H^{1/2+\epsilon}[0,T]} + \|h_2(t)\|_{H^{1/2+\epsilon}[0,T]} + \|h_3(t)\|_{H^{1/2+\epsilon}[0,T]}) \\ &\le 3CC_\epsilon\|(\phi, \overrightarrow{h})\|_{E_{0,T}},\end{aligned}$$

and

$$\|\phi - \psi\|_{H^1(0,L)} \le 3CC_\epsilon\|(\phi, \overrightarrow{h})\|_{E_{0,T}}.$$

By Lemma 3.2, we have

$$\sup_{0\le t\le T} \|v(\cdot,t)\|_{H^{3/2+\epsilon}(0,L)} \le C\|(\psi, \overrightarrow{h})\|_{X_{1/2+\epsilon,T}} \le 4CC_\epsilon\|(\phi, \overrightarrow{h})\|_{E_{0,T}}.$$

Multiply both sides of Eq. (4.3) by w and integrate over $(0, L)$ with respect to x, we have

$$\frac{d}{dt}(\|w(\cdot,t)\|^2_{L^2(0,L)} + \|w_x(\cdot,t)\|^2_{L^2(0,L)}) \le C\Big(\int_0^L |w^2(\cdot,t)v_x(\cdot,t)|dx + \int_0^L |w(\cdot,t)v(\cdot,t)v_x(\cdot,t)|dx\Big).$$

Using the Sobolev embedding theorem, we have

$$\begin{aligned}\int_0^L |w^2(\cdot,t)v_x(\cdot,t)|dx &\le \sup_{0\le x\le L} |v_x(\cdot,t)|\|w(\cdot,t)\|^2_{L^2(0,L)} \\ &\le C_\epsilon\|v(\cdot,t)\|_{H^{3/2+\epsilon}(0,L)}\|w(\cdot,t)\|^2_{L^2(0,L)},\end{aligned}$$

and

$$\begin{aligned}\int_0^L |v_x(\cdot,t)v(\cdot,t)w(\cdot,t)|dx &\le \sup_{0\le x\le L} |v_x(\cdot,t)|\|v(\cdot,t)\|_{L^2(0,L)}\|w(\cdot,t)\|_{L^2(0,L)} \\ &\le C_\epsilon\|v(\cdot,t)\|^2_{H^{3/2+\epsilon}(0,L)}\|w(\cdot,t)\|_{L^2(0,L)},\end{aligned}$$

where ϵ is any positive constant. So we have the following estimate

$$\frac{d}{dt}\|w(\cdot,t)\|_{H^1(0,L)} \le C_\epsilon\|v(\cdot,t)\|_{H^{3/2+\epsilon}(0,L)}\|w(\cdot,t)\|_{H^1(0,L)} + C_\epsilon\|v(\cdot,t)\|^2_{H^{3/2+\epsilon}(0,L)}.$$

Using Gronwall's inequality, we have

$$\|w(\cdot,t)\|_{H^1(0,L)} \le \big(\|w(\cdot,0)\|_{H^1(0,L)} + C_\epsilon\int_0^t \|v(\cdot,\tau)\|^2_{H^{3/2+\epsilon}(0,L)}d\tau\big)exp\big\{C_\epsilon\int_0^t \|v(\cdot,\tau)\|_{H^{3/2+\epsilon}(0,L)}d\tau\big\},$$

$$\begin{aligned}\sup_{0\le t\le T}\|w(\cdot,t)\|_{H^1(0,L)} \le \big(&\|\phi-\psi\|_{H^1(0,L)} + C_\epsilon T\sup_{0\le t\le T}\|v(\cdot,t)\|^2_{H^{3/2+\epsilon}(0,L)}\big) \\ &\cdot exp\big\{C_\epsilon T\sup_{0\le t\le T}\|v(\cdot,t)\|_{H^{3/2+\epsilon}(0,L)}\big\}.\end{aligned}$$

With the estimates above, we have the following estimate

$$\sup_{0\le t\le T}\|w(\cdot,t)\|_{H^1(0,L)} \le \big(3CC_\epsilon\|(\phi,\vec{h})\|_{E_{0,T}} + 16CC_\epsilon T\|(\phi,\vec{h})\|^2_{E_{0,T}}\big)exp\{4CC_\epsilon T\|(\phi,\vec{h})\|_{E_{0,T}}\},$$

we have proved (4.1) is true for $s=0$.

Next, we consider the case $s=1$. Let $w=u_x$, then we have

$$\begin{cases} w_t - w_{txx} + w_{xxx} + w_x + (uw)_x = 0, \quad w(x,0)=\phi_x(x), \quad x\in[0,L], \quad t\ge 0,\\ w(0,t)=g_1(t), \quad w(L,t)=g_2(t), \quad w_x(L,t)=g_3(t), \end{cases}$$

where $(g_1,g_2,g_3)=(u_x(0,t),u_x(L,t),u_{xx}(L,t))=\overrightarrow{g}$. For any $0<T'\le T$, we have

$$\|w\|_{K_{0,T'}} \le C\|(\phi_x,\vec{g})\|_{X_{0,T}} + CT'^2\|u\|_{K_{0,T}}\|w\|_{K_{0,T'}}.$$

Choose proper $T'\le T$ such that $0<CT'\|u\|_{K_{0,T}}\le \frac{1}{2}$, then we have

$$\|w\|_{K_{0,T'}} \le 2C\|(\phi'_x,\vec{g})\|_{X_{0,T}}.$$

Since T' only depends on $\|u\|_{K_{0,T}}$ and $\|u\|_{K_{0,T}}$ only depends on $\|(\phi,\vec{h})\|_{E_{0,T}}$, we know that T' only depends on $\|(\phi,\vec{h})\|_{E_{0,T}}$. Using Hahn–Banach extension theorem, we have the following estimate

$$\|w\|_{K_{0,T}} \le C_1\|(\phi,\vec{h})\|_{E_{0,T}}.$$

So, we have proved that (4.1) is true for $s=1$.

We will use the nonlinear interpolation theory presented by Bona and Scott to prove (4.1) is true when $0<s<1$. Let's first introduce the nonlinear interpolation theory briefly, more details can be found in [2]. Let K_0 and K_1 be two Banach spaces satisfying $K_1\in K_0$ with the inclusion map being continuous. Let $f\in K_0$ and we define

$$G(f,t)=\inf_{g\in K_1}\{\|f-g\|_{K_0}+t\|g\|_{K_1}\}, \text{ for any } t>0.$$

For any $\theta\in(0,1)$ and $p\in[0,+\infty]$, we define

$$[K_0,K_1]_{\theta,p}=K_{\theta,p}=\big\{f\in K_0 : \|f\|_{\theta,p}=\big(\int_0^{+\infty} G(f,t)^p t^{-\theta p-1}dt\big)^{1/p}<+\infty\big\}.$$

It's obvious that $K_{\theta,p}$ is a Banach space. Let its norm be given as $\|\|_{\theta,p}$. We point out that $(\theta_1,p_1)<(\theta_2,p_2)$ means that

$$\begin{cases} \theta_1<\theta_2, \quad or\\ \theta_1=\theta_2, \quad and \quad p_1>p_2. \end{cases}$$

So, $(\theta_1,p_1)<(\theta_2,p_2)$ implies $E_{\theta_2,p_2}\subset K_{\theta_1,p_1}$ with the inclusion map being continuous. The following result Lemma 4.1 is cited from [2].

Lemma 4.1. *Let K_0^j and K_1^j be two Banach spaces satisfying $K_1^j\subset K_0^j$ with inclusion map being continuous, $j=1,2$. For any $\lambda\in(0,1)$ and $p\in[1,+\infty]$, we define a map Ψ satisfying the following conditions (i) and (ii)*

$$(i) \qquad \Psi : K^1_{\lambda,p} \to K^2_0 \text{ and for any } f, g \in K^1_{\lambda,p}$$
$$\|\Psi f - \Psi g\|_{K^2_0} \leq C_0(\|f\|_{K^1_{\lambda,p}} + \|g\|_{K^1_{\lambda,p}})\|f - g\|_{K^1_0};$$

and

$$(ii) \qquad \Psi : K^1_1 \to K^2_1 \text{ and for any } h \in K^1_1$$
$$\|\Psi h\|_{K^2_1} \leq C_1(\|h\|_{K^1_{\lambda,p}})\|h\|_{K^1_1},$$

where $C_j : \mathbb{R}+ \to \mathbb{R}^+$ *are continuous and nondecreasing functions,* $j = 0, 1$.

Then if $(\lambda, p) \leq (\theta, q)$*,* Ψ *maps* $K^1_{\theta,q}$ *into* $K^2_{\theta,q}$ *and*

$$\|\Psi f\|_{K^2_{\theta,q}} \leq C(\|f\|_{K^1_{\lambda,p}})\|f\|_{K^1_{\theta,q}}, \text{ for any } f \in K^1_{\theta,q},$$

where $C(r) = 4C_0(4r)^{1-\theta}C_1(3r)^\theta$*, for any* $r > 0$.

In order to use Lemma 4.1 to prove (4.1) is true for any $0 < s < 1$, we give the following declaration:

$$K^1_0 = E_{0,T}, \quad K^1_1 = E_{1,T}, \quad K^2_0 = C([0,T]; H^1(0,L)), \quad K^2_1 = C([0,T]; H^2(0,L)).$$

Let Γ be the solution map of the Eq. (3.1) and denote the solution u of Eq. (3.1) as $u = \Gamma(\phi, \overrightarrow{h})$. For any give $s \in (0, 1)$, let $q = 2$ and $\theta = s$, then

$$K^2_{\theta,q} = C([0,T]; H^{s+1}(0,L)), \qquad K^1_{\theta,q} = E_{s,T}.$$

It's obvious that (4.1) satisfies (ii) of Lemma 4.1 when $s = 1$ which has already been proved. We need only to verify the condition (i) of Lemma 4.1. Let $u_1 = \Gamma(\phi_1, \vec{h}_1)$ and $u_2 = \Gamma(\phi_2, \vec{h}_2)$ and $v = u_1 - u_2$, then v is the solution of the following equation

$$\begin{cases} v_t - v_{txx} + v_{xxx} + v_x + (\xi v)_x = 0, \quad v(x,0) = \phi_1(x) - \phi_2(x), \quad x \in [0,L], \quad t \geq 0, \\ v(0,t) = h_{1,1}(t) - h_{2,1}(t), \quad v(L,t) = h_{1,2}(t) - h_{2,2}(t), \quad v_x(L,t) = h_{1,3}(t) - h_{2,3}(t), \end{cases}$$

where $\xi = (u_1 + u_2)/2$. Using Lemma 3.2 for the case $s = 0$, for any $0 \leq T' \leq T$, we have

$$\begin{aligned} \|v\|_{K_{0,T'}} &\leq C\big(\|(\phi_1, \vec{h}_1) - (\phi_2, \vec{h}_2)\|_{X_{0,T}} + \|(\xi v)_x\|_{L^1(0,T';H^{-1}(0,L))}\big) \\ &\leq C\big(\|(\phi_1, \vec{h}_1) - (\phi_2, \vec{h}_2)\|_{X_{0,T}} + CT'^2\|\xi\|_{K_{0,T}}\|v\|_{K_{0,T'}}. \end{aligned}$$

Since

$$\|\xi\|_{K_{0,T}} \leq \Lambda_0(\|(\phi_1, \vec{h}_1)\|_{E_{0,T}} + \|(\phi_2, \vec{h}_2)\|_{E_{0,T}}),$$

we choose proper T', such that $0 < CT'^2\|\xi\|_{K_{0,T}} \leq \frac{1}{2}$, then we have

$$\|v\|_{K_{0,T'}} \leq \frac{1}{2}\|(\phi_1, \vec{h}_1) - (\phi_2, \vec{h}_2)\|_{E_{0,T}}.$$

Since T' only depends on $\|\xi\|_{K_{0,T}}$ and $\|\xi\|_{K_{0,T}}$ only depends on $\|(\phi_1, \vec{h}_1)\|_{E_{0,T}} + \|(\phi_2, \vec{h}_2)\|_{E_{0,T}}$, we know that T' only depends on $\|(\phi_1, \vec{h}_1)\|_{E_{0,T}} + \|(\phi_2, \vec{h}_2)\|_{E_{0,T}}$. Using Hahn–Banach extension theorem, we have the following estimate

$$\|v\|_{K_{0,T}} \leq \Lambda_0(\|(\phi_1,\vec{h}_1)\|_{E_{0,T}} + \|(\phi_2,\vec{h}_2)\|_{E_{0,T}})\|(\phi_1,\vec{h}_1) - (\phi_2,\vec{h}_2)\|_{E_{0,T}},$$

and the condition (i) in Lemma 4.1 is satisfied. Then, using Lemma 4.1, we know that (4.1) is true for any $0 < s < 1$.

When $1 < s < 2$, we let $v = u_x$. According to Eq. (3.1), we have

$$\begin{cases} v_t - v_{txx} + v_{xxx} = (uv)_x, \quad u(x,0) = \phi_x(x), \quad x \in [0,L], \quad t \geq 0, \\ v(0,t) = g_1(t), \quad v(L,t) = g_2(t), \quad v_x(L,t) = g_3(t), \end{cases}$$

where $\vec{g} = (g_1, g_2, g_3) = (u_x(0,t), u_x(L,t), u_{xx}(L,t))$. Applying Lemma 3.2, for any $T' \in (0,T]$, we have the following estimate

$$\|v\|_{K_{s-1,T'}} \leq C\|(\phi,\vec{h})\|_{E_{s,T}} + CT'^2\|u\|_{K_{s-1,T}}\|v\|_{K_{s-1,T'}},$$

where C is a positive constant independent of T' and $(\phi,\vec{h})$. We choose some proper T' such that $0 < CT'^2\|u\|_{K_{s-1,T}} \leq \frac{1}{2}$, then we have

$$\|v\|_{K_{s-1,T'}} \leq 2C\|(\phi,\vec{h})\|_{E_{s,T}}.$$

Since T' only depends on $\|u\|_{K_{s-1,T}}$ and according to the result we have proved that (4.1) is true for any $0 < s < 1$, we know that T' only depends on $\|(\phi,\vec{h})\|_{E_{s-1,T}}$ and we have

$$\|v\|_{K_{s-1,T}} \leq \Lambda_{s-1}(\|(\phi,\vec{h})\|_{E_{s-1,T}})\|(\phi,\vec{h})\|_{E_{s,T}}.$$

So we have the following estimate

$$\|u\|_{K_{s,T}} \leq C\Lambda_{s-1}(\|(\phi,\vec{h})\|_{E_{s-1,T}})\|(\phi,\vec{h})\|_{E_{s,T}}.$$

For the case $s \geq 2$, estimate (4.1) can be proved using the iteration method. The proof is complete. □

The global well-posedness of Eq. (3.1) follows directly from Proposition 4.1.

Theorem 4.1. *For any $T > 0$, $L > 0$, $s \geq 0$ and $(\phi, \vec{h}) = (\phi, h_1, h_2, h_3) \in E_{s,T}$, there exists a unique solution u of Eq. (3.1) with $u \in C([0,T], H^{s+1}(0,L))$.*

Acknowledgments

This work was supported by the Natural Science Foundation of China (No. 11231007). The authors deeply thank Professor Bingyu, Zhang in Cincinnati University for some very important advice on this problem. The authors also deeply thank the reviewers for their helpful comments on this paper.

References

[1] T.B. Benjamin, J.L. Bona, J.J. Mahony, Model equations for long waves in nonlinear dispersive systems, Philos. Trans. R. Soc. Lond. 272 (1972) 47–78.
[2] J.L. Bona, L.R. Scott, Solutions of the Korteweg–de Vries equation in fractional order Sobolev spaces, Duke Math. J. 43 (1976) 87–99.
[3] J.L. Bona, S.M. Sun, B.Y. Zhang, A nonhomogeneous boundary-value problem for the Korteweg–de Vries equation posed on a finite domain, Comm. Partial Differential Equations 28 (2003) 1391–1436.
[4] J.L. Bona, S.M. Sun, B.Y. Zhang, A non-homogeneous boundary-value problem for the Korteweg–de Vries equation posed on a finitedomain II, J. Differential Equations 247 (2009) 2558–2596.

[5] J.L. Bona, N. Tzvetkov, Sharp well-posedness results for the BBM equation, Discrete Contin. Dyn. Syst. 23 (2009) 1241–1252.

[6] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, United States, 2011.

[7] J.E. Colliander, C.E. Kenig, A generalized KdV-equation on the half-line, Comm. Partial Differential Equations 27 (2002) 2187–2266.

[8] M. Davila, G. Perla Menzala, Unique continuation for the Benjamin–Bona–Mahony and Boussinesq's equations, NoDEA Nonlinear Differential Equations Appl. 5 (3) (1998) 367–382.

[9] J.K. Engelbrecht, V.E. Fridman, E.N. Pelinovski, Nonlinear Evolution Equations, Longman, 1988.

[10] M. Francius, E.N. Pelinovsky, A.V. Slunyaev, Wave dynamics in nonlinear media with two dispersionless limits for long and short waves, Phys. Lett. A 280 (1–2) (2001) 53–57.

[11] C.E. Kenig, G. Ponce, L. Vega, Well-posedness of the Korteweg–de Vries equation, J. Amer. Math. Soc. 4 (1991) 33–69.

[12] E. Kramer, I. Rivas, B.Y. Zhang, Well-posedness of a class of non-homogeneous boundary value problems of the Korteweg–de Vries equation on a finite domain, ESAIM Control Optim. Calc. Var. 19 (2013) 358–384.

[13] N.A. Larkin, M.P. Vishnevskii, Dissipative initial boundary value problem for the BBM-equation, Electron. J. Differential Equations 149 (2008) 1–10.

[14] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci., vol. 44, Springer-Verlag, New York–Berlin–Heidelberg–Tokyo, 1983.

[15] S.I. Pokhozhaev, On the nonexistence of global solutions of some initial-boundary value problems for the Korteweg–de Vries equation, Differ. Equ. 47 (2010) 488–493.

[16] L. Rosier, B.Y. Zhang, Unique continuation property and control for the Benjamin–Bona–Mahony equation on a periodic domain, J. Differential Equations 254 (2013) 141–178.

[17] T. Tao, Local and Global Analysis of Nonlinear Dispersive and Wave Equations, CBMS Reg. Conf. Ser. Math., vol. 106, AMS, 2006.

[18] X. Zhang, E. Zuazua, Unique continuation for the linearized Benjamin–Bona–Mahony equation with space-dependent potential, Math. Ann. 325 (2003) 543–582.

Please cite this article in press as: J. Li, K. Liu, Well-posedness of Korteweg–de Vries–Benjamin Bona Mahony equation on a finite domain, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.02.038