



Radial continuous valuations on star bodies



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ABSTRACT

We show that a radial continuous valuation defined on the n -dimensional star bodies extends uniquely to a continuous valuation on the n -dimensional bounded star sets. Moreover, we provide an integral representation of every such valuation, in terms of the radial function, which is valid on the dense subset of the simple Borel star sets. Along the way, we also show that every radial continuous valuation defined on the n -dimensional star bodies can be decomposed as a sum $V = V^+ - V^-$, where both V^+ and V^- are positive radial continuous valuations.

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1. Introduction

This note continues the study of valuations on star bodies started in [18]. A valuation is a function V , defined on a class of sets, with the property that

$$V(A \cup B) + V(A \cap B) = V(A) + V(B).$$

As a generalization of the notion of measure, valuations have become a relevant area of study in convex geometry. In fact, this notion played a critical role in M. Dehn's solution to Hilbert's third problem, asking whether an elementary definition for volume of polytopes was possible. See, for instance, [15,16] and the references there included for a broad vision of the field.

Valuations on convex bodies belong to the Brunn–Minkowski theory. This theory has been extended in several important ways, and in particular, to the dual Brunn–Minkowski theory, where convex bodies, Minkowski addition and Hausdorff metric are replaced by star bodies, radial addition and radial metric, respectively. The dual Brunn–Minkowski theory, initiated in [17], has been broadly developed and successfully

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applied to several areas, such as integral geometry, local theory of Banach spaces and geometric tomography (see [5,10] for these and other applications). In particular, it played a key role in the solution of the Busemann–Petty problem [9,11,19].

D. A. Klain initiated in [13,14] the study of rotationally invariant valuations on a certain class of star sets, namely those whose radial function is n -th power integrable.

In [18], the second named author started the study of valuations on star bodies, characterizing positive rotation invariant valuations as those described by a certain integral representation.

The assumption of rotational invariance strongly simplifies the analysis in [18]. In this note, we drop that assumption and study continuous valuations on star bodies without further restrictions.

The main question in this context is whether radial continuous valuations in general admit an integral representation in the spirit of the representation valid for rotation invariant valuations. Such a representation would provide a detailed understanding of valuations. We do not fully answer this question, but we do give a partially positive answer which is probably already sufficient for many applications.

Our main result states that every radial continuous valuation can be extended to a continuous valuation on the bounded star sets, and this extension provides an integral representation of the valuation on the star sets with a simple Borel radial function. Note that the star sets with simple Borel radial function are dense, with the radial metric, in the space of bounded star sets.

For the sake of clarity, we split the result in two statements.

Theorem 1.1. *Let $V : \mathcal{S}_0^n \rightarrow \mathbb{R}$ be a radial continuous valuation on the n -dimensional star bodies \mathcal{S}_0^n . Then, there exists a unique radial continuous extension of V to a valuation $\bar{V} : \mathcal{S}_b^n \rightarrow \mathbb{R}$ on the bounded Borel star sets of \mathbb{R}^n .*

Theorem 1.2. *Let $V : \mathcal{S}_0^n \rightarrow \mathbb{R}$ be a radial continuous valuation, and let \bar{V} be its extension mentioned in Theorem 1.1. Then, there exists a Borel measure μ on S^{n-1} and a function $K : \mathbb{R}^+ \times S^{n-1} \rightarrow \mathbb{R}$ such that, for every star body L whose radial function ρ_L is a simple function, we have*

$$\bar{V}(L) = \int_{S^{n-1}} K(\rho_L(t), t) d\mu(t).$$

The main technical difficulties arise in the proof of Theorem 1.1. In the rotation invariant case, the uniqueness of the Lebesgue measure among normalized rotation invariant measures on the unit sphere of \mathbb{R}^n greatly simplified the study of the problem. In the general case, we do not have an equivalent result. Mimicking the techniques of [18], it is not too difficult to define a new valuation on the simple star sets. Difficulties arise when trying to extend this valuation to the bounded star sets, in order to check that it coincides with the original one. We do not know whether radial continuous valuations are uniformly continuous on bounded sets. For that reason, we do not know a priori that the valuation defined on the simple star sets preserves Cauchy sequences and can, therefore, be extended to its completion. We need to go through elaborate reasonings, especially in Section 6, to overcome this problem.

To prove Theorems 1.1 and 1.2, we also need an independent auxiliary result: a Jordan-like decomposition which will probably find applications elsewhere. We show that every continuous valuation $V : \mathcal{S}_0^n \rightarrow \mathbb{R}$ on the n -dimensional star bodies can be decomposed as the difference of two positive continuous valuations. With this structural result at hand, the study of continuous valuations on star bodies reduces to the simpler case of positive continuous valuations.

Theorem 1.3. *Let $V : \mathcal{S}_0^n \rightarrow \mathbb{R}$ be a radial continuous valuation on the n -dimensional star bodies \mathcal{S}_0^n such that $V(\{0\}) = 0$. Then, there exist two radial continuous valuations $V^+, V^- : \mathcal{S}_0^n \rightarrow \mathbb{R}_+$ such that $V^+(\{0\}) = V^-(\{0\}) = 0$ and such that*

$$V = V^+ - V^-.$$

Moreover, if V is rotationally invariant, then so are V^+ and V^- .

In the next paragraphs we describe the structure of the paper.

In Section 2 we describe our notation, framework and some known facts that we will need. Then, in Section 3 we show that continuous valuations are bounded on bounded sets, and prove some preliminary results needed later. In Section 4 we prove Theorem 1.3. As a simple application we solve a question left open in [18].

Section 5 is devoted to the construction of control and representing measures associated to a general valuation. It is based on similar work done in [18]. Once we have the representing measures, we can easily define a new valuation, \bar{V} , on the simple star sets.

In Section 6 we prove our main results, Theorems 1.1 and 1.2. This is the most technical part of the paper. As we said before, the main difficulties follow from the fact that we do not know whether V , or \bar{V} , are uniformly continuous on bounded sets. So, we have to prove “by hand” that \bar{V} preserves Cauchy sequences and, therefore, can be continuously extended to the bounded star sets. Once we have that, it is relatively simple to show that the restriction of this extension to the star bodies coincides with our original V .

Finally, in Section 7 we relate our results with existing previous work ([1,7,8]). In these papers, uniform continuity on bounded sets is assumed a priori. Hence, many of our difficulties are not present.

2. Notation and known facts

A set $L \subset \mathbb{R}^n$ is a *star set* if it contains the origin and every line through 0 that meets L does so in a (possibly degenerate) line segment. Let \mathcal{S}^n denote the set of the star sets of \mathbb{R}^n .

Given $L \in \mathcal{S}^n$, we define its *radial function* ρ_L by

$$\rho_L(t) = \sup\{c \geq 0 : ct \in L\},$$

for each $t \in \mathbb{R}^n$. Clearly, radial functions are completely characterized by their restriction to S^{n-1} , the euclidean unit sphere in \mathbb{R}^n , so from now on we consider them defined on S^{n-1} .

A star set L is called a *star body* if ρ_L is continuous. Conversely, given a positive continuous function $f : S^{n-1} \rightarrow \mathbb{R}^+ = [0, \infty)$ there exists a star body L_f such that f is the radial function of L_f . We denote by \mathcal{S}_0^n the set of n -dimensional star bodies and we denote by $C(S^{n-1})^+$ the set of positive continuous functions on S^{n-1} .

Analogously, a star set L is a *bounded Borel star set* if ρ_L is a bounded Borel function. Note that star bodies are always bounded. We denote by \mathcal{S}_b^n the set of n -dimensional bounded Borel star sets, Σ_n the σ -algebra of Borel subsets of S^{n-1} , and $B(S^{n-1})^+$ the set of positive bounded Borel functions on S^{n-1} .

Given two sets $K, L \in \mathcal{S}^n$, we define their *radial sum* $K \tilde{+} L$ as the star set whose radial function is $\rho_K + \rho_L$. Note that $K \tilde{+} L \in \mathcal{S}_0^n$ (respectively, \mathcal{S}_b^n) whenever $K, L \in \mathcal{S}_0^n$ (respectively, \mathcal{S}_b^n).

The dual analog for the Hausdorff metric of convex bodies is the so-called *radial metric*, which is defined by

$$\delta(K, L) = \inf\{\lambda \geq 0 : K \subset L \tilde{+} \lambda B_n, L \subset K \tilde{+} \lambda B_n\},$$

where B_n denotes the euclidean unit ball of \mathbb{R}^n . It is easy to check that

$$\delta(K, L) = \|\rho_K - \rho_L\|_\infty.$$

In this paper, *radial continuous* will always mean *continuous for the radial metric*.

A function $V : \mathcal{S}^n \rightarrow \mathbb{R}$ is a *valuation* if for any $K, L \in \mathcal{S}^n$,

$$V(K \cup L) + V(K \cap L) = V(K) + V(L).$$

It is clear that a linear combination of valuations is a valuation.

Given two functions $f_1, f_2 \in B(S^{n-1})^+$, we denote their maximum and minimum by

$$\begin{aligned}(f_1 \vee f_2)(t) &= \max\{f_1(t), f_2(t)\}, \\ (f_1 \wedge f_2)(t) &= \min\{f_1(t), f_2(t)\}.\end{aligned}$$

Given $K, L \in \mathcal{S}_0^n$ (respectively \mathcal{S}_b^n), both $K \cup L$ and $K \cap L$ are in \mathcal{S}_0^n (respectively \mathcal{S}_b^n), and it is easy to see that

$$\rho_{K \cup L} = \rho_K \vee \rho_L, \quad \rho_{K \cap L} = \rho_K \wedge \rho_L.$$

With this notation, a valuation $V : \mathcal{S}_0^n \rightarrow \mathbb{R}$ induces a function $\tilde{V} : C(S^{n-1})^+ \rightarrow \mathbb{R}$ given by

$$\tilde{V}(f) = V(L_f),$$

where L_f is the star body whose radial function satisfies $\rho_{L_f} = f$. If V is continuous, then \tilde{V} is continuous with respect to the $\|\cdot\|_\infty$ norm in $C(S^{n-1})^+$ and satisfies

$$\tilde{V}(f) + \tilde{V}(g) = \tilde{V}(f \vee g) + \tilde{V}(f \wedge g)$$

for every $f, g \in C(S^{n-1})^+$. Conversely, every such function \tilde{V} induces a continuous valuation on \mathcal{S}_0^n . Similarly, a valuation $V : \mathcal{S}_b^n \rightarrow \mathbb{R}$ induces a function $\tilde{V} : B(S^{n-1})^+ \rightarrow \mathbb{R}$ with analogous properties, and vice versa.

Given $A \subset S^{n-1}$, we denote the closure of A by \overline{A} . Given a function $f : S^{n-1} \rightarrow \mathbb{R}$, we define the support of f by

$$\text{supp}(f) = \overline{\{t \in S^{n-1} : f(t) \neq 0\}},$$

and for any set $G \subset S^{n-1}$, we will write $f \prec G$ if $\text{supp}(f) \subset G$. Conversely, $G \prec f$ denotes that $f(t) \geq 1$ for every $t \in G$. Throughout, given $A \subset S^{n-1}$, $\chi_A : S^{n-1} \rightarrow \mathbb{R}$ denotes the characteristic function of A , and $\mathbb{1} = \chi_{S^{n-1}}$ denotes the function identically equal to 1.

For completeness, we state now a result of [18] which will be needed later on several occasions.

Lemma 2.1. [18, Lemmas 3.3 and 3.4] *Let $\{G_i : i \in I\}$ be a family of open subsets of S^{n-1} . Let $G = \cup_{i \in I} G_i$. Then, for every $i \in I$ there exists a function $\varphi_i : G \rightarrow [0, 1]$ continuous in G satisfying $\varphi_i \prec G_i$ and such that $\bigvee_{i \in I} \varphi_i = \chi_G$. Moreover, let $f \in C(S^{n-1})^+$ satisfy $f \prec G$. Then, for every $i \in I$, the function $f_i = \varphi_i f$ belongs to $C(S^{n-1})^+$. Also, $f_i \prec G_i$ and $\bigvee_{i \in I} f_i = f$. In particular, for every $i \in I$, $0 \leq f_i \leq f$.*

It should be noted that most of the results stated in this work referring to $C(S^{n-1})$ could also be given for $C(K)$ where K is a compact metrizable space, with the same proofs. An adaptation of the results to the non-metrizable setting should also be possible by a standard application of uniformities (cf. [6]).

3. Preliminary results

To prove our main results we will need to control the maximum value of V on certain sets. The first step in this direction is to show that V is bounded on bounded sets:

We say that a valuation $V : \mathcal{S}_0^n \rightarrow \mathbb{R}$ is *bounded on bounded sets* if for every $\lambda > 0$ there exists a real number $R > 0$ such that, for every star body $L \subset \lambda B_n$, $|V(L)| \leq R$.

Equivalently, V is bounded on bounded sets if for every $\lambda > 0$ there exists $R > 0$ such that for every $f \in C(S^{n-1})^+$ with $\|f\|_\infty \leq \lambda$ we have $\tilde{V}(f) \leq R$.

Lemma 3.1. *Every radial continuous valuation $V : \mathcal{S}_0^n \rightarrow \mathbb{R}$ is bounded on bounded sets.*

Proof. We reason by contradiction. If the result is not true, there exists $\lambda > 0$ and a sequence $(f_i)_{i \in \mathbb{N}} \subset C(S^{n-1})^+$, with $\|f_i\|_\infty \leq \lambda$ for every $i \in \mathbb{N}$ and such that $|\tilde{V}(f_i)| \rightarrow +\infty$.

Consider the function

$$\theta : \mathbb{R}^+ \rightarrow \mathbb{R}$$

defined by

$$\theta(c) = \tilde{V}(c\mathbb{1}).$$

The continuity of \tilde{V} implies that θ is continuous. Therefore, θ is uniformly continuous on $[0, \lambda]$. In particular, it is bounded on that interval. Therefore, there exists $M > 0$ such that, for every $c \in [0, \lambda]$,

$$|\tilde{V}(c\mathbb{1})| \leq M.$$

We define inductively two sequences $(a_j)_{j \in \mathbb{N}}, (b_j)_{j \in \mathbb{N}} \subset \mathbb{R}^+$: First define $a_0 = 0$, $b_0 = \lambda$. Let $c_0 = \frac{a_0 + b_0}{2}$. We note that

$$\tilde{V}(f_i \vee c_0\mathbb{1}) + \tilde{V}(f_i \wedge c_0\mathbb{1}) = \tilde{V}(f_i) + \tilde{V}(c_0\mathbb{1}).$$

Since $|\tilde{V}(c_0\mathbb{1})| \leq M$ and $|\tilde{V}(f_i)| \rightarrow +\infty$, we know that there must exist an infinite set $M_1 \subset \mathbb{N}$ such that for $i \in M_1$ either $|\tilde{V}(f_i \vee c_0\mathbb{1})| \rightarrow +\infty$ or $|\tilde{V}(f_i \wedge c_0\mathbb{1})| \rightarrow +\infty$ as i grows to ∞ . In the first case, we set $a_1 = c_0$, $b_1 = \lambda$ and $f_i^1 = f_i \vee c_0\mathbb{1}$. In the second case, we set $a_1 = 0$ and $b_1 = c_0$ and $f_i^1 = f_i \wedge c_0\mathbb{1}$. Now we define $c_1 = \frac{a_1 + b_1}{2}$ and proceed similarly.

Inductively, we construct two sequences $(a_j), (b_j) \subset \mathbb{R}^+$, a decreasing sequence of infinite subsets $M_j \subset \mathbb{N}$, and sequences $(f_i^j)_{i \in M_j} \subset C(S^{n-1})^+$ such that, for every $j \in \mathbb{N}$,

$$|a_j - b_j| = \frac{\lambda}{2^j},$$

and for every $i \in M_j$, for every $t \in S^{n-1}$,

$$a_j \leq f_i^j(t) \leq b_j,$$

and with the property that

$$\lim_{i \rightarrow \infty} |\tilde{V}(f_i^j)| = +\infty.$$

Passing to a further subsequence we may assume without loss of generality that, for every $i \in \mathbb{N}$,

$$|\tilde{V}(f_i^i)| \geq i.$$

Call $d = \lim_i a_i$. If we consider now the sequence $(f_i^i)_{i \in \mathbb{N}} \subset C(S^{n-1})^+$, we have that

$$\|f_i^i - d\mathbb{1}\|_\infty \rightarrow 0$$

but

$$|\tilde{V}(f_i^i)| \geq i,$$

in contradiction to the continuity of \tilde{V} at $d\mathbb{1}$. \square

We thank the anonymous referee of [18] for suggesting a procedure very similar to this as an alternative reasoning to show a statement in that paper.

In the rest of this note we will repeatedly use the fact that S^{n-1} is a compact metric space. We will write d to denote the euclidean metric in S^{n-1} .

We need to recall an additional concept for our next result:

Definition 3.2. Given a set $A \subset S^{n-1}$, and $\omega > 0$, the *outer parallel band* around A is the set

$$A_\omega = \{t \in S^{n-1} : 0 < d(t, A) < \omega\}.$$

Note that, for every $A \subset S^{n-1}$ and $\omega > 0$, A_ω is an open set.

In our next result we use the fact that V is bounded on bounded sets to control V on these bands.

Lemma 3.3. Let $V : S_0^n \rightarrow \mathbb{R}$ be a radial continuous valuation. Let $A \subset S^{n-1}$ be any Borel set and $\lambda \in \mathbb{R}^+$. Then

$$\lim_{\omega \rightarrow 0} \sup\{|\tilde{V}(f)| : f \prec A_\omega, \|f\|_\infty \leq \lambda\} = 0.$$

Proof. We reason by contradiction. Suppose the result is not true. Then there exist $A \subset S^{n-1}$, $\lambda \in \mathbb{R}^+$, $\epsilon > 0$, a sequence $(\omega_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ and a sequence $(f_i)_{i \in \mathbb{N}} \subset C(S^{n-1})^+$ such that $\lim_{i \rightarrow \infty} \omega_i = 0$ and, for every $i \in \mathbb{N}$, the following conditions hold: $\omega_i > 0$, $f_i \prec A_{\omega_i}$, $\|f_i\|_\infty \leq \lambda$, and $|\tilde{V}(f_i)| \geq \epsilon$.

Therefore, there exists an infinite subset $I \subset \mathbb{N}$ such that either $\tilde{V}(f_i) > \epsilon$ for every $i \in I$ or $\tilde{V}(f_i) < -\epsilon$ for every $i \in I$. So, we assume without loss of generality that $\tilde{V}(f_i) > \epsilon$ for every $i \in I$. The case $\tilde{V}(f_i) < -\epsilon$ is totally analogous.

Consider f_1 . Using the continuity of \tilde{V} at f_1 , we get the existence of $\delta > 0$ such that for every $g \in C(S^{n-1})^+$ with $\|f_1 - g\|_\infty < \delta$,

$$|\tilde{V}(f_1) - \tilde{V}(g)| \leq \frac{\epsilon}{2}.$$

Since f_1 is uniformly continuous and $f_1(t) = 0$ for every $t \in A \subset S^{n-1} \setminus A_{\omega_1}$, there exists $0 < \rho < \omega_1$ such that, for every $t \in S^{n-1}$ with $d(t, A) < \rho$, $f_1(t) < \delta$. We consider the disjoint closed sets

$$C_1 = \{t \in S^{n-1} : d(t, A) \leq \frac{\rho}{2}\}$$

and

$$C_2 = f_1^{-1}([\delta, \lambda]).$$

By Urysohn's Lemma, we can consider a continuous function ψ_1 with $\psi_1|_{C_1} = 0$, $\psi_1|_{C_2} = 1$ and $0 \leq \psi_1(t) \leq 1$ for every $t \in S^{n-1}$. We consider now the function $\psi_1 f_1 \in C(S^{n-1})^+$. On the one hand, $\|f_1 - \psi_1 f_1\|_\infty \leq \delta$ and, therefore,

$$|\tilde{V}(\psi_1 f_1)| \geq |\tilde{V}(f_1)| - |\tilde{V}(f_1) - \tilde{V}(\psi_1 f_1)| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}.$$

On the other hand, $\psi_1 f_1 \prec A_{\omega_1} \setminus A_{\frac{\rho}{2}}$. Now, we can choose $\omega_{i_2} < \frac{\rho}{2}$ and we can reason similarly as above with the function f_{i_2} .

Inductively, we construct a sequence of functions $(\psi_j f_{i_j})_{j \in \mathbb{N}} \subset C(S^{n-1})^+$ with disjoint support such that $\tilde{V}(\psi_j f_{i_j}) > \frac{\epsilon}{2}$. Noting that

$$\tilde{V}\left(\bigvee_j \psi_j f_{i_j}\right) = \sum_j \tilde{V}(\psi_j f_{i_j}),$$

and that

$$\left\| \bigvee_j \psi_j f_{i_j} \right\|_{\infty} \leq \lambda,$$

we get a contradiction with the fact that V is bounded on bounded sets. \square

4. Proof of Theorem 1.3

In this section we prove Theorem 1.3 and, as a simple application, we complete the main result of [18].

Proof of Theorem 1.3. Let $V : \mathcal{S}_0^n \rightarrow \mathbb{R}$ be as in the hypothesis and consider the associated $\tilde{V} : C(S^{n-1})^+ \rightarrow \mathbb{R}$. For every $f \in C(S^{n-1})^+$, we define

$$\tilde{V}^+(f) = \sup\{\tilde{V}(g) : 0 \leq g \leq f\},$$

and we consider the function $V^+ : \mathcal{S}_0^n \rightarrow \mathbb{R}$ defined by $V^+(K) = \tilde{V}^+(\rho_K)$.

Assume for the moment that V^+ is a radial continuous valuation. In that case, the result follows easily:

First we note that it follows from $\tilde{V}(0) = 0$ that $V^+(\{0\}) = 0$ and that, for every $f \in C(S^{n-1})^+$, one has $\tilde{V}^+(f) \geq 0$. Therefore, $V^+(K) \geq 0$ for every $K \in \mathcal{S}_0^n$.

We next define $V^- = V^+ - V$. Clearly, V^- is a radial continuous valuation and $V^-(\{0\}) = 0$. By the definition of V^+ , it follows that, for every $K \in \mathcal{S}_0^n$, one has $V(K) \leq V^+(K)$. Thus, $V^-(K) \geq 0$. And clearly we have

$$V = V^+ - V^-.$$

Therefore, we will finish if we show that V^+ is a radial continuous valuation. First, we see that it is a valuation. Let $f_1, f_2 \in C(S^{n-1})^+$. We have to check that

$$\tilde{V}^+(f_1 \vee f_2) + \tilde{V}^+(f_1 \wedge f_2) = \tilde{V}^+(f_1) + \tilde{V}^+(f_2). \quad (1)$$

Fix $\epsilon > 0$. We choose $0 \leq g_1 \leq f_1$ such that $\tilde{V}^+(f_1) \leq \tilde{V}(g_1) + \epsilon$, and $0 \leq g_2 \leq f_2$ such that $\tilde{V}^+(f_2) \leq \tilde{V}(g_2) + \epsilon$.

Then,

$$\begin{aligned} \tilde{V}^+(f_1) + \tilde{V}^+(f_2) &\leq \tilde{V}(g_1) + \tilde{V}(g_2) + 2\epsilon = \tilde{V}(g_1 \vee g_2) + \tilde{V}(g_1 \wedge g_2) + 2\epsilon \\ &\leq \tilde{V}^+(f_1 \vee f_2) + \tilde{V}^+(f_1 \wedge f_2) + 2\epsilon, \end{aligned}$$

where the last inequality follows from the fact that $0 \leq g_1 \vee g_2 \leq f_1 \vee f_2$ and $0 \leq g_1 \wedge g_2 \leq f_1 \wedge f_2$. Since $\epsilon > 0$ was arbitrary, this proves one of the inequalities in (1).

For the other one, fix again $\epsilon > 0$. We choose $0 \leq g \leq f_1 \vee f_2$ such that $\tilde{V}^+(f_1 \vee f_2) \leq \tilde{V}(g) + \epsilon$, and $0 \leq h \leq f_1 \wedge f_2$ such that $\tilde{V}^+(f_1 \wedge f_2) \leq \tilde{V}(h) + \epsilon$. Consider the sets

$$A = \{t \in S^{n-1} : f_1(t) \geq f_2(t)\}$$

and

$$B = \{t \in S^{n-1} : f_1(t) < f_2(t)\}.$$

Let $\lambda = \|f_1 \vee f_2\|_\infty$. According to [Lemma 3.3](#), there exists $\omega_1 > 0$ such that, for every $f \prec A_{\omega_1}$ with $\|f\|_\infty \leq \lambda$ we have $|\tilde{V}(f)| \leq \epsilon$.

Since \tilde{V} is continuous at g , there exists $\delta > 0$ such that, $|\tilde{V}(g) - \tilde{V}(g')| < \epsilon$ for every g' such that $\|g - g'\|_\infty < \delta$. We define $g' = (g - \frac{\delta}{2}) \vee 0$. Then, for every $t \in A$, it follows that

$$g'(t) = \max \left\{ g(t) - \frac{\delta}{2}, 0 \right\} \leq g(t) \leq (f_1 \vee f_2)(t) = f_1(t).$$

Now, we can apply the uniform continuity of g and f_1 to find ω_2 such that for every $t, s \in S^{n-1}$, if $|t-s| < \omega_2$, then $|f_1(t) - f_1(s)| < \delta/4$ and $|g(t) - g(s)| < \delta/4$. In particular, this implies that for every $t \in A \cup A_{\omega_2}$, $g'(t) \leq f_1(t)$. On the other hand, it is clear that $g'(t) \leq f_2(t)$ for $t \in B$.

Let $\omega = \min\{\omega_1, \omega_2\}$, and let

$$J(A, \omega) = A \cup A_\omega$$

be the open ω -outer parallel set of the closed set A . Note that $S^{n-1} = J(A, \omega) \cup B$, where both $J(A, \omega)$ and B are open sets. Moreover, we clearly have $J(A, \omega) \cap B = A_\omega$.

We consider the functions $\varphi_1 \prec J(A, \omega)$, $\varphi_2 \prec B$ associated to the decomposition $S^{n-1} = J(A, \omega) \cup B$ by [Lemma 2.1](#). Then $\varphi_1 \vee \varphi_2 = \mathbb{1}$. Let us define $g'_1 = \varphi_1 g'$, $g'_2 = \varphi_2 g'$, $h_1 = \varphi_1 h$, $h_2 = \varphi_2 h$ as in [Lemma 2.1](#).

A simple verification yields

- $g' = g'_1 \vee g'_2$, $h = h_1 \vee h_2$,
- $g'_1 \wedge g'_2 \prec A_\omega$, $h_1 \wedge h_2 \prec A_\omega$,
- $g'_1 \wedge h_2 \prec A_\omega$, $h_1 \wedge g'_2 \prec A_\omega$,
- $0 \leq g'_1 \vee h_2 \leq f_1$,
- $0 \leq g'_2 \vee h_1 \leq f_2$.

Therefore, we get

$$\begin{aligned} \tilde{V}^+(f_1 \vee f_2) + \tilde{V}^+(f_1 \wedge f_2) &\leq \tilde{V}(g) + \tilde{V}(h) + 2\epsilon \leq \tilde{V}(g') + \tilde{V}(h) + 3\epsilon \\ &= \tilde{V}(g'_1) + \tilde{V}(g'_2) - \tilde{V}(g'_1 \wedge g'_2) + \tilde{V}(h_1) + \tilde{V}(h_2) - \tilde{V}(h_1 \wedge h_2) + 3\epsilon \\ &\leq \tilde{V}(g'_1) + \tilde{V}(h_2) + \tilde{V}(g'_2) + \tilde{V}(h_1) + 5\epsilon \\ &= \tilde{V}(g'_1 \vee h_2) + \tilde{V}(g'_1 \wedge h_2) + \tilde{V}(g'_2 \vee h_1) + \tilde{V}(g'_2 \wedge h_1) + 5\epsilon \\ &\leq \tilde{V}(g'_1 \vee h_2) + \tilde{V}(g'_2 \vee h_1) + 7\epsilon \leq \tilde{V}^+(f_1) + \tilde{V}^+(f_2) + 7\epsilon. \end{aligned}$$

Again, since $\epsilon > 0$ was arbitrary, this finishes the proof of [\(1\)](#).

Let us see now that \tilde{V}^+ is continuous. Let us consider $f_0 \in C(S^{n-1})^+$ and take $\epsilon > 0$. There exists $g_0 \in C(S^{n-1})^+$ with $0 \leq g_0 \leq f_0$ such that $\tilde{V}^+(f_0) \leq \tilde{V}(g_0) + \epsilon$.

Since \tilde{V} is continuous at f_0 and g_0 , there exists $\delta > 0$ such that for every $f, g \in C(S^{n-1})^+$ with $\|f_0 - f\|_\infty < \delta$ and $\|g_0 - g\|_\infty < \delta$, we have $|\tilde{V}(f_0) - \tilde{V}(f)| < \epsilon$ and $|\tilde{V}(g_0) - \tilde{V}(g)| < \epsilon$.

Let now $f \in C(S^{n-1})^+$ be such that $\|f_0 - f\|_\infty < \delta$. Pick $g \in C(S^{n-1})^+$ with $0 \leq g \leq f$ such that $\tilde{V}^+(f) \leq \tilde{V}(g) + \epsilon$.

Note that $\|g_0 \wedge f - g_0\| < \delta$ and $\|g \vee f_0 - f_0\| < \delta$. Then, we have

$$\tilde{V}^+(f) \geq \tilde{V}(g_0 \wedge f) \geq \tilde{V}(g_0) - \epsilon \geq \tilde{V}^+(f_0) - 2\epsilon,$$

and

$$\begin{aligned} \tilde{V}^+(f) &\leq \tilde{V}(g) + \epsilon = \tilde{V}(g \wedge f_0) + \tilde{V}(g \vee f_0) - \tilde{V}(f_0) + \epsilon \\ &\leq \tilde{V}(g \wedge f_0) + |\tilde{V}(g \vee f_0) - \tilde{V}(f_0)| + \epsilon \leq \tilde{V}^+(f_0) + 2\epsilon. \end{aligned}$$

Hence,

$$|\tilde{V}^+(f_0) - \tilde{V}^+(f)| < 2\epsilon$$

and \tilde{V}^+ is continuous as claimed.

The last statement follows immediately from the proof. \square

As an application of [Theorem 1.3](#), we can complete the main result of [\[18\]](#). In that paper, *positive* rotationally invariant continuous valuations V on the star bodies of \mathbb{R}^n , satisfying that $V(\{0\}) = 0$, are characterized by an integral representation as in [Corollary 4.1](#) below. The question of whether a similar description is valid for the case of real-valued (not necessarily positive or negative) continuous rotationally invariant valuations was left open.

Now, [Theorem 1.3](#) immediately gives a positive answer to this question:

Corollary 4.1. *Let $V : \mathcal{S}_0^n \rightarrow \mathbb{R}$ be a rotationally invariant radial continuous valuation on \mathcal{S}_0^n . Then, there exists a continuous function $\theta : [0, \infty) \rightarrow \mathbb{R}$ such that, for every $K \in \mathcal{S}_0^n$,*

$$V(K) = \int_{S^{n-1}} \theta(\rho_K(t)) dm(t),$$

where m is the Lebesgue measure on S^{n-1} normalized so that $m(S^{n-1}) = 1$.

Conversely, let $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous function. Then the function $V : \mathcal{S}_0^n \rightarrow \mathbb{R}$ given by

$$V(K) = \int_{S^{n-1}} \theta(\rho_K(t)) dm(t)$$

is a radial continuous rotationally invariant valuation.

Proof. Let $V : \mathcal{S}_0^n \rightarrow \mathbb{R}$ be a rotationally invariant radial continuous valuation. Then, the function defined by $V'(L) = V(L) - V(\{0\})$ is easily seen to be a rotationally invariant radial continuous valuation such that $V'(\{0\}) = 0$. We decompose it as $V' = V^+ - V^-$ as in [Theorem 1.3](#).

According to [\[18, Theorem 1.1\]](#), there exist two continuous functions $\theta^+, \theta^- : [0, \infty) \rightarrow \mathbb{R}$ such that, for every $K \in \mathcal{S}_0^n$,

$$V'(K) = V^+(K) - V^-(K) = \int_{S^{n-1}} \theta^+(\rho_K(t)) dm(t) - \int_{S^{n-1}} \theta^-(\rho_K(t)) dm(t).$$

We now define $\theta = \theta^+ - \theta^- + V(\{0\})$ and the first part of the result follows.

The converse statement was proven in [\[18, Theorem 1.1\]](#) (for that implication, the positivity is not needed). \square

Remark 4.2. As in [\[18\]](#), the function θ in [Corollary 4.1](#) is nothing but $\theta(\lambda) = V(\lambda S^{n-1})$.

5. Construction of the control measure and the representing measures

As in [18], one of the difficulties we face in the rest of the paper is the fact that V is not defined on star sets, so that we cannot a priori assign a meaning to $V(\chi_A)$. In order to assign a meaning to it, we proceed in two steps as in [18]: for each $\lambda \geq 0$, first we need a *control measure* μ_λ that will allow us next to define the *representing measure* ν_λ meant to extend V in the sense that $\nu_\lambda(A)$ is the natural assignation for the (not yet defined) $\bar{V}(\lambda\chi_A)$. Even when this measure ν_λ is defined, it is still not obvious how to extend V to the Borel measurable functions. This will be done in the next section.

For each $\lambda \geq 0$ we construct a control measure associated to a positive radial continuous valuation $V : \mathcal{S}_0^n \rightarrow \mathbb{R}^+$ exactly as it was done in [18], since rotational invariance did not play a role in that construction. We sketch the reasonings here and we refer the reader to [18] for a more detailed description.

For every $\lambda \geq 0$ we define the outer measure μ_λ^* as follows: For every open set $G \subset S^{n-1}$ we define

$$\mu_\lambda^*(G) = \sup\{\tilde{V}(f) : f \prec G, \|f\|_\infty \leq \lambda\}.$$

Next, for every $A \subset S^{n-1}$, we define

$$\mu_\lambda^*(A) = \inf\{\mu_\lambda^*(G) : A \subset G, G \text{ an open set}\}.$$

It is easy to see that both definitions coincide on open sets. It is not difficult to see that μ_λ^* is an outer measure [18, Proposition 3.5] and that the Borel sets of S^{n-1} are μ_λ^* measurable [18, Proposition 3.6], so that μ_λ , the restriction of μ_λ^* to the Borel σ -algebra of S^{n-1} , is a measure: for $\lambda \geq 0$ and any Borel set $A \subset S^{n-1}$,

$$\mu_\lambda(A) = \inf\left\{\sup\{\tilde{V}(f) : f \prec G, \|f\|_\infty \leq \lambda\} : A \subset G \text{ open}\right\}. \quad (2)$$

In [18], the rotational invariance of V was used to show that μ_λ was finite. Now we do not have rotational invariance, but Lemma 3.1 yields that, for every λ , μ_λ is finite.

We make explicit the control role of the μ_λ 's in the following observation:

Observation 5.1. *Let V be a positive radial continuous valuation and let μ_λ be the previously defined measure associated to it. For every $\lambda \geq 0$ and $\epsilon > 0$, if $G \subset S^{n-1}$ is an open set such that $\mu_\lambda(G) \leq \epsilon$, and $f \in C(S^{n-1})^+$ is such that $f \prec G$ and $\|f\|_\infty \leq \lambda$, then*

$$\tilde{V}(f) \leq \epsilon.$$

Observation 5.2. *For every $\lambda \geq 0$, μ_λ is a finite Borel measure on the compact metric space S^{n-1} . Hence, by Ulam's Theorem, μ_λ is regular (cf. [3, Theorem 7.1.4]). That is, for every Borel set $A \subset S^{n-1}$ we have*

$$\mu_\lambda(A) = \sup\{\mu_\lambda(K) : K \subset A, K \text{ compact}\} = \inf\{\mu_\lambda(G) : A \subset G, G \text{ open}\}.$$

As in [18], we will now define, for each $\lambda \geq 0$, a measure ν_λ which we will use to represent \tilde{V} . Again, we only sketch the construction here and we refer the reader to [18] for further details.

We recall that a *content* in S^{n-1} is a non-negative, finite, monotone set function defined on the family of all closed subsets of S^{n-1} , which is finitely subadditive and finitely additive on disjoint sets [12, §53]. For each $\lambda \geq 0$ we define a content in the following way:

Definition 5.3. For every closed set $K \subset S^{n-1}$, we define

$$\zeta_\lambda(K) = \inf\{\tilde{V}(f) : K \prec \frac{f}{\lambda}, \|f\|_\infty \leq \lambda\}.$$

Thus defined, ζ_λ can be well approximated from above by decreasing open sets:

Lemma 5.4. *Let $K \subset G \subset S^{n-1}$ be such that K is closed and G is open. Then*

$$\zeta_\lambda(K) = \inf\{\tilde{V}(f) : K \prec \frac{f}{\lambda} \prec G, \|f\|_\infty \leq \lambda\}.$$

Proof. One of the inequalities is trivial. We only need to check that $\zeta_\lambda(K) \geq \inf\{\tilde{V}(f) : K \prec \frac{f}{\lambda} \prec G, \|f\|_\infty \leq \lambda\}$. To see this, we choose $\epsilon > 0$. We now pick $f \in C(S^{n-1})^+$ with $K \prec \frac{f}{\lambda}$ and $\|f\|_\infty \leq \lambda$ such that $\zeta_\lambda(K) \geq \tilde{V}(f) - \epsilon$. The set $C = \text{supp}(f) \setminus G$ is closed (it could be empty, in which case the next reasonings are trivial). Therefore C is compact, and $K \cap C = \emptyset$.

Since μ_λ is regular, there exists an open set $H \supset C$, with $H \cap K = \emptyset$, such that $\mu_\lambda(H \setminus C) \leq \epsilon$. Therefore, $\mu_\lambda(G \cap H) \leq \mu_\lambda(H \setminus C) \leq \epsilon$. We apply now Lemma 2.1 to the open sets G, H and we obtain the functions φ_G, φ_H . We define $f_G = f\varphi_G$ and $f_H = f\varphi_H$. We have that $f = f_G \vee f_H$ and $\text{supp}(f_G \wedge f_H) \subset G \cap H$. Therefore, Observation 5.1 tells us that $\tilde{V}(f_G \wedge f_H) \leq \epsilon$. So, we have

$$\begin{aligned} \zeta_\lambda(K) &\geq \tilde{V}(f) - \epsilon = \tilde{V}(f_G \vee f_H) - \epsilon \geq \tilde{V}(f_G \vee f_H) - \epsilon + \tilde{V}(f_G \wedge f_H) - \epsilon \\ &= \tilde{V}(f_G) + \tilde{V}(f_H) - 2\epsilon \geq \tilde{V}(f_G) - 2\epsilon \geq \zeta_\lambda(K) - 2\epsilon, \end{aligned}$$

due to the positivity of \tilde{V} . Since $K \prec \frac{f_G}{\lambda} \prec G$ and $\|f_G\|_\infty \leq \lambda$, our result follows. \square

Now, the fact that ζ_λ is a content, and indeed a regular content, can be seen exactly as in [18, Lemmas 4.2 and 4.3]. Therefore, we can define a regular measure ν_λ associated to ζ_λ in a standard way (see [12, §53]) by setting, for each Borel set $A \subset S^{n-1}$,

$$\nu_\lambda(A) = \inf\{\sup\{\zeta_\lambda(K) : K \subset G\} : G \text{ open, } A \subset G\}. \quad (3)$$

It is easy to see that, for every closed set $K \subset S^{n-1}$, $\zeta_\lambda(K) = \nu_\lambda(K)$.

The measures ν_λ immediately provide the extension of the valuation to simple Borel star sets. The next two lemmas will allow us to have good control of the measures ν_λ .

Lemma 5.5. *Let $C \subset S^{n-1}$ be a closed set, $\lambda \geq 0$, $\epsilon > 0$ and let $G \subset S^{n-1}$ be an open set such that $C \subset G$ and such that $\mu_\lambda(G \setminus C) < \epsilon$. Then, for every pair of positive continuous functions f_1, f_2 such that, for $j = 1, 2$, $C \prec \frac{f_j}{\lambda} \prec G$ with $\|f_j\|_\infty \leq \lambda$, we have*

$$|\tilde{V}(f_1) - \tilde{V}(f_2)| \leq 6\epsilon.$$

Proof. Since \tilde{V} is continuous at f_1 and f_2 , there exists $\delta_1 > 0$ such that, for every $f \in C(S^{n-1})^+$, if $\|f - f_j\| \leq \delta_1$, then $|\tilde{V}(f) - \tilde{V}(f_j)| < \epsilon$, for $j = 1, 2$. We define $\delta = \min\{\delta_1, \frac{\lambda}{2}\}$ (this is just needed to make sure that $\lambda - \delta$ below is strictly greater than 0). Now, using the fact that both f_1 and f_2 are uniformly continuous, we get the existence of ρ such that, for every $t, s \in S^{n-1}$, $|t - s| < \rho$ implies that $|f_j(t) - f_j(s)| < \delta$, $j = 1, 2$.

Let

$$J(C, \rho) = \{t \in S^{n-1} : d(t, C) < \rho\}.$$

The paragraph above implies that, for $j = 1, 2$, for every $t \in J(C, \rho)$, $f_j(t) > \lambda - \delta$. For $j = 1, 2$ we define the functions

$$\tilde{f}_j = \lambda \mathbb{1} \wedge \left(\frac{f_j}{1 - \frac{\delta}{\lambda}} \right).$$

We clearly have that $\tilde{f}_j \in C(S^{n-1})^+$, $\tilde{f}_j \prec G$ and, for every $t \in J(C, \rho)$, $\tilde{f}_j(t) = \lambda$. Also, we have that

$$\|\tilde{f}_j - f_j\|_\infty \leq \delta.$$

(For this last inequality, note that if $f_j(t) \geq \lambda(1 - \frac{\delta}{\lambda})$, then $\tilde{f}_j(t) - f_j(t) = \lambda - f_j(t) \leq \delta$. Otherwise, if $f_j(t) < \lambda(1 - \frac{\delta}{\lambda})$, we have $\tilde{f}_j(t) - f_j(t) = f_j(t) \left(\frac{1}{1 - \frac{\delta}{\lambda}} - 1 \right) = f_j(t) \frac{\delta}{\lambda - \delta} < \delta$.)

We consider now the open sets $G_1 = G \cap \{t \in S^{n-1} : d(t, C) < \frac{2\rho}{3}\}$ and $G_2 = G \cap \{t \in S^{n-1} : \frac{\rho}{3} < d(t, C)\}$. We consider two functions $\varphi_i \prec G_i$, $i = 1, 2$ as in Lemma 2.1 and for $i = 1, 2$, $j = 1, 2$ we define the function $\tilde{f}_j^i = \varphi_i \tilde{f}_j$.

Then, for $j = 1, 2$, $\tilde{f}_j = \tilde{f}_j^1 \vee \tilde{f}_j^2$. Also, for every $t \in G_1$, $\tilde{f}_1(t) = \tilde{f}_2(t)$. Therefore, $\tilde{f}_1^1 = \tilde{f}_2^1$. Moreover, for $j = 1, 2$, $\tilde{f}_j^2 \prec G_2 \subset G \setminus C$ and, therefore, also $\tilde{f}_j^1 \wedge \tilde{f}_j^2 \prec G \setminus C$. Hence, by Observation 5.1, we have that, for $j = 1, 2$, $\tilde{V}(\tilde{f}_j^2) \leq \epsilon$ and $\tilde{V}(\tilde{f}_j^1 \wedge \tilde{f}_j^2) \leq \epsilon$.

Recalling that, for $j = 1, 2$,

$$\tilde{V}(\tilde{f}_j) = \tilde{V}(\tilde{f}_j^1) + \tilde{V}(\tilde{f}_j^2) - \tilde{V}(\tilde{f}_j^1 \wedge \tilde{f}_j^2),$$

we get

$$\begin{aligned} |\tilde{V}(f_1) - \tilde{V}(f_2)| &\leq |\tilde{V}(f_1) - \tilde{V}(\tilde{f}_1)| + |\tilde{V}(\tilde{f}_1) - \tilde{V}(\tilde{f}_2)| + |\tilde{V}(f_2) - \tilde{V}(\tilde{f}_2)| \\ &\leq |\tilde{V}(\tilde{f}_1) - \tilde{V}(\tilde{f}_2)| + 2\epsilon \\ &= |\tilde{V}(\tilde{f}_1^2) - \tilde{V}(\tilde{f}_1^1 \wedge \tilde{f}_1^2) - \tilde{V}(\tilde{f}_2^2) + \tilde{V}(\tilde{f}_2^1 \wedge \tilde{f}_2^2)| + 2\epsilon \leq 6\epsilon. \quad \square \end{aligned}$$

As an immediate corollary, we have:

Lemma 5.6. *Let $C \subset S^{n-1}$ be a closed set, $\lambda \geq 0$, $\epsilon > 0$ and let $G \subset S^{n-1}$ be an open set such that $C \subset G$ and such that $\mu_\lambda(G \setminus C) < \epsilon$. Then, for every $f \in C(S^{n-1})^+$ such that $\|f\|_\infty \leq \lambda$ and $C \prec \frac{f}{\lambda} \prec G$,*

$$\nu_\lambda(C) \leq \tilde{V}(f) \leq \nu_\lambda(C) + 7\epsilon.$$

Therefore, if for every $\omega > 0$ we choose $f_\omega \in C(S^{n-1})^+$ such that $\|f_\omega\|_\infty \leq \lambda$ and $C \prec \frac{f_\omega}{\lambda} \prec C \cup C_\omega$,

$$\lim_{\omega \rightarrow 0} \tilde{V}(f_\omega) = \nu_\lambda(C).$$

Proof. Using Lemma 5.4 and the fact that $\nu_\lambda(C) = \zeta_\lambda(C)$, we can choose $g \in C(S^{n-1})^+$ such that $\|g\|_\infty \leq \lambda$ and $C \prec \frac{g}{\lambda} \prec G$ and $\tilde{V}(g) \leq \nu_\lambda(C) + \epsilon$. Lemma 5.5 proves now the first part of the statement.

For the second part, it is enough to note that Lemma 3.3 implies that $\mu_\lambda((C \cup C_\omega) \setminus C) = \mu_\lambda(C_\omega)$ tends to 0 as ω tends to 0. \square

6. Proof of the main result

In this section we prove Theorems 1.1 and 1.2. The main technical difficulty is to prove that V , and its extension to the simple functions defined through the measures ν_λ , not only are continuous, but preserve Cauchy sequences.

To do this, we first show how a positive radial continuous valuation $V : \mathcal{S}_0^n \rightarrow \mathbb{R}^+$ can be extended to a positive radial continuous valuation $\bar{V} : \mathcal{S}_b^n \rightarrow \mathbb{R}^+$ on the bounded Borel star sets of \mathbb{R}^n . Once this is done, the positivity assumption can be removed using Theorem 1.3.

As we mentioned in the introduction, the star bodies of \mathbb{R}^n can be identified, by means of their radial functions, with the cone $C(S^{n-1})^+$ of the positive continuous functions defined on S^{n-1} . Similarly, the star sets of \mathbb{R}^n can be identified with $B(S^{n-1})^+$, the positive bounded Borel functions. We use Σ_n to denote the σ -algebra of the Borel subsets of S^{n-1} and $S(\Sigma_n)^+$ denotes the space of positive Borel simple functions, that is, functions of the form $\sum_{i=1}^n a_i \chi_{A_i}$ with $A_i \in \Sigma_n$ and $a_i \geq 0$ for $i = 1, \dots, n$. Recall that every bounded Borel function is the uniform limit of Borel simple functions.

For simplicity, in the rest of the paper we slightly abuse the notation and, whenever X is one of the spaces $C(S^{n-1})^+$, $B(S^{n-1})^+$ or $S(\Sigma_n)^+$, we say that a function $V : X \rightarrow \mathbb{R}$ is a valuation if, for every $f, g \in X$,

$$V(f \vee g) + V(f \wedge g) = V(f) + V(g).$$

With this notation, the result we need to prove can be stated as:

Theorem 6.1. *Let $\tilde{V} : C(S^{n-1})^+ \rightarrow \mathbb{R}^+$ be a continuous valuation with $\tilde{V}(0) = 0$. Then \tilde{V} admits a unique continuous extension $\bar{V} : B(S^{n-1})^+ \rightarrow \mathbb{R}^+$ which is also a valuation.*

Also for simplicity, we use the same notation for the valuation $\bar{V} : \mathcal{S}_b^n \rightarrow \mathbb{R}^+$ and its associated function $\bar{V} : B(S^{n-1})^+ \rightarrow \mathbb{R}^+$. It will be clear from the context to which of them we refer every time.

We will start by defining \bar{V} on simple functions: given a simple function $g = \sum_{i=1}^M a_i \chi_{A_i} \in S(\Sigma_n)^+$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, we set

$$\bar{V}(g) = \sum_{i=1}^M \nu_{a_i}(A_i).$$

We want to extend \bar{V} now to $B(S^{n-1})^+$, the closure of $S(\Sigma_n)^+$. To do this, we need to show that $\bar{V} : S(\Sigma_n)^+ \rightarrow \mathbb{R}^+$ preserves Cauchy sequences. In order to prove this, we need several prior technical results.

The following lemma is a refinement of [Lemma 5.5](#).

Lemma 6.2. *Let $C \subset S^{n-1}$ be a closed set. Let $\epsilon > 0$, $\lambda \geq 0$ and let $G \subset S^{n-1}$ be an open set such that $C \subset G$ and $\mu_{\lambda+1}(G \setminus C) < \epsilon$. For $j = 1, 2$, let $f_j \in C(S^{n-1})^+$ be such that $f_j \prec G$, $\|f_j\|_\infty \leq \lambda$ and $f_1(t) = f_2(t)$ for every $t \in C$. Then*

$$|\tilde{V}(f_1) - \tilde{V}(f_2)| \leq 8\epsilon.$$

Proof. Since \tilde{V} is continuous at f_1 and f_2 , there exists $d_1 > 0$ such that, for every $f \in C(S^{n-1})^+$, if $\|f - f_j\|_\infty \leq d_1$, then $|\tilde{V}(f) - \tilde{V}(f_j)| < \epsilon$, for $j = 1, 2$.

We define $d = \min\{d_1, 1\}$ and by Urysohn's lemma we consider a function $h \in C(S^{n-1})^+$ such that $0 \leq h \leq d$, $h|_C = d$ and $h|_{S^{n-1} \setminus G} = 0$. For $j = 1, 2$, let $\hat{f}_j = f_j + h$. We have that, for $j = 1, 2$, $\hat{f}_j \in C(S^{n-1})^+$ and satisfy

- $\|\hat{f}_j\|_\infty \leq \lambda + 1$
- $\min_{t \in C} \{\hat{f}_j(t)\} \geq d$
- $\hat{f}_j \prec G$
- $|\tilde{V}(\hat{f}_j) - \tilde{V}(f_j)| < \epsilon$.

Now we use again the continuity of \tilde{V} to find $\delta > 0$ such that, for $j = 1, 2$, for every $g \in C(S^{n-1})^+$, if $\|g - \hat{f}_j\|_\infty \leq \delta$ then $|\tilde{V}(g) - \tilde{V}(\hat{f}_j)| < \epsilon$.

We choose a real number α such that

$$\frac{1}{1 + \frac{\delta}{\lambda+1}} < \alpha < 1.$$

For $j = 1, 2$ we define the functions

$$\tilde{f}_j = (\hat{f}_1 \vee \hat{f}_2) \wedge \left(\frac{\hat{f}_j}{\alpha} \right).$$

Then, for $j = 1, 2$, we have that $\tilde{f}_j \in C(S^{n-1})^+$ with $\tilde{f}_j \prec G$ and $\|\tilde{f}_j\|_\infty \leq \lambda + 1$.

We define $\sigma = \frac{\delta}{2} \left(\frac{1-\alpha}{1+\alpha} \right) > 0$ and, using the fact that both \hat{f}_1 and \hat{f}_2 are uniformly continuous, we get the existence of ρ such that, for $j = 1, 2$, and for every $t, s \in S^{n-1}$, $|t - s| < \rho$ implies that $|\hat{f}_j(t) - \hat{f}_j(s)| < \sigma$.

Now, we have that for every t such that $d(t, C) < \rho$,

$$\tilde{f}_1(t) = \tilde{f}_2(t). \quad (4)$$

Indeed, take t such that $d(t, C) < \rho$. Then, there exists $s_0 \in C$ such that $|t - s_0| < \rho$. Therefore, for $j = 1, 2$, $\hat{f}_j(t) < \hat{f}_j(s_0) + \sigma$ and, hence,

$$\hat{f}_1(t) \vee \hat{f}_2(t) < \hat{f}_j(s_0) + \sigma.$$

Moreover

$$\frac{\hat{f}_j(t)}{\alpha} > \frac{1}{\alpha} \left(\hat{f}_j(s_0) - \sigma \right).$$

Now, the definition of σ implies that $\hat{f}_1(t) \vee \hat{f}_2(t) < \frac{\hat{f}_j(t)}{\alpha}$ and, hence, $\tilde{f}_1(t) = \tilde{f}_2(t) = \hat{f}_1(t) \vee \hat{f}_2(t)$.

Next, we show that

$$\|\tilde{f}_j - \hat{f}_j\|_\infty < \delta.$$

To see this, note first that if $\hat{f}_1(t) \vee \hat{f}_2(t) = \hat{f}_j(t)$, then $\tilde{f}_j(t) = \hat{f}_j(t) \wedge \frac{\hat{f}_j(t)}{\alpha} = \hat{f}_j(t)$, so we only need to consider the case when $\hat{f}_1(t) \vee \hat{f}_2(t) = \hat{f}_k(t)$, with $k \neq j$. In that case, $\tilde{f}_j(t) = \hat{f}_k(t) \wedge \frac{\hat{f}_j(t)}{\alpha}$ and we have

$$\begin{aligned} \left| \tilde{f}_j(t) - \hat{f}_j(t) \right| &= \left| \left(\hat{f}_k(t) \wedge \frac{\hat{f}_j(t)}{\alpha} \right) - \hat{f}_j(t) \right| = \left(\hat{f}_k(t) \wedge \frac{\hat{f}_j(t)}{\alpha} \right) - \hat{f}_j(t) \\ &\leq \frac{\hat{f}_j(t)}{\alpha} - \hat{f}_j(t) = \hat{f}_j(t) \left(\frac{1}{\alpha} - 1 \right) \\ &\leq (\lambda + 1) \left(\frac{1}{\alpha} - 1 \right) < \delta, \end{aligned}$$

where the last inequality follows from our choice of α .

We consider now the open sets $G_1 = \{t \in G : d(t, C) < \frac{2\rho}{3}\}$ and $G_2 = \{t \in G : \frac{\rho}{3} < d(t, C)\}$. We consider two functions $\varphi_i \prec G_i$, $i = 1, 2$ as in [Lemma 2.1](#) and for $i = 1, 2$, $j = 1, 2$ we define the function $\tilde{f}_j^i = \varphi_i \tilde{f}_j$.

Thus defined, the functions \tilde{f}_j^i satisfy the following conditions:

- since $\tilde{f}_1(t) = \tilde{f}_2(t)$ for every $t \in G_1$, we get that $\tilde{f}_1^1 = \tilde{f}_2^1$,
- $\text{supp}(\tilde{f}_j^2) \subset G \setminus C$, for $j = 1, 2$,

- $\tilde{f}_j = \tilde{f}_j^1 \vee \tilde{f}_j^2$, for $j = 1, 2$,
- $\text{supp}(\tilde{f}_j^1 \wedge \tilde{f}_j^2) \subset G \setminus C$,
- for $i, j = 1, 2$, $\|\tilde{f}_j^i\|_\infty \leq \lambda + 1$.

We note that, for $j = 1, 2$,

$$\tilde{V}(\tilde{f}_j) = \tilde{V}(\tilde{f}_j^1) + \tilde{V}(\tilde{f}_j^2) - \tilde{V}(\tilde{f}_j^1 \wedge \tilde{f}_j^2).$$

Finally, using the fact that $\mu_{\lambda+1}(G \setminus C) < \epsilon$ and [Observation 5.1](#), we have

$$\begin{aligned} |\tilde{V}(f_1) - \tilde{V}(f_2)| &\leq |\tilde{V}(\hat{f}_1) - \tilde{V}(\hat{f}_2)| + 2\epsilon \leq |\tilde{V}(\tilde{f}_1) - \tilde{V}(\tilde{f}_2)| + 4\epsilon \\ &\leq |\tilde{V}(\tilde{f}_1^2) - \tilde{V}(\tilde{f}_1^1 \wedge \tilde{f}_1^2) - \tilde{V}(\tilde{f}_2^2) + \tilde{V}(\tilde{f}_2^1 \wedge \tilde{f}_2^2)| + 4\epsilon \leq 8\epsilon. \quad \square \end{aligned}$$

Lemma 6.3. *Let $g = \sum_{i=1}^M a_i \chi_{A_i}$ be a positive simple function with $A_i \cap A_j = \emptyset$ for every $i \neq j$. Let $\lambda = \|g\|_\infty$. Then, for every $\rho > 0$, $\epsilon > 0$ there exists $f \in C(S^{n-1})^+$ and a set $A \subset S^{n-1}$ with $\mu_\lambda(S^{n-1} \setminus A) < \epsilon$ such that $|\tilde{V}(f) - \sum_{i=1}^M \nu_{a_i}(A_i)| < \rho$ and $f(t) = g(t)$ for every $t \in A$. Moreover, f can be chosen so that $\|f\|_\infty \leq \lambda$.*

Proof. Without loss of generality we can assume that $\bigcup_{i=1}^M A_i = S^{n-1}$. Let $N = \sum_{k=2}^M \binom{M}{k}$. Using the regularity of μ_λ , for every $1 \leq i \leq M$, we choose a closed set K_i and an open set G'_i such that $K_i \subset A_i \subset G'_i$ and such that

$$\mu_\lambda(G'_i \setminus K_i) < \min\left\{\frac{\epsilon}{M}, \frac{\rho}{21M}, \frac{\rho}{2N}\right\}.$$

Next, for $1 \leq i \leq M$, we define

$$G_i = G'_i \cap \bigcap_{j \neq i} K_j^c.$$

Note that we still have

$$\mu_\lambda(G_i \setminus K_i) < \min\left\{\frac{\epsilon}{M}, \frac{\rho}{21M}, \frac{\rho}{2N}\right\}. \quad (5)$$

Clearly, $\bigcup_{i=1}^M G_i = S^{n-1}$. We use [Lemma 2.1](#) to choose a lattice partition of unity $(\varphi_i)_{i=1}^M$ with $\varphi_i \prec G_i$ and $\bigvee_{i=1}^M \varphi_i = \mathbb{1}$.

We define $f_i = a_i \varphi_i$ and $f = \bigvee_{i=1}^M f_i$. Then, on the one hand, for every $t \in A = \bigcup_{i=1}^M K_i$, $f(t) = g(t)$. Note that

$$\mu_\lambda(S^{n-1} \setminus A) \leq \sum_{i=1}^M \mu_\lambda(G_i \setminus K_i) < \epsilon.$$

On the other hand, for every $1 \leq i \leq M$, $K_i \prec \frac{f_i}{a_i} \prec G_i$. Therefore it follows from [Lemma 5.6](#) and condition (5) that

$$|\tilde{V}(f_i) - \nu_{a_i}(K_i)| < \frac{\rho}{3M}.$$

Also, for every $i \neq j$, $\text{supp}(f_i \wedge f_j) \subset G_i \setminus K_i$, which, by [Observation 5.1](#), implies that, for $k \geq 2$ and $1 \leq i_1 < i_2 < \dots < i_k \leq M$, we have that

$$\tilde{V}(f_{i_1} \wedge f_{i_2} \wedge \dots \wedge f_{i_k}) < \rho/2N.$$

We can now apply [18, Lemma 3.1] and we get

$$\begin{aligned} \left| \tilde{V}(f) - \sum_{j=1}^M \nu_{a_j}(A_j) \right| &= \left| \sum_{k=1}^M (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq M} \tilde{V}(f_{i_1} \wedge \dots \wedge f_{i_k}) - \sum_{j=1}^M \nu_{a_j}(A_j) \right| \\ &\leq \left| \sum_{j=1}^M \tilde{V}(f_j) - \sum_{j=1}^M \nu_{a_j}(A_j) \right| + \sum_{k=2}^M \sum_{1 \leq i_1 < \dots < i_k \leq M} \frac{\rho}{2N} \\ &\leq \sum_{j=1}^M |\tilde{V}(f_j) - \nu_{a_j}(K_j)| + \sum_{j=1}^M \nu_{a_j}(A_j \setminus K_j) + \frac{\rho}{2} \\ &< \rho. \end{aligned}$$

For the last part of the statement, note that, for $1 \leq i \leq M$, $\|f_i\|_\infty \leq a_i$. \square

Given a Borel set $A \subset S^{n-1}$ and $g \in B(S^{n-1})$, we define $\|g\|_A = \sup_{t \in A} |g(t)|$.

The following lemma is a simple consequence of the fact that for $g \in C(S^{n-1})^+$, and any $A \subset S^{n-1}$, we have $\sup_{t \in A} |g(t)| = \sup_{t \in \bar{A}} |g(t)|$.

Lemma 6.4. *Let $(f_i)_{i \in \mathbb{N}} \subset C(S^{n-1})^+$, and let $A \subset S^{n-1}$ be a Borel set. If the sequence of restrictions $(f_i|_A)_{i \in \mathbb{N}}$ is a Cauchy sequence for the norm $\|\cdot\|_A$, then the sequence $(f_i|_{\bar{A}})_{i \in \mathbb{N}}$ (the sequence of restrictions to \bar{A}) is also a Cauchy sequence for the norm $\|\cdot\|_{\bar{A}}$.*

We will need the following result of Dugundji which we state for completeness ([4, Theorem 5.1]).

Theorem 6.5. *Let K be a compact metric space, and let $A \subset K$ be a closed subset. Then there exists a norm one simultaneous extender, that is, a norm one injective continuous linear mapping $T : C(A) \rightarrow C(K)$ such that, for every $f \in C(A)$, $T(f)|_A = f$. Moreover, T can be chosen so that, for every $f \in C(A)^+$, $T(f) \in C(K)^+$.*

Proof. Only the last statement is not explicitly stated in [4], but it follows immediately from the proof. \square

We can now prove the following.

Lemma 6.6. *Let $\lambda \geq 0$, $\epsilon > 0$. Let $B \subset S^{n-1}$ be a Borel set with $\mu_{\lambda+1}(B) < \epsilon$. Let $A = S^{n-1} \setminus B$ and let $(f_i)_{i \in \mathbb{N}} \subset C(S^{n-1})^+$ be a sequence such that $(f_i|_A)_{i \in \mathbb{N}}$ is a Cauchy sequence for the norm $\|\cdot\|_A$ and such that $\|f_i\|_\infty \leq \lambda$ for every $i \in \mathbb{N}$. Then, for every $\rho > 0$ there exists $N \in \mathbb{N}$ such that, for every $p, q > N$,*

$$|\tilde{V}(f_p) - \tilde{V}(f_q)| \leq 16\epsilon + \rho.$$

Proof. Using Lemma 6.4 we may assume that A is a closed set, thus B is open. We consider the simultaneous extender $T : C(A) \rightarrow C(S^{n-1})$ of Theorem 6.5. Then, for every $i \in \mathbb{N}$, $\|T(f_i)\|_\infty \leq \lambda$ and $(T(f_i))_{i \in \mathbb{N}} \subset C(S^{n-1})^+$ is a Cauchy sequence for the supremum norm, hence converges to some $f \in C(S^{n-1})^+$. Therefore, there exists i_0 such that, for every $p, q \geq i_0$,

$$|\tilde{V}(T(f_p)) - \tilde{V}(T(f_q))| < \rho.$$

Lemma 6.2 implies that, for every $i \in \mathbb{N}$,

$$|\tilde{V}(T(f_i)) - \tilde{V}(f_i)| \leq 8\epsilon.$$

Therefore, for every $i \geq i_0$, we have

$$\begin{aligned} |\tilde{V}(f_p) - \tilde{V}(f_q)| &\leq |\tilde{V}(f_p) - \tilde{V}(T(f_p))| + |\tilde{V}(T(f_p)) - \tilde{V}(T(f_q))| + \\ &\quad + |\tilde{V}(T(f_q)) - \tilde{V}(f_q)| < \rho + 16\epsilon. \quad \square \end{aligned}$$

Finally, we can prove the result that will allow us to extend \bar{V} to $B(S^{n-1})^+$:

Proposition 6.7. *Let $(g_i)_{i \in \mathbb{N}}$ be a Cauchy sequence of simple functions. Then $(\bar{V}(g_i))_{i \in \mathbb{N}}$ is a Cauchy sequence of real numbers.*

Proof. Since $(g_i)_{i \in \mathbb{N}}$ is Cauchy, there exists $\sup_{i \in \mathbb{N}} \|g_i\|_\infty = \lambda < \infty$. We fix $\epsilon > 0$. According to [Lemma 6.3](#), for every $i \in \mathbb{N}$ there exist $f_i^\epsilon \in C(S^{n-1})^+$ such that $|\tilde{V}(f_i^\epsilon) - \bar{V}(g_i)| < \epsilon$ and a Borel set $B_i^\epsilon \subset S^{n-1}$, with $\mu_{\lambda+1}(B_i^\epsilon) < \frac{\epsilon}{2^i}$, such that, for every $t \in A_i^\epsilon := S^{n-1} \setminus B_i^\epsilon$, $f_i^\epsilon(t) = g_i(t)$.

Let $B_\epsilon = \bigcup_{i \in \mathbb{N}} B_i^\epsilon$. Then $\mu_{\lambda+1}(B_\epsilon) < \epsilon$, and let $A_\epsilon = S^{n-1} \setminus B_\epsilon$.

We can apply [Lemma 6.6](#) and we get the existence of $N \in \mathbb{N}$ such that, for every $p, q \geq N$,

$$|\tilde{V}(f_p^\epsilon) - \tilde{V}(f_q^\epsilon)| < 17\epsilon.$$

Therefore, for $p, q \geq N$ we have

$$|\bar{V}(g_p) - \bar{V}(g_q)| \leq |\bar{V}(g_p) - \tilde{V}(f_p^\epsilon)| + |\tilde{V}(f_p^\epsilon) - \tilde{V}(f_q^\epsilon)| + |\tilde{V}(f_q^\epsilon) - \bar{V}(g_q)| \leq 19\epsilon. \quad \square$$

Therefore, $\bar{V} : S(\Sigma_n)^+ \rightarrow \mathbb{R}^+$ can be extended uniquely to a continuous function, which we will denote equally $\bar{V} : B(S^{n-1})^+ \rightarrow \mathbb{R}^+$. Namely, given $f \in B(S^{n-1})^+$ and any sequence $(f_n) \subset S(\Sigma_n)^+$ such that $\|f_n - f\|_\infty \rightarrow 0$ we can set

$$\bar{V}(f) = \lim_n \bar{V}(f_n). \quad (6)$$

By [Proposition 6.7](#), the limit above always exists and does not depend on the choice of $(f_n) \subset S(\Sigma_n)^+$.

Moreover, note that given $f, g \in B(S^{n-1})^+$ and $(f_n), (g_n) \subset S(\Sigma_n)^+$ such that $\|f_n - f\|_\infty \rightarrow 0$ and $\|g_n - g\|_\infty \rightarrow 0$ it follows that $f_n \vee g_n, f_n \wedge g_n \in S(\Sigma_n)^+$, $\|f_n \vee g_n - f \vee g\|_\infty \rightarrow 0$ and $\|f_n \wedge g_n - f \wedge g\|_\infty \rightarrow 0$. Thus we have

$$\begin{aligned} \bar{V}(f \vee g) + \bar{V}(f \wedge g) &= \lim_n \bar{V}(f_n \vee g_n) + \lim_n \bar{V}(f_n \wedge g_n) \\ &= \lim_n \bar{V}(f_n \vee g_n) + \bar{V}(f_n \wedge g_n) \\ &= \lim_n \bar{V}(f_n) + \bar{V}(g_n) \\ &= \lim_n \bar{V}(f_n) + \lim_n \bar{V}(g_n) \\ &= \bar{V}(f) + \bar{V}(g). \end{aligned}$$

This means that \bar{V} is a continuous valuation on $B(S^{n-1})^+$.

We show next that \bar{V} is actually an extension of \tilde{V} .

Proposition 6.8. *Let $\tilde{V} : C(S^{n-1})^+ \rightarrow \mathbb{R}^+$ and $\bar{V} : B(S^{n-1})^+ \rightarrow \mathbb{R}^+$ be as above. Then, for every $f \in C(S^{n-1})^+$, $\tilde{V}(f) = \bar{V}(f)$.*

Proof. Let $f \in C(S^{n-1})^+$. We will construct two sequences $(g_j)_{j \in \mathbb{N}} \subset S(\Sigma_n)^+$, $(f_j)_{j \in \mathbb{N}} \subset C(S^{n-1})^+$ such that, for every $j \in \mathbb{N}$,

$$\|g_j - f\|_\infty \leq \frac{1}{j}, \quad (7)$$

$$\|f_j - f\|_\infty \leq \frac{2}{j}, \quad (8)$$

$$|\bar{V}(g_j) - \tilde{V}(f_j)| \leq \frac{1}{j}. \quad (9)$$

The proof will be finished once we have constructed such sequences $(g_j)_{j \in \mathbb{N}}$, $(f_j)_{j \in \mathbb{N}}$, since, in that case,

$$\bar{V}(f) = \lim_j \bar{V}(g_j) = \lim_j \tilde{V}(f_j) = \tilde{V}(f).$$

We proceed to the construction of the sequences $(g_j)_{j \in \mathbb{N}}$, $(f_j)_{j \in \mathbb{N}}$. Let $\lambda = \|f\|_\infty$.

For each $j \in \mathbb{N}$ we make the following construction: Let $\delta = \frac{1}{j}$. We define $M = \left\lfloor \frac{\|f\|_\infty}{\delta} \right\rfloor + 1$, where $[x]$ denotes the integer part of x . Let $A_1 = f^{-1}([0, \delta])$ and, for $2 \leq i \leq M$,

$$A_i = f^{-1}(((i-1)\delta, i\delta]).$$

Now, we define $g_j = \sum_{i=1}^M i\delta\chi_{A_i}$. Clearly, (7) follows, since

$$\|g_j - f\|_\infty \leq \delta = \frac{1}{j}.$$

For $1 \leq i \leq M$, we proceed as in Lemma 6.3 to choose a closed set K'_i and an open set G'_i such that $K'_i \subset A_i \subset G'_i$, and $\mu_{\lambda+1}(G'_i \setminus K'_i) < \frac{\delta}{14M}$.

Next, define $K_1 = A_1$, and, for $2 \leq i \leq M$,

$$K_i = K'_i \cup f^{-1}\left(\left[\left(i - \frac{99}{100}\right)\delta, i\delta\right]\right).$$

Now, for $1 \leq i \leq M$, define

$$G''_i = G'_i \cap \bigcap_{k \neq i} K_k^c.$$

Finally, define

$$G_1 = G''_1 \cap f^{-1}\left(\left[0, \left(i + \frac{1}{100}\right)\delta\right]\right)$$

and, for $2 \leq i \leq M$,

$$G_i = G''_i \cap f^{-1}\left(\left((i-1)\delta, \left(i + \frac{1}{100}\right)\delta\right)\right).$$

Then we have that:

- $K_i \subset A_i \subset G_i$ for $1 \leq i \leq M$,
- $\mu_{\lambda+1}(G_i \setminus K_i) < \frac{\delta}{14M}$ for $1 \leq i \leq M$,
- $K_i \cap G_{i'} = \emptyset$ if $i \neq i'$,
- $\bigcup_{i=1}^M G_i = S^{n-1}$, and
- $G_i \cap G_{i'} = \emptyset$ if $|i - i'| > 1$.

We apply again [Lemma 2.1](#) to choose a lattice partition of unity $(\varphi_i)_{i=1}^M$ with $\varphi_i \prec G_i$ and $\bigvee_{i=1}^M \varphi_i = 1$. Then, we define $h_i = i\delta\varphi_i$ and set

$$f_j = \bigvee_{i=1}^M h_i.$$

Note that for every $t \in K_i$, since $K_i \cap G_{i'} = \emptyset$ for $i \neq i'$, we have

$$f_j(t) = h_i(t) = i\delta = g_j(t).$$

Otherwise, for $t \in G_i \setminus K_i$, since $G_i \cap G_{i'} = \emptyset$ if $|i - i'| > 1$, there are only two possibilities: $t \in G_i \cap G_{i-1}$ or $t \in G_i \cap G_{i+1}$. In the former case we have $f_j(t), g_j(t) \in [(i-1)\delta, i\delta]$, while in the latter we have $f_j(t), g_j(t) \in [i\delta, (i+1)\delta]$. Therefore,

$$\|f_j - g_j\|_\infty \leq \delta.$$

From this together with [\(7\)](#), we get [\(8\)](#):

$$\|f_j - f\|_\infty \leq 2\delta.$$

The coincidence of h_i and g_j on K_i , together with the fact that $\mu_{i\delta}(G_i \setminus K_i) < \frac{\delta}{14M}$, implies, by [Lemma 5.6](#) that

$$|\tilde{V}(h_i) - \nu_{i\delta}(K_i)| < \frac{\delta}{2M}.$$

Moreover, note that if $i' \notin \{i-1, i, i+1\}$, then $h_i \wedge h_{i'} = 0$. Otherwise, if $i' \in \{i-1, i+1\}$, $\text{supp}(h_i \wedge h_{i'}) \subset G_i \setminus K_i$. Also, for every three different indexes i, i', i'' , we have $h_i \wedge h_{i'} \wedge h_{i''} = 0$. Therefore, applying [\[18, Lemma 3.1\]](#) again we get

$$\begin{aligned} |\tilde{V}(f_j) - \overline{V}(g_j)| &= \left| \tilde{V}\left(\bigvee_{i=1}^M h_i\right) - \sum_{i=1}^M \nu_{i\delta}(A_i) \right| \\ &\leq \left| \sum_{i=1}^M \tilde{V}(h_i) - \sum_{i=1}^M \nu_{i\delta}(A_i) \right| + \left| \sum_{i=1}^{M-1} \tilde{V}(h_i \wedge h_{i+1}) \right| \\ &\leq \frac{\delta}{2} + \frac{\delta}{14} < \delta = \frac{1}{j}. \end{aligned}$$

This proves [\(9\)](#) and the result follows. \square

This finishes the proof of [Theorem 6.1](#) and, hence, also the proof of [Theorem 1.1](#).

Now we can prove [Theorem 1.2](#). The precise statement is

Theorem 6.9. *Let $\tilde{V} : C(S^{n-1})^+ \rightarrow \mathbb{R}$ be a continuous valuation. If we consider its extension $\overline{V} : B(S^{n-1})^+ \rightarrow \mathbb{R}$ given by [Theorem 1.1](#), then there exists a measure μ defined on the Borel σ -algebra of S^{n-1} and a function $K : \mathbb{R}^+ \times S^{n-1} \rightarrow \mathbb{R}$ such that, for every $g \in S(\Sigma_n)^+$, we have*

$$\overline{V}(g) = \int_{S^{n-1}} K(g(t), t) d\mu(t).$$

Proof. We first consider the radial continuous valuation $\tilde{V}'(f) = \tilde{V}(f) - \tilde{V}(0)$ together with its extension \overline{V}' . Using Theorem 1.3 we can write $\tilde{V}' = \tilde{V}_1 - \tilde{V}_2$, both of them positive valuations with $\tilde{V}_i(0) = 0$.

For $i = 1, 2$ and $\lambda \geq 0$, we consider the corresponding representing and control measures ν_λ^i and μ_λ^i as in Section 5.

For every $\lambda \geq 0$, we define the measure $\mu_\lambda = \mu_\lambda^1 + \mu_\lambda^2$, and we also define the normalized control measure μ by

$$\mu = \sum_{k=1}^{\infty} \frac{\mu_k}{2^k \mu_k(S^{n-1})}.$$

It is clear from the definitions that, for every $\lambda \geq 0$, for $i = 1, 2$, the measure ν_λ^i is continuous with respect to μ_λ^i . Since the control measures μ_λ are clearly monotonous with respect to λ , it follows that, for each $\lambda \geq 0$, μ_λ is continuous with respect to μ and, hence, also $\nu_\lambda := \nu_\lambda^1 - \nu_\lambda^2$ is continuous with respect to μ . By Radon–Nikodym's theorem, for every $\lambda \geq 0$ there exists a function $K'_\lambda \in L^1(\mu)$ such that

$$\nu_\lambda(A) = \int_A K'_\lambda(t) d\mu(t),$$

for every $A \in \Sigma_n$. Let $K' : \mathbb{R}^+ \times S^{n-1} \rightarrow \mathbb{R}$ be the function given by

$$K'(\lambda, t) = K'_\lambda(t).$$

Using the fact that $K'(0, t) = 0$ μ -a.e. t , for every $A \in \Sigma_n$ we have

$$\nu_\lambda(A) = \int_{S^{n-1}} K'(\lambda \chi_A(t), t) d\mu(t).$$

Therefore, for $g = \sum_{j=1}^n a_j \chi_{A_j} \in S(\Sigma_n)^+$ with pairwise disjoint $(A_j)_{j=1}^n$, we have

$$\overline{V}'(g) = \sum_{j=1}^n \nu_{a_j}(A_j) = \sum_{j=1}^n \int_{A_j} K'(a_j, t) d\mu(t) = \int_{S^{n-1}} K'(g(t), t) d\mu(t).$$

Defining $K(\lambda, t) = K'(\lambda, t) + \tilde{V}(0)$, we finish the proof. \square

7. Previous work and open questions

Integral representations in the spirit of Riesz's theorem have been previously considered for certain classes of (not necessarily linear) functionals on spaces $C(T)$, T a compact Hausdorff space. In particular, in a series of papers [1, 7, 8], N. Friedman et al. studied integral representations for additive functionals in spaces $C(T)$. Let us briefly recall their main result and terminology:

Definition 7.1. Given a compact Hausdorff space T , a functional $\phi : C(T) \rightarrow \mathbb{R}$ is called:

(1) *Additive*, if for any $f_1, f_2, f \in C(T)$ with $|f_1| \wedge |f_2| = 0$, it follows that

$$\phi(f_1 + f_2 + f) = \phi(f_1 + f) + \phi(f_2 + f) - \phi(f).$$

(2) *Bounded on bounded sets*, if for each $m > 0$, there is $M(m) > 0$ such that $|\phi(f)| \leq M(m)$ whenever $\|f\|_\infty \leq m$.

- (3) *Uniformly continuous on bounded sets*, if for every $\epsilon > 0$ and $m > 0$, there is $\delta(\epsilon, m) > 0$ such that $|\phi(f) - \phi(g)| \leq \epsilon$ whenever $\|f - g\|_\infty < \delta(\epsilon, m)$ with $\|f\|_\infty, \|g\|_\infty \leq m$.

Theorem 7.2. *Given a compact Hausdorff space T , and a functional $\phi : C(T) \rightarrow \mathbb{R}$, the following are equivalent:*

- (1) ϕ is additive, bounded on bounded sets and uniformly continuous on bounded sets.
- (2) There exist a measure μ of finite variation defined on the Borel σ -algebra of T , and a function $f : \mathbb{R} \times T \rightarrow \mathbb{R}$ such that
 - (a) $f(x, \cdot)$ is measurable for every x ,
 - (b) $f(\cdot, t)$ is continuous for μ -almost every t ,
 - (c) for each $m > 0$ there is $C_m > 0$ such that $|f(x, t)| \leq C_m$ for μ -almost every t , whenever $|x| \leq m$, such that for every $g \in C(T)$

$$\phi(g) = \int_T f(g(t), t) d\mu(t).$$

We will see now that additivity is the same as the property defining a valuation:

Lemma 7.3. *A mapping $\phi : C(T)^+ \rightarrow \mathbb{R}$ is an additive functional if and only if for every $f, g \in C(T)^+$,*

$$\phi(f) + \phi(g) = \phi(f \vee g) + \phi(f \wedge g).$$

Proof. Suppose first $\phi : C(T)^+ \rightarrow \mathbb{R}$ is an additive functional, that is

$$\phi(f_1 + f_2 + f) = \phi(f_1 + f) + \phi(f_2 + f) - \phi(f)$$

whenever $f_1 \wedge f_2 = 0$. Given $f, g \in C(T)^+$, let $f_1 = f - f \wedge g$ and $f_2 = g - f \wedge g$. It is clear that $f_1 \wedge f_2 = 0$, hence

$$\begin{aligned} \phi(f \vee g) &= \phi(f + g - f \wedge g) = \phi(f_1 + f_2 + f \wedge g) \\ &= \phi(f_1 + f \wedge g) + \phi(f_2 + f \wedge g) - \phi(f \wedge g) \\ &= \phi(f) + \phi(g) - \phi(f \wedge g). \end{aligned}$$

Therefore, ϕ is a valuation.

Conversely, let us suppose that $\phi : C(T)^+ \rightarrow \mathbb{R}$ is a valuation and take $f_1, f_2, f \in C(T)^+$ with $f_1 \wedge f_2 = 0$. We have that

$$\begin{aligned} \phi(f_1 + f) + \phi(f_2 + f) &= \phi((f_1 + f) \vee (f_2 + f)) + \phi((f_1 + f) \wedge (f_2 + f)) \\ &= \phi((f_1 \vee f_2) + f) + \phi((f_1 \wedge f_2) + f) \\ &= \phi(f_1 + f_2 + f) + \phi(f), \end{aligned}$$

which yields that ϕ is an additive functional. \square

As a side remark, note that every valuation clearly defines an *orthogonally additive functional*, that is $\phi(f + g) = \phi(f) + \phi(g)$ whenever $f \wedge g = 0$. However, not every orthogonally additive functional is a valuation, as the following simple example shows:

Example 7.4. Suppose T is a connected compact Hausdorff space. Let $\phi : C(T)^+ \rightarrow \mathbb{R}$ be given by $\phi(f) = \min\{f(t) : t \in K\}$. If $f \wedge g = 0$, then we have that $\phi(f) = \phi(g) = 0$, and as T is connected, ϕ is orthogonally additive. However, we can consider a partition of T into two sets A, B with $A \cap B = \emptyset$ and functions $f_A, g_B \in C(T)^+$ such that $f_A(t) = 1$ for every $t \in A$, $g_B(t) = 1$ for every $t \in B$ and for some $t_A \in A$ and $t_B \in B$ we have $f_A(t_B) = 0$ and $g_B(t_A) = 0$. It follows that

$$\phi(f_A) = \phi(g_B) = \phi(f_A \wedge g_B) = 0,$$

while $\phi(f_A \vee g_B) = 1$. Therefore, ϕ cannot be a valuation. Note that ϕ is continuous and satisfies $\phi(0) = 0$.

As we mentioned above the functionals under consideration in the work of Friedman et al. satisfy the additional assumptions of being bounded and uniformly continuous on bounded sets. Note that by [Lemmas 3.1 and 7.3](#), being bounded on bounded sets follows from continuity. We do not know whether continuity for valuations is actually enough to obtain uniform continuity on bounded sets. Note this last hypothesis is heavily used in [\[1,7,8\]](#) to obtain the desired integral representation.

Our main open questions now are the following

Question 7.5. *Is every radial continuous valuation on the star bodies of S^{n-1} uniformly continuous on bounded sets?*

Question 7.6. *Is the integral representation in [Theorem 1.2](#) valid for every star body?*

If [Question 7.5](#) would be true, then [Theorem 7.2](#) would imply a positive answer to [Question 7.6](#). So far, we do not even know that the function $K(\lambda, t)$ is measurable in the first variable for μ almost every t . Uniform continuity in bounded sets would imply that $K(\lambda, t)$ would be continuous μ -almost everywhere in the first variable [\[1\]](#), and the validity of the integral representation for continuous functions would follow.

The “2 dimensional densities” $K(\lambda, t)$ appearing in the integral representation of [Theorem 6.9](#) have certain continuity in the first variable, detailed in the lemma below, which is however not yet sufficient to answer the questions above.

Lemma 7.7. *Given $\lambda \geq 0$, for every $\epsilon > 0$ there exists $\delta > 0$ such that for every Borel set $A \subset S^{n-1}$, if $|\lambda - \lambda'| < \delta$ then*

$$|\nu_\lambda(A) - \nu_{\lambda'}(A)| < \epsilon.$$

Proof. The continuity of \overline{V} implies that, given ϵ , there exists δ such that, for every $g \in B(S^{n-1})^+$, if $\|\lambda \mathbb{1} - g\|_\infty < \delta$ then $|\overline{V}(\lambda \mathbb{1}) - \overline{V}(g)| < \delta$.

Let $A \subset S^{n-1}$ be a Borel set. Using that $\nu_\lambda(S^{n-1}) = \overline{V}(\lambda \mathbb{1})$, and defining $g = \lambda' \chi_A + \lambda \chi_{A^c}$, we get

$$|\nu_\lambda(A) - \nu_{\lambda'}(A)| = |\nu_\lambda(A) + \nu_\lambda(A^c) - \nu_{\lambda'}(A) - \nu_\lambda(A^c)| = |\overline{V}(\lambda \mathbb{1}) - \overline{V}(g)| < \epsilon. \quad \square$$

Proposition 7.8. *Let $\lambda \geq 0$. For every $\epsilon > 0$ there exists $\delta > 0$ such that, if $|\lambda - \lambda'| < \delta$ then*

$$\|K_\lambda - K_{\lambda'}\|_{L_1(\mu)} < 2\epsilon.$$

Proof. Given $\varphi \in L_1(\mu)$, if we define $A = \varphi^{-1}([0, \infty))$, then

$$\|\varphi\|_{L_1(\mu)} = \int_{S^{n-1}} \varphi(t)(\chi_A(t) - \chi_{A^c}(t)) d\mu(t).$$

Given $\epsilon > 0$, we take δ as in [Lemma 7.7](#) and considering $\varphi = K_\lambda - K_{\lambda'}$, we have

$$\begin{aligned} \|K_\lambda - K_{\lambda'}\|_{L_1(\mu)} &= \int_{S^{n-1}} \varphi(t)(\chi_A(t) - \chi_{A^c}(t))d\mu(t) \\ &= |\nu_\lambda(A) - \nu_\lambda(A^c) - (\nu_{\lambda'}(A) - \nu_{\lambda'}(A^c))| \\ &\leq |\nu_\lambda(A) - \nu_{\lambda'}(A)| + |\nu_\lambda(A^c) - \nu_{\lambda'}(A^c)| < 2\epsilon. \quad \square \end{aligned}$$

Finally, the next fact provides some additional information related to [Question 7.5](#).

Proposition 7.9. *Let $V : C(S^{n-1})^+ \rightarrow \mathbb{R}$ be a radial continuous valuation. Then, for every weakly compact subset $W \subset C(S^{n-1})^+$, V is uniformly continuous on W .*

Proof. Suppose the contrary. That is, there exist a weakly compact set $W \subset C(S^{n-1})^+$, $\epsilon > 0$ and two sequences $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \subset W$ such that

$$\|f_n - g_n\|_\infty \rightarrow 0, \quad (10)$$

while for every $n \in \mathbb{N}$

$$|V(f_n) - V(g_n)| \geq \epsilon. \quad (11)$$

Taking into account that W is weakly compact, by the Eberlein–Smulian Theorem (cf. [\[2\]](#)), passing to a further subsequence we can assume that $f_n \rightarrow f$ in the weak topology, for certain $f \in C(S^{n-1})^+$.

Since the point evaluations are continuous linear functionals in $C(S^{n-1})$, we thus have that $f_n \rightarrow f$ pointwise. Let $\lambda = \sup_{h \in W} \|h\|_\infty$. Now, by Egoroff’s Theorem (cf. [\[3\]](#)), there is a Borel set $A \subset S^{n-1}$ with $\mu_\lambda(S^{n-1} \setminus A) < \epsilon/17$ such that $\|f_n - f\|_A = \sup_{t \in A} |f_n(t) - f(t)| \rightarrow 0$. By [\(10\)](#), we also have $\|g_n - f\|_A \rightarrow 0$. Therefore, [Lemma 6.6](#) yields in particular that for some $N \in \mathbb{N}$ and every $n \geq N$

$$|V(f_n) - V(g_n)| < \epsilon,$$

which is a contradiction with [\(11\)](#). \square

In connection with [Question 7.5](#), if $V : C(S^{n-1})^+ \rightarrow \mathbb{R}$ is a radial continuous valuation which is not uniformly continuous on bounded sets, then there must be some bounded sequence $(f_n)_{n \in \mathbb{N}} \subset C(S^{n-1})^+$ and perturbations $(\tilde{f}_n)_{n \in \mathbb{N}}$ with $\|f_n - \tilde{f}_n\|_\infty \rightarrow 0$ but $|V(f_n) - V(\tilde{f}_n)| \geq \epsilon$ for some $\epsilon > 0$. [Proposition 7.9](#) yields that no subsequence of $(f_n)_{n \in \mathbb{N}}$ can be weakly Cauchy, hence, by Rosenthal’s ℓ_1 Theorem (cf. [\[2, Chapter XI\]](#)), the sequence $(f_n)_{n \in \mathbb{N}}$ must be equivalent to the unit basis of ℓ_1 in the sense that $\|\sum_n a_n f_n\| \approx \sum_n |a_n|$ (and should be a Rademacher-like sequence, see [\[2, Chapter XI\]](#) for details).

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