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Blow-up criteria for three-dimensional compressible radiation hydrodynamics equations with vacuum

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ABSTRACT

We consider the blow-up criteria for the Cauchy problems of three-dimensional compressible radiation fluids with vacuum. It is shown to own the same BKM-type criterion as the compressible Navier–Stokes equations [14], while the $L^{\tilde{p}}$ ($\tilde{p} \in [2, 3]$) norm of the density gradient should be involved for the Serrin-type criterion.

1. Introduction

This paper is concerned with the blow-up criteria to the strong solution with vacuum for the three-dimensional compressible isentropic radiation hydrodynamics (RHD) equations. The system is used in various astrophysical contexts [18] and in high-temperature plasma physics [17]. The couplings between fluid field and radiation field involve momentum source and energy source depending strongly on the specific radiation intensity derived by the so called radiative transfer integro-differential equation [28]. In particular, if the matter is in the local thermodynamical equilibrium (LTE), the system can be governed by the following Navier–Stokes–Boltzmann equations:

$$\begin{cases} \frac{1}{c} I_t + \Omega \cdot \nabla I = A_r, \quad (v, \Omega, t, x) \in \mathbb{R}^+ \times S^2 \times \mathbb{R}^+ \times \mathbb{R}^3, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ \left(\rho u + \frac{1}{c^2} F_r \right)_t + \operatorname{div}(\rho u \otimes u + P_r) + \nabla p_m = \operatorname{div}\mathbb{T}, \end{cases} \tag{1.1}$$

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where $\rho(t,x)$, $u(t,x)=(u_1,u_2,u_3)$ and $I(v,\Omega,t,x)$ denote the density, velocity field and specific radiation intensity, respectively.

In this system, S^2 is the unit sphere in $\mathbb{R}^3$, v is the frequency of photon and Ω is the travel direction of photon. The associated material pressure p_m is given by the state equation

$$p_m = A\rho^\gamma \tag{1.2}$$

for some positive constant A and adiabatic index $\gamma > 1$. Meanwhile, the stress tensor $\mathbb{T}$ equals

$$\mathbb{T} = \mu\left(\nabla u + (\nabla u)^\top\right) + \lambda\,(\mathrm{div} u)\,\mathbb{I}_3, \tag{1.3}$$

where $\mathbb{I}_3$ is the 3×3 unit matrix, $\mu > 0$ and $\lambda + \frac{2}{3}\mu \geq 0$ are the shear viscosity coefficient and bulk viscosity coefficient, respectively. These ensure the ellipticity of the Lamé operator.

As to the radiation part, the radiation flux F_r and the radiation pressure tensor P_r are defined by

$$F_r = \int_0^\infty \int_{S^2} I(v,\Omega,t,x)\Omega \mathrm{d}\Omega \mathrm{d}v, \;\; P_r = \frac{1}{c}\int_0^\infty \int_{S^2} I(v,\Omega,t,x)\Omega\otimes\Omega \mathrm{d}\Omega \mathrm{d}v,$$

and the collision term A_r in radiation transfer equation can be expressed as

$$A_r = S - \sigma_a I + \int_0^\infty \int_{S^2} \left(\frac{v}{v'}\sigma_s I' - \sigma_s' I\right)\mathrm{d}\Omega' \mathrm{d}v',$$

in which $I' = I(v',\Omega',t,x)$; $S = S(v,\Omega,t,x) \geq 0$ denotes the rate of energy emission due to spontaneous process; $\sigma_a = \sigma_a(v,\Omega,t,x,\rho) \geq 0$ is the absorption coefficient that may also depend on the mass density ρ; The differential scattering coefficient σ_s has two different state transitions:

$$\sigma_s \equiv \sigma_s(v' \to v, \Omega'\cdot\Omega, \rho) = O(\rho), \;\; \sigma_s' \equiv \sigma_s(v \to v', \Omega\cdot\Omega', \rho) = O(\rho).$$

Studying the radiation hydrodynamics equations is challenging because of its complexity and mathematical difficulty. For Navier–Stokes–Boltzmann equations, under some physical assumptions with the mass density away from vacuum, the local classical solution of the Cauchy problems was studied by Chen–Wang [6]. Ducomet and Nečasová [7,8] established the global weak solutions and the large time behavior in 1-D space. The local existence of strong solutions with vacuum was first established by Li–Zhu [23] when the initial data are arbitrarily large. They [22] also considered the formation of singularities to classical solutions when the initial mass density is compactly supported. For the inviscid radiation hydrodynamics equations (i.e., Euler–Boltzmann equations), we refer to [15–17,24].

As it was shown in [22], if we assume $\sigma_s = 0$, from the induced process and LTE assumption, the Navier–Stokes–Boltzmann system (1.1) can be rewritten into

$$\begin{cases} \dfrac{1}{c}I_t + \Omega\cdot\nabla I = -K_a\left(I - \bar{B}(v)\right), \quad (v,\Omega,t,x) \in \mathbb{R}^+ \times S^2 \times \mathbb{R}^+ \times \mathbb{R}^3, \\ \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u\otimes u) + \nabla p_m = \dfrac{1}{c}\displaystyle\int_0^\infty \int_{S^2} K_a\left(I - \bar{B}(v)\right)\Omega \mathrm{d}\Omega \mathrm{d}v + \mathrm{div}\mathbb{T}, \end{cases} \tag{1.4}$$

where $\bar{B}(v)$ represents the energy density of black-body radiation and $K_a(v,t,x,\rho) \geq 0$ ($K_a(v,t,x,0) = 0$) is the absorption coefficient. So, the impact of radiation on dynamical properties of fluid vanishes when $I \equiv \bar{B}(v)$, i.e., the system serves as the Navier–Stokes equations. For the rigorous justification of this type of limit, we refer to [9]. In the study of the Navier–Stokes equations, there are two well-known blow-up criteria for the strong (smooth) solutions: the Serrin-type (also called Ladyženskaja–Prodi–Serrin-type) criterion and the Beal–Kato–Majda type criterion.

We first present some results about incompressible fluid. Assuming T^* is the maximal existence time of the local strong (smooth) solutions, Serrin [29] first gave a Serrin-type criterion: if $T^* < \infty$, then

$$\lim_{T \to T^*} \|u\|_{L^s(0,T;L^r)} = \infty,$$

for any (r,s) with $\frac{2}{s} + \frac{3}{r} < 1$, $3 < r < \infty$. Later, Fabes, Jones and Riviere [10] extended the above criterion to the case $\frac{2}{s} + \frac{3}{r} = 1$. Beirão da Veiga also obtained a Serrin-type criterion by using the velocity and the pressure [2] or the gradient of velocity [3].

The BKM-type criterion was first established by Beal, Kato and Majda in [1]: if $T^* < \infty$, then

$$\lim_{T \to T^*} \|\omega(t)\|_{L^1(0,T;L^\infty)} = \infty,$$

where $\omega = \nabla \times u$ is the vorticity. Although it was shown for the Euler equations describing the motion of inviscid fluids, the same assertion also holds for Navier–Stokes equations. To see more BKM-type criteria, we refer to Chae–Choe [5] for controlling only two components of the vorticity, Kozono–Taniuchi [19] for extending to the marginal space BMO, He [11] for controlling any single component of the vorticity.

For the Cauchy problem of compressible Navier–Stokes equations, Huang, Li and Xin [13] established a BKM-type criterion: If $T^* < \infty$, then

$$\lim_{T \to T^*} \|D(u)\|_{L^1(0,T;L^\infty)} = \infty,$$

where $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the deformation tensor, and the Serrin-type criterion [14]: If $T^* < \infty$, then

$$\lim_{T \to T^*} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\sqrt{\rho}u\|_{L^s(0,T;L^r)}) = \infty,$$

where $\frac{2}{s} + \frac{3}{r} \leq 1$, $3 < r < \infty$. Moreover, if $\lambda < 7\mu$, then

$$\lim_{T \to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} = \infty.$$

By applying the Lamé operator, Sun et al. [30] obtained a blow-up criterion in bounded domain under condition $\lambda < 7\mu$: if $T^* < \infty$, then

$$\lim_{T \to T^*} \|\rho(t)\|_{L^\infty(0,T;L^\infty)} = \infty.$$

Wen–Zhu [32] extended this to non-isentropic flow with

$$\lim_{T \to T^*} \left(\|\rho(t)\|_{L^\infty(0,T;L^\infty)} + \|\theta(t)\|_{L^\infty(0,T;L^\infty)}\right) = \infty$$

under the assumption $3\lambda < 29\mu$.

On the other hand, when coupled with parabolic equations, some similar blow-up criteria can be proved for models like liquid crystal fluids, magnetoelectrical fluids and viscoelastic fluids, see [12,25], [27,33] and [26], respectively. Whereas, there are few results on the blow-up mechanism for Navier–Stokes equations coupled with a hyperbolic equation. The main purpose of our paper is to give both a Serrin-type and a BKM-type criterion for strong solutions to system (1.1), which can be viewed as the Navier–Stokes equations coupled with a transport equation. The interesting point is that we will give a same BKM-type criterion but a different Serrin-type criterion for using the $L^{\tilde{p}}(\tilde{p} \in [2,3])$ norm of the density gradient to control the higher order radiation terms.

Throughout this paper, we use the following notations

Notations.

(1) Some simplified notations for standard homogeneous and inhomogeneous Sobolev spaces:

$$\begin{aligned}
&\|(f,g)\|_X = \|f\|_X + \|g\|_X,\ \|f\|_{X_1 \cap X_2} = \|f\|_{X_1} + \|f\|_{X_2},\\
&|f|_p = \|f\|_{L^p(\mathbb{R}^3)},\ \|f\|_{W^{m,p}} = \|f\|_{W^{m,p}(\mathbb{R}^3)},\ \|f\|_s = \|f\|_{H^s(\mathbb{R}^3)},\\
&\|f(v,\Omega,t,x,\rho(t,x))\|_{X_1(\mathbb{R}^+\times S^2;X_2(\mathbb{R}^3))} = \big\|\|f(v,\Omega,t,\cdot,\rho(t,\cdot))\|_{X_2(\mathbb{R}^3)}\big\|_{X_1(\mathbb{R}^+\times S^2)},\\
&\|f(v,\Omega,t,x,\rho(t,x))\|_{X_1(\mathbb{R}^+\times S^2;X_2([0,T]\times\mathbb{R}^3))} = \big\|\|f(v,\Omega,\cdot,\cdot,\rho(\cdot,\cdot))\|_{X_2([0,T]\times\mathbb{R}^3)}\big\|_{X_1(\mathbb{R}^+\times S^2)},\\
&D^k = \{f \in L^1_{loc}(\mathbb{R}^3) : |f|_{D^k} = |\nabla^k f|_{L^2} < +\infty\},\quad |f|_{D^k} = \|f\|_{D^k(\mathbb{R}^3)},\\
&\mathbb{D}^1 = \{f \in L^6(\mathbb{R}^3) : |f|_{\mathbb{D}^1} = |\nabla f|_{L^2} < \infty\},
\end{aligned}$$

where $0 < T < \infty$ is a constant, X, X_1, and X_2 are some Sobolev spaces.

(2) $G = (2\mu+\lambda)\mathrm{div}u - p_m$ is called the effective viscous flux.

(3) $\dot{f} = f_t + u\cdot\nabla f$ denotes the material derivative.

Next we make some assumptions on the physical coefficients σ_a and σ_s.

Assumption A. Let $\sigma_s = \overline{\sigma}_s(v' \to v, \Omega'\cdot\Omega)\rho$, $\sigma'_s = \overline{\sigma}'_s(v \to v', \Omega\cdot\Omega')\rho$, where the functions $\overline{\sigma}_s \geq 0$ and $\overline{\sigma}'_s \geq 0$ satisfy

$$\begin{cases}
\displaystyle\int_0^\infty\int_{S^2}\Big(\int_0^\infty\int_{S^2}\Big|\frac{v}{v'}\Big|^2\overline{\sigma}_s^2\mathrm{d}\Omega'\mathrm{d}v'\Big)^{\lambda_1}\mathrm{d}\Omega\mathrm{d}v \leq C,\\
\displaystyle\int_0^\infty\int_{S^2}\Big(\int_0^\infty\int_{S^2}\overline{\sigma}'_s\mathrm{d}\Omega'\mathrm{d}v'\Big)^{\lambda_2}\mathrm{d}\Omega\mathrm{d}v + \int_0^\infty\int_{S^2}\overline{\sigma}'_s\mathrm{d}\Omega'\mathrm{d}v' \leq C,
\end{cases} \tag{1.5}$$

where $\lambda_1 = 1$ or $\frac{1}{2}$, and $\lambda_2 = 1$ or 2. Here we denote by C a generic positive constant depending only on the fixed constants μ, λ, γ, $\tilde{q}$, T and the norms of S, where $\tilde{q} \in (3,6]$ and will be given in (2.3).

Assumption B. Let $\sigma_a(v,\Omega,t,x,\rho) = \overline{\sigma}_a(v,\Omega,t,x,\rho)\rho$, for

$$|\rho(t)|_\infty + |\nabla\rho(t)|_r < +\infty,\ \text{for } r \in [2,\tilde{q}], \tag{1.6}$$

then we have

$$\begin{cases}
\|\overline{\sigma}_a(v,\Omega,t,x,\rho(t,x))\|_{L^2\cap L^\infty(\mathbb{R}^+\times S^2;L^\infty(\mathbb{R}^3))} \leq M(|\rho(t)|_\infty),\\
\|\nabla\overline{\sigma}_a(v,\Omega,t,x,\rho)\|_{L^2\cap L^\infty(\mathbb{R}^+\times S^2;L^r(\mathbb{R}^3))} \leq M(|\rho(t)|_\infty)(|\nabla\rho(t)|_r + 1),\\
\|(\overline{\sigma}_a)_t(v,\Omega,t,x,\rho)\|_{L^2(\mathbb{R}^+\times S^2;L^r(\mathbb{R}^3))} \leq M(|\rho(t)|_\infty)(|\rho_t(t)|_r + 1),
\end{cases} \tag{1.7}$$

where $M = M(\cdot)$ denotes a strictly increasing continuous function from $[0,\infty)$ to $[1,\infty)$, and $\overline{\sigma}_a(v,\Omega,t,x,\rho) \in C([0,T];L^2(\mathbb{R}^+\times S^2;L^\infty(\mathbb{R}^3)))$.

Remark 1.1. These assumptions are similar to the assumptions in [17] for the local existence of classical solutions to the Euler–Boltzmann equations with initial mass density away from vacuum and the assumptions in [24] for the local existence of regular solutions to the Euler–Boltzmann equations with vacuum. The evaluation of these radiation quantities is a difficult problem of quantum mechanics and their general form is not known. However, a general expression of σ_a and σ_s used for describing Compton Scattering process in [28] satisfies Assumptions A and B.

The rest of this paper is organized as follows. In Section 2, we give basic lemmas and review on the results of local existence of smooth solutions to (1.1). The main result of this paper is Theorem 2.2 which includes a Serrin-type and a BKM-type blow-up criterion and is stated in the end of Section 2. Section 3 is devoted to detailed energy estimates to prove the Serrin-type criterion. In Section 4, we will give the outline of the proof of the BKM-type criterion.

2. Preliminaries and main results

In this section, we will give our main results. Before stating our main results, we would like to give some preliminaries which will be used throughout the paper.

2.1. Preliminaries

In this section, we recall some known facts and elementary inequalities that will be used later. We begin with the following well-known Gagliardo–Nirenberg inequality which can be found in [21].

Lemma 2.1. *(Gagliardo–Nirenberg inequality.) For $p \in [2,6]$, $q \in (1,\infty)$ and $r \in (3,\infty)$, there exists a positive constant C, depending only on q and r, such that for any $f \in H^1$ and $g \in L^q \cap D^{1,r}$, we have*

$$|f|_p \le C|f|_2^{\frac{6-p}{2p}} |\nabla f|_2^{\frac{3p-6}{2p}},$$
$$|g|_\infty \le C|g|_q^{\frac{q(r-3)}{3r+q(r-3)}} |\nabla g|_r^{\frac{3r}{3r+q(r-3)}}.$$

The following inequalities are direct inference from the Gagliardo–Nirenberg inequality, which will be used in our paper frequently:

$$|u|_6 \le C|u|_{\mathbb{D}^1}, \quad |u|_\infty \le C|u|_{\mathbb{D}^1 \cap D^2}, \quad |u|_\infty \le C\|u\|_{W^{1,q}}.$$

We now state Minkowski's inequality [4], which will be used to estimate terms with integral over $(0,\infty)$ like $\|I\|_{L^2(\mathbb{R}^+\times S^2; C([0,T]; L^p(\mathbb{R}^3)))}$.

Lemma 2.2. *(Minkowski's inequality.) For all $1 \le p \le q \le \infty$, we have*

$$\big\| \|f(\cdot, x_2)\|_{L^p(X_1)} \big\|_{L^q(X_2)} \le \big\| \|f(x_1, \cdot)\|_{L^q(X_2)} \big\|_{L^p(X_1)}.$$

The following BKM-type inequality was proved in [1] for impressible case and in [14] for compressible case. This inequality will be used later to estimate $|\nabla u|_\infty$ and $\|\nabla u\|_{L^2 \cap L^q}$.

Lemma 2.3. *For $3 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in L^2 \cap D^{1,q}$:*

$$|\nabla u|_\infty \le C(q) \left(|\mathrm{div} u|_\infty + |\nabla \times u|_\infty\right) \ln\left(e + |\nabla^2 u|_q\right) + C(q)|\nabla u|_2 + C(q).$$

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Then, we recall some elementary facts on the weak L^p-space [4] which is defined as

$$L^p_w := \left\{ f \in L^1_{\text{loc}} \Big| \|f\|_{L^p_w} = \sup_{r>0} \left(r \left(\text{meas}\left[|f(x)| > r\right]\right)^{\frac{1}{p}} \right) < \infty \right\}.$$

It is worth mentioning that

$$L^p \subsetneqq L^p_w, \quad \|f\|_{L^p_w} \le \|f\|_{L^p}, \quad L^\infty_w = L^\infty.$$

We also know that the weak L^p-space belongs to the family of Lorentz spaces $L^{p,q}$. It turns out that the weak L^p-space coincides with $L^{p,\infty}$. For the details of the real interpolation and Lorentz space, we refer to the book of Triebel [31]. By Proposition 2.1 in [20] we have the following Hölder inequality in Lorentz space.

Lemma 2.4. *Let $1 < p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1$ and let $1 < q_1, q_2 \le \infty$. Then for $f \in L^{p_1,q_1}$ and $g \in L^{p_2,q_2}$, it holds that $fg \in L^{p,q}$ with*

$$\|fg\|_{L^{p,q}} \le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}} \qquad \text{for } q = \min\{q_1, q_2\},$$

where C is a positive constant depending only on p_1, p_2, q_1 and q_2.

2.2. The main result

Via the radiative transfer integro-differential equation in (1.1) combining with the definitions of F_r and P_r, the system (1.1) can be written as

$$\begin{cases} \frac{1}{c} I_t + \Omega \cdot \nabla I = A_r, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p_m + Lu = -\frac{1}{c} \int_0^\infty \int_{S^2} A_r \Omega \mathrm{d}\Omega \mathrm{d}v, \end{cases} \tag{2.1}$$

where the Lamé operator L is defined by

$$Lu = -\mu \triangle u - (\lambda + \mu)\nabla \operatorname{div} u.$$

The initial data are given by

$$I|_{t=0} = I_0(v, \Omega, x), \quad (\rho, u)|_{t=0} = (\rho_0(x), u_0(x)),\ (v, \Omega, x) \in \mathbb{R}^+ \times S^2 \times \mathbb{R}^3, \tag{2.2}$$

where

$$\begin{aligned} & I_0(v, \Omega, x) \in L^2(\mathbb{R}^+ \times S^2; H^1 \cap W^{1,\tilde{q}}(\mathbb{R}^3)), \quad \rho_0 \ge 0, \\ & \rho_0 \in H^1 \cap W^{1,\tilde{q}} \cap L^1, \quad u_0 \in \mathbb{D}^1 \cap D^2, \\ & (I_0, \rho_0, u_0) \to (0,0,0), \text{ as } |x| \to \infty,\ \forall\ (v, \Omega) \in \mathbb{R}^+ \times S^2. \end{aligned} \tag{2.3}$$

For simplicity, we look for solutions with far field behavior

$$I(x,t) \to 0,\ \rho(x,t) \to 0,\ u(x,t) \to 0 \text{ as } |x| \to \infty,\ t \ge 0,\ \forall\ (v, \Omega) \in \mathbb{R}^+ \times S^2. \tag{2.4}$$

First, we recall the local existence of strong solutions to Cauchy problem (2.1)–(2.2).

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Theorem 2.1. [23] *Let the Assumptions A and B hold, and*

$$\|S(v,\Omega,t,x)\|_{L^2(\mathbb{R}^+\times S^2;C^1([0,\infty);H^1\cap W^{1,\tilde{q}}(\mathbb{R}^3)))\cap C^1([0,\infty);L^1(\mathbb{R}^+\times S^2;L^1\cap L^2(\mathbb{R}^3)))} < +\infty.$$

Assume the initial data (I_0,ρ_0,u_0) satisfy (2.3) as well as the initial layer compatibility condition

$$Lu_0 + \nabla p_m^0 + \frac{1}{c}\int_0^\infty\int_{S^2} A_r^0 \Omega\, d\Omega\, dv = \sqrt{\rho_0} g_1 \tag{2.5}$$

for some $g_1 \in L^2$. Then there exists a time T and a unique strong solution (I,ρ,u) on $\mathbb{R}^+\times S^2\times[0,T]\times\mathbb{R}^3$ to Cauchy problem (2.1)–(2.2). That is (I,ρ,u) is the solution of (2.1)–(2.2) in sense of distribution. Moreover, (I,ρ,u) satisfies the following regularities, for some $\tilde{q}\in(3,6]$,

$$\begin{aligned}
&I \in L^2(\mathbb{R}^+\times S^2; C([0,T];H^1\cap W^{1,\tilde{q}})),\ \ I_t \in L^2(\mathbb{R}^+\times S^2; C([0,T];L^2\cap L^{\tilde{q}})),\\
&\rho \in C([0,T];H^1\cap W^{1,\tilde{q}}\cap L^1), \quad \rho_t \in C([0,T];L^2\cap L^{\tilde{q}}),\\
&u \in C([0,T];\mathbb{D}^1\cap D^2)\cap L^2([0,T];D^{2,\tilde{q}}),\ \ u_t \in L^2([0,T];\mathbb{D}^1),\ \ \sqrt{\rho}u_t \in L^\infty([0,T];L^2).
\end{aligned}$$

Theorem 2.2. *Let the assumption of Theorem 2.1 hold and (I,ρ,u) be a strong solution to (2.1)–(2.2) in $\mathbb{R}^+\times S^2\times[0,T]\times\mathbb{R}^3$. If $0<T^*<+\infty$ is the maximum time of existence of the strong solution, then we have*

- **1. Serrin-type:**

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\sqrt{\rho}u\|_{L^s(0,T;L^r_w)} + \|\nabla\rho\|_{L^2(0,T;L^{\tilde{p}})}\right) = \infty \tag{2.6}$$

  for any $\tilde{p}\in[2,3]$ and any r and s satisfying

$$\frac{2}{s}+\frac{3}{r}\le 1, \quad 3<r<\infty. \tag{2.7}$$

- **2. BKM-type:**

$$\limsup_{T\to T^*}\|D(u)\|_{L^1(0,T;L^\infty)} = \infty, \tag{2.8}$$

  where $D(u)=\frac{1}{2}(\nabla u+\nabla u^T)$ is the deformation tensor.

Remark 2.1. From $\dot{\rho} = -\rho \mathrm{div} u$, we know that if

$$\|\mathrm{div} u\|_{L^1(0,T;L^\infty)} \le C,$$

we can get

$$\|\rho\|_{L^\infty(0,T;L^\infty)} \le C.$$

Thus, we can replace $\|\mathrm{div} u\|_{L^1(0,T;L^\infty)}$ in (2.6) by $\|\rho\|_{L^\infty(0,T;L^\infty)}$.

Remark 2.2. It should be pointed that our criteria depend only on the fluid field, in other words, the radiation terms can be controlled by the fluid terms. More precisely, the BKM-type criterion is the same as the isentropic compressible Navier–Stokes equations, while, for the Serrin-type criterion, the $L^{\tilde{p}}$ norm of the density gradient should be involved to control some radiation terms.

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3. Proof of Serrin-type criterion

Let $0 < T < T^*$ be arbitrary but fixed. Assume that (ρ, u, I) is the strong solution to the problem (2.1)–(2.2). Suppose that $T^* < \infty$. We will prove the Serrin-type part of Theorem 2.2 by a contradiction argument. We assume, for any $T < T^*$

$$\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\sqrt{\rho}u\|_{L^s(0,T;L^r_w)} + \|\nabla\rho\|_{L^2(0,T;L^{\bar{p}})} \le M_0 < \infty, \tag{3.1}$$

where M_0 is independent of T.

The first key step in the proof of Serrin-type criterion is to derive some norms of I under the conditions of Theorem 2.2 and assumption (3.1). To show this, we need the following lemma which is concerned with the estimates of the effective viscous flux G and vorticity ω.

From the first and second equations of (1.1), we have

$$\triangle G = \mathrm{div}\left(\rho\dot{u} + \frac{1}{c}\int_0^\infty\int_{S^2} A_r\Omega \mathrm{d}\Omega \mathrm{d}v\right),$$

$$\mu\triangle\omega = \nabla\times\left(\rho\dot{u} + \frac{1}{c}\int_0^\infty\int_{S^2} A_r\Omega \mathrm{d}\Omega \mathrm{d}v\right).$$

Thus, following the standard L^p-estimate of elliptic system, we have

Lemma 3.1. *Under assumption (3.1), it holds that*

$$|\nabla G|_2 + |\nabla\omega|_2 \le C\left(|\rho u_t|_2 + |\rho u\cdot\nabla u|_2 + \left|\int_0^\infty\int_{S^2} A_r\Omega d\Omega dv\right|_2\right) \tag{3.2}$$

and

$$\begin{aligned}|\nabla G|_6 + |\nabla\omega|_6 &\le C\left(|\rho\dot{u}|_6 + \left|\int_0^\infty\int_{S^2} A_r\Omega d\Omega dv\right|_6\right)\\ &\le C\left(|\nabla\dot{u}|_2 + \left|\int_0^\infty\int_{S^2} A_r\Omega d\Omega dv\right|_6\right).\end{aligned} \tag{3.3}$$

Here and in what follows, the same letter C denotes various generic positive constants depending only on μ, λ, p, γ, c, T^*, M_0 and the initial data.

The next lemma is to estimate the radiation intensity I.

Lemma 3.2. *Under the assumption that $\|\rho\|_{L^\infty(0,T;L^\infty)} \le M_0$, it holds for any $0 \le T < T^*$ that*

$$\sup_{0\le t\le T}\int_0^\infty\int_{S^2} |I|_p^2 + |A_r|_p + |A_r|_p^2 d\Omega dv \le C \quad \text{for } p\in[2,\infty). \tag{3.4}$$

Proof. Multiplying the radiative transfer equation in (2.1) by $p|I|^{p-2}I$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}\frac{1}{2cp}\frac{d}{dt}|I|_p^p \leq |S|_p|I|_p^{p-1} - \int_{\mathbb{R}^3}\left(\sigma_a + \int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)|I|^p\mathrm{d}x \\ + |I|_p^{p-1}\int_0^\infty\int_{S^2}\frac{v}{v'}|\sigma_s|_\infty|I'|_p\mathrm{d}\Omega'\mathrm{d}v'.\end{aligned} \tag{3.5}$$

Noting that

$$\sigma_a + \int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v' \geq 0,$$

we get

$$\begin{aligned}\frac{1}{2cp}\frac{d}{dt}|I|_p^2 &\leq |S|_p^2 + |I|_p^2 + |I|_p\int_0^\infty\int_{S^2}\frac{v}{v'}|\sigma_s|_\infty|I'|_p\mathrm{d}\Omega'\mathrm{d}v' \\ &\leq |S|_p^2 + C|I|_p^2 + \left(\int_0^\infty\int_{S^2}\frac{v}{v'}|\sigma_s|_\infty|I'|_p\mathrm{d}\Omega'\mathrm{d}v'\right)^2 \\ &\leq |S|_p^2 + C|I|_p^2 + \int_0^\infty\int_{S^2}\frac{v^2}{v'^2}|\sigma_s|_\infty^2\mathrm{d}\Omega'\mathrm{d}v'\cdot\int_0^\infty\int_{S^2}|I|_p^2\mathrm{d}\Omega\mathrm{d}v.\end{aligned} \tag{3.6}$$

Then integrating (3.6) over $(0,\infty)\times S^2$, we get

$$\begin{aligned}\frac{d}{dt}\int_0^\infty\int_{S^2}|I|_p^2\mathrm{d}\Omega\mathrm{d}v \leq &C\int_0^\infty\int_{S^2}|S|_p^2\mathrm{d}\Omega\mathrm{d}v + C\int_0^\infty\int_{S^2}|I|_p^2\mathrm{d}\Omega\mathrm{d}v \\ &+ C\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\frac{v^2}{v'^2}|\sigma_s|_\infty^2\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v\cdot\int_0^\infty\int_{S^2}|I|_p^2\mathrm{d}\Omega\mathrm{d}v.\end{aligned} \tag{3.7}$$

From (1.5) and (1.7), we have

$$\frac{d}{dt}\int_0^\infty\int_{S^2}|I|_p^2\mathrm{d}\Omega\mathrm{d}v \leq C + C\int_0^\infty\int_{S^2}|I|_p^2\mathrm{d}\Omega\mathrm{d}v.$$

Applying Gronwall's inequality to the above inequality we obtain the first item of (3.4).

For the rest two items we have the fact that

$$\begin{aligned}\int_0^\infty\int_{S^2}|A_r|_p^2\mathrm{d}\Omega\mathrm{d}v \leq &C\int_0^\infty\int_{S^2}|S|_p^2\mathrm{d}\Omega\mathrm{d}v \\ &+ C\max_{(v,\Omega)\in[0,\infty)\times S^2}\left|\sigma_a + \int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right|_\infty^2\int_0^\infty\int_{S^2}|I|_p^2\mathrm{d}\Omega\mathrm{d}v \\ &+ C\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\frac{v^2}{v'^2}|\sigma_s|_\infty^2\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v\cdot\int_0^\infty\int_{S^2}|I|_p^2\mathrm{d}\Omega\mathrm{d}v \leq C\end{aligned}$$

and

$$
\begin{aligned}
&\int_0^\infty\int_{S^2} |A_r|_p \mathrm{d}\Omega \mathrm{d}v \\
&\le \int_0^\infty\int_{S^2} |S|_p \mathrm{d}\Omega \mathrm{d}v + \Big(\int_0^\infty\int_{S^2} |I|_p^2 \mathrm{d}\Omega \mathrm{d}v\Big)^{\frac{1}{2}} \Big(\int_0^\infty\int_{S^2} \Big|\sigma_a + \int_0^\infty\int_{S^2} \sigma_s' \mathrm{d}\Omega' \mathrm{d}v'\Big|_\infty^2 \mathrm{d}\Omega \mathrm{d}v\Big)^{\frac{1}{2}} \\
&\quad + \int_0^\infty\int_{S^2} \Big(\int_0^\infty\int_{S^2} \frac{v^2}{v'^2} |\sigma_s|_\infty^2 \mathrm{d}\Omega' \mathrm{d}v'\Big)^{\frac{1}{2}} \mathrm{d}\Omega \mathrm{d}v \Big(\int_0^\infty\int_{S^2} |I|_p^2 \mathrm{d}\Omega \mathrm{d}v\Big)^{\frac{1}{2}} \le C. \qquad \square
\end{aligned}
$$

Next we give the estimate for velocity u of fluids. From the standard energy estimate, it can be easily obtained that

$$
\sup_{0\le t\le T} |\rho(t)|_1 \le |\rho_0|_1 \quad 0 \le T \le T^* \tag{3.8}
$$

and

$$
\sup_{0\le t\le T} \Big(|\sqrt{\rho}u|_2^2 + |\rho|_\gamma^\gamma\Big) + \int_0^T |\nabla u|_2^2 \, dt \le C, \quad 0 \le T \le T^*. \tag{3.9}
$$

The estimate on ∇u will be given in the following lemma.

Lemma 3.3. *Under assumption (3.1), it holds for any $0 \le T < T^*$ that*

$$
\sup_{0\le t\le T} |\nabla u|_2 + \int_0^T |\sqrt{\rho}u_t(t)|_2^2 dt \le C. \tag{3.10}
$$

Proof. Multiplying the momentum equations in (2.1) by u_t and integrating the result equations over $\mathbb{R}^3$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} (\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}u)^2)\mathrm{d}x + \int_{\mathbb{R}^3} \rho|u_t|^2 \mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \rho u\cdot\nabla u\cdot u_t \mathrm{d}x + \int_{\mathbb{R}^3} p_m \mathrm{div}u_t \mathrm{d}x + \frac{1}{c}\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2} A_r u_t\cdot\Omega \mathrm{d}\Omega \mathrm{d}v \mathrm{d}x.
\end{aligned} \tag{3.11}
$$

For the first term on the right-hand side of (3.11), Cauchy's inequality yields

$$
-\int_{\mathbb{R}^3} \rho u\cdot\nabla u\cdot u_t \mathrm{d}x \le \frac{1}{2}\int_{\mathbb{R}^3} \rho|u_t|^2 \mathrm{d}x + C\int_{\mathbb{R}^3} \rho|u\cdot\nabla u|^2 \mathrm{d}x. \tag{3.12}
$$

Keeping in mind that L_w^r coincides with the Lorentz space $L^{r,\infty}$, it follows from Lemma 2.4 that

$$
\int_{\mathbb{R}^3} \rho|u\cdot\nabla u|^2 \mathrm{d}x = \|\nabla u\cdot\sqrt{\rho}u\|_{L^{2,2}}^2 \le C\|\sqrt{\rho}u\|_{L_w^r}^2 \|\nabla u\|_{L^{\frac{2r}{r-2},2}}^2 \quad \text{for any } r > 3.
$$

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Recalling that $L^{\frac{2r}{r-2},2}$ with $3 < r < \infty$ is a real interpolation space of $L^{\frac{2r_1}{2r_1-2}}$ and $L^{\frac{2r_2}{2r_2-2}}$, where r_1, r_2 and r satisfy $3 < r_1 < r < r_2 < \infty$ and $\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, we thus deduce from Lemma 2.1, Sobolev embedding inequality and $\frac{2}{s} + \frac{3}{r} \leq 1$ that

$$\begin{aligned}
\int_{\mathbb{R}^3} \rho|u\cdot\nabla u|^2 \mathrm{d}x &\leq C\|\sqrt{\rho}u\|_{L^r_w}^2 |\nabla u|_{\frac{2r_1}{2r_1-2}} |\nabla u|_{\frac{2r_2}{2r_2-2}} \\
&\leq C\|\sqrt{\rho}u\|_{L^r_w}^2 |\nabla u|_2^{\frac{r_1-3}{r_1}} |\nabla u|_6^{\frac{3}{r_1}} |\nabla u|_2^{\frac{r_2-3}{r_2}} |\nabla u|_6^{\frac{3}{r_2}} \\
&\leq C\|\sqrt{\rho}u\|_{L^r_w}^2 |\nabla u|_2^{\frac{2r-6}{r}} |\nabla u|_6^{\frac{6}{r}} \\
&\leq C(\eta)\left(\|\sqrt{\rho}u\|_{L^r_w}^s + 1\right)|\nabla u|_2^2 + \eta|\nabla u|_6^2, \quad \text{for } 3 < r < \infty,
\end{aligned} \tag{3.13}$$

where $\eta > 0$ is a constant.

Noting that $L^\infty_w = L^\infty$, the same inequality certainly holds for the case $r = \infty$.

By the standard L^p-estimate, we deduce from (3.1), Lemma 2.1 and Lemma 3.1 that

$$\begin{aligned}
|\nabla u|_6 &\leq C\left(|\mathrm{div}u|_6 + |\omega|_6\right) \\
&\leq C\left(|G|_6 + |p_m|_6 + |\omega|_6\right) \\
&\leq C\left(|\nabla G|_2 + |p_m|_6 + |\nabla\omega|_2\right) \\
&\leq C\left(|\rho\dot{u}|_2 + \left|\int_0^\infty\int_{S^2} A_r\Omega \mathrm{d}\Omega\mathrm{d}v\right|_2 + 1\right) \\
&\leq C\left(|\sqrt{\rho}u_t|_2 + |\sqrt{\rho}u\cdot\nabla u|_2 + 1\right).
\end{aligned} \tag{3.14}$$

Insert (3.14) into (3.13), we get

$$\begin{aligned}
\int_{\mathbb{R}^3} \rho|u\cdot\nabla u|^2 \mathrm{d}x &\leq C(\eta)\left(\|\sqrt{\rho}u\|_{L^r_w}^s + 1\right)|\nabla u|_2^2 + \eta|\sqrt{\rho}u_t|_2^2 + C \\
&\leq C(\eta)|\nabla u|_2^2 + \eta|\sqrt{\rho}u_t|_2^2 + C.
\end{aligned} \tag{3.15}$$

For the second term on the right-hand side of (3.11), $(2.1)_2$ yields

$$\begin{aligned}
\int_{\mathbb{R}^3} p_m \mathrm{div}u_t \mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} p_m \mathrm{div}u \mathrm{d}x - \int_{\mathbb{R}^3} (p_m)_t \mathrm{div}u \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} p_m \mathrm{div}u \mathrm{d}x + \int_{\mathbb{R}^3} \gamma A\rho^{\gamma-1}\left(\nabla\rho\cdot u + \rho\mathrm{div}u\right)\mathrm{div}u \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} p_m \mathrm{div}u \mathrm{d}x - \int_{\mathbb{R}^3} p_m \mathrm{div}(u\mathrm{div}u)\mathrm{d}x + \gamma\int_{\mathbb{R}^3} p_m(\mathrm{div}u)^2 \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} p_m \mathrm{div}u \mathrm{d}x - \frac{1}{2\mu+\lambda}\int_{\mathbb{R}^3} p_m u\cdot\nabla G \mathrm{d}x + \frac{1}{2(2\mu+\lambda)}\int_{\mathbb{R}^3} {p_m}^2 \mathrm{div}u \mathrm{d}x \\
&\quad + (\gamma-1)\int_{\mathbb{R}^3} p_m(\mathrm{div}u)^2 \mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3} p_m \mathrm{div}u\mathrm{d}x + \eta|\nabla G|_2^2 + C|\mathrm{div}u|_2^2 + C(\eta) \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3} p_m \mathrm{div}u\mathrm{d}x + C(\eta)|\nabla u|_2^2 + \eta|\sqrt{\rho}u_t|_2^2 + C|\mathrm{div}u|_2^2 + C(\eta).
\end{aligned} \tag{3.16}$$

For the third term on the right-hand side of (3.11), one has

$$\begin{aligned}
&-\frac{1}{c}\int\limits_{\mathbb{R}^3}\int\limits_0^\infty\int\limits_{S^2} A_r u_t\cdot\Omega\mathrm{d}\Omega\mathrm{d}v\mathrm{d}x \\
&= -\frac{1}{c}\int\limits_0^\infty\int\limits_{S^2}\int\limits_{\mathbb{R}^3}\left(S - \sigma_a I + \int\limits_0^\infty\int\limits_{S^2}\frac{v}{v'}\sigma_s I'\mathrm{d}\Omega'\mathrm{d}v'\right) u_t\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v \\
&\quad + \frac{1}{c}\int\limits_0^\infty\int\limits_{S^2}\int\limits_0^\infty\int\limits_{S^2}\int\limits_{\mathbb{R}^3}\sigma_s' I u_t\cdot\Omega\mathrm{d}x\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v \triangleq \sum_{j=1}^4 I_j.
\end{aligned} \tag{3.17}$$

We estimate I_j term by term. From Gagliardo–Nirenberg inequality, Hölder's inequality and Young's inequality, we get

$$\begin{aligned}
I_1 &= -\frac{1}{c}\int\limits_0^\infty\int\limits_{S^2}\int\limits_{\mathbb{R}^3} S u_t\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v \\
&= -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_0^\infty\int\limits_{S^2}\int\limits_{\mathbb{R}^3} S u\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v + \frac{1}{c}\int\limits_0^\infty\int\limits_{S^2}\int\limits_{\mathbb{R}^3} S_t u\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v \\
&\le -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_0^\infty\int\limits_{S^2}\int\limits_{\mathbb{R}^3} S u\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v + C|\nabla u|_2\int\limits_0^\infty\int\limits_{S^2}|S_t|_{\frac{6}{5}}\mathrm{d}\Omega\mathrm{d}v \\
&\le -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_0^\infty\int\limits_{S^2}\int\limits_{\mathbb{R}^3} S u\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v + C|\nabla u|_2^2 + C,
\end{aligned}$$

$$\begin{aligned}
I_2 =&\frac{1}{c}\int\limits_0^\infty\int\limits_{S^2}\int\limits_{\mathbb{R}^3}\sigma_a I u_t\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v \le C|\rho|_\infty^{\frac{1}{2}}|\sqrt{\rho}u_t|_2\int\limits_0^\infty\int\limits_{S^2}|\overline{\sigma}_a|_\infty|I|_2\mathrm{d}\Omega\mathrm{d}v \\
\le&\eta|\sqrt{\rho}u_t|_2^2 + C(\eta)\int\limits_0^\infty\int\limits_{S^2}|I|_2^2\mathrm{d}\Omega\mathrm{d}v\int\limits_0^\infty\int\limits_{S^2}|\overline{\sigma}_a|_\infty^2\mathrm{d}\Omega\mathrm{d}v \\
\le&\eta|\sqrt{\rho}u_t|_2^2 + C(\eta),
\end{aligned}$$

$$\begin{aligned}
I_3 &= -\frac{1}{c}\int\limits_0^\infty\int\limits_{S^2}\int\limits_0^\infty\int\limits_{S^2}\int\limits_{\mathbb{R}^3}\frac{v}{v'}\sigma_s I' u_t\cdot\Omega\mathrm{d}x\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v \\
&\le C|\rho|_\infty^{\frac{1}{2}}|\sqrt{\rho}u_t|_2\int\limits_0^\infty\int\limits_{S^2}\int\limits_0^\infty\int\limits_{S^2}\frac{v}{v'}\overline{\sigma}_s|I'|_2\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v
\end{aligned}$$

$$\le \eta|\sqrt{\rho}u_t|_2^2 + C(\eta)\int_0^\infty\int_{S^2}|I'|_2^2 d\Omega' dv' \Big(\int_0^\infty\int_{S^2}\Big(\int_0^\infty\int_{S^2}\Big|\frac{v}{v'}\Big|^2\overline{\sigma}_s^2 d\Omega' dv'\Big)^{\frac{1}{2}} d\Omega dv\Big)^2$$
$$\le \eta|\sqrt{\rho}u_t|_2^2 + C(\eta),$$

$$\begin{aligned} I_4 =& \frac{1}{c}\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\sigma_s' I u_t\cdot\Omega dx d\Omega' dv' d\Omega dv \\ \le& C|\rho|_\infty^{\frac{1}{2}}|\sqrt{\rho}u_t|_2\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\overline{\sigma}_s'|I|_2 d\Omega' dv' d\Omega dv \\ \le& \eta|\sqrt{\rho}u_t|_2^2 + C(\eta)\int_0^\infty\int_{S^2}|I|_2^2 d\Omega dv\int_0^\infty\int_{S^2}\Big(\int_0^\infty\int_{S^2}\overline{\sigma}_s' d\Omega' dv'\Big)^2 d\Omega dv \\ \le& \eta|\sqrt{\rho}u_t|_2^2 + C(\eta). \end{aligned}$$

Substituting (3.15), (3.16) and estimates of I_j into (3.11), one has after choosing η suitably small,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3}\big(\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}u)^2\big) dx + |\sqrt{\rho}u_t|_2^2 \\ &\quad\le C|\nabla u|_2^2 + C|\mathrm{div}u|_2^2 + C - \frac{2}{c}\frac{d}{dt}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3} Su\cdot\Omega dx d\Omega dv \\ &\qquad + \frac{d}{dt}\int_{\mathbb{R}^3} p_m \mathrm{div}u dx. \end{aligned} \tag{3.18}$$

Integrating (3.18) over $(0,T)$, we get

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}(\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}u)^2) dx + \int_0^T|\sqrt{\rho}u_t|_2^2 dt \\ &\quad\le C\int_0^T|\nabla u|_2^2 dt + C\int_0^T|\mathrm{div}u|_2^2 dt + C(T,\eta) + \eta\left(|\mathrm{div}u|_2^2 + |\nabla u|_2^2\right), \end{aligned} \tag{3.19}$$

where we have used the fact that

$$\frac{2}{c}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3} Su\cdot\Omega dx d\Omega dv \le C|\nabla u|\int_0^\infty\int_{S^2}|S|_{\frac{6}{5}} d\Omega dv \le \eta|\nabla u|_2^2 + C(\eta).$$

For η small enough, (3.19) together with Gronwall inequality gives (3.10). □

Now we will estimate $\nabla\dot{u}$.

Lemma 3.4. *Under assumption (3.1), it holds for any $0 \le T < T^*$ that*

$$\sup_{0\le t\le T}|\sqrt{\rho}\dot{u}|_2^2 + \int_0^T|\nabla\dot{u}|_2^2 dt \le C. \tag{3.20}$$

Proof. Taking the material derivative to (1.1) and using the fact

$$\dot{f} = f_t + \operatorname{div}(f \otimes u) - f\operatorname{div}u,$$

we have

$$\begin{aligned}
&\rho\dot{u}_t + \rho u \cdot \nabla\dot{u} + \operatorname{div}(\nabla p_m \otimes u) + \nabla\left(p_m\right)_t + \frac{1}{c}\int_0^\infty\int_{S^2}(A_r)_t\Omega \mathrm{d}\Omega\mathrm{d}v \\
&\quad + \frac{1}{c}\int_0^\infty\int_{S^2}\operatorname{div}(A_r\Omega \otimes u)\mathrm{d}\Omega\mathrm{d}v \\
&\quad = \mu\triangle u_t + \mu\operatorname{div}(\triangle u \otimes u) + (\mu+\lambda)\nabla\operatorname{div}u_t + (\mu+\lambda)\operatorname{div}(\nabla\operatorname{div}u \otimes u).
\end{aligned}$$

Multiplying the equation above by $\dot{u}$ and integrating the resulting over $\mathbb{R}^3$, we obtain after integration by parts

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho|\dot{u}|^2\mathrm{d}x \\
&= \mu\int_{\mathbb{R}^3}\dot{u}\cdot\left(\triangle u_t + \operatorname{div}(\triangle u \otimes u)\right)\mathrm{d}x + (\mu+\lambda)\int_{\mathbb{R}^3}\dot{u}\cdot\left(\nabla\operatorname{div}u_t + \operatorname{div}(\nabla\operatorname{div}u \otimes u)\right)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3}\dot{u}\cdot\left(\operatorname{div}(\nabla p_m \otimes u) + \nabla\left(p_m\right)_t\right)\mathrm{d}x + \frac{1}{c}\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2}\nabla\dot{u} : \left(A_r\Omega \otimes u\right)\mathrm{d}\Omega\mathrm{d}v\mathrm{d}x \\
&\quad - \frac{1}{c}\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2}(A_r)_t\dot{u}\cdot\Omega\mathrm{d}\Omega\mathrm{d}v\mathrm{d}x. \qquad (3.21)
\end{aligned}$$

As in [14], one can estimate the first three terms in the right side of (3.21) as follows.

$$\begin{aligned}
&\mu\int_{\mathbb{R}^3}\dot{u}\cdot\left(\triangle u_t + \operatorname{div}(\triangle u \otimes u)\right)\mathrm{d}x \\
&= \mu\int_{\mathbb{R}^3}\dot{u}\cdot\left(\triangle\dot{u} - \triangle(u\cdot\nabla u) + \operatorname{div}(\triangle u \otimes u)\right)\mathrm{d}x \\
&= -\mu|\nabla\dot{u}|_2^2 + \mu\int_{\mathbb{R}^3}\left(\nabla\dot{u} : \nabla(u\cdot\nabla u) - \nabla\dot{u} : (\triangle u \otimes u)\right)\mathrm{d}x \\
&= -\mu|\nabla\dot{u}|_2^2 + \mu\int_{\mathbb{R}^3}\left(\partial_i\dot{u}^j\partial_i u^k\partial_k u^j + \partial_i\dot{u}^j u^k\partial_{ik}u^j - \partial_k\dot{u}^j u^k\partial_{ii}u^j\right)\mathrm{d}x \\
&= -\mu|\nabla\dot{u}|_2^2 + \mu\int_{\mathbb{R}^3}\left(\partial_i\dot{u}^j\partial_i u^k\partial_k u^j - \partial_i\dot{u}^j\partial_k u^k\partial_i u^j + \partial_i u^j\partial_i u^k\partial_k\dot{u}^j\right)\mathrm{d}x \\
&\le -\frac{8\mu}{9}|\nabla\dot{u}|_2^2 + C\int_{\mathbb{R}^3}|\nabla u|^4\,\mathrm{d}x, \qquad (3.22)
\end{aligned}$$

$$
\begin{aligned}
&(\mu+\lambda)\int_{\mathbb{R}^3} \dot{u}\cdot(\nabla \mathrm{div} u_t + \mathrm{div}(\nabla \mathrm{div} u \otimes u))\,\mathrm{d}x \\
&\quad = -(\mu+\lambda)\int_{\mathbb{R}^3} \mathrm{div}\dot{u}\,\mathrm{div} u_t + \nabla\dot{u} : (\nabla \mathrm{div} u\otimes u)\,\mathrm{d}x \\
&\quad = -(\mu+\lambda)|\mathrm{div}\dot{u}|_2^2 + (\mu+\lambda)\int_{\mathbb{R}^3}\left(\partial_j \dot{u}^j \partial_k u^i \partial_i u^k + \partial_k \dot{u}^j \partial_j u^k \partial_i u^i - \partial_j \dot{u}^j \partial_k u^k \partial_i u^i\right)\mathrm{d}x \\
&\quad \le -\frac{\mu+\lambda}{2}|\mathrm{div}\dot{u}|_2^2 + \frac{\mu}{9}|\nabla\dot{u}|_2^2 + C\int_{\mathbb{R}^3}|\nabla u|^4\,\mathrm{d}x,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^3} \dot{u}\cdot(\mathrm{div}(\nabla p_m\otimes u) + \nabla (p_m)_t)\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} (\nabla\dot{u} : (\nabla p_m\otimes u) + (p_m)_t\,\mathrm{div}\dot{u})\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} \left(\partial_k \dot{u}^j \partial_j p_m u^k - (\mathrm{div}(p_m u) + (\gamma-1)p_m \mathrm{div} u)\,\mathrm{div}\dot{u}\right)\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} \left(-\partial_j(\partial_k \dot{u}^j u^k)p_m - \partial_k(p_m u^k)\partial_j\dot{u}^j - (\gamma-1)p_m \mathrm{div} u\,\mathrm{div}\dot{u}\right)\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} \left(-\partial_k \dot{u}^j \partial_j u^k p_m - (\gamma-1)p_m \mathrm{div} u\,\mathrm{div}\dot{u}\right)\mathrm{d}x \\
&\quad \le C|\nabla u|_2^2 + \frac{\mu}{9}|\nabla\dot{u}|_2^2
\end{aligned}
\tag{3.24}
$$

and

$$
\begin{aligned}
&\frac{1}{c}\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2} \nabla\dot{u} : (A_r\Omega\otimes u)\,\mathrm{d}\Omega\mathrm{d}v\mathrm{d}x \\
&\quad \le C|\nabla\dot{u}|_2|u|_6\int_0^\infty\int_{S^2}|A_r|_3\mathrm{d}\Omega\mathrm{d}v \\
&\quad \le C|\nabla u|_2^2 + \frac{\mu}{9}|\nabla\dot{u}|_2^2,
\end{aligned}
\tag{3.25}
$$

where i, j, $k = 1, 2, 3$.

Then we have to estimate the last term in the right of (3.21)

$$
\begin{aligned}
&\frac{1}{c}\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2} (A_r)_t\dot{u}\cdot\Omega\mathrm{d}\Omega\mathrm{d}v\mathrm{d}x \\
&\quad = -\frac{1}{c}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\dot{u}\cdot\Omega\Big(S_t - \left(\sigma_a + \int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)_t I
\end{aligned}
\tag{3.26}
$$

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$$-\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)I_t+\int_0^\infty\int_{S^2}\frac{v}{v'}(\sigma_s)_tI'\mathrm{d}\Omega'\mathrm{d}v'\Big)\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v$$

$$-\frac{1}{c}\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\dot{u}\cdot\Omega\frac{v}{v'}\sigma_sI_t'\mathrm{d}x\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v\triangleq\sum_{j=1}^5J_j.$$

According to Gagliardo–Nirenberg inequality, Holder's inequality and Young's inequality, for $0\le t\le T_2$, we have

$$J_1=-\frac{1}{c}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}S_t\dot{u}\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\le C|\dot{u}|_6\int_0^\infty\int_{S^2}|S_t|_{\frac{6}{5}}\mathrm{d}\Omega\mathrm{d}v\le\frac{\mu}{9}|\nabla\dot{u}|_2^2+C,$$

$$\begin{aligned}
J_2=&\frac{1}{c}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)_tI\dot{u}\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
\le& C|\dot{u}|_6\int_0^\infty\int_{S^2}\left|\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)_t\right|_{\tilde{p}}|I|_{\frac{6\tilde{p}}{5\tilde{p}-6}}\mathrm{d}\Omega\mathrm{d}v\\
\le&\frac{\mu}{9}|\nabla\dot{u}|_2^2\\
&+C\int_0^\infty\int_{S^2}|I|^2_{\frac{6\tilde{p}}{5\tilde{p}-6}}\mathrm{d}\Omega\mathrm{d}v\int_0^\infty\int_{S^2}\left|\rho(\overline{\sigma}_a)_t+\rho_t\left(\overline{\sigma}_a+\int_0^\infty\int_{S^2}\overline{\sigma}_s'\mathrm{d}\Omega'\mathrm{d}v'\right)\right|^2_{\tilde{p}}\mathrm{d}\Omega\mathrm{d}v\\
\le&\frac{\mu}{9}|\nabla\dot{u}|_2^2+CM(|\rho|_\infty)(|\rho_t|^2_{\tilde{p}}+1)\\
\le&\frac{\mu}{9}|\nabla\dot{u}|_2^2+CM(|\rho|_\infty)(|\rho\mathrm{div}u|^2_{\tilde{p}}+|\nabla\rho u|^2_{\tilde{p}}+1)\\
\le&\frac{\mu}{9}|\nabla\dot{u}|_2^2+CM(|\rho|_\infty)(|\rho|^2_\infty|\mathrm{div}u|^2_{\tilde{p}}+|\nabla\rho|^2_{\tilde{p}}|u|^2_\infty+1)\\
\le&\frac{\mu}{9}|\nabla\dot{u}|_2^2+C\left(C\left(|\nabla u|_2^2+|\nabla u|_6^2\right)+|\nabla\rho|^2_{\tilde{p}}\left(|\nabla u|_2^2+|\nabla u|_6^2\right)\right)+C\\
\le&\frac{\mu}{9}|\nabla\dot{u}|_2^2+C\left(|\nabla\rho|^2_{\tilde{p}}+1\right)\left(|\sqrt{\rho}\dot{u}|_2^2+1\right),
\end{aligned}$$

where we have used (3.14) and $\frac{6\tilde{p}}{5\tilde{p}-6}\ge2$.

$$\begin{aligned}
J_3=&\frac{1}{c}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)I_t\dot{u}\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
=&\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)A_r\dot{u}\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
&-\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)\Omega\cdot\nabla I\right)(\dot{u}\cdot\Omega)\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
\le&\frac{\mu}{27}|\nabla\dot{u}|_2^2+CM(|\rho|_\infty)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\Omega\cdot\nabla\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)I\right)(\dot u\cdot\Omega)\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
&+\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)I(\Omega\cdot\nabla\dot u\cdot\Omega)\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
\le&\frac{\mu}{27}|\nabla\dot u|_2^2+CM\left(|\rho|_\infty\right)\\
&+C|\dot u|_6\int_0^\infty\int_{S^2}\left|\rho\nabla\overline{\sigma}_a+\nabla\rho\left(\overline{\sigma}_a+\int_0^\infty\int_{S^2}\overline{\sigma}_s'\mathrm{d}\Omega'\mathrm{d}v'\right)\right|_{\tilde p}|I|_{\frac{6\tilde p}{5\tilde p-6}}\mathrm{d}\Omega\mathrm{d}v\\
&+\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)I(\Omega\cdot\nabla\dot u\cdot\Omega)\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
\le&\frac{\mu}{27}|\nabla\dot u|_2^2+CM\left(|\rho|_\infty\right)+\frac{\mu}{27}|\nabla\dot u|_2^2+CM\left(|\rho|_\infty\right)\left(|\nabla\rho|_{\tilde p}^2+1\right)\\
&+C\int_0^\infty\int_{S^2}|I|^2_{\frac{6\tilde p}{5\tilde p-6}}\mathrm{d}\Omega\mathrm{d}v+\frac{\mu}{27}|\nabla\dot u|_2^2\\
&+C\max_{(v,\Omega)\in[0,\infty)\times S^2}\left|\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right|_\infty^2\int_0^\infty\int_{S^2}|I|_2^2\mathrm{d}\Omega\mathrm{d}v\\
\le&\frac{\mu}{9}|\nabla\dot u|_2^2+CM\left(|\rho|_\infty\right)\left(|\nabla\rho|_{\tilde p}^2+1\right),
\end{aligned}
$$

$$
\begin{aligned}
J_4=&-\frac{1}{c}\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\frac{v}{v'}(\sigma_s)_tI'\dot u\cdot\Omega\mathrm{d}x\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v\\
\le&C|\dot u|_6\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\left|\frac{v}{v'}\right|\bar\sigma_s|\rho_t|_{\tilde p}|I'|_{\frac{6\tilde p}{5\tilde p-6}}\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v\\
\le&\frac{\mu}{9}|\nabla\dot u|_2^2+CM\left(|\rho|_\infty\right)\left(|\rho_t|_{\tilde p}^2+1\right)\\
\le&\le\frac{\mu}{9}|\nabla\dot u|_2^2+C\left(|\nabla\rho|_{\tilde p}^2+1\right)\left(|\sqrt{\rho}\dot u|_2^2+1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
J_5=&\frac{1}{c}\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\dot u\cdot\Omega\frac{v}{v'}\sigma_sI_t'\mathrm{d}x\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v\\
=&\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\dot u\cdot\Omega\frac{v}{v'}\sigma_sA_r'\mathrm{d}x\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v\\
&-\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}(\dot u\cdot\Omega)\frac{v}{v'}\sigma_s(\nabla I'\cdot\Omega)\mathrm{d}x\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v\\
\le&\frac{\mu}{9}|\nabla\dot u|_2^2+CM\left(|\rho|_\infty\right)+\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}(\dot u\cdot\Omega)\frac{v}{v'}I'(\nabla\sigma_s\cdot\Omega)\mathrm{d}x\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}(\Omega\cdot\nabla\dot{u}\cdot\Omega)\frac{v}{v'}\sigma_s I'\mathrm{d}x\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v\\
&\le\frac{\mu}{9}|\nabla\dot{u}|_2^2+CM\left(|\rho|_\infty\right)\left(|\nabla\rho|_{\tilde{p}}^2+1\right).
\end{aligned}
$$

Hence,

$$
\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\rho}\dot{u}|_2^2+|\nabla\dot{u}|_2^2+|\mathrm{div}\dot{u}|_2^2\le C|\nabla u|_4^4+C\left(|\nabla\rho|_{\tilde{p}}^2+1\right)\left(|\sqrt{\rho}\dot{u}|_2^2+1\right). \tag{3.27}
$$

To conclude the estimates by the Gronwall's inequality, we remain to estimate $|\nabla u|_4^4$. By Lemma 3.1, we have

$$
\begin{aligned}
|\nabla u|_4^4\le&|G|_4^4+|\omega|_4^4+|p_m|_4^4\le|G|_2^{\frac{5}{2}}|\nabla G|_6^{\frac{3}{2}}+|\omega|_2^{\frac{5}{2}}|\nabla\omega|_6^{\frac{3}{2}}+C\\
\le&C\left(|\nabla\dot{u}|_2+\left|\int_0^\infty\int_{S^2}A_r\Omega\mathrm{d}\Omega\mathrm{d}v\right|_6\right)^{\frac{3}{2}}+C\\
\le&\frac{1}{2}|\nabla\dot{u}|_2^2+C.
\end{aligned} \tag{3.28}
$$

Substituting this estimate into (3.27), we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\rho}\dot{u}|_2^2+|\nabla\dot{u}|_2^2\le C\left(|\nabla\rho|_{\tilde{p}}^2+1\right)\left(|\sqrt{\rho}\dot{u}|_2^2+1\right). \tag{3.29}
$$

Integrating (3.29) over (τ,t) $(\tau\in(0,t))$, for $\tau\le t\le T$, we have

$$
|\sqrt{\rho}\dot{u}(t)|_2^2+\int_\tau^t|\nabla\dot{u}|_2^2\mathrm{d}s\le|\sqrt{\rho}\dot{u}(\tau)|_2^2+C\int_\tau^t|\nabla\rho|_{\tilde{p}}^2(1+|\sqrt{\rho}\dot{u}|_2)\mathrm{d}s+C. \tag{3.30}
$$

From the momentum equations $(2.1)_3$, we easily have

$$
|\sqrt{\rho}\dot{u}(\tau)|_2^2\le C\int_{\mathbb{R}^3}\frac{|\nabla p_m+Lu+\frac{1}{c}\int_0^\infty\int_{S^2}A_r\Omega\mathrm{d}\Omega\mathrm{d}v|^2}{\rho}(\tau)\mathrm{d}x. \tag{3.31}
$$

Due to the initial layer compatibility condition (2.5), letting $\tau\to0$ in (3.31), we have

$$
\limsup_{\tau\to0}|\sqrt{\rho}\dot{u}(\tau)|_2^2\le C\int_{\mathbb{R}^3}|g_1|^2\mathrm{d}x\le C. \tag{3.32}
$$

Then, letting $\tau\to0$ in (3.30), we have

$$
|\sqrt{\rho}\dot{u}(t)|_2^2+\int_0^t|\nabla\dot{u}|_2^2\mathrm{d}s\le C\int_0^t|\nabla\rho|_{\tilde{p}}^2(1+|\sqrt{\rho}\dot{u}|_2)\mathrm{d}s+C. \tag{3.33}
$$

Noting that $|\nabla\rho|_{\tilde{p}}^2\in L^1[0,T]$ by the blow-up criterion, we conclude by Gronwall's inequality that

$$
\sup_{0\le t\le T}|\sqrt{\rho}\dot{u}|_2^2+\int_0^T|\nabla\dot{u}|_2^2dt\le C.\qquad\square \tag{3.34}
$$

In the next lemma, we will show that the density gradient and the velocity gradient are bounded.

Lemma 3.5. *Under assumption (3.1), it holds for any $0 \le T < T^*$ that*

$$\sup_{0\le t\le T} \|\nabla\rho\|_{L^2\cap L^{\tilde q}} + \int_0^T |\nabla u|_\infty \, dt \le C. \tag{3.35}$$

Proof. Taking the derivative with respect to x for the second equation of (2.1) to obtain:

$$\partial_t \nabla\rho + (u\cdot\nabla)\nabla\rho + (\nabla u \nabla)\,\rho + \mathrm{div}u\nabla\rho + \rho\nabla\mathrm{div}u = 0. \tag{3.36}$$

Multiplying (3.36) by $p|\nabla\rho|^{p-2}\nabla\rho$ and integrating the resulting equation on $\mathbb{R}^3$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\rho|_p^p \le & C\int_{\mathbb{R}^3} |\nabla u||\nabla\rho|^p \mathrm{d}x + p\int_{\mathbb{R}^3} \rho|\nabla\mathrm{div}u||\nabla\rho|^{p-1}\mathrm{d}x \\ \le & C\left(1+|\nabla u|_\infty\right)|\nabla\rho|_p^p + C|\nabla\mathrm{div}u|_p^p \\ \le & C\left(1+|\nabla u|_\infty\right)|\nabla\rho|_p^p + C|\nabla\dot u|_2^p + C,\end{aligned} \tag{3.37}$$

where we have used, for $p\in[2,6]$

$$\begin{aligned}|\nabla\mathrm{div}u|_p \le & |\nabla^2 u|_p \le C\left(|\rho\dot u|_p + |\nabla p_m|_p + \left|\int_0^\infty\int_{S^2} A_r\Omega\mathrm{d}\Omega\mathrm{d}v\right|_p\right) \\ \le & C\left(|\rho\dot u|_2 + |\rho\dot u|_6 + |\nabla\rho|_p + \left|\int_0^\infty\int_{S^2} A_r\Omega\mathrm{d}\Omega\mathrm{d}v\right|_2 + \left|\int_0^\infty\int_{S^2} A_r\Omega\mathrm{d}\Omega\mathrm{d}v\right|_6\right) \\ \le & C\left(1+|\nabla\dot u|_2 + |\nabla\rho|_p\right).\end{aligned} \tag{3.38}$$

It thus follows from Lemma 2.3 and (3.38) that

$$\begin{aligned}|\nabla u|_\infty \le & C\left(|\mathrm{div}u|_\infty + |\omega|_\infty\right)\ln\left(e+|\nabla^2 u|_{\tilde q}\right) + C \\ \le & C\left(|\mathrm{div}u|_\infty + |\omega|_\infty\right)\ln\left(e+|\nabla\dot u|_2\right) \\ & + C\left(|\mathrm{div}u|_\infty + |\omega|_\infty\right)\ln\left(e+|\nabla\rho|_{\tilde q}\right) + C \\ \le & C\left(|\nabla\dot u|_2 + 1\right)\ln\left(e+|\nabla\dot u|_2\right) + C\left(|\nabla\dot u|_2 + 1\right)\ln\left(e+|\nabla\rho|_{\tilde q}\right) + C,\end{aligned} \tag{3.39}$$

where we have used,

$$\begin{aligned}|\mathrm{div}u|_\infty + |\omega|_\infty & \le C\left(|G|_\infty + |p_m|_\infty + |\omega|_\infty\right) \\ & \le C\left(|G|_2 + |\omega|_2 + |\nabla G|_6 + |\nabla\omega|_6\right) \\ & \le C\left(|\nabla\dot u|_2 + \left|\int_0^\infty\int_{S^2} A_r\Omega\mathrm{d}\Omega\mathrm{d}v\right|_6 + 1\right).\end{aligned} \tag{3.40}$$

Insert (3.40) into (3.37) and choose $p=\tilde q$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\rho|_{\tilde{q}}^{\tilde{q}} \le C\left(1+\ln\left(e+|\nabla\rho|_{\tilde{q}}^{\tilde{q}}\right)\right)|\nabla\rho|_{\tilde{q}}^{\tilde{q}}+C. \tag{3.41}$$

It follows from Gronwall's inequality that

$$\sup_{0\le t\le T}|\nabla\rho|_{\tilde{q}} \le C, \tag{3.42}$$

which combined with (3.39), (3.40) and (3.27), we get

$$\int_0^T |\nabla u|_\infty \le C. \tag{3.43}$$

Taking $p=2$ in (3.37), we have, after using the Gronwall's inequality,

$$\sup_{0\le t\le T}|\nabla\rho|_2 \le C. \qquad \square \tag{3.44}$$

Then, due to $\rho_t = -\mathrm{div}(\rho u)$, the desired conclusions for ρ_t are obvious.

Next we give the estimate for the ∇I.

Lemma 3.6. *Under assumption (3.1), it holds that*

$$\int_0^\infty\int_{S^2}|\nabla I|_p^2 d\Omega dv \le C \quad \textit{for} \quad p\in[2,\tilde{q}]. \tag{3.45}$$

Proof. Differentiating the radiative transfer equation with respect to x_j, multiplying the result equation by $p|\partial_{x_j}I|^{p-2}\partial_{x_j}I$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
\frac{1}{2cp}\frac{d}{dt}|\partial_{x_j}I|_p^2 \le{}& |\partial_{x_j}S|_p^2+|\partial_{x_j}I|_p^2+\left|\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right|_\infty|\partial_{x_j}I|_p^2\\
&+|\partial_{x_j}I|_p\int_0^\infty\int_{S^2}\frac{v}{v'}|\sigma_s|_\infty|\partial_{x_j}I'|_p\mathrm{d}\Omega'\mathrm{d}v'\\
&+\left(|\nabla\sigma_a|_{\tilde{q}}+\left|\int_0^\infty\int_{S^2}\bar{\sigma}_s'\mathrm{d}\Omega'\mathrm{d}v'\right|_\infty|\nabla\rho|_{\tilde{q}}\right)|I|_{\frac{\tilde{q}p}{\tilde{q}-p}}|\partial_{x_j}I|_p\\
&+|\partial_{x_j}I|_p\int_0^\infty\int_{S^2}\frac{v}{v'}|\bar{\sigma}_s|_\infty|\nabla\rho|_{\tilde{q}}|I'|_{\frac{\tilde{q}p}{\tilde{q}-p}}\mathrm{d}\Omega'\mathrm{d}v' \qquad (3.46)\\
\le{}& |\nabla S|_p^2+\left(C+\left|\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right|_\infty\right)|\nabla I|_p^2\\
&+\int_0^\infty\int_{S^2}\frac{v^2}{v'^2}|\sigma_s|_\infty^2\mathrm{d}\Omega'\mathrm{d}v'\cdot\int_0^\infty\int_{S^2}|\nabla I|_p^2\mathrm{d}\Omega\mathrm{d}v
\end{aligned}$$

$$+\left(M(|\rho(t)|_\infty)+\left|\int_0^\infty\int_{S^2}\bar{\sigma}_s'\mathrm{d}\Omega'\mathrm{d}v'\right|_\infty\right)|I|^2_{\frac{\tilde{q}p}{\tilde{q}-p}}$$

$$+\int_0^\infty\int_{S^2}\frac{v^2}{v'^2}|\bar{\sigma}_s|^2_\infty\mathrm{d}\Omega'\mathrm{d}v'\cdot\int_0^\infty\int_{S^2}|I|^2_{\frac{\tilde{q}p}{\tilde{q}-p}}\mathrm{d}\Omega\mathrm{d}v,$$

where we have used

$$|\nabla\rho I|_p\leq|\nabla\rho|_{\tilde{q}}|I|_{\frac{\tilde{q}p}{\tilde{q}-p}}.$$

Then integrating (3.46) over $(0,\infty)\times S^2$ and noting that

$$|I|_{\frac{\tilde{q}p}{\tilde{q}-p}}\leq C|I|_2+C|\nabla I|_p\ \text{ hold for }\ p=2\ \text{ and }\ p=\tilde{q},$$

we have

$$\frac{d}{dt}\int_0^\infty\int_{S^2}|\nabla I|_p^2\mathrm{d}\Omega\mathrm{d}v\leq C+C\int_0^\infty\int_{S^2}|\nabla I|_p^2\mathrm{d}\Omega\mathrm{d}v\ \text{ for }\ p=2,\,\tilde{q}.$$

Applying Gronwall's inequality and interpolation inequality, we get the result. □

Similarly, due to $\frac{1}{c}I_t=A_r-\Omega\cdot\nabla I$ and Lemma 3.2, the desired conclusions for I_t are obvious.

4. Outline of the proof of BKM-type criterion

In this section, we will give the outline of the proof of BKM-type criterion. We also assume that the opposite holds, i.e.,

$$\limsup_{T\to T^*}|D(u)|_{L^1(0,T;L^\infty)}=M_0<\infty. \tag{4.1}$$

For the BKM-type criterion, the coupled system does not pose any significant new challenges compared with Navier–Stokes equations [13]. Indeed, equation $\rho_t+\mathrm{div}(\rho u)=0$ is treated exactly the same. So, we can also get the estimates of $|\rho|_p$ $(1\leq p\leq\infty)$. Then we obtain (3.4) and (3.9).

The next lemma will give a key estimate on $\nabla\rho$ and ∇u.

Lemma 4.1.

$$\sup_{0\leq t\leq T}\left(|\nabla u|_2^2+|\nabla\rho|_2^2\right)+\int_0^T|\nabla^2u|_2^2dt\leq C,\quad 0\leq T<T^*,$$

where C only depends on M_0 and T.

Proof. Firstly, multiplying $(2.1)_3$ by $\rho^{-1}\left(Lu+\nabla p_m+\frac{1}{c}\int_0^\infty\int_{S^2}A_r\Omega\mathrm{d}\Omega\mathrm{d}v\right)$ and integrating the result equation over $\mathbb{R}^3$, then we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\mu}{2}|\nabla u|_2^2+\frac{\mu+\lambda}{2}|\mathrm{div}u|_2^2\right)+\int_{\mathbb{R}^3}\rho^{-1}\left(Lu+\nabla p_m+\frac{1}{c}\int_0^\infty\int_{S^2}A_r\Omega\mathrm{d}\Omega\mathrm{d}v\right)^2\mathrm{d}x\\
=&-\mu\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\nabla\times(\mathrm{rot}u)\mathrm{d}x+(2\mu+\lambda)\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\nabla\mathrm{div}u\mathrm{d}x\\
&-\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\nabla p_m(\rho)\mathrm{d}x-\frac{1}{c}\int_{\mathbb{R}^3}(u\cdot\nabla u)\int_0^\infty\int_{S^2}A_r\Omega\mathrm{d}\Omega\mathrm{d}v\mathrm{d}x\\
&-\int_{\mathbb{R}^3}u_t\cdot\nabla p_m(\rho)\mathrm{d}x-\frac{1}{c}\int_{\mathbb{R}^3}u_t\cdot\int_0^\infty\int_{S^2}A_r\Omega\mathrm{d}\Omega\mathrm{d}v\mathrm{d}x\equiv:\sum_{i=1}^6L_i,
\end{aligned}
\tag{4.2}
$$

where we have used the fact that $\triangle u=\nabla\mathrm{div}u-\nabla\times\mathrm{rot}u$.

We now estimate each term in (4.2). Due to the fact that $\rho^{-1}\geq C^{-1}>0$, we find that the second term on the left hand side of (4.2) admits

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\rho^{-1}\left|Lu+\nabla p_m+\frac{1}{c}\int_0^\infty\int_{S^2}A_r\Omega\mathrm{d}\Omega\mathrm{d}v\right|^2\mathrm{d}x\\
\geq&C^{-1}|Lu|_2^2-C\left(|\nabla p_m|_2^2+|\nabla u|_2^2+\frac{1}{c}\int_0^\infty\int_{S^2}|A_r|_2^2\Omega\mathrm{d}\Omega\mathrm{d}v\right)\\
\geq&C^{-1}|u|_{D^2}^2-C\left(|\nabla p_m|_2^2+|\nabla u|_2^2+\frac{1}{c}\int_0^\infty\int_{S^2}|A_r|_2^2\Omega\mathrm{d}\Omega\mathrm{d}v\right),
\end{aligned}
\tag{4.3}
$$

where we have used the standard L^2-theory of elliptic system and Lemma 2.2. Note that L is a strong elliptic operator. Next according to

$$
\begin{cases}
u\times\mathrm{rot}u=\frac{1}{2}\nabla(|u|^2)-u\cdot\nabla u,\\
\nabla\times(a\times b)=(b\cdot\nabla)a-(a\cdot\nabla)b+(\mathrm{div}b)a-(\mathrm{div}a)b,
\end{cases}
$$

and Holder's inequality, Gagliardo–Nirenberg inequality and Young's inequality, we deduce

$$
\begin{aligned}
|L_1|=&\mu\Big|\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\nabla\times(\mathrm{rot}u)\mathrm{d}x\Big|=\mu\Big|\int_{\mathbb{R}^3}\nabla\times(u\cdot\nabla u)\cdot\mathrm{rot}u\mathrm{d}x\Big|\\
=&\mu\Big|\int_{\mathbb{R}^3}\nabla\times(u\times\mathrm{rot}u)\cdot\mathrm{rot}u\mathrm{d}x\Big|\\
=&\mu\Big|\frac{1}{2}\int_{\mathbb{R}^3}(\mathrm{rot}u)^2\mathrm{div}u\mathrm{d}x-\int_{\mathbb{R}^3}\mathrm{rot}u\cdot D(u)\cdot\mathrm{rot}u\mathrm{d}x\Big|\\
\leq&C|D(u)|_\infty|\nabla u|_2^2,
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
|L_2|=&(2\mu+\lambda)\Big|\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\nabla\mathrm{div}u\mathrm{d}x\Big|\\
=&(2\mu+\lambda)\Big|-\int_{\mathbb{R}^3}\nabla u:(\nabla u)^\top\mathrm{div}u\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}(\mathrm{div}u)^3\mathrm{d}x\Big|\\
\leq&C|D(u)|_\infty|\nabla u|_2^2,
\end{aligned}
\tag{4.5}
$$

$$\begin{aligned}
|L_3| =&\Big|\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\nabla p_m \mathrm{d}x\Big| \le C|\nabla u|_2|\nabla u|_3|\nabla p_m|_2\\
\le& C(|\nabla p_m|_2^2+1)|\nabla u|_2^2+\epsilon\left(|u|_{D^2}^2+|\nabla u|_2^2\right)\\
\le& C(|\nabla \rho|_2^2+1)|\nabla u|_2^2+\epsilon|u|_{D^2}^2,
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
L_4 =& -\frac{1}{c}\int_{\mathbb{R}^3}(u\cdot\nabla u)\int_0^\infty\int_{S^2} A_r\Omega \mathrm{d}\Omega \mathrm{d}v \mathrm{d}x\\
\le& C\int_0^\infty\int_{S^2}|A_r|_2\Omega \mathrm{d}\Omega \mathrm{d}v|\nabla u|_3|u|_6\\
\le& C\left(\int_0^\infty\int_{S^2}|A_r|_2\Omega \mathrm{d}\Omega \mathrm{d}v\right)^2|\nabla u|_2^2+\epsilon\left(|u|_{D^2}^2+|\nabla u|_2^2\right)\\
\le& C|\nabla u|_2^2+\epsilon|u|_{D^2}^2,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
L_5 =& -\int_{\mathbb{R}^3} u_t\cdot\nabla p_m \mathrm{d}x = \frac{d}{dt}\int_{\mathbb{R}^3} p_m \mathrm{div}u \mathrm{d}x - \int_{\mathbb{R}^3} p_{mt}\mathrm{div}u \mathrm{d}x\\
=&\frac{d}{dt}\int_{\mathbb{R}^3} p_m \mathrm{div}u \mathrm{d}x + \int_{\mathbb{R}^3}\left(u\cdot\nabla p_m \mathrm{div}u+\gamma p_m \mathrm{div}u^2\right)\mathrm{d}x\\
\le&\frac{d}{dt}\int_{\mathbb{R}^3} p_m \mathrm{div}u \mathrm{d}x + C|\nabla p_m|_2|u|_6|\nabla u|_3 + C|p_m|_\infty|\nabla u|_2^2\\
\le&\frac{d}{dt}\int_{\mathbb{R}^3} p_m \mathrm{div}u \mathrm{d}x + C|\nabla u|_2^2(1+|\nabla\rho|_2^2)+\epsilon|u|_{D^2}^2
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
L_6 =& -\frac{1}{c}\int_{\mathbb{R}^3} u_t\cdot\int_0^\infty\int_{S^2} A_r\Omega \mathrm{d}\Omega \mathrm{d}v \mathrm{d}x\\
=& -\frac{1}{c}\frac{d}{dt}\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2} u\cdot A_r\Omega \mathrm{d}\Omega \mathrm{d}v \mathrm{d}x + \frac{1}{c}\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2} u\cdot (A_r)_t\Omega \mathrm{d}\Omega \mathrm{d}v \mathrm{d}x,
\end{aligned} \tag{4.9}$$

where $\epsilon > 0$ is a sufficiently small constant. To deal with the last term on the right-hand side of L_6, we need to use

$$\begin{aligned}
&\frac{1}{c}\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2} u\cdot (A_r)_t\Omega \mathrm{d}\Omega \mathrm{d}v \mathrm{d}x\\
&= -\frac{1}{c}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3} u\cdot\Omega\Big(S_t - \left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s' \mathrm{d}\Omega' \mathrm{d}v'\right)_t I
\end{aligned} \tag{4.10}$$

$$
-\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)I_t+\int_0^\infty\int_{S^2}\frac{v}{v'}(\sigma_s)_tI'\mathrm{d}\Omega'\mathrm{d}v'\Big)\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v
$$

$$
-\frac{1}{c}\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}u\cdot\Omega\frac{v}{v'}\sigma_sI_t'\mathrm{d}x\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v\triangleq\sum_{j=1}^5J_j'.
$$

Similar to the proof of estimate (3.26), we also have

$$
J_1'=-\frac{1}{c}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}S_tu\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\le C|u|_6\int_0^\infty\int_{S^2}|S_t|_{\frac{6}{5}}\mathrm{d}\Omega\mathrm{d}v\le|\nabla u|_2^2+C,
$$

$$
\begin{aligned}
J_2'=&\frac{1}{c}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)_tIu\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
&\le C|u|_\infty\int_0^\infty\int_{S^2}\left|\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)_t\right|_{\frac{3}{2}}|I|_3\mathrm{d}\Omega\mathrm{d}v\\
&\le\epsilon\left(|\nabla u|_2^2+|\nabla u|_6^2\right)\\
&\quad+C\int_0^\infty\int_{S^2}|I|_3^2\mathrm{d}\Omega\mathrm{d}v\int_0^\infty\int_{S^2}\left|\rho(\overline{\sigma}_a)_t+\rho_t\left(\overline{\sigma}_a+\int_0^\infty\int_{S^2}\overline{\sigma}_s'\mathrm{d}\Omega'\mathrm{d}v'\right)\right|_{\frac{3}{2}}^2\mathrm{d}\Omega\mathrm{d}v\\
&\le\epsilon\left(|\nabla u|_2^2+|\nabla u|_6^2\right)+CM\left(|\rho|_\infty\right)\left(|\rho_t|_{\frac{3}{2}}^2+1\right)\\
&\le\epsilon\left(|\nabla u|_2^2+|\nabla u|_6^2\right)+CM\left(|\rho|_\infty\right)\left(|\rho\mathrm{div}u|_{\frac{3}{2}}^2+|\nabla\rho u|_{\frac{3}{2}}^2+1\right)\\
&\le\epsilon\left(|\nabla u|_2^2+|\nabla u|_6^2\right)+CM\left(|\rho|_\infty\right)\left(|\rho|_6^2|\mathrm{div}u|_2^2+|\nabla\rho|_2^2|u|_6^2+1\right)\\
&\le\epsilon|u|_{D^2}^2+C\left(|\nabla\rho|_2^2+1\right)\left(|\nabla u|_2^2+1\right),
\end{aligned}
$$

$$
\begin{aligned}
J_3'=&\frac{1}{c}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)I_tu\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
=&\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)A_ru\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
&-\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)\Omega\cdot\nabla I\right)(u\cdot\Omega)\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
\le&C|\nabla u|_2^2+CM\left(|\rho|_\infty\right)\\
&+\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\Omega\cdot\nabla\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)I\right)(u\cdot\Omega)\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
&+\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(\sigma_a+\int_0^\infty\int_{S^2}\sigma_s'\mathrm{d}\Omega'\mathrm{d}v'\right)I(\Omega\cdot\nabla u\cdot\Omega)\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v\\
\le&C|\nabla u|_2^2+CM\left(|\rho|_\infty\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ C|u|_6 \int_0^\infty \int_{S^2} \left| \rho \nabla \overline{\sigma}_a + \nabla \rho \left(\overline{\sigma}_a + \int_0^\infty \int_{S^2} \overline{\sigma}'_s \mathrm{d}\Omega' \mathrm{d}v' \right) \right|_2 |I|_3 \mathrm{d}\Omega \mathrm{d}v \\
&+ \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} \left(\sigma_a + \int_0^\infty \int_{S^2} \sigma'_s \mathrm{d}\Omega' \mathrm{d}v' \right) I (\Omega \cdot \nabla u \cdot \Omega) \mathrm{d}x \mathrm{d}\Omega \mathrm{d}v \\
\le & C|\nabla u|_2^2 + CM(|\rho|_\infty) + C|\nabla u|_2^2 + CM(|\rho|_\infty)(|\nabla \rho|_2^2 + 1) \\
&+ C \int_0^\infty \int_{S^2} |I|_3^2 \mathrm{d}\Omega \mathrm{d}v + C|\nabla u|_2^2 \\
&+ C \max_{(v,\Omega)\in[0,\infty)\times S^2} \left| \sigma_a + \int_0^\infty \int_{S^2} \sigma'_s \mathrm{d}\Omega' \mathrm{d}v' \right|_\infty^2 \int_0^\infty \int_{S^2} |I|_2^2 \mathrm{d}\Omega \mathrm{d}v \\
\le & \epsilon |u|_{D^2}^2 + C\left(|\nabla \rho|_2^2 + 1\right)\left(|\nabla u|_2^2 + 1\right),
\end{aligned}
$$

$$
\begin{aligned}
J'_4 = & -\frac{1}{c} \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} \frac{v}{v'} (\sigma_s)_t I' u \cdot \Omega \mathrm{d}x \mathrm{d}\Omega' \mathrm{d}v' \mathrm{d}\Omega \mathrm{d}v \\
\le & C|u|_\infty \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \left| \frac{v}{v'} \right| \bar{\sigma}_s |\rho_t|_{\frac{3}{2}} |I'|_3 \mathrm{d}\Omega' \mathrm{d}v' \mathrm{d}\Omega \mathrm{d}v \\
\le & \epsilon \left(|\nabla u|_2^2 + |\nabla u|_6^2 \right) + CM(|\rho|_\infty)(|\rho_t|_{\frac{3}{2}}^2 + 1) \\
\le & \epsilon |u|_{D^2}^2 + C\left(|\nabla \rho|_2^2 + 1\right)\left(|\nabla u|_2^2 + 1\right)
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
J'_5 = & \frac{1}{c} \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} u \cdot \Omega \frac{v}{v'} \sigma_s I'_t \mathrm{d}x \mathrm{d}\Omega' \mathrm{d}v' \mathrm{d}\Omega \mathrm{d}v \\
= & \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} u \cdot \Omega \frac{v}{v'} \sigma_s A'_r \mathrm{d}x \mathrm{d}\Omega' \mathrm{d}v' \mathrm{d}\Omega \mathrm{d}v \\
& - \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} (u \cdot \Omega) \frac{v}{v'} \sigma_s (\nabla I' \cdot \Omega) \mathrm{d}x \mathrm{d}\Omega' \mathrm{d}v' \mathrm{d}\Omega \mathrm{d}v \\
\le & C|\nabla u|_2^2 + CM(|\rho|_\infty) + \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} (u \cdot \Omega) \frac{v}{v'} I' (\nabla \sigma_s \cdot \Omega) \mathrm{d}x \mathrm{d}\Omega' \mathrm{d}v' \mathrm{d}\Omega \mathrm{d}v \\
& + \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} (\Omega \cdot \nabla u \cdot \Omega) \frac{v}{v'} \sigma_s I' \mathrm{d}x \mathrm{d}\Omega' \mathrm{d}v' \mathrm{d}\Omega \mathrm{d}v \\
\le & \epsilon |u|_{D^2}^2 + C\left(|\nabla \rho|_2^2 + 1\right)\left(|\nabla u|_2^2 + 1\right)
\end{aligned}
\tag{4.12}
$$

Then combining (4.2)–(4.12), we have

$$
\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u|^2 + \frac{\mu + \lambda}{2} |\mathrm{div} u|^2 - p_m \mathrm{div} u + \frac{1}{c} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} u \cdot A_r \Omega \mathrm{d}\Omega \mathrm{d}v \mathrm{d}x \right) \mathrm{d}x + C|\nabla^2 u|_2^2 \\
\le & C(|\nabla u|_2^2 + 1)(|\nabla \rho|_2^2 + |D(u)|_\infty + 1).
\end{aligned}
\tag{4.13}
$$

Secondly, applying ∇ to $(2.1)_2$ and multiplying the result equation by $2\nabla\rho$, we have

$$\begin{aligned}
&(|\nabla\rho|^2)_t + \mathrm{div}(|\nabla\rho|^2 u) + |\nabla\rho|^2 \mathrm{div}u \\
=& -2(\nabla\rho)^\top \nabla u \nabla\rho - 2\rho\nabla\rho\cdot\nabla\mathrm{div}u \\
=& -2(\nabla\rho)^\top D(u)\nabla\rho - 2\rho\nabla\rho\cdot\nabla\mathrm{div}u.
\end{aligned} \tag{4.14}$$

Then integrating (4.14) over $\mathbb{R}^3$, we have

$$\frac{d}{dt}|\nabla\rho|_2^2 \le C(|D(u)|_\infty + 1)|\nabla\rho|_2^2 + \epsilon|\nabla^2 u|_2^2. \tag{4.15}$$

Thirdly, similar to the proof of estimate (3.17), we get

$$\begin{aligned}
&-\frac{1}{c}\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2} A_r u\cdot\Omega \mathrm{d}\Omega\mathrm{d}v\mathrm{d}x \\
&= -\frac{1}{c}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\left(S - \sigma_a I + \int_0^\infty\int_{S^2}\frac{v}{v'}\sigma_s I'\mathrm{d}\Omega'\mathrm{d}v'\right) u\cdot\Omega\mathrm{d}x\mathrm{d}\Omega\mathrm{d}v \\
&\quad + \frac{1}{c}\int_0^\infty\int_{S^2}\int_0^\infty\int_{S^2}\int_{\mathbb{R}^3}\sigma_s' I u\cdot\Omega\mathrm{d}x\mathrm{d}\Omega'\mathrm{d}v'\mathrm{d}\Omega\mathrm{d}v \\
&\le C|\sqrt{\rho}u|_2^2 + C|\nabla u|_2^2 \le C + C|\nabla u|_2^2.
\end{aligned} \tag{4.16}$$

Adding (4.15) and (4.16) to (4.13), from Gronwall's inequality we immediately obtain

$$|\nabla u(t)|_2^2 + |\nabla\rho(t)|_2^2 + \int_0^t |\nabla^2 u(s)|_2^2 \mathrm{d}t \le C, \quad 0 \le t < T.$$

The rest of the proof is same to the Section 3. □

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