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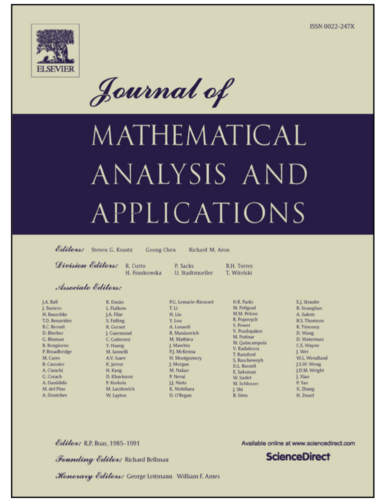

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Abstract

In this paper, we present a complete general solution of the unsteady Brinkman equations. To this end, we introduce a representation for velocity and pressure in terms of two scalar functions. One of these scalar functions satisfies a second order partial differential equation (PDE) while the other satisfies a fourth order PDE which can be factorized into a pair of second order PDEs. We show that the solution of this fourth order PDE is indeed the sum of the solutions of the two second order PDEs. We also use these solutions to obtain a complete general solution of the unsteady Brinkman equations.

Keywords: Stokes flows, Brinkman equations, heat operator, factorization of operators.
2008 MSC: 35Q35, 76D07.

1. Introduction

Consider a homogeneous porous medium characterized by a permeability parameter κ/L^2, where κ is the Darcy permeability of the medium and L is the particle length scale of a cloud of spherical particles in the medium. This kind of a medium is said to be a Brinkman medium. Moreover, in a Brinkman medium the size of the particles is smaller than the characteristic length scale of the flow. Therefore, they occupy a negligible volume (see [5]). Flows through porous media are governed by either the Darcy model or the Brinkman model. The latter is found to be more suitable when the permeability of the medium is high. Consider an incompressible flow of a viscous fluid through a Brinkman medium with a permeability κ which is modeled by the Brinkman equations,

$$\rho \frac{\partial \mathbf{q}}{\partial t}(\mathbf{x}, t) = -\nabla p(\mathbf{x}, t) + \mu \left(\Delta - \frac{1}{\kappa} \right) \mathbf{q}(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^3, \ t > 0, \tag{1}$$

$$\operatorname{div}(\mathbf{q}(\mathbf{x}, t)) = 0, \quad \mathbf{x} \in \mathbb{R}^3, \ t > 0. \tag{2}$$

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where $\mathbf{q}, p, \mu$ and ρ are the average fluid velocity, average pressure, coefficient of dynamic viscosity and density of the fluid respectively. It is assumed that the averages that we have considered occur over many realizations of particle arrangements which satisfy the volume fraction constraint, permeability of the medium and size of the particles in the Brinkman medium (see [5], [12]). Notice that (1)–(2) reduce to the unsteady Stokes equations if the drag term $\mu\mathbf{q}/\kappa$ is neglected. On the other hand, the Brinkman equations reduce to the unsteady Darcy equations if the diffusion term $\mu\Delta\mathbf{q}$ is neglected.

Flow past an aggregate with radially varying solid fraction and permeability has been studied in [25] using the Brinkman equations for the internal flow and the Stokes equations for the external flow. In [7], the author has used the Brinkman equations to model flows through an array of fixed cylindrical fibers. Many biological phenomena are modeled using the Brinkman equations (see, for instance [6], [19]).

Transmission problems for Brinkman and Stokes equations have attracted attention of many mathematicians in recent times (see [8], [9], [10], [13], [14]). In particular, methods of potential theory which are used to study the elliptic boundary value problems are employed to obtain the existence and uniqueness of a solution of the models under consideration. The authors in [8] have studied the existence and uniqueness results for the transmission problems with Lipschitz interface for the Darcy-Forchheimer-Brinkman and Stokes systems using the layer potential method for the linear Stokes and Brinkman system, and fixed point theory. In [9], the integral layer potentials of the Stokes and Brinkman systems along with Leray-Schauder degree theory is used to establish the existence result for a nonlinear Neumann transmission problem for the Stokes and Brinkman systems in two adjacent bounded Lipschitz domains. In [14] the author has obtained existence and uniqueness results for Robin-transmission problem and Dirichlet-transmission problem for the Brinkman system using integral equation method.

Most of the work on Brinkman flows are restricted to the steady case, i.e., time independent flows. Moreover, most of the unsteady cases considered in the literature are of oscillatory type (see [1], [16], [17], [24]). The advantage of considering oscillatory flow is that the governing equations reduce to the steady Brinkman equations. Many problems which are solved in either Stokes' flows or Brinkman flows use some form of complete general solutions of these equations. For instance, complete general solutions of the steady Stokes equations are due to Lamb (see [11]) and Padmavati et al (see [20]) have been extensively used to discuss flows at low Reynolds number (see [2], [3], [4], [15], [21]).

In general, in order to derive a complete general solution, the velocity vector is expressed in terms of two scalar functions in a particular form in such a way that the equation of continuity holds for every choice of the scalar functions. Then one needs to find the suitable equations which are satisfied by these scalar functions such that every solution of the equations of motion can be expressed in that particular form (see [18], [20], [22], [26]).

One of the main difficulties faced in this method is to solve the equations satisfied by the scalar functions. This is because one of the scalar functions satisfies a higher order PDE (fourth order in most cases). In order to overcome this difficulty we employ the technique of factorizing the operator. Recently the authors of [23] have shown that by factorizing the operator, of course under suitable hypotheses on the operators involved, any solution of the higher order equation can be split into a sum of two functions which satisfy the PDEs of lower order.

In this paper, we consider the unsteady Brinkman equations and express the velocity vector in terms of two scalar functions. One of the functions satisfies a sixth order PDE and the other one a fourth order PDE. The sixth and fourth order operators involved are factorized into second order operators. Further, we show that any solution of these sixth and fourth order PDEs can be split into sum of functions each satisfying a second order PDE. We exploit the structure of the solutions of these second order operators involved to obtain a complete general solution of the unsteady Brinkman equations.

2. Method of Solution

Let L denote the two dimensional Laplace operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. Let $\hat{\imath},\ \hat{\jmath}$, and $\hat{k}$ be the unit vectors of along $x,\ y$, and z axes respectively, in $\mathbb{R}^3$. It has been proved in [26], that any divergence free vector field $\mathbf{q}$ can be written as

$$\mathbf{q}(x,y,z,t) = CurlCurl\big(A(x,y,z,t)\hat{k}\big) + Curl\big(B(x,y,z,t)\hat{k}\big) \tag{3}$$

where A and B are smooth functions defined on $\mathbb{R}^3\times(0,\infty)$ satisfy, for every fixed $z,\ t$,

$$LA = -\hat{k}.\mathbf{q}, \tag{4}$$

$$LB = -\hat{k}.Curl(\mathbf{q}). \tag{5}$$

So far we have dealt with only a divergence free vector field to obtain equations (4) and (5). If the divergence free $\mathbf{q}$ given in (3) is also a solution of (1) then A and B have to solve some more equations apart from (4)–(5). This is given in the following Lemma.

Lemma 1. *Assume that* $(\mathbf{q},p)$, *where* $\mathbf{q}$ *is given in* (3), *solves system* (1)–(2). *Then the functions* A *and* B *given in* (3) *solve*

$$L\Delta\left(\mu\Delta-\frac{\mu}{\kappa}-\rho\frac{\partial}{\partial t}\right)A = 0, \tag{6}$$

and

$$L\left(\mu\Delta-\frac{\mu}{\kappa}-\rho\frac{\partial}{\partial t}\right)B = 0, \tag{7}$$

respectively.

Proof. We begin with the following observations regarding $\mathbf{q}$ and p. We take divergence of (1) and use the equation of continuity, i.e., (2) to establish that the pressure p is harmonic. Moreover, it is straightforward to observe that

$$\Delta\left(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\right)\mathbf{q} = \mathbf{0}, \tag{8}$$

$$\left(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\right)Curl(\mathbf{q}) = \mathbf{0}. \tag{9}$$

The required result immediately follows from equations (4)–(5) and (8)–(9). □

Substituting (3) into (1), we obtain

$$\begin{aligned}\nabla\Big(p - \frac{\partial}{\partial z}(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t})A\Big) \quad &= \quad \hat{i}\frac{\partial}{\partial y}(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t})B \\ &\quad -\hat{j}\frac{\partial}{\partial x}(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t})B \\ &\quad -\hat{k}\Delta(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t})A. \end{aligned} \tag{10}$$

If we have to find a complete general solution of (1)–(2), then we should write the pressure p in terms of A in (10). This can be done easily if the right hand side of (10) is zero. Therefore one of our main objectives is to choose the scalar functions A and B in (3) such that (i) they satisfy (4)–(7), (ii) all the components on the right hand side of (10) vanish. To this end, we present the following factorization theorem which plays a vital role. Let $L_1,\ L_2$ and L_3 be linear operators on a normed linear space X such that L_2 is invertible.

Theorem 1. *Assume that L_2^{-1} commutes with both L_1 and L_3, i.e., $L_1L_2^{-1} = L_2^{-1}L_1$ and $L_3L_2^{-1} = L_2^{-1}L_3$. Then for any given u such that $L_1(L_3L_1+L_2)u = 0$, there exists $u_1,\ u_2 \in X$ such that $u = u_1+u_2$ and $L_1u_1 = 0$, $(L_3L_1+L_2)u_2 = 0$.*

Proof. In order to prove the theorem, we use the similar technique introduced in [23]. First, define $u_1 = L_2^{-1}(L_3L_1 + L_2)u$. Since L_2^{-1} commutes with L_1, we obtain $L_1u_1 = 0$. On the other hand, consider

$$\begin{aligned}(L_3L_1 + L_2)(u - u_1) \quad &= \quad (L_3L_1 + L_2)u - (L_3L_1 + L_2)L_2^{-1}(L_3L_1 + L_2)u \\ &= \quad (L_3L_1 + L_2)u - L_3L_1L_2^{-1}(L_3L_1 + L_2)u - (L_3L_1 + L_2)u \\ &= \quad -L_2^{-1}L_3L_1(L_3L_1 + L_2)u = 0.\end{aligned}$$

Therefore, if $u_2 = u - u_1$ then we have $(L_3L_1 + L_2)u_2 = 0$. This completes the proof. □

In Lemma 1, we have proved that the scalar functions A and B which give the velocity field via (3) satisfy the sixth order and fourth order equations given in

(6) and (7) respectively. Now we use the factorization given in Theorem 1 to write A and B as the sum of the functions which satisfy lower order equations that help to make the right hand side of (10) vanish. This is given in the next result.

Theorem 2. *Assume that A and B solve* (6) *and* (7) *respectively. Then there exist A_1, A_2, B_1 and B_2 such that:*

(i) $A = A_1 + A_2$, *with* $LA_1 = 0$ *and* $\Delta\big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big)A_2 = 0.$

(ii) $B = B_1 + B_2$, *with* $LB_1 = 0$ *and* $\big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big)B_2 = 0.$

Proof. We begin with the proof of (i). First, we rewrite (6) as

$$L\big(L + \frac{\partial^2}{\partial z^2}\big)\Big(\mu L + \mu\frac{\partial^2}{\partial z^2} - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\Big)A = 0.$$

This is same as

$$L\left[\Big(\mu\Delta + \mu\frac{\partial^2}{\partial z^2} - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\Big)L + \frac{\partial^2}{\partial z^2}\Big(\mu\frac{\partial^2}{\partial z^2} - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\Big)\right]A = 0. \tag{11}$$

It is easy to observe that the operator acting on A in (11) satisfies the hypotheses of Theorem 1. Therefore we get by Theorem 1, that there exist A_1 and A_2 such that $LA_1 = 0$ and $\Delta\big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big)A_2 = 0$.

By using similar arguments one can easily prove (ii). Therefore we omit the details. ☐

From the representation of the solution given in (3) and Theorem 2, we get

$$\begin{aligned}\mathbf{q}(x,y,z,t) &= CurlCurl\big(A_1\hat{k}\big) + CurlCurl\big(A_2\hat{k}\big)\\ &\quad + Curl\big(B_1\hat{k}\big) + Curl\big(B_2\hat{k}\big),\end{aligned} \tag{12}$$

where $A_1,\ A_2,\ B_1$ and B_2 are as in Theorem 2. Now in the next two results we are going demonstrate that, it is possible to find two functions A_3 and B_3 such that

$$\mathbf{q}(x,y,z,t) = CurlCurl\big(A_3\hat{k}\big) + Curl\big(B_3\hat{k}\big),$$

where A_3 and B_3 solve the same equations that A_2 and B_2 solve respectively.

Theorem 3. *Let $\mathbf{v} = CurlCurl(u\hat{k})$, with $Lu = 0$ be such that $(\mathbf{v}, p)$ solves* (1)–(2). *Then there exists U such that $CurlCurl(U\hat{k}) = \mathbf{0}$ and*

$$\Delta\big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big)(u + U) = 0. \tag{13}$$

Proof. We substitute $CurlCurl(u\hat{k}) = (u_{xz}, u_{yz}, 0)$ into (1) to get

$$\rho u_{xzt} - \mu u_{xzzz} + \frac{\mu}{\kappa}u_{xz} + p_x = 0, \tag{14a}$$

$$\rho u_{yzt} - \mu u_{yzzz} + \frac{\mu}{\kappa} u_{yz} + p_y = 0, \tag{14b}$$

$$p_z = 0. \tag{14c}$$

Differentiate (14a) and (14b) with respect to z and use (14c) to obtain

$$\frac{\partial^3}{\partial x \partial z^2} \mathcal{H}_{t,z} u = 0, \ \frac{\partial^3}{\partial y \partial z^2} \mathcal{H}_{t,z} u = 0,$$

where $\mathcal{H}_{t,z} u := \rho u_t - \mu u_{zz} + \frac{\mu}{\kappa} u$. It is easy to observe that there exists three functions $F(z,t)$, $g(x,y,t)$, and $h(x,y,t)$ such that

$$\mathcal{H}_{t,z} u = F(z,t) + g(x,y,t) z + h(x,y,t).$$

Since $Lu = 0$, we get $\mathcal{H}_{t,z} Lu = 0$. This immediately implies that $Lg = 0$ and $Lh = 0$.
On the other hand, let $U(z,t)$ be a solution of $\mathcal{H}_{t,z} U(z,t) = -F(z,t)$. Since U does not depend on x and y, we have,

$$LU = 0, \quad \text{and} \quad CurlCurl(U\hat{k}) = \mathbf{0}.$$

It is straightforward to compute

$$\begin{aligned}
\Delta\big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big)(u+U) &= -\frac{\partial^2}{\partial z^2}\mathcal{H}_{t,z}\big(u+U\big) \\
&= -\frac{\partial^2}{\partial z^2}F(z,t) + \frac{\partial^2}{\partial z^2}F(z,t) = 0.
\end{aligned}$$

This completes the proof the theorem. □

An immediate consequence of Theorem 3 is that the first term on the right hand side of (12) can be absorbed into the second term of the same. In other words, there exists, U such that

$$\mathbf{q} = CurlCurl(A_1\hat{k}) = CurlCurl\big((A_1 + U)\hat{k})\big)$$

where $A_1 + U$ solves (13) whenever $(\mathbf{q}, p)$ solves (1)–(2). Therefore, we have $\mathbf{q} = CurlCurl(A_3\hat{k})$ for some A_3 which solves (13) provided $(\mathbf{q}, p)$ is a solution of (1)–(2).
Now, we turn our attention towards the third term in the representation given in (12). In this context, we have the following theorem.

Theorem 4. *Let $\mathbf{v} = Curl(w\hat{k})$, with $Lw = 0$. If $(\mathbf{v}, p)$ solves (1)–(2), then there exist V and W_1 such that*

$$\mathbf{v} = CurlCurl(V\hat{k}) + Curl(W_1\hat{k}),$$

and

$$\Delta\big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big)V = 0, \ \big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big)W_1 = 0.$$

Proof. We begin the proof with the same argument that is used in Theorem 3. On substituting $\mathbf{v} = (w_y, -w_x, 0)$ into (1), we obtain

$$\rho w_{yt} - \mu w_{yzz} + \frac{\mu}{\kappa} w_y + p_x = 0, \tag{15a}$$

$$\rho w_{xt} - \mu w_{xzz} + \frac{\mu}{\kappa} w_x - p_y = 0, \tag{15b}$$

$$p_z = 0. \tag{15c}$$

Recall the operator $\mathcal{H}_{t,z}$ that was introduced in Theorem 3. Differentiate (15a) and (15b) with respect to z and use (15c) to obtain

$$\frac{\partial^2}{\partial x \partial z} \mathcal{H}_{t,z} w = 0, \;\; \frac{\partial^2}{\partial y \partial z} \mathcal{H}_{t,z} w = 0.$$

Therefore we have,

$$\mathcal{H}_{t,z} w = f(z,t) + \eta(x,y,t)$$

for some smooth functions f and η. Since $Lw = 0$, we get $\mathcal{H}_{t,z} Lw = 0$. This implies $L\eta = 0$.
Now define $W = w - u_1$ where $u_1(z,t)$ solves $\mathcal{H}_{t,z} u_1(z,t) = f(z,t)$.
It is easy to observe that

$$Curl(W\hat{k}) = Curl(w\hat{k}), \quad LW = 0, \quad \mathcal{H}_{t,z} W = \eta(x,y,t). \tag{16}$$

This W can be written as the sum $W_1 + W_2$ where $\mathcal{H}_{t,z} W_1 = 0$ and

$$W_2(x,y,t) := \frac{1}{\rho} \exp\big(\frac{-\mu t}{\rho\kappa}\big) \int^t \exp\big(\frac{\mu s}{\rho\kappa}\big) \eta(x,y,s) ds,$$

because $\mathcal{H}_{t,z} W_2 = \eta$. Further, observe that $LW_2 = 0$. This immediately gives us, $LW_1 = 0$ and therefore

$$(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}) W_1 = 0. \tag{17}$$

With this decomposition of W, we have

$$Curl(W\hat{k}) = Curl(W_1\hat{k}) + Curl(W_2\hat{k}) \tag{18}$$

with W_1 satisfying (17). It remains to find V such that $Curl(W_2\hat{k}) = CurlCurl(V\hat{k})$. In other words, we need $V(x,y,z,t)$ that satisfies

$$(W_2)_y = V_{xz}, \;\; -(W_2)_x = V_{yz}, \;\; LV = 0.$$

Notice that one can take V as a 2-D harmonic conjugate, (with respect to x and y) for a fixed z, of zW_2. Therefore we get $LV = 0$. In particular, we can take

$$V(x,y,z,t) = \int^x z(W_2)_y dx - \int^y z(W_2)_x dy - \int^x \int^y z(W_2)_{yy} dxdy.$$

This in turn gives us

$$Curl(W_2\hat{k}) = CurlCurl(V\hat{k}). \tag{19}$$

Since $LV = 0$, we have

$$\Delta\big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big) V = -\frac{\partial^2}{\partial z^2} \mathcal{H}_{t,z} V = 0. \tag{20}$$

The announced result follows from (16)–(20). □

This proves that any given solution $(\mathbf{q}, p)$ of (1)–(2) can be written as

$$\mathbf{q}(x,y,z,t) = CurlCurl\big(A_3\hat{k}\big) + Curl\big(B_3\hat{k}\big), \tag{21}$$

where

$$\Delta\big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big)A_3 = 0, \tag{22}$$

and

$$\big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big)B_3 = 0. \tag{23}$$

Therefore, equation (10) reduces to

$$\nabla\Big(p - \frac{\partial}{\partial z}\big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big)A_3\Big) = \mathbf{0}.$$

Hence the given pressure has a representation

$$p = p_0 + \frac{\partial}{\partial z}\big(\mu\Delta - \frac{\mu}{\kappa} - \rho\frac{\partial}{\partial t}\big)A_3 \tag{24}$$

where p_0 is a nonnegative constant.
Moreover, using the factorization result, i.e., Theorem 1, we can show that any solution A_3 of (22) can be written as $A_3 = A_4 + A_5$ where A_4 is harmonic and A_5 solves (23). Therefore one can write A_3 as the sum of two terms which solve two standard second order equations. Notice that after multiplication with $\exp\left(\frac{\mu t}{\rho\kappa}\right)$ (23) reduces to the heat equation.
Hence a complete general solution of (1)–(2) is given by

$$\mathbf{q}(x,y,z,t) = CurlCurl\big(A_4\hat{k}\big) + CurlCurl(A_5\hat{k}) + Curl\big(B_3\hat{k}\big), \tag{25}$$

where A_4 is harmonic, A_5 solves the same equation as B_3 which is (23) and p is given by (24).

3. Conclusions

One of the key ingredients which are used to establish that (21)–(24) give a complete general solution of the Brinkman equations is the factorization of the sixth and fourth order operators in (6)–(7). We have shown in Theorem 2 that any solution of (6) can be written as the sum of the solutions of 2-D Laplace equation and (22). Similarly any solution of (7) is the sum of a function which is harmonic in x, y and a solution of (23). Another important technique used to prove the completeness of the representation (21) is presented in Theorems 3 and 4. In particular, we have shown that the first and third terms on the right hand side of (12) can be absorbed into the remaining terms of the same. Therefore we have established that any solution of (1)–(2) is expressed, in terms of two scalar functions, as in (21)–(24). Further, if we use the factorization result (see [23]) for the operator in (22) we can write any solution of (22) as the sum of solutions of Laplace equation and (23). This enables us to express any solution of the Brinkman equations as given in (25) in terms of three scalar functions which satisfy Laplace equation and (23) whose general solutions are well known.

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