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Stability of ordered equilibria

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A B S T R A C T

In this article we give an elementary method to investigate linear stability of equilibria of finite dimensional dynamical systems. In particular, under general hypotheses, the equilibria can be organised in an *ordered chain* along which the determinant of the associated Jacobian matrix has alternating sign. We develop the idea in two and three-dimensional cases, and then give a result for general n-dimensional systems. We also apply the technique to some particular, well known dynamical systems.

1. Introduction

The stability of the equilibria of a given dynamical system is a fundamental information on that system. Typically, a dynamical system models a natural phenomenon and depends on a number of parameters, and hence the determination of the regions in the parameter space in which a particular equilibrium is stable or unstable is a fundamental question. For example, in epidemiology, the function of parameters that discriminates the stability of the disease free equilibrium is the *basic reproduction number*. In this article we try to give a method to deduce information about the stability of an equilibrium only having information on the stability of some other equilibria. In particular we show that, under very general hypotheses, the equilibria of a dynamical system can be ordered to form a *chain of equilibria* $E_1 < E_2 < \cdots < E_m$. When such a chain of ordered equilibria is determined, we can show that whenever one of them, say $E_{\bar{i}}$, is stable, than the neighbouring equilibria $E_{\bar{i}-1}$, $E_{\bar{i}+1}$ must necessarily be unstable. This is a generalisation of the elementary fact that the equilibria of a 1-dimensional system $\dot{x} = f(x)$, $x \in \mathbb{R}$, must necessarily be of alternating type stable/unstable (if the given function f changes sign in a neighbourhood of a zero).

The construction relies on very elementary arguments but, as far as we know, this result is not explicitly described in the literature. Moreover, as we will show, this result makes it very simple to determine the stability of equilibria without analysing the positivity of some functions, e.g. those used in the Routh–Hurwitz

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criterion [6]. This result is very useful in particular when the dynamical system has high dimension or it involves many parameters.

We prove our results first in dimension two (Section 2) and three (Section 4). Then, we give the general n-dimensional result (Section 6). We apply the method to well known two (Section 3) and three dimensional dynamical systems (Section 4). We also give a 4-dimensional example at the end of Section 6. In Section 7 we give some concluding remarks, emphasising in particular the strengths and weaknesses of the proposed method.

2. The two-dimensional case

Let us consider a 2-dimensional ordinary differential system

$$\dot{\boldsymbol{x}} = \boldsymbol{X}(\boldsymbol{x}) \tag{1}$$

with $\boldsymbol{x} = (x, y)^T$ in Ω, a domain of $\mathbb{R}^2$, and $\boldsymbol{X} = (X, Y)^T$ a function defined in Ω with values in $\mathbb{R}^2$. The above system, in components, may be written

$$\begin{cases} \dot{x} = X(x, y) \\ \dot{y} = Y(x, y). \end{cases} \tag{2}$$

Assume that $X, Y : \Omega \to \mathbb{R}$ are differentiable functions, and let

$$\boldsymbol{\gamma} : (a, b) \subset \mathbb{R} \to \Omega \subset \mathbb{R}^2$$

be the parametrisation of a regular differentiable curve in Ω with $\boldsymbol{\gamma}'(s) \neq 0$ for every $s \in (a, b)$, $\boldsymbol{\gamma}(a) \neq \boldsymbol{\gamma}(b)$ and such that

$$X(\boldsymbol{\gamma}(s)) = 0, \qquad \nabla X(\boldsymbol{\gamma}(s)) \neq 0,$$

for every $s \in (a, b)$. These conditions imply that $\boldsymbol{\gamma}$ is a *regular parametrization* of a null-cline of X, i.e. is a curve in the set $\{(x, y) \,|\, X(x, y) = 0\}$.

Definition 1. A chain of ordered equilibria is an ordered list of points

$$E_1 = \boldsymbol{\gamma}(s_1) < \cdots < E_m = \boldsymbol{\gamma}(s_m)$$

where $s_1 < \cdots < s_m$ are all the $s \in (a, b)$ such that $Y(\boldsymbol{\gamma}(s_i)) = 0$.

Let $\boldsymbol{J}(\boldsymbol{x})$ be the Jacobian matrix of the vector field $\boldsymbol{X}$, and $J(\boldsymbol{x})$ its determinant (the Jacobian determinant). By hypothesis, the function $X(\boldsymbol{\gamma}(s))$ is identically zero, while the function

$$h(s) = Y(\boldsymbol{\gamma}(s)) \tag{3}$$

not only is not identically zero, but it plays an important role in the determination of the stability of the equilibria. The equilibria correspond in fact to the values $s_1, ..., s_m$ such that $h(s_i) = 0$, and the derivative of h at such points is related to the Jacobian determinant.

Lemma 1. *Let $\boldsymbol{X}$ be a vector field on the plane, $\boldsymbol{\gamma}(s)$ be a parametrization of a null-cline of the first component of $\boldsymbol{X}$, and let $h(s) = Y(\boldsymbol{\gamma}(s))$ be the composition of $\boldsymbol{\gamma}$ with the other component of $\boldsymbol{X}$. If the curve $\boldsymbol{\gamma}(s)$*

is a regular parametrization and if $\nabla X(\boldsymbol{\gamma}(s))$ *is a non-vanishing vector, then there exists a non-vanishing function* $k(s)$ *such that*

$$J(\boldsymbol{\gamma}(s)) = k(s)\, h'(s). \tag{4}$$

Proof. The equilibria of the chain correspond to the values of s such that $h(s) = 0$. Since $X(\boldsymbol{\gamma}(s)) \equiv 0$ it follows that ∇X is orthogonal to the vector $\boldsymbol{\gamma}(s)$. Being ∇X never zero, it follows that there exists a nonvanishing function $k(s)$ such that

$$\nabla X(\boldsymbol{\gamma}(s)) = k(s)(\gamma_2'(s), -\gamma_1'(s)),$$

with γ_1, γ_2 the components of $\boldsymbol{\gamma}(s)$. Hence

$$\begin{aligned}\det \boldsymbol{J}(\boldsymbol{\gamma}(s)) = J(\boldsymbol{\gamma}(s)) &= X_x(\boldsymbol{\gamma}(s))Y_y(\boldsymbol{\gamma}(s)) - X_y(\boldsymbol{\gamma}(s))Y_x(\boldsymbol{\gamma}(s)) = \\ &= k(s)\Big(\gamma_2'(s)Y_y(\boldsymbol{\gamma}(s)) + \gamma_1'(s)Y_x(\boldsymbol{\gamma}(s))\Big) \\ &= k(s)\frac{d}{ds}(Y(\boldsymbol{\gamma}(s))) = k(s)\, h'(s). \quad \square\end{aligned}$$

Observation 1. In many cases, the parametrisation $\boldsymbol{\gamma}$ of the null-cline can be chosen in such a way that $k(s) \neq 0$ is a constant. Moreover, the role of X and Y can be exchanged.

Example 1. 1-degree of freedom conservative mechanical systems.

We are dealing with vector fields of the form $(y, -V'(x))$, were $V(x)$ is the potential energy of the mechanical system. In this case $\boldsymbol{\gamma}(s) = (s, 0)$ is a parametrization of the null-cline of the first component of the vector field. Moreover,

$$\nabla X(\boldsymbol{\gamma}(s)) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad h(s) = Y(\boldsymbol{\gamma}(s)) = -V'(s).$$

The Jacobian determinant along the curve parametrized by $\boldsymbol{\gamma}$ is the function $V''(s)$, the derivative of h is $-V''(s)$. They correspond with a coefficient of proportionality $k(s) = -1$.

Example 2. Vector fields of the form

$$\boldsymbol{X} = \begin{pmatrix} \alpha(x)y - \varphi(x) \\ Y(x, y) \end{pmatrix}$$

where $\alpha(x), \varphi(x) \in C^1(a, b)$, and $\alpha(x) \neq 0$, $\forall x \in (a, b)$.

A possible choice of a parametrisation of a null-cline for one component of $\boldsymbol{X}$ is

$$\boldsymbol{\gamma}(s) = \left(s, \frac{\varphi(s)}{\alpha(s)}\right).$$

The curve $\boldsymbol{\gamma}$ is regular and

$$\nabla X(\boldsymbol{\gamma}) = \begin{pmatrix} \alpha'(s)\dfrac{\varphi(s)}{\alpha(s)} - \varphi'(s) \\ \alpha(s) \end{pmatrix}$$

is never zero. In this case $h(s) = Y\left(s, \dfrac{\varphi(s)}{\alpha(s)}\right)$, and hence

$$
\begin{aligned}
h'(s) &= Y_x\left(s, \frac{\varphi(s)}{\alpha(s)}\right) + Y_y\left(s, \frac{\varphi(s)}{\alpha(s)}\right)\frac{\varphi'(s)\alpha(s) - \varphi(s)\alpha'(s)}{\alpha^2(s)} = \\
&= \frac{1}{\alpha(s)}\left(\alpha(s)Y_x\left(s, \frac{\varphi(s)}{\alpha(s)}\right) - Y_y\left(s, \frac{\varphi(s)}{\alpha(s)}\right)\left(-\varphi'(s) + \alpha'(s)\frac{\varphi(s)}{\alpha(s)}\right)\right) = \\
&= -\frac{1}{\alpha(s)} J(\boldsymbol{\gamma}(s)).
\end{aligned}
$$

In this case $k(s) = -1/\alpha(s)$. □

The main result of this paper rests on the trivial observation that if a function $h(s)$ has only *simple zeroes* (i.e. $h(s_i) = 0\,, h'(s_i) \neq 0$), then its derivative $h'(s)$ along the zeroes must have alternating sign. It follows that *if the function $h(s)$ has simple zeroes* then the Jacobian determinant of the vector field must have sign that alternates from positive to negative along the chain of equilibria. We summarise this argument in the following theorem.

Theorem 1. *Assume that the function $h(s)$ given in (3) has simple zeroes in the points s_i. Given the chain of ordered equilibria $E_1 < \ldots < E_m$, two consecutive equilibria cannot be both stable.*

One can immediately deduce the type of linear stability of the equilibria in the two previous examples. In the first example the trace of the Jacobian matrix is always zero, it follows that the equilibria must alternate between saddles and centres. In the second example the trace of the Jacobian evaluated at the equilibria $\boldsymbol{\gamma}(s_i)$ is

$$
\operatorname{tr} J(\boldsymbol{\gamma}(s)) = X_x(\boldsymbol{\gamma}(s)) + Y_y(\boldsymbol{\gamma}(s)) = \alpha'(s)\frac{\varphi(s)}{\alpha(s)} - \varphi'(s) + Y_y\left(s, \frac{\varphi(s)}{\alpha(s)}\right)
$$

Nothing can be said in such a generality, but one infers without computations that *if a stable equilibrium exists, the immediately surrounding equilibria (the adjacent equilibria) in the chain must be unstable.* This will be extremely useful in all cases in which one equilibrium is particularly simple (the origin, for example) while the other equilibria have complicate expressions [5,9,10].

3. Applications to two-dimensional systems

There are a number of two dimensional systems to which the theorem above can be applied. A non-exhaustive list of them is

Example 3. The *FitzHugh–Nagumo model* [2], [9, p. 241], [11] arises in neurobiology. In this model the variables v and w are related to a potential of a membrane and a 'leakage' current. The equations are

$$
\begin{cases}
\dot v = v(a - v)(v - 1) - w + I \\
\dot w = bv - \gamma w,
\end{cases}
\tag{5}
$$

where $0 < a < 1$, $0 < b << 1$, $\gamma > 0$, and I is a non-negative constant.

Let us begin considering $I = 0$. Equating to zero the second component of the vector field we can choose the curve $\boldsymbol{\gamma}(v) = (v, \frac{b}{\gamma}v)^T$. By substituting in the first component of the vector field, we obtain the function

$$h(v) = v\left((a-v)(v-1) - \frac{b}{\gamma}\right).$$

The equation $h(v) = 0$ gives the equilibrium $E_0 = (0,0)$, and, if $4b/\gamma < (a-1)^2$, other two equilibria $E_- = (v_-, bv_-/\gamma)$ and $E_+ = (v_+, bv_+/\gamma)$, where

$$v_\pm = \frac{1 + a \pm \sqrt{(a-1)^2 - \dfrac{4b}{\gamma}}}{2}.$$

We may order the equilibria $E_0 < E_- < E_+$. The Jacobian matrix at the generic equilibrium $E = (\hat{v}, \hat{w})$ is

$$J(\hat{v}, \hat{w}) = \begin{pmatrix} -3\hat{v}^2 + 2(1+a)\hat{v} - a & -1 \\ b & -\gamma \end{pmatrix}.$$

At the equilibrium E_0 one has

$$J(0,0) = \begin{pmatrix} -a & -1 \\ b & -\gamma \end{pmatrix}.$$

From this it immediately follows that E_0 is stable. In fact, $J(0,0) = a\gamma + b > 0$ and $\mathrm{tr}(\boldsymbol{J}(0,0)) = -a-\gamma < 0$. From Theorem 1 it follows that E_- is unstable (the determinant of the Jacobian is negative). Moreover, in this case, one can see that E_+ is stable (the determinant of the Jacobian, by Theorem 1, is positive and the trace $\mathrm{tr}(\boldsymbol{J}(E_+)) = h'(v_+) + b/\gamma - \gamma$ is negative).

When $I \neq 0$ one can proceed in the same manner and obtain the alternating sign of the Jacobian determinant at the equilibria.

Example 4. The two-dimensional *Hindmarsh–Rose system* [4] for neuronal activity, where x and y represent the membrane potential of a neuron and the neuronal signal (spiking variable). In this system the equations are

$$\begin{cases} \dot{x} = 3x^2 - x^3 + y + I \\ \dot{y} = -5x^2 + 1 - y, \end{cases} \tag{6}$$

where $I \geq 0$.

Solving the equation $\dot{y} = 0$ with respect to y and substituting y in the RHS of the first equation, we obtain $h(x) = X(x, y(x)) = -x^3 - 2x^2 + 1 + I$, moreover $h'(x) = -\det J(x, y(x)) = -3x^2 - 4x$. If we choose $I = 0$, then we have the equilibria with x-component equal to $x_0 = -1$ or $x_\pm = (-1 \pm \sqrt{5})/2$, that correspond to the equilibria $E_0 = (-1, -4)$ and

$$E_\pm = \left(\frac{-1 \pm \sqrt{5}}{2}, \frac{-13 \pm 5\sqrt{5}}{2}\right).$$

One has that $E_- < E_0 < E_+$. Since $h'(x_-) < 0$ (and the trace is negative) the equilibrium E_- is stable. It follows that E_0 is unstable. Moreover it is easy to prove that E_+ is also unstable (the trace is positive). The case $I \neq 0$ can be studied in a similar way.

Example 5. A system that models drinking with an information function is given in [3]. It models a population of non-drinkers S, binge drinkers B and subject to an influence information M. The equations are

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$$\begin{cases} \dot S = \mu N - \mu S - \frac{\gamma(M)}{N} BS + \eta B \\ \dot B = -\mu B + \frac{\gamma(M)}{N} BS - \eta B \\ \dot M = -\alpha M + \psi B^2 , \end{cases}$$

with $\gamma(M)$ is suitable function depending on the information M.

Since the total population is constant in this model, the system can be reduced to a two-dimensional system which, in non-dimensional form, can be written

$$\begin{cases} \dot b = (T_0 - 1 + T_1 m)b - (T_0 + T_1 m)b^2 , \\ \dot m = \zeta(b^2 - m) , \end{cases} \tag{7}$$

where T_0, T_1 and ζ are positive parameters.

The equilibria are the points $(\hat b, \hat m)$ such that $\hat m = \hat b^2$ and

$$(T_0 - 1 + T_1 \hat m)\hat b - (T_0 + T_1 \hat m)\hat b^2 = 0 .$$

Substituting $\hat m = \hat b^2$ one obtains the equation

$$\hat b \left(T_1 \hat b^3 - T_1 \hat b^2 + T_0 \hat b + 1 - T_0 \right) = 0 .$$

Here $\boldsymbol{\gamma} = (b, b^2)$ and $h(b) = -b(T_1 b^3 - T_1 b^2 + T_0 b + 1 - T_0)$.

The solutions to this equation are $\hat b = 0$ and the three solutions to the cubic polynomial

$$p = T_1 b^3 - T_1 b^2 + T_0 b + 1 - T_0 \tag{8}$$

under the condition that they are real and in $[0, 1]$.

It can be proved that the system has the disease-free equilibrium $E_0 = (0, 0, 0)$ and at most three equilibria depending on the parameters T_0, T_1. Applying Lemma 1, we have $\zeta h'(\hat b) = J(\hat b, \hat b^2)$, where $(\hat b, \hat b^2)$ is the generic equilibrium. The Jacobian matrix of the vector field X is

$$\boldsymbol{J}(b, m) = \begin{pmatrix} T_0 - 1 + T_1 m - 2b(T_0 + T_1 m) & T_1 b(1 - b) \\ 2\zeta b & -\zeta \end{pmatrix} .$$

By computing the Jacobian matrix at E_0 we obtain the matrix

$$\boldsymbol{J}(E_0) = \begin{pmatrix} T_0 - 1 & 0 \\ 0 & -\zeta \end{pmatrix} .$$

It follows that E_0 is stable if and only if $T_0 < 1$. The other equilibria E_i ($i = 1, 2, 3$, when they exist) have the form $E_i = (\hat b_i, \hat b_i^2)$ with $\hat b_i$ positive real solutions of the cubic polynomial (8). Since they are equilibria, they satisfy the equation $(T_0 - 1 + T_1 \hat b^2) = (T_0 + T_1 \hat b^2)\hat b$. Hence the Jacobian in such points is given by

$$\boldsymbol{J}(E_i) = \begin{pmatrix} -\hat b_i(T_0 + T_1 \hat b_i^2) & T_1 \hat b_i(1 - \hat b_i) \\ 2\zeta \hat b_i & -\zeta \end{pmatrix} ,$$

and it has negative trace. Thus, by Theorem 1, ordering the equilibria as above one has that E_0 is stable while the other are unstable, stable, unstable according to their order.

4. The three-dimensional case

Consider the 3×3 dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{X}(\boldsymbol{x})$ which in components is

$$\begin{cases} \dot{x} = X(x, y, z) \\ \dot{y} = Y(x, y, z) \\ \dot{z} = Z(x, y, z)\,, \end{cases} \tag{9}$$

with X, Y, Z differentiable real functions defined in a domain Ω of $\mathbb{R}^3$. Let $\boldsymbol{\gamma} : (a, b) \subset \mathbb{R} \to \Omega$ be $C^1(a, b)$ with $\boldsymbol{\gamma}'(s) \neq 0$ for every $s \in (a, b)$, $\boldsymbol{\gamma}(a) \neq \boldsymbol{\gamma}(b)$.

Assume that $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)^T$ is a *regular parametrization* of a null-cline of two components of the vector field $\boldsymbol{X}$, for instance of the first and the third components, i.e. $X(\gamma_1(s), \gamma_2(s), \gamma_3(s)) = 0$, $Z(\gamma_1(s), \gamma_2(s), \gamma_3(s)) = 0$. Let

$$h(s) = Y(\gamma_1(s), \gamma_2(s), \gamma_3(s)).$$

The equilibria of the vector field correspond to isolated values $s_1 < s_2 < \cdots < s_m$ such that $h(s_i) = 0$. Consider the Jacobian matrix

$$\boldsymbol{J}(X, Y, Z) = \begin{pmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{pmatrix},$$

and assume that its submatrix obtained by deleting the second line has rank 2. We denote A, B and C the cofactors of Y_x, Y_y and Y_z,

$$A = -(X_y Z_z - X_z Z_y), \quad B = (X_x Z_z - X_z Z_x), \quad C = -(X_x Z_y - X_y Z_x).$$

We observe that the rank 2 condition implies $(A, B, C) \neq (0, 0, 0)$. Moreover, from $X(\gamma_1(s), \gamma_2(s), \gamma_3(s)) = 0$, $Z(\gamma_1(s), \gamma_2(s), \gamma_3(s)) = 0$, it follows that ∇X and ∇Z are orthogonal to $\boldsymbol{\gamma}'(s)$. From the definition of $J(X, Y, Z)$ and the previous identities, we have that there exists a nonvanishing function $k(s)$ such that

$$\gamma_1'(s) = k(s)A(\gamma(s)),\ \gamma_2'(s) = k(s)B(\gamma(s)),\ \gamma_3'(s) = k(s)C(\gamma(s)).$$

Therefore, as in the planar case, we have

Lemma 2. *Let $\boldsymbol{X} = (X, Y, Z)$ be a vector field on the 3-space, $\boldsymbol{\gamma}(s)$ be a null-cline of two components of $\boldsymbol{X}$, for instance the first and the third components X and Z. Let $h(s) = Y(\boldsymbol{\gamma}(s))$ be the composition of $\boldsymbol{\gamma}$ with the last component of $\boldsymbol{X}$. If the curve $\boldsymbol{\gamma}(s)$ is a regular curve and if the rank of the Jacobian of (X, Z) along $\boldsymbol{\gamma}$ is always two, then there exists a non-vanishing function $k(s)$ such that*

$$J(\boldsymbol{\gamma}(s)) = k(s)h'(s).$$

Proof. The proof is an easy application of the chain rule to the derivative of the function $h(s)$ and the definition of the Jacobian determinant of the vector field $\boldsymbol{X}(x, y, z)$. $\square$

Also in the three-dimensional case the following theorem holds:

Theorem 2. *Under the hypothesis that the function h has simple zeroes, then whenever an equilibrium $E_i = \gamma(s_i)$ of system (9) is stable then the adjacent equilibria E_{i-1} and E_{i+1} must be unstable.*

5. Applications to three-dimensional systems

In this section we give two three-dimensional examples.

Example 6. Consider the system $\dot{\boldsymbol{x}} = \boldsymbol{X}(\boldsymbol{x})$ with

$$\boldsymbol{X}(x, y, z) = \begin{pmatrix} X(x, y, z) \\ \alpha x + \varphi(y, z) \\ \psi(y) + \beta z \end{pmatrix},$$

where α and β are nonzero real numbers.

Here

$$h(y) = X(x(y), y, z(y)),\ x(y) = -\frac{\varphi(y, -\psi(y)/\beta)}{\alpha},\ z(y) = -\psi(y)/\beta$$

and

$$\det J(\hat{x}, \hat{y}, \hat{z}) = -\alpha\beta h'(\hat{y}).$$

An application can easily be done to the system studied in Mulone and Straughan [8], to model binge drinking:

$$\begin{cases} \dot{a}_1 = -\beta(a_1 + a_2)^2 - \beta r(a_1 + a_2) + (\beta - \gamma - \mu)a_1 + \beta a_2 \\ \dot{a}_2 = \gamma a_1 - (\zeta + \mu)a_2 + \rho r \\ \dot{r} = \zeta a_2 - (\rho + \mu)r. \end{cases} \tag{10}$$

The system has two equilibria: the disease-free, and an endemic that is stable if the reproduction number

$$R_0 = \frac{\mu(\gamma + \mu)(\rho + \zeta + \mu)}{\beta[\mu(\zeta + \rho + \mu) + \gamma(\rho + \mu)]}$$

is bigger than 1. By ordering the equilibria according to the second component a_2, we have $E_0 < E_+$. It can be verified that, since E_+ is stable, then Theorem 2 implies that E_0 is unstable.

The same method can be applied in the case of the *anorexia and bulimia model* studied in [1].

Example 7. The well known Lorenz system [7] is given by the equations

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = -xz + rx - y \\ \dot{z} = xy - bz, \end{cases} \tag{11}$$

where the Prandtl number σ, the Rayleigh number r and the aspect ratio b are positive numbers.

Following the scheme of the above example, there exists a curve $\boldsymbol{\gamma}(y) = (y, y, y^2/b)^T$, that annihilates first and third equations. The equilibria of this system are $E_0 = (0, 0, 0)$ for any r, b and σ, and

$$E_\pm = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r - 1)$$

for $r > 1$. They form an ordered chain $E_- < E_0 < E_+$. It is easy to see that $J(\hat{x}, \hat{y}, \hat{z}) = b\,\sigma\, h'(\hat{y})$, where

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$$h(y) = -\frac{y^3}{b} + ry - y.$$

If $r < 1$ the sole equilibrium is E_0 and it is stable. When $r > 1$ then $J(E_0) > 0$, and E_0 is unstable. Hence the equilibria $E_\pm$ have negative Jacobian determinant, and they can be stable or unstable according to the sign of other minors. By choosing $r = 2$, $b = 4$, $\sigma = 1$, it is easy to prove, that the equilibria E_- and E_+ are stable. In this case we have a chain of equilibria with an alternating stability behaviour.

6. The n-dimensional case

Consider the $n \times n$ dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{X}(\boldsymbol{x}),$$

with

$$\boldsymbol{X}(\boldsymbol{x}) = \begin{pmatrix} X_1(x_1, x_2, \dots, x_n) \\ X_2(x_1, x_2, \dots, x_n) \\ \dots \\ X_n(x_1, x_2, \dots, x_n) \end{pmatrix}.$$

The system can be written in components in the following way:

$$\begin{cases} \dot{x}_1 = X_1(x_1, x_2, \dots, x_n) \\ \dot{x}_2 = X_2(x_1, x_2, \dots, x_n) \\ \dots \\ \dot{x}_n = X_n(x_1, x_2, \dots, x_n)\,, \end{cases} \tag{12}$$

with $X_1, X_2, \dots, X_n$ sufficiently smooth real functions defined in a domain Ω of $\mathbb{R}^n$.

Let $\boldsymbol{\gamma} : (a, b) \subset \mathbb{R} \to \Omega$ be $C^1(a, b)$ with $\boldsymbol{\gamma}'(s) \neq 0$ for every $s \in (a, b)$, $\boldsymbol{\gamma}(a) \neq \boldsymbol{\gamma}(b)$. Assume that $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots \gamma_n)$ is a *regular parametrization* of the null-cline of $n - 1$ components of the vector field $\boldsymbol{X}$, for instance of $X_1, \dots, X_{j-1}$, $X_{j+1}, \dots, X_n$, i.e.,

$$X_1(\boldsymbol{\gamma}) = 0, \cdots, \quad X_{j-1}(\boldsymbol{\gamma}) = 0, \quad X_{j+1}(\boldsymbol{\gamma}) = 0, \cdots, \quad X_n(\boldsymbol{\gamma}) = 0,$$

with j a fixed integer in $\{1, ..., n\}$. Defining

$$h(s) = X_j(\gamma_1(s), \gamma_2(s), \dots, \gamma_n(s)),$$

we assume that equilibria of the vector field correspond to isolated values $s_1 < \cdots < s_m$, such that $h(s_i) = 0$ for every $i = 1, ..., m$. The equilibria can be ordered $E_1 = \boldsymbol{\gamma}(s_1) < \cdots < E_m = \boldsymbol{\gamma}(s_m)$. Let us consider the Jacobian matrix $\boldsymbol{J}(\boldsymbol{x})$ and its $(n-1) \times n$ submatrix $\boldsymbol{K}(\boldsymbol{x})$ obtained by deleting from $\boldsymbol{J}$ the line of index j. Assume that the matrix $\boldsymbol{K}$ has rank $n - 1$ along the curve $\boldsymbol{\gamma}(s)$, and denote by $\boldsymbol{K}_1, ..., \boldsymbol{K}_i, ..., \boldsymbol{K}_n$ the cofactors of the elements of line j of the Jacobian matrix.

Since we have assumed that $\boldsymbol{K}$ has rank $n-1$, it follows that $\boldsymbol{K}_1^2 + \cdots + \boldsymbol{K}_n^2 > 0$. By the definition of determinant, one obtains that

$$J(\boldsymbol{\gamma}(s)) = \nabla X_j(\boldsymbol{\gamma}(s)) \cdot \boldsymbol{\gamma}'(s) = k(s)h'(s),$$

where $k(s) \neq 0$ for every $s \in (a, b)$. Choosing therefore for simplicity $j = n$, we have

Lemma 3. *Let $\boldsymbol{X} = (X_1, ..., X_n)$ be a vector field, $\boldsymbol{\gamma}(s)$ be a null-cline of the first $n-1$ components of $\boldsymbol{X}$, and let $h(s) = X_n(\boldsymbol{\gamma}(s))$ be the composition of $\boldsymbol{\gamma}$ with the last component of $\boldsymbol{X}$. If the curve $\boldsymbol{\gamma}(s)$ is a regular curve and if the rank of the Jacobian of $(X_1, ..., X_{n-1})$ along $\boldsymbol{\gamma}$ is always $n-1$, then there exists a nonvanishing function $k(s)$ such that*

$$J(\boldsymbol{X}(\boldsymbol{\gamma}(s)) = k(s)h'(s)\,.$$

From this lemma, by assuming that $h(s)$ has simple zeroes at s_i, as in the two and three dimensional cases, *the alternating stability of equilibria can be proved: if an equilibrium E_i is stable then the adjacent equilibria E_{i-1} and E_{i+1} (if they exist) are unstable.* This is a very useful tool in some applications because, for high dimensions, may be very difficult to apply the Routh–Hurwitz criteria for stability/instability.

Example 8. The dynamical system investigated in [12], related to a drinking epidemic model in $\mathbb{R}^4$:

$$\begin{pmatrix} \dot{T} \\ \dot{R} \\ \dot{S} \\ \dot{D} \end{pmatrix} = \begin{pmatrix} -\beta_1 DS + \Lambda + \eta R - \mu S \\ -D\left(\gamma + \delta_1 + \mu\right) + \beta_1 DS + \beta_2 DT \\ \gamma D - \beta_2 DT - \left(\delta_2 + \mu + \sigma\right) T \\ \sigma T - R(\eta + \mu) \end{pmatrix}.$$

The zeroes of this system can determined by equating to zero the components of the vector field. From the third component one has that

$$T = T(D) = \frac{\gamma D}{\beta_2 D + \delta_2 + \mu + \sigma}.$$

Substituting this equation in the remaining three components and equating to zero the fourth component one obtains

$$R = R(D) = \frac{\sigma}{\eta + \mu} \frac{\gamma\, D}{\beta_2 D + \delta_2 + \mu + \sigma} = \frac{\sigma}{\eta + \mu} T(D).$$

Substituting in the remaining two components and equating to zero the first component one obtains

$$S = S(D) = \frac{\Lambda(\eta + \mu)(\delta_2 + \mu + \sigma) + D(\beta_2 \Lambda(\eta + \mu) + \gamma\eta\sigma)}{(\eta + \mu)(\beta_1 D + \mu)(\beta_2 D + \delta_2 + \mu + \sigma)}.$$

It follows that there exists a curve on which all but the second component of the vector field vanish. Such component, composed with the curve $D \to (T(D), R(D), S(D), D)$, gives a function $h(D)$ which is a rational expression whose numerator is a cubic polynomial whose factors are D and a quadratic polynomial. It follows that this system has three equilibria which create a chain of equilibria that can be labelled E_0 (corresponding to $D = 0$) and $E_\pm$ corresponding to the two choices of solutions of the above quadratic polynomial in D. The stability of E_0 is easily investigated, and there are two possibilities: when E_0 is stable it is between the other two equilibria $E_\pm$ which are necessarily both unstable; when E_0 is unstable it is at one end of the chain of equilibria, hence the neighbouring is unstable.

With the particular choice

$$\Lambda = 0.5, \quad \beta_1 = 0.04, \quad \beta_2 = 0.99, \quad \mu = 0.025, \quad \sigma = 0.01, \quad \eta = 0.1,$$
$$\gamma = 0.9, \quad \delta_1 = 0.035, \quad \delta_2 = 0.03$$

one has that the solutions of the equation $h(D) = 0$ are

$$D = 0, \quad 0.0161807, \quad 6.76286.$$

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The equilibrium with $D = 0$ can be easily shown to be stable (the parameters are chosen so that the reproduction number $R_0 < 1$, and hence the equilibrium with $D = 0.0161807$ is necessarily unstable. In fact, the eigenvalues of the Jacobian matrix are respectively $-0.16, -0.125, -0.065, -0.025$ in the equilibrium with $D = 0$, $-0.154704, -0.124207, -0.0237624, 0.0710075$ in the equilibrium with $D = 0.0161807$ and $-6.76879, -0.211032, -0.125083, -0.0758437$ in the equilibrium with $D = 6.76286$.

7. Conclusions

In this article we give a method to order the equilibria to obtain their alternating stability behaviour. The equilibria are ordered by making the hypothesis that they belong to a curve $\boldsymbol{\gamma}$ along which all but one of the components of the vector field (for example the component X_j) vanish. Observe that the curve $\boldsymbol{\gamma}$ may not be a solution of the system.

By assuming that the function $h(s) = X_j(\boldsymbol{\gamma}(s))$ has only simple zeroes, the best result we can obtain with minimal hypotheses is the alternance of the sign of the Jacobian determinant. This is enough to prove that the equilibria neighbouring a stable equilibrium must be unstable. This result allows us to determine the instability of equilibria without computing cumbersome Routh–Hurwitz conditions on them.

We have not been able to give analogous conditions when the equilibrium is unstable, and it can be shown with examples that almost anything can happen. It is nonetheless possible that stronger, but natural conditions along the curve $\boldsymbol{\gamma}$ can allow to prove some stronger type of alternance which typically takes place e.g. in epidemiological models.

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References

[1] C. Ciarcià, A. Giacobbe, P. Falsaperla, G. Mulone, A mathematical model of anorexia and bulimia, Math. Methods Appl. Sci. 38 (14) (2015) 2937–2952.
[2] R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane, Biophys. J. 1 (1961) 445–466.
[3] A. Giacobbe, G. Mulone, B. Straughan, W. Wang, Modelling binge drinking with information, Math. Methods Appl. Sci. 40 (2017) 4400–4411.
[4] J.L. Hindmarsh, R.M. Rose, A model of neuronal bursting using three coupled first order differential equations, Proc. R. Soc. Lond., Ser. B 221 (1984) 87–102.
[5] M.W. Hirsch, S. Smale, Differential Equations, Dynamical Systems, and Linear Algebra, Pure and Applied Mathematics, vol. 60, Academic Press, New York–London, 1974.
[6] A. Hurwitz, On the conditions under which an equation has only roots with negative real parts, Math. Ann. 46 (1895) 273–284.
[7] E.N. Lorenz, Deterministic nonperiodic flow, J. Atmos. Sci. 20 (1963) 131–141.
[8] G. Mulone, B. Straughan, Modelling binge drinking, Int. J. Biomath. 5 (1) (January 2012) 1250005.
[9] J.D. Murray, Mathematical Biology I. An Introduction, third edition, Interdisciplinary Applied Mathematics, vol. 17, Springer-Verlag, New York, 2003.
[10] J.D. Murray, Mathematical Biology. II. Spatial Models and Biomedical Applications, third edition, Interdisciplinary Applied Mathematics, vol. 18, Springer-Verlag, New York, 2003.
[11] J.S. Nagumo, S. Arimoto, S. Yoshizawa, An active pulse transmission line simulating nerve axon, Proc. IRE 50 (1962) 2061–2071.
[12] S. Sharma, G.P. Samanta, Analysis of a drinking epidemic model, Int. J. Dynam. Control 3 (2015) 288–305, doi.org/10.1007/s40435-015-0151-8.

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