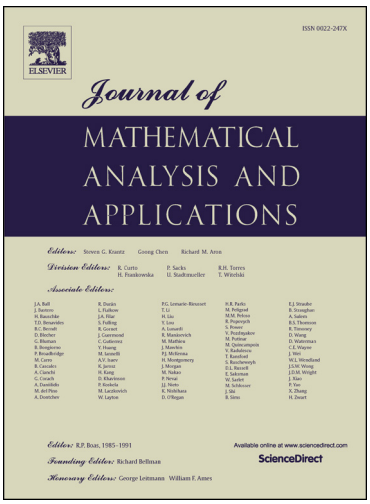
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ON THE SUM OF TWO SUBNORMAL KERNELS

SOUMITRA GHARA AND SURJIT KUMAR

Abstract. We show, by means of a class of examples, that if K_1 and K_2 are two positive definite kernels on the unit disc such that the multiplication by the coordinate function on the corresponding reproducing kernel Hilbert space is subnormal, then the multiplication operator on the Hilbert space determined by their sum $K_1 + K_2$ need not be subnormal. This settles a recent conjecture of Gregory T. Adams, Nathan S. Feldman and Paul J. McGuire in the negative. We also discuss some cases for which the answer is affirmative.

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space. Let $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators on $\mathcal{H}$. Recall that an operator T in $B(\mathcal{H})$ is said to be *subnormal* if there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator N in $B(\mathcal{K})$ such that $N(\mathcal{H}) \subset \mathcal{H}$ and $N|_{\mathcal{H}} = T$. For the basic theory of subnormal operators, we refer to [12].

Completely hyperexpansive operators were introduced in [6]. An operator $T \in B(\mathcal{H})$ is said to be completely hyperexpansive if

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}T^{*j}T^j \leq 0 \qquad (n \geq 1).$$

The theory of subnormal and completely hyperexpansive operators are closely related with the theory *completely monotone* and *completely alternating* sequences (cf. [5], [6]).

Let $\mathbb{Z}_+$ denote the set of non-negative integers. A sequence $\{a_k\}_{k\in\mathbb{Z}_+}$ of positive real numbers is said to be a completely monotone if

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}a_{m+j} \geq 0 \qquad (m, n \geq 0).$$

It is well-known that a sequence $\{a_k\}_{k\in\mathbb{Z}_+}$ is completely monotone if and only if the sequence $\{a_k\}_{k\in\mathbb{Z}_+}$ is a Hausdorff moment sequence, that is, there exists a positive measure ν supported in $[0, 1]$ such that $a_k = \int_{[0,1]} x^k d\nu(x)$ for all $k \in \mathbb{Z}_+$ (cf. [9]).

Similarly, a sequence $\{a_k\}_{k\in\mathbb{Z}_+}$ of positive real numbers is said to be completely alternating if

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}a_{m+j} \leq 0 \qquad (m \geq 0, n \geq 1).$$

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Note that $\{a_k\}_{k\in\mathbb{Z}_+}$ is completely alternating if and only if the sequence $\{\Delta a_k\}_{k\in\mathbb{Z}_+}$ is completely monotone, where $\Delta a_k := a_{k+1} - a_k$.

Let $\mathcal{H}(K)$ be a reproducing kernel Hilbert space consisting of holomorphic functions on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with reproducing kernel $K(z,w) = \sum_{k\in\mathbb{Z}_+} a_k(z\bar{w})^k$. Thus K is holomorphic in the first variable and anti-holomorphic in the second. We make the assumption that a kernel function is always holomorphic in the first variable and anti-holomorphic in the second throughout this note. Consider the operator M_z of multiplication by the coordinate function z on $\mathcal{H}(K)$. As is well-known, such a multiplication operator is unitarily equivalent to a weighted shift operator W on the sequence space $\ell^2(\mathbb{Z}_+)$ with $We_n = \sqrt{\frac{a_n}{a_{n+1}}}e_{n+1}$, where $\{e_n : n \in \mathbb{Z}_+\}$ is the standard orthonormal basis of $\ell^2(\mathbb{Z}_+)$. Assume that M_z is bounded. Then M_z is a contractive subnormal if and only if the sequence $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}_+}$ is a Hausdorff moment sequence. On the other hand, M_z on $\mathcal{H}(K)$ is completely hyperexpansive if and only if the sequence $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}_+}$ is completely alternating (cf. [6, Proposition 3]).

For any two positive definite kernels K_1 and K_2 defined on $\mathbb{D}\times\mathbb{D}$, their sum K_1+K_2 is again a positive definite kernel and therefore determines a Hilbert space $\mathcal{H}(K_1+K_2)$ of functions on $\mathbb{D}$. It was shown in [4] that

$$\mathcal{H}(K_1+K_2) = \{f = f_1 + f_2 : f_1 \in \mathcal{H}(K_1), f_2 \in \mathcal{H}(K_2)\},$$

is a Hilbert space with the norm given by

$$\|f\|^2_{\mathcal{H}(K_1+K_2)} := \inf\left\{\|f_1\|^2_{\mathcal{H}(K_1)} + \|f_2\|^2_{\mathcal{H}(K_2)} : f = f_1 + f_2,\ f_1 \in \mathcal{H}(K_1),\ f_2 \in \mathcal{H}(K_2)\right\}.$$

The sum of two kernel functions is also discussed by Salinas in [19]. He proved that if K_1 and K_2 are generalized Bergman kernels (for definition, refer to [16]), then so is $K_1 + K_2$. Although not explicitly stated in [4], it is not hard to verify that the multiplication operator M_z on $\mathcal{H}(K_1+K_2)$ is unitarily equivalent to the operator $P_{\mathcal{M}^\perp}(M_{z,1}\oplus M_{z,2})|_{\mathcal{M}^\perp}$, where $M_{z,i}$ is the operator of multiplication by the coordinate function z on $\mathcal{H}(K_i)$ and

$$\mathcal{M} = \{(g,-g) \in \mathcal{H}(K_1)\oplus\mathcal{H}(K_2) : g \in \mathcal{H}(K_1)\cap\mathcal{H}(K_2)\} \subseteq \mathcal{H}(K_1)\oplus\mathcal{H}(K_2).$$

Evidently, if $M_{z,1}$ and $M_{z,2}$ are subnormal, then so is $M_{z,1}\oplus M_{z,2}$. Here, we discuss the subnormality of the compression $P_{\mathcal{M}^\perp}(M_{z,1}\oplus M_{z,2})|_{\mathcal{M}^\perp}$ for a class of kernels. In particular, we show that the subnormality of $M_{z,1}$ and $M_{z,2}$ need not imply $P_{\mathcal{M}^\perp}(M_{z,1}\oplus M_{z,2})|_{\mathcal{M}^\perp}$ is subnormal.

A similar question on subnormality involving the point-wise product of two positive definite kernels was raised in [19]. Recall that the product K_1K_2 of two positive definite kernels defined on, say $\mathbb{D}\times\mathbb{D}$, is also a positive definite kernel. Indeed, if $\mathcal{H}(K_1)\otimes\mathcal{H}(K_2) \subseteq \mathrm{Hol}(\mathbb{D}\times\mathbb{D})$ is the usual tensor product of the two Hilbert spaces $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$, and $\mathcal{N}$ is the subspace $\{h \in \mathcal{H}(K_1)\otimes\mathcal{H}(K_2) : h(z,z) = 0, z \in \mathbb{D}\}$, then the operator $M_{z,1}\otimes I$ acting on the Hilbert space $\mathcal{H}(K_1)\otimes\mathcal{H}(K_2)$ compressed to $\mathcal{N}^\perp$ is unitarily equivalent to the multiplication operator M_z on the Hilbert space $\mathcal{H}(K_1K_2)$. If $M_{z,1}$ is subnormal on $\mathcal{H}(K_1)$, then so is the operator $M_{z,1}\otimes I$.

The answer to the question of subnormality, both in the case of the sum as well as the product, is affirmative in several examples. For over thirty years, the question of whether the

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compression of $M_{z,1} \otimes I$ to $\mathcal{N}^\perp$ is subnormal had remained open. Recently, a counter-example has been found, see [3, Theorem 1.5]. The conjecture below is similar except that it involves the sum of two kernels.

Conjecture 1.1. **(**[1, pp. 22]**).** *Let $K_1(z,w) = \sum_{k\in\mathbb{Z}_+} a_k(z\bar{w})^k$ and $K_2(z,w) = \sum_{k\in\mathbb{Z}_+} b_k(z\bar{w})^k$ be any two reproducing kernels satisfying:*

(a) $\lim \frac{a_k}{a_{k+1}} = \lim \frac{b_k}{b_{k+1}} = 1$
(b) $\lim a_k = \lim b_k = \infty$
(c) $\frac{1}{a_k} = \int_{[0,1]} t^k d\nu_1(t)$ *and* $\frac{1}{b_k} = \int_{[0,1]} t^k d\nu_2(t)$ *for all* $k \in \mathbb{Z}_+$, *where* ν_1 *and* ν_2 *are two positive measures supported in* $[0,1]$.

Then the multiplication operator M_z on $\mathcal{H}(K_1 + K_2)$ is a subnormal operator.

An equivalent formulation, in terms of the moment sequence criterion, of the conjecture is the following. If $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}_+}$ and $\{\frac{1}{b_k}\}_{k\in\mathbb{Z}_+}$ are Hausdorff moment sequences, does it necessarily follow that $\{\frac{1}{a_k+b_k}\}_{k\in\mathbb{Z}_+}$ is also a Hausdorff moment sequence?

Like the case of the product of two kernels, here we give a class of counter examples to the conjecture stated above. Paul McGuire, in a private communication, has informed the authors of an example that they had found. In fact, he says that their example is one of the examples discussed in this note.

These two cases suggest that it may be fruitful to ask when the compression of a subnormal operator to a co-invariant subspace is again subnormal.

The paper is organized as follows. In section 2, we provide a few classes of counter-examples which settles Conjecture 1.1. We also discuss some cases for which answer to this conjecture is affirmative. In the last section, we try to answer analogously in a certain class of weighted multi-shifts.

2. The Sum of two subnormal reproducing kernels need not be subnormal

For the construction of counter-examples to the conjecture, we make use of the following result, borrowed from [2, Proposition 4.3].

Proposition 2.1. *For distinct positive real numbers $a_0, ..., a_n$ and non-zero real numbers $b_0, ..., b_n$, consider the polynomial $p(x) = \Pi_{k=0}^n (x + a_k + ib_k)(x + a_k - ib_k)$. Then the sequence $\{\frac{1}{p(l)}\}_{l\in\mathbb{Z}_+}$ is never a Hausdorff moment sequence.*

For $r > 0$, let K_r be a positive definite kernel given by

$$K_r(z,w) := \sum_{k\in\mathbb{Z}_+} \frac{k+r}{r}(z\bar{w})^k \qquad (z,w \in \mathbb{D}).$$

The case $r = 1$ corresponds to the Bergman kernel. It is easy to see that the multiplication operator M_z on $\mathcal{H}(K_r)$ is contractive and subnormal and the representing measure is $rx^{r-1}dx$.

For $s,t > 0$, consider the multiplication operator M_z on $\mathcal{H}(K_{s,t})$, where

$$K_{s,t}(z,w) := \sum_{k\in\mathbb{Z}_+} \frac{(k+s)(k+t)}{st}(z\bar{w})^k \qquad (z,w \in \mathbb{D}).$$

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The case $s = 1$ and $t = 2$, corresponds to the kernel $(1 - z\bar{w})^{-3}$. Note that M_z is a contractive subnormal with the representing measure ν is given by

$$d\nu(x) \quad = \quad \begin{cases} -s^2 x^{s-1} \log x \; dx & \text{if } s = t \\ st\frac{x^{t-1} - x^{s-1}}{s-t} dx & \text{if } s \neq t. \end{cases}$$

One easily verifies that K_r and $K_{s,t}$ both satisfy all the conditions $(a), (b)$ and (c) of the Conjecture 1.1. But the multiplication operator on their sum need not be subnormal for all possible choices of $r, s, t > 0$. This follows from the following theorem.

Theorem 2.2. *The multiplication operator M_z on $\mathcal{H}(K_r + K_{s,t})$ is subnormal if and only if*

$$(rs + st + tr)^2 \geq 8r^2 st. \tag{2.1}$$

Proof. Notice that

$$(K_r + K_{s,t})(z, w) = \sum_{k \in \mathbb{Z}_+} \frac{k^2 + (s + t + \frac{st}{r})k + 2st}{st} (z\bar{w})^k \qquad (z, w \in \mathbb{D}).$$

The roots of the polynomial $x^2 + (s + t + \frac{st}{r})x + 2st$ are

$$\begin{aligned} x_1 \quad &:= \quad \frac{-(s + t + \frac{st}{r}) + \sqrt{(s + t + \frac{st}{r})^2 - 8st}}{2} \text{ and} \\ x_2 \quad &:= \quad \frac{-(s + t + \frac{st}{r}) - \sqrt{(s + t + \frac{st}{r})^2 - 8st}}{2}. \end{aligned}$$

Suppose that $(rs + st + tr)^2 \geq 8r^2 st$. Then the kernel $K_r + K_{s,t}$ will be of the form $2K_{s',t'}$ where $s' = -x_1$ and $t' = -x_2$. Hence, M_z on $\mathcal{H}(K_r + K_{s,t})$ is a subnormal operator.

Conversely, assume that $(rs + st + tr)^2 < 8r^2 st$. By Proposition 2.1, it follows that M_z on $\mathcal{H}(K_r + K_{s,t})$ can not be subnormal. □

Remark 2.3 : If we choose $s = 1,\ t = 2$ and $r > 2$, then the inequality (2.1) is not valid.

We also point out that if K_1 and K_2 are any two reproducing kernels such that the multiplication operators on $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ are hyponormal, then the multiplication operator on $\mathcal{H}(K_1 + K_2)$ need not be hyponormal. An example illustrating this is given below.

Example 2.4. *For any $s, t > 0$, consider the reproducing kernel $K^{s,t}$ given by*

$$K^{s,t}(z, w) := 1 + sz\bar{w} + s^2(z\bar{w})^2 + t\frac{(z\bar{w})^3}{1 - z\bar{w}}.$$

Note that $K^{s,t}$ defines a reproducing kernel on the unit disc $\mathbb{D}$ and the multiplication operator M_z on $\mathcal{H}(K^{s,t})$ can be seen as a weighted shift operator with weights $(\sqrt{\frac{1}{s}}, \sqrt{\frac{1}{s}}, \sqrt{\frac{s^2}{t}}, 1, 1, \cdots)$. Thus, it follows that M_z on $\mathcal{H}(K^{s,t})$ is hyponormal if and only if $s^2 \leq t \leq s^3$ (cf. [20]).

Observe that M_z on $\mathcal{H}(K^{s,t} + K^{s',t'})$ can be realized as a weighted shift operator with weights $(\sqrt{\frac{2}{s+s'}}, \sqrt{\frac{s+s'}{s^2+s'^2}}, \sqrt{\frac{s^2+s'^2}{t+t'}}, 1, 1, \cdots)$. For the hyponormality of this weighted shift operator, it is necessary that $\frac{2}{s+s'} \leq \frac{s+s'}{s^2+s'^2}$, which is true only when $s = s'$.

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We remark that this is different from the case of the product of two kernels, where, the hyponormality of the multiplication operator on the Hilbert space $\mathcal{H}(K_1K_2)$ follows as soon as we assume they are hyponormal on the two Hilbert spaces $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$, see [10].

If $T \in B(\mathcal{H})$ is left invertible then the operator T' given by $T' = T(T^*T)^{-1}$ is said to be the operator *Cauchy dual* to T. The following result has been already recorded in [6, Proposition 6], which may be paraphrased as follows:

Theorem 2.5. *Let $K(z,w) = \sum_{k\in\mathbb{Z}_+} a_k(z\bar{w})^k$ be a positive definite kernel on $\mathbb{D}$ and M_z be the multiplication operator on $\mathcal{H}(K)$. Assume that M_z is left invertible. Then the followings are equivalent:*

(i) *$\{a_k\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence.*
(ii) *The Cauchy dual M_z' of M_z is completely hyperexpansive.*
(iii) *For all $t > 0$, $\{\frac{1}{t(a_k-1)+1}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence.*
(iv) *For all $t > 0$, the multiplication operator M_z on $\mathcal{H}(tK + (1-t)S)$ is contractive subnormal, where S is the Szegö kernel on $\mathbb{D}$.*

Remark 2.6 : If $\{a_k\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence then by putting $t = 1$ in part (iii) of Theorem 2.5, it follows that $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence.

Corollary 2.7. *Let $K_1(z,w) = \sum_{k\in\mathbb{Z}_+} a_k(z\bar{w})^k$ and $K_2(z,w) = \sum_{k\in\mathbb{Z}_+} b_k(z\bar{w})^k$ be any two reproducing kernels such that $\{a_k\}_{k\in\mathbb{Z}_+}$ and $\{b_k\}_{k\in\mathbb{Z}_+}$ are completely alternating sequences, then the multiplication operator M_z on $\mathcal{H}(K_1 + K_2)$ is subnormal.*

Proof. It is easy to verify that the sum of two completely alternating sequences is completely alternating. The desired conclusion follows immediately from Remark 2.6. □

Remark 2.8 : Note that $\{\frac{k+r}{r}\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence but the sequence $\{\frac{(k+s)(k+t)}{st}\}_{k\in\mathbb{Z}_+}$ is not completely alternating. So, the reproducing kernels K_r and $K_{s,t}$ discussed in Theorem 2.2 does not satisfy the hypothesis of Corollary 2.7.

Proposition 2.9. *Let $K(z,w) = \sum_{k\in\mathbb{Z}_+} a_k(z\bar{w})^k$ be any positive definite kernel such that the multiplication operator M_z on $\mathcal{H}(S + K)$ is subnormal. Then the multiplication operator on $\mathcal{H}(K)$ is subnormal.*

Proof. From subnormality of M_z on $\mathcal{H}(S + K)$, it follows that $\{\frac{1}{1+a_k}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence. Thus, $\{1 - \frac{1}{1+a_k}\}_{k\in\mathbb{Z}_+}$ is completely alternating. Note that

$$\begin{aligned}(a_k)^{-1} &= (1+a_k)^{-1}(1-\tfrac{1}{1+a_k})^{-1} \\ &= \sum_{j=1}^{\infty}\left(\tfrac{1}{1+a_k}\right)^j.\end{aligned}$$

Observe that $\{\frac{1}{(1+a_k)^j}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence for all $j \geq 1$. Now, being the limit of completely monotone sequences, $\{a_k^{-1}\}_{k\in\mathbb{Z}_+}$ is completely monotone. □

Remark 2.10 : We have following remarks:

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(i) The converse of the Proposition 2.9 is not true (see the example discussed in part (ii) of the Remark 2.13).

(ii) If we replace Szegö kernel S by Bergman kernel then the conclusion of the Proposition 2.9 need not be true. For example, by using Proposition 2.1, one may choose $\alpha > 0$ such that the sequence $\{\frac{1}{k^2+\alpha k+1}\}_{k\in\mathbb{Z}_+}$ is not completely monotone but the sequence $\{\frac{1}{k^2+(\alpha+1)k+2}\}_{k\in\mathbb{Z}_+}$ is completely monotone.

We use the convenient Pochhammer symbol given by $(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)}$, where Γ denotes the gamma function.

For $\lambda, \mu > 0$, consider the positive definite kernel

$$K_{\lambda,\mu}(z,w) = \sum_{k\in\mathbb{Z}_+} \frac{(\lambda)_k}{(\mu)_k}(z\bar{w})^k \qquad (z,w \in \mathbb{D}).$$

It is easy to see that the case $\mu = 1$ corresponds to the kernel $(1 - z\bar{w})^{-\lambda}$. Note that the multiplication operator M_z on $\mathcal{H}(K_{\lambda,\mu})$ may be realized as a weighted shift operator with weight sequence $\{\sqrt{\frac{k+\mu}{k+\lambda}}\}_{k\in\mathbb{Z}_+}$.

The first part of the following theorem is proved in [15] and the representing measure is given in [13, Lemma 2.2]. Here, we provide a proof for the second part only.

Theorem 2.11. *The multiplication operator M_z on $\mathcal{H}(K_{\lambda,\mu})$ is*

(i) *subnormal if and only if $\lambda \geq \mu$. In this case, the representing measure ν of M_z is given by*

$$d\nu(x) = \begin{cases} \frac{\Gamma(\lambda)}{\Gamma(\mu)\Gamma(\lambda-\mu)} x^{\mu-1}(1-x)^{\lambda-\mu-1}dx & \text{if } \lambda > \mu \\ \delta_1(x)dx & \text{if } \lambda = \mu, \end{cases}$$

where δ_1 is the Dirac delta function.

(ii) *completely hyperexpansive if and only if $\lambda \leq \mu \leq \lambda + 1$.*

Proof. The multiplication operator M_z on $\mathcal{H}(K_{\lambda,\mu})$ is completely hyperexpansive if and only if the sequence $\{\frac{(\mu)_k}{(\lambda)_k}\}_{k\in\mathbb{Z}_+}$ is completely alternating. Here

$$\Delta\left(\frac{(\mu)_k}{(\lambda)_k}\right) = \frac{(\mu)_{k+1}}{(\lambda)_{k+1}} - \frac{(\mu)_k}{(\lambda)_k} = \frac{(\mu)_{k+1} - (\mu)_k(\lambda+k-1)}{(\lambda)_{k+1}}$$
$$= \frac{\mu-\lambda}{\lambda}\frac{(\mu)_k}{(\lambda+1)_k}.$$

By the first part of this theorem, $\{\frac{\mu-\lambda}{\lambda}\frac{(\mu)_k}{(\lambda+1)_k}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence if and only if $\lambda \leq \mu \leq \lambda + 1$. This completes the proof. □

The following proposition gives a sufficient condition for the subnormality of the multiplication operator on Hilbert space determined by the sum of two kernels belonging to the class $K_{\lambda,\mu}$.

Proposition 2.12. *Let $0 < \mu \leq \lambda' \leq \lambda \leq \lambda' + 1$. Then the multiplication operator M_z on $\mathcal{H}(K_{\lambda,\mu} + K_{\lambda',\mu})$ is contractive subnormal.*

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Proof. Observe that

$$(K_{\lambda,\mu} + K_{\lambda',\mu})(z,w) = \sum_{k\in\mathbb{Z}_+} \frac{(\lambda)_k + (\lambda')_k}{(\mu)_k}(z\bar{w})^k \qquad (z,w \in \mathbb{D})$$

and

$$\frac{(\mu)_k}{(\lambda)_k + (\lambda')_k} = \frac{(\mu)_k}{(\lambda')_k}\frac{1}{1+\frac{(\lambda)_k}{(\lambda')_k}}.$$

Since $\mu \leq \lambda'$, it follows from part (i) of Theorem 2.11 that $\{\frac{(\mu)_k}{(\lambda')_k}\}_{k\in\mathbb{Z}_+}$ is completely monotone.

If $\lambda' \leq \lambda \leq \lambda'+1$, then by part (ii) of Theorem 2.11, the sequence $\{\frac{(\lambda)_k}{(\lambda')_k}\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence. So is the sequence $\{1+\frac{(\lambda)_k}{(\lambda')_k}\}_{k\in\mathbb{Z}_+}$. Hence, by Remark 2.6, $\left\{\frac{1}{1+\frac{(\lambda)_k}{(\lambda')_k}}\right\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence. Thus, being a product of two completely monotone sequences, $\{\frac{(\mu)_k}{(\lambda)_k+(\lambda')_k}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence. This completes the proof. □

Remark 2.13 : Here are some remarks:

(i) The case when $\mu < \lambda'$ and $\lambda = \lambda' + 1$. The representing measure for the sequence $\left\{\frac{1}{1+\frac{(\lambda)_k}{(\lambda')_k}}\right\}_{k\in\mathbb{Z}_+}$ is $\lambda' x^{2\lambda'-1}dx$. The representing measure for the sequence $\{\frac{(\mu)_k}{(\lambda')_k}\}_{k\in\mathbb{Z}_+}$ is given in part (i) of Theorem 2.11. Thus, using Remark 2.4 of [2], one may obtain the representing measure for M_z on $\mathcal{H}(K_{\lambda,\mu} + K_{\lambda',\mu})$ to be given by

$$d\nu(x) = \frac{\lambda'\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\lambda'-\mu)}x^{2\lambda'-1}\left(\int_0^{1-x} t^{\lambda'-\mu-1}(1-t)^{\mu-2\lambda'-1}dt\right)dx.$$

But in general, when $\lambda < \lambda' + 1$, we do not know the representing measure for the sequence $\left\{\frac{1}{1+\frac{(\lambda)_k}{(\lambda')_k}}\right\}_{k\in\mathbb{Z}_+}$ as well as for the sequence $\left\{\frac{(\mu)_k}{(\lambda)_k+(\lambda')_k}\right\}_{k\in\mathbb{Z}_+}$.

(ii) Consider the kernel $(K_{1,1} + K_{3,1})(z,w) = \sum_{k\in\mathbb{Z}_+} \frac{k^2+3k+4}{2}(z\bar{w})^k$ for all $z,w \in \mathbb{D}$. It follows from Proposition 2.1 that the sequence $\left\{\frac{2}{k^2+3k+4}\right\}_{k\in\mathbb{Z}_+}$ is not completely monotone. Consequently, the multiplication operator M_z on $\mathcal{H}(K_{1,1} + K_{3,1})$ is not subnormal. For $\lambda > 1$, consider the kernel $K_{\lambda,1} + K_{3,1}$. We claim that there exists a $\lambda_0 > 1$ such that $\left\{\frac{(1)_k}{(\lambda_0)_k+(3)_k}\right\}_{k\in\mathbb{Z}_+}$ is not completely monotone. If not, assume that $\left\{\frac{(1)_k}{(\lambda)_k+(3)_k}\right\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence for all $\lambda > 1$. As λ goes to 1, one may get that $\left\{\frac{2}{k^2+3k+4}\right\}_{k\in\mathbb{Z}_+}$ is completely monotone, which is a contradiction. Therefore, we conclude that there exists a $\lambda_0 > 1$ such that the multiplication operator M_z on $\mathcal{H}(K_{\lambda_0,1} + K_{3,1})$ is not subnormal. By using the properties of the gamma function, one may verify that $K_{\lambda_0,1}$ and $K_{3,1}$ both satisfy $(a), (b)$ and (c) of the Conjecture 1.1. Hence this also provides a class of counterexamples for the Conjecture 1.1.

Proposition 2.14. *Let $0 < p \leq q \leq p+1$. Suppose $K_1(z,w) = \sum_{k\in\mathbb{Z}_+} a_k^p(z\bar{w})^k$ and $K_2(z,w) = \sum_{k\in\mathbb{Z}_+} a_k^q(z\bar{w})^k$ are any two reproducing kernels such that $\{a_k\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence. Then the multiplication operator M_z on $\mathcal{H}(K_1 + K_2)$ is subnormal.*

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Proof. Note that

$$\frac{1}{a_k^p + a_k^q} = \frac{1}{a_k^p(1 + a_k^{q-p})}.$$

Since $0 \leq q - p \leq 1$ and $\{a_k\}_{k\in\mathbb{Z}_+}$ is completely alternating, it follows from [7, Corollary 1] that $\{a_k^{q-p}\}_{k\in\mathbb{Z}_+}$ is also completely alternating. Thus so is $\{1 + a_k^{q-p}\}_{k\in\mathbb{Z}_+}$. Hence, by Remark 2.6, $\{(1+a_k^{q-p})^{-1}\}_{k\in\mathbb{Z}_+}$ is completely monotone. By [8, Corollary 4.1], $\{a_k^{-p}\}_{k\in\mathbb{Z}_+}$ is completely monotone. Now the proof follows as the product of two completely monotone sequences is also completely monotone. □

Example 2.15. *For any $p > 0$, let $K_p(z, w)$ be the positive definite kernel given by*

$$K_p(z, w) := \sum_{k\in\mathbb{Z}_+} (k + 1)^p (z\bar{w})^k \qquad (z, w \in \mathbb{D}).$$

Then it is known that the multiplication operator M_z on $\mathcal{H}(K_p)$ is subnormal with the representing measure $d\nu(x) = \frac{(-\log x)^{p-1}}{\Gamma(p)}dx$ (cf. [14, Theorem 4.3]*). By Proposition 2.14, it follows that M_z on $\mathcal{H}(K_p + K_q)$ is subnormal if $p \leq q \leq p + 1$.*

The next result also provides a class of counter-examples to the Conjecture 1.1.

Theorem 2.16. *Consider the positive definite kernel K_p given in Example 2.15. Then the multiplication operator M_z on $\mathcal{H}(K_p + K_{p+2})$ is subnormal if and only if $p \geq 1$.*

Proof. For $x \in (0, 1]$, let $g(x) := \frac{1}{\Gamma(p)} \int_0^{-\log x} (-\log x - y)^{p-1} \sin y \; dy$ and $d\nu(x) = g(x)dx$. Then

$$\begin{aligned}
\int_0^1 x^k d\nu(x) &= \frac{1}{\Gamma(p)} \int_{y=0}^{\infty} \int_{x=0}^{e^{-y}} x^k (-\log x - y)^{p-1} dx \sin y \; dy \\
&= \frac{1}{\Gamma(p)} \int_{y=0}^{\infty} e^{-(k+1)y} \int_{u=0}^{\infty} e^{-(k+1)u} u^{p-1} du \sin y \; dy \\
&= \frac{1}{\Gamma(p)} \int_{y=0}^{\infty} e^{-(k+1)y} \frac{\Gamma(p)}{(k+1)^p} \sin y \; dy \\
&= \frac{1}{(k+1)^p} \frac{1}{(k+1)^2 + 1}.
\end{aligned}$$

Thus the sequence $\{\frac{1}{(k+1)^p} \frac{1}{(k+1)^2+1}\}_{k\in\mathbb{Z}_+}$ is completely monotone if and only if the function $g(x)$ is non-negative a.e. Note that the function $g(x)$ is non-negative on $(0, 1]$ a.e. if and only if the function $h(x) := g(e^{-x})$ is non-negative a.e. on $(0, \infty)$. Now

$$h(x) = \frac{1}{\Gamma(p)} \int_0^x (x - y)^{p-1} \sin y \; dy = \frac{x^p}{\Gamma(p)} \int_0^1 (1 - y)^{p-1} sin(xy) \; dy.$$

By [17, Chapter 3, pp 439], we have $h(x) = \frac{\sqrt{x}}{\Gamma(p)} s_{p-\frac{1}{2},\frac{1}{2}}(x)$, where $s_{p-\frac{1}{2},\frac{1}{2}}(x)$ is the Lommel function of first kind. Thus, the sequence $\{\frac{1}{(k+1)^p} \frac{1}{(k+1)^2+1}\}_{k\in\mathbb{Z}_+}$ being completely monotone is equivalent to the non-negativity of the function $s_{p-\frac{1}{2},\frac{1}{2}}(x)$. If $p \geq 1$ then by [21, Theorem A], we get that $s_{p-\frac{1}{2},\frac{1}{2}}(x) \geq 0$ for all $x > 0$. The converse follows from [21, Theorem 2], which completes the proof. □

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3. Multi-variable case

Let $\mathbb{Z}_+^d$ denote the cartesian product $\mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+$ (d times). Let $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$, we write $|\alpha| := \alpha_1 + \cdots + \alpha_d$ and $\alpha! = \alpha_1! \cdots \alpha_d!$.

If $T = (T_1, \cdots, T_d)$ is a d-tuple of commuting bounded linear operators T_j $(1 \leq j \leq d)$ on $\mathcal{H}$ then we set T^* to be $(T_1^*, \cdots, T_d^*)$ and T^α to be $T_1^{\alpha_1} \cdots T_d^{\alpha_d}$.

Given a commuting d-tuple T of bounded linear operators $T_1, \cdots, T_d$ on $\mathcal{H}$, set

$$Q_T(X) := \sum_{i=1}^{d} T_i^* X T_i \qquad (X \in B(\mathcal{H})).$$

For $X \in B(\mathcal{H})$ and $k \geq 1$, one may define $Q_T^k(X) := Q_T(Q_T^{k-1}(X))$, where $Q_T^0(X) = X$.

Recall that T is said to be

(i) *spherical contraction* if $Q_T(I) \leq I$.
(ii) *jointly left invertible* if there exists a positive number c such that $Q_T(I) \geq cI$.

For a jointly left invertible T, the spherical Cauchy dual $T^{\mathfrak{s}}$ of T is the d-tuple $(T_1^{\mathfrak{s}}, T_2^{\mathfrak{s}} \cdots, T_d^{\mathfrak{s}})$, where $T_i^{\mathfrak{s}} := T_i(Q_T(I))^{-1}$ $(i = 1, 2, \cdots, d)$. We say that T is a *joint complete hyperexpansion* if

$$B_n(T) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} Q_T^k(I) \leq 0 \qquad (n \geq 1).$$

Throughout this section $\mathbb{B}$ denotes the open unit ball $\{z \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 < 1\}$ and $\partial\mathbb{B}$ denotes the unit sphere $\{z \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 = 1\}$ in $\mathbb{C}^d$.

Let $\{\beta_\alpha\}_{\alpha \in \mathbb{Z}_+^d}$ be a multi-sequence of positive numbers. Consider the Hilbert space $H^2(\beta)$ of formal power series $f(z) = \sum_{\alpha \in \mathbb{Z}_+^d} \hat{f}(\alpha) z^\alpha$ such that

$$\|f\|_{H^2(\beta)}^2 = \sum_{\alpha \in \mathbb{Z}_+^d} |\hat{f}(\alpha)|^2 \beta_\alpha^2 < \infty.$$

The Hilbert space $H^2(\beta)$ is said to be *spherically balanced* if the norm on $H^2(\beta)$ admits the slice representation $[\nu, H^2(\gamma)]$, that is, there exist a *Reinhardt measure* ν and a Hilbert space $H^2(\gamma)$ of formal power series in one variable such that

$$\|f\|_{H^2(\beta)}^2 = \int_{\partial\mathbb{B}} \|f_z\|_{H^2(\gamma)}^2 d\nu(z) \qquad (f \in H^2(\beta)),$$

where $\gamma = \{\gamma_k\}_{k \in \mathbb{Z}_+}$ is given by the relation $\beta_\alpha = \gamma_{|\alpha|} \|z^\alpha\|_{L^2(\partial\mathbb{B}, \nu)}$ for all $\alpha \in \mathbb{Z}_+^d$. Here, by the Reinhardt measure, we mean a $\mathbb{T}^d$-invariant finite positive Borel measure supported in $\partial\mathbb{B}$, where $\mathbb{T}^d$ denotes the the unit d-torus $\{z \in \mathbb{C}^d : |z_1| = 1, \cdots, |z_d| = 1\}$. For more details on spherically balanced Hilbert spaces, we refer to [11].

The following lemma has been already recorded in [11, Lemma 4.3]. We include a statement for ready reference.

Lemma 3.1. *Let $H^2(\beta)$ be a spherically balanced Hilbert space and let $[\nu, H^2(\gamma)]$ be the slice representation for the norm on $H^2(\beta)$. Consider the d-tuple $M_z = (M_{z_1}, \cdots, M_{z_d})$ of multiplication*

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by the co-ordinate functions $z_1, \cdots, z_d$ *on* $H^2(\beta)$. *Then for every* $n \in \mathbb{Z}_+$ *and* $\alpha \in \mathbb{Z}_+^d$,

$$\langle B_n(M_z)z^\alpha,\, z^\alpha\rangle = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\langle Q_{M_z}^k(I)z^\alpha,\, z^\alpha\rangle = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\gamma_{k+|\alpha|}^2\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu)}^2.$$

If the interior of the point spectrum $\sigma_p(M_z^*)$ of M_z^* is non-empty then $H^2(\beta)$ may be realized as a reproducing kernel Hilbert space $\mathcal{H}(K)$ [18, Propositions 19 and 20], where the reproducing kernel K is given by

$$K(z,w) = \sum_{\alpha\in\mathbb{Z}_+^d}\frac{z^\alpha\bar{w}^\alpha}{\beta_\alpha^2} \qquad (z,w \in \sigma_p(M_z^*)).$$

This leads to the following definition.

Definition 3.2 : Let $\mathcal{H}(K)$ be a reproducing kernel Hilbert space defined on the open unit ball $\mathbb{B}$ with reproducing kernel $K(z,w) = \sum_{\alpha\in\mathbb{Z}_+^d} a_\alpha z^\alpha\bar{w}^\alpha$ for all $z, w \in \mathbb{B}$. We say that K is a *balanced kernel* if $\mathcal{H}(K)$ is a spherically balanced Hilbert space. Further, the multiplication d-tuple M_z on $\mathcal{H}(K)$ may be called a *balanced multiplication tuple*.

Remark 3.3 : The spherical Cauchy dual $M_z^{\mathfrak{s}}$ of a jointly left invertible balanced multiplication tuple M_z can be seen as a multiplication d-tuple $M_z^{\mathfrak{s}} = (M_{z_1}^{\mathfrak{s}}, \cdots, M_{z_d}^{\mathfrak{s}})$ of multiplication by the co-ordinate functions $z_1, \cdots, z_d$ on $H^2(\beta^{\mathfrak{s}})$, where

$$\beta_\alpha^{\mathfrak{s}} = \frac{1}{\gamma_{|\alpha|}}\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu)} \qquad (\alpha \in \mathbb{Z}_+^d).$$

In other words, the norm on $H^2(\beta^{\mathfrak{s}})$ admits the slice representation $[\nu, H^2(\gamma')]$, where $\gamma_k' = 1/\gamma_k$ for all $k \in \mathbb{Z}_+$.

Proposition 3.4. *If* $K_1(z,w) = \sum_{\alpha\in\mathbb{Z}_+^d} a_\alpha z^\alpha\bar{w}^\alpha$ *and* $K_2(z,w) = \sum_{\alpha\in\mathbb{Z}_+^d} b_\alpha z^\alpha\bar{w}^\alpha$ *are any two balanced kernels with the slice representations* $[\nu, H^2(\gamma_1)]$ *and* $[\nu, H^2(\gamma_2)]$ *respectively, then* $K_1 + K_2$ *is a balanced kernel with the slice representation* $[\nu/2, H^2(\gamma)]$, *where* $\gamma = \{\gamma_k\}$ *is given by the relation*

$$\gamma_k = \frac{\sqrt{2}\gamma_{k,1}\gamma_{k,2}}{(\gamma_{k,1}^2 + \gamma_{k,2}^2)^{1/2}} \qquad (k \in \mathbb{Z}_+).$$

Proof. For every $\alpha \in \mathbb{Z}_+^d$, we have

$$a_\alpha + b_\alpha = \frac{1}{\gamma_{|\alpha|,1}^2\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu)}^2} + \frac{1}{\gamma_{|\alpha|,2}^2\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu)}} = \frac{(\gamma_{|\alpha|,1}^2 + \gamma_{|\alpha|,2}^2)}{\gamma_{|\alpha|,1}^2\gamma_{|\alpha|,2}^2\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu)}^2}.$$

Therefore

$$\|z^\alpha\|_{\mathcal{H}(K_1+K_2)}^2 = \frac{2\gamma_{|\alpha|,1}^2\gamma_{|\alpha|,2}^2}{(\gamma_{|\alpha|,1}^2 + \gamma_{|\alpha|,2}^2)}\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu/2)}^2 = \gamma_{|\alpha|}^2\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu/2)}^2$$

for all $\alpha \in \mathbb{Z}_+^d$. Since $\{z^\alpha\}_{\alpha\in\mathbb{Z}_+^d}$ forms an orthogonal subset of $L^2(\partial\mathbb{B}, \nu/2)$, the conclusion follows immediately. □

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Remark 3.5 : The conclusion of the Proposition 3.4 still holds even if we chose two different Reinhardt measures ν_1 and ν_2 in the slice representations of K_1 and K_2, such that for some sequence of positive real numbers $\{h_k\}_{k\in\mathbb{Z}_+}$, $\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu_1)} = h_{|\alpha|}\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu_2)}$ for all $\alpha \in \mathbb{Z}_+^d$. For every $j = 1, 2$, it is easy to verify that

$$\sum_{i=1}^{d} \frac{\|z^{\alpha+\varepsilon_i}\|^2_{L^2(\partial\mathbb{B},\nu_j)}}{\|z^\alpha\|^2_{L^2(\partial\mathbb{B},\nu_j)}} = 1.$$

This implies that $\{h_k\}_{k\in\mathbb{Z}_+}$ is a constant sequence, say c. Now, by a routine argument, using the Stone-Weierstrass theorem, we conclude that $\nu_1 = c^2\nu_2$.

Let $\mathcal{K}_\nu$ denote the class of all balanced kernels with the following properties:

(i) For all $K \in \mathcal{K}_\nu$, the norm on $\mathcal{H}(K)$ admits the slice representations with fixed Reinhardt measure ν.

(ii) For every member K of $\mathcal{K}_\nu$, the multiplication operator M_z defined on $\mathcal{H}(K)$ is jointly left invertible.

(iii) The Cauchy dual tuple $M_z^{\mathfrak{s}}$ of M_z is a joint complete hyperexpansion.

Lemma 3.6. *For every member K of $\mathcal{K}_\nu$, the multiplication operator tuple M_z defined on $\mathcal{H}(K)$ is a subnormal spherical contraction.*

Proof. Let $K \in \mathcal{K}_\nu$ and $[\nu, H^2(\gamma)]$ be the slice representation for the norm on $\mathcal{H}(K)$. Note that the Cauchy dual $M_z^{\mathfrak{s}}$ of M_z is a balanced multiplication tuple with slice representation $[\nu, H^2(1/\gamma)]$ (see Remark 3.3). Since $M_z^{\mathfrak{s}}$ is a joint complete hyperexpansion, it follows from Lemma 3.1 that $\{1/\gamma_k^2\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence. Therefore, by Remark 2.6, $\{\gamma_k^2\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence. Now again by applying Lemma 3.1, we conclude that the multiplication operator M_z is a subnormal spherical contraction. □

Theorem 3.7. *If K_1 and K_2 are any two members of $\mathcal{K}_\nu$ then the multiplication operator M_z on $\mathcal{H}(K_1 + K_2)$ is a subnormal spherical contraction.*

Proof. Note that the norm on $\mathcal{H}(K_1 + K_2)$ admits the slice representation $[\nu/2, H^2(\gamma)]$, where $\gamma_k^2 = 2\frac{\gamma_{k,1}^2\gamma_{k,2}^2}{\gamma_{k,1}^2+\gamma_{k,2}^2}$ for all $k \in \mathbb{Z}_+$ (see Proposition 3.4). It follows from the proof of Lemma 3.6 that $\{1/\gamma_{k,1}^2\}_{k\in\mathbb{Z}_+}$ and $\{1/\gamma_{k,2}^2\}_{k\in\mathbb{Z}_+}$ are completely alternating. So their sum, and hence $\{1/\gamma_k^2\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence. Now the conclusion follows by imitating the argument given in Lemma 3.6. □

For $\lambda > 0$, consider the positive definite kernel K_λ given by

$$K_\lambda(z, w) = \frac{1}{(1 - \langle z,\, w\rangle)^\lambda} \qquad (z, w \in \mathbb{B}).$$

The norm on $H(K_\lambda)$ admits the slice representation $[\sigma, H^2(\gamma)]$, where σ denotes the normalized surface area measure on $\partial\mathbb{B}$ and $\gamma_k^2 = \frac{(d)_k}{(\lambda)_k}$ for all $k \in \mathbb{Z}_+$. It is well known that the multiplication operator $M_{z,\lambda}$ on $H(K_\lambda)$ is a subnormal contraction if and only if $\lambda \geq d$. The same can also be verified by using Lemma 3.1 and part (i) of Theorem 2.11. Similarly, by using Lemma 3.1 and part (ii) of Theorem 2.11, one may conclude that the Cauchy dual tuple $M_{z,\lambda}^{\mathfrak{s}}$ is a joint

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complete hyperexpansion if and only if $d \leq \lambda \leq d+1$. Thus, if we choose λ and λ' are such that $d \leq \lambda, \lambda' \leq d+1$. Then K_λ and $K_{\lambda'} \in \mathcal{K}_\sigma$. It now follows from Theorem 3.7 that the multiplication operator M_z on $\mathcal{H}(K_\lambda + K_{\lambda'})$ is subnormal. This is also included in the following example.

Example 3.8. *Let $0 < d \leq \lambda' \leq \lambda \leq \lambda' + 1$. Note that the norm on $\mathcal{H}(K_\lambda + K_{\lambda'})$ admits the slice representation $[\sigma/2, H^2(\gamma)]$, where $\gamma_k^2 = \frac{2(d)_k}{(\lambda)_k + (\lambda')_k}$ for all $k \in \mathbb{Z}_+$. From the proof of Proposition 2.12, it is clear that $\{\gamma_k^2\}_{k\in\mathbb{Z}_+}$ is completely monotone. Hence, the multiplication operator M_z on $\mathcal{H}(K_\lambda + K_{\lambda'})$ is subnormal.*

A d-tuple $S = (S_1, \cdots, S_d)$ of commuting bounded linear operators $S_1, \cdots, S_d$ in $\mathcal{B}(\mathcal{H})$ is a *spherical isometry* if $S_1^* S_1 + \cdots + S_d^* S_d = I$. In other words, $Q_S(I) = I$. The most interesting example of a spherical isometry is the Szegö d-shift; that is, the d-tuple M_z of multiplication operators $M_{z_1}, \cdots, M_{z_d}$ on the Hardy space $H^2(\partial\mathbb{B})$ of the unit ball.

Let ν be a Reinhardt measure. Consider the multiplication d-tuple M_z on a reproducing kernel Hilbert space $\mathcal{H}(K^\nu)$ determined by the reproducing kernel

$$K^\nu(z,w) = \sum_{\alpha\in\mathbb{Z}_+^d} \frac{z^\alpha \bar{w}^\alpha}{\|z^\alpha\|^2_{L^2(\partial\mathbb{B},\nu)}} \qquad (z, w \in \mathbb{B}). \tag{3.2}$$

Note that M_z is a spherical isometry. In this case, the norm on $\mathcal{H}(K^\nu)$ admits the slice representation $[\nu, H^2(\mathbb{D})]$, where $H^2(\mathbb{D})$ is the Hardy space of the unit disc.

Theorem 3.9. *Let K^ν be the reproducing kernel given as in equation (3.2) and $\tilde{K}$ be any balanced kernel with the slice representation $[\nu, H^2(\tilde{\gamma})]$. Assume that the multiplication operator M_z on $\mathcal{H}(K^\nu + \tilde{K})$ is subnormal. Then the multiplication operator on $\mathcal{H}(\tilde{K})$ is subnormal.*

Proof. Observe that the norm on $\mathcal{H}(K^\nu + \tilde{K})$ admits the slice representation $[\nu/2, H^2(\gamma)]$, where $\gamma_k^2 = 2(1 + 1/\tilde{\gamma}_k^2)^{-1}$ for all $k \in \mathbb{Z}_+$. Since M_z on $\mathcal{H}(K^\nu + \tilde{K})$ is subnormal, it follows from Lemma 3.1 that $\{\gamma_k^2\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence. Hence, $\{(1 + 1/\tilde{\gamma}_k^2)^{-1}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence. If we replace a_k by $1/\tilde{\gamma}_k^2$ in the proof of the Proposition 2.9, we get that $\{\tilde{\gamma}_k^2\}_{k\in\mathbb{Z}_+}$ is completely monotone. Now, by applying Lemma 3.1, we conclude that the multiplication operator on $\mathcal{H}(\tilde{K})$ is subnormal. □

We conclude the paper with the following questions:

Question 3.10. *In view of Proposition 2.12 and Theorem 2.16, it is natural to ask that*

(i) *what is the necessary and sufficient condition for the multiplication operator M_z on $\mathcal{H}(K_{\lambda,\mu} + K_{\lambda',\mu})$ to be subnormal?*
(ii) *what is the necessary and sufficient condition for the multiplication operator M_z on $\mathcal{H}(K_p + K_q)$ to be subnormal?*

Question 3.11. *Let K^ν be the reproducing kernel given as in equation (3.2) and $\tilde{K}$ be any positive definite kernel given by*

$$\tilde{K}(z,w) := \sum_{\alpha\in\mathbb{Z}_+^d} a_\alpha z^\alpha \bar{w}^\alpha \qquad (z, w \in \mathbb{B}).$$

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Assume that the d-tuple $M_z = (M_{z_1}, \cdots, M_{z_d})$ of multiplication by the co-ordinate functions $z_1, \cdots, z_d$ on $\mathcal{H}(K^\nu + \tilde{K})$ is subnormal. Is it necessary that the multiplication operator on $\mathcal{H}(\tilde{K})$ subnormal?

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