J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Note on the variable exponent Lebesgue function spaces close to L^∞ ☆

Tengiz Kopaliani, Shalva Zviadadze *

Faculty of Exact and Natural Sciences, Javakhishvili Tbilisi State University, 13, University St., Tbilisi, 0143, Georgia

A R T I C L E I N F O

Article history:
Received 11 September 2018
Available online xxxx
Submitted by R.H. Torres

Keywords:
Variable exponent Lebesgue space
Decreasing rearrangement

A B S T R A C T

In this paper we characterize those exponents $p(\cdot)$ for which corresponding variable exponent Lebesgue space $L^{p(\cdot)}([0;1])$ has in common with L^∞ the property that the space of continuous functions is a closed linear subspace in it. In particular, we obtain necessary and sufficient condition on decreasing rearrangement of exponent $p(\cdot)$ for which exists equimeasurable exponent of $p(\cdot)$ which corresponding variable exponent Lebesgue space have the above mentioned property.

1. Introduction

Recently Edmunds, Gogatishvili and Kopaliani [5] show that there is a variable exponent space $L^{p(\cdot)}([0;1])$, with $1 < p(x) < \infty$ a.e., which has in common with $L^\infty([0;1])$ the property that the space $C([0;1])$ of continuous functions on $[0;1]$ is a closed linear subspace in it. Moreover, both the Kolmogorov and the Marcinkiewicz examples of functions with a.e. divergence Fourier series belong to $L^{p'(\cdot)}([0;1])$, where $p'(\cdot)$ conjugate function of $p(\cdot)$.

It is interesting to some ways characterize such exponents for which the space of continuous functions is closed in corresponding variable Lebesgue space. We give a necessary and sufficient condition on the decreasing rearrangement $p^*(\cdot)$ of exponent $p(\cdot)$ for existence of equimeasurable exponent function of $p(\cdot)$ whose corresponding variable Lebesgue space has the property that the space of continuous functions is closed in it.

Let $W(p)$ denote set of all functions equimeasurable with $p(\cdot)$. Below we will find the conditions on the function $p(\cdot)$ for which exists $\bar{p}(\cdot) \in W(p)$ such that the space $C([0;1])$ continuous functions is closed subspace in $L^{p(\cdot)}([0;1])$. Particularly we prove the following

☆ This work was supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG) FR17_589.

* Corresponding author.
E-mail addresses: tengizkopaliani@gmail.com (T. Kopaliani), sh.zviadadze@gmail.com (Sh. Zviadadze).

https://doi.org/10.1016/j.jmaa.2019.02.029
0022-247X/

Theorem 1.1. *For the existence of $\bar{p}(\cdot) \in W(p)$ for which $C([0;1])$ is closed subspace in $L^{\bar{p}(\cdot)}([0;1])$ it is necessary and sufficient that*

$$\limsup_{t\to 0+} \frac{p^*(t)}{\ln(e/t)} > 0. \tag{1.1}$$

2. Definitions and auxiliary results

Let $\Omega \subset \mathbb{R}^n$ and let $\mathcal{M}$ be the space of all equivalence classes of Lebesgue measurable real-valued functions endowed with the topology of convergence in measure relative to each set of finite measure.

Definition 2.1. A Banach subspace X of $\mathcal{M}$ is called a Banach function space (BFS) on Ω if

1) the norm $\|f\|_X$ is defined for every measurable function f and $f \in X$ if and only if $\|f\|_X < \infty$. $\|f\|_X = 0$ if and only if $f = 0$ a.e.;
2) $\||f|\|_X = \|f\|_X$ for all $f \in X$;
3) if $0 \le f \le g$ a.e., then $\|f\|_X \le \|g\|_X$;
4) if $0 \le f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$;
5) if E is measurable subset of Ω such that $|E| < \infty$, (below we denote the Lebesgue measure of E by $|E|$) then $\|\chi_E\|_X < \infty$;
6) for every measurable set E, $|E| < \infty$, there is a constant $C_E < \infty$ such that $\int_E f(t)dt \le C_E\|f\|_X$.

We now introduce various interesting subspaces of a BFS X. A function f in X is said to have absolutely continuous norm in X if $||f\chi_{E_n}||_X \to 0$ whenever $\{E_n\}$ is a sequence of measurable subsets of Ω such that $\chi_{E_n} \downarrow 0$ a.e. The set of all such functions is denoted by X_A.

By X_B is meant the closure of the set of all bounded functions in X. Following Lai and Pick [10], a function $f \in X$ is said to have continuous norm in X if for every $x \in \Omega$, $\lim\limits_{\varepsilon\to 0} ||f\chi_{B(x,\varepsilon)}||_X = 0$, where $B(x,\varepsilon)$ is a ball centred in x and radius ε; the set of all these functions is written as X_C. The connection between this notion and the compactness of Hardy operators from a weighted BFS (X, w) to L^∞ is explored in [10]; for a connection with unconditional bases in BFSs see [7,8]. In general, the relation between the subspaces X_A, X_B and X_C is complicated: for example (see [11]), there is a BFS X for which $\{0\} = X_A \subsetneq X_C = X$.

Theorem 2.2 *(Edmunds, Gogatishvili, Kopaliani). Let X be a BFS on $[0;1]$. The space $C([0;1])$ of continuous functions is a closed linear subspace of X if and only if there exists a positive constant c satisfying*

$$c \le ||\chi_{(a;b)}||_X, \qquad \textit{whenever} \quad 0 \le a < b \le 1. \tag{2.1}$$

Let $\mathcal{P}$ through whole paper denotes the family of all measurable functions $p(\cdot) : [0;1] \to [1;+\infty)$. When $p(\cdot) \in \mathcal{P}$ we denote by $L^{p(\cdot)}([0;1])$ the set of all measurable functions f on $[0;1]$ such that for some $\lambda > 0$

$$\int\limits_{[0;1]} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a BFS when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int\limits_{[0;1]} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

The variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the corresponding variable exponent Sobolev spaces $W^{k,p(\cdot)}$ are of interest for their applications to the problems in fluid dynamics, partial differential equations with non-standard growth conditions, calculus of variations, image processing and etc. (see [2,3]).

For the particular BFS $X = L^{p(\cdot)}([0;1])$ the relation between it and its subspaces X_A, X_B and X_C was investigated in [6]: we give some of the results of that paper next.

Proposition 2.3 *(Edmunds, Lang, Nekvinda). Let $p(\cdot) \in \mathcal{P}$ and set $X = L^{p(\cdot)}([0;1])$. Then*

(i) $X_A = X_C$;
(ii) $X_B = X$ *if and only if* $p(\cdot) \in L^\infty([0;1])$;
(iii) $X_A = X_B$ *if and only if*

$$\int_0^1 c^{p^*(t)} dt < \infty, \quad \text{for all } \; c > 1,$$

where p^* is the decreasing rearrangement of $p(\cdot)$.

The decreasing rearrangement of the measurable function f is defined by

$$f^*(x) = \inf\{\lambda \geq 0 \; : \; |\{|f| > \lambda\}| \leq x\}.$$

By construction of f^*, it is a decreasing right-continuous function. More over functions $|f|$ and f^* are equimeasurable.

Recall that a nonnegative function φ defined on $[0;+\infty)$ is called quasiconcave if $\varphi(0) = 0$, $\varphi(t)$ increases and $\varphi(t)/t$ decreases.

The Marcinkiewicz space M_φ is the set of all $f \in \mathcal{M}([0;1])$ such that

$$||f||_{M_\varphi} = \sup_{0<t\leq 1} \frac{1}{\varphi(t)} \int_0^t f^*(u)du < +\infty.$$

Note that $(M_\varphi)_A = (M_\varphi)_B$ and $(M_\varphi)_A$ can be characterized as the set of functions $f \in \mathcal{M}([0;1])$ such that (see [9])

$$\lim_{t\to 0+} \frac{1}{\varphi(t)} \int_0^t f^*(u)du = 0. \tag{2.2}$$

If ψ is an increasing convex function on $[0;+\infty)$, $\psi(0) = 0$, then the Orlicz space L_ψ consist of all $f \in \mathcal{M}([0;1])$ such that

$$||f||_{L_\psi} = \inf\left\{\lambda > 0 \; : \; \int_0^1 \psi\left(\frac{|f(t)|}{\lambda}\right) dt \leq 1\right\} < +\infty.$$

Note that when $\psi(t) = e^t - 1$ and $\varphi(t) = t\ln(e/t)$ the corresponding Orlicz and Marcinkiewicz spaces coincide (see [1]) and in the sequel we denote the corresponding spaces by e^L and $M_{\ln}$. Also note that (see [4, Corollary 3.4.28])

$$||f||_{e^L} \asymp ||f||_{M_{\ln}} \asymp \sup_{0<t\leq 1} \frac{f^*(t)}{\ln(e/t)}. \tag{2.3}$$

Please cite this article in press as: T. Kopaliani, Sh. Zviadadze, Note on the variable exponent Lebesgue function spaces close to L^∞, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.02.029

3. Poof of Theorem 1.1

Necessity. Since the space $C([0;1])$ is closed in $L^{p(\cdot)}([0;1])$, then by Theorem 2.2 there exists positive constant d such that $d \leq ||\chi_{(a;b)}||_{p(\cdot)}$ for all intervals $(a;b)$. This implies $X_A \neq X_B$. Then by Proposition 2.3 there exists $c > 1$ such that

$$\int_0^1 c^{p^*(t)} dt = +\infty. \tag{3.1}$$

Consider two cases:

Case 1) $p^*(\cdot) \in e^L$. Since (3.1) holds then function $p^*(\cdot)$ does not have absolute continuous norm that is $p^*(\cdot) \in e^L \backslash \left(e^L\right)_A$. Then by (2.3) we get that $p^*(\cdot) \in M_{\ln} \backslash (M_{\ln})_A$ then by (2.2) we obtain

$$\limsup_{t \to 0+} \frac{1}{t \ln(e/t)} \cdot \int_0^t p^*(u) du > 0.$$

From the last estimation we get (1.1). Indeed, suppose opposite

$$\limsup_{t \to 0+} \frac{p^*(t)}{\ln(e/t)} = 0.$$

Then by (2.3) we get

$$\sup_{t \in (0;\varepsilon)} \frac{p^*(t)}{\ln(e/t)} \asymp ||p^*(\cdot) \cdot \chi_{(0;\varepsilon)}||_{M_{\ln}} \to 0, \quad \varepsilon \to 0+.$$

Thus we get that $p^*(\cdot)$ has absolute continuous norm in $M_{\ln}$ (and also e^L) space, which is contradiction.

Case 2) $p^*(\cdot) \notin e^L$. Then by (2.3)

$$\sup_{0 < t \leq 1} \frac{p^*(t)}{\ln(e/t)} = +\infty,$$

consequently (1.1) holds. The necessity part of the theorem proved.

Sufficiency. Let (1.1) hold. For all $t \in [0;1]$ define function $h(t) = \min\{p^*(t), \ln(e/t)\}$. It is obvious that in this case holds

$$\limsup_{t \to 0+} \frac{h(t)}{\ln(e/t)} > 0,$$

then there exists a sequence $t_k \downarrow 0$, such that

$$\frac{h(t_k)}{\ln(e/t_k)} \geq d, \quad k \in \mathbb{N}, \tag{3.2}$$

for some positive number d. Now choose subsequence (t_{k_n}) such that $2t_{k_{n+1}} < t_{k_n}$, for all natural n. Since $t_k \downarrow 0$, we can always choose such subsequence, so without loss of generality we can assume that sequence (t_k) is already such.

Let given function f defined by

$$f(t) = d \cdot \ln(e/t_k), \quad t \in (t_{k+1}; t_k], \quad k \in \mathbb{N} \quad \text{and} \quad f(t) = 1, \quad t \in (t_1; 1].$$

Please cite this article in press as: T. Kopaliani, Sh. Zviadadze, Note on the variable exponent Lebesgue function spaces close to L^∞, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.02.029

Since function h is decreasing it is clear that $h(t) \geq f(t)$ for all $t \in [0;1]$. Now choose positive number c such that $c > e^{1/d}$ then we get

$$\int_0^1 c^{h(t)}dt = +\infty. \tag{3.3}$$

Indeed,

$$\int_0^1 c^{h(t)}dt \geq \int_0^1 c^{f(t)}dt > \int_{t_{k+1}}^{t_k} c^{d\cdot\ln(e/t_k)}dt =$$
$$= (t_k - t_{k+1}) \cdot e^{d\cdot\ln c\cdot\ln(e/t_k)} > \frac{t_k}{2} \cdot \left(\frac{e}{t_k}\right)^{d\cdot\ln c} \to +\infty, \quad k \to +\infty.$$

Choose decreasing sequence $\{a_k\}_{k\in\mathbb{N}}$, such that

$$\int_{a_{k+1}}^{a_k} c^{h(t)}dt = 1.$$

By (3.3) such sequence always can be chosen. Now let $\Delta_k = [a_{k+1}; a_k]$, and $\{r_k : k \in \mathbb{N}\}$ is a countable dense set in $[0;1]$. Define $b_k = -a_{k+1} + r_k$. Now let $A_k := \Delta_k + b_k = [r_k; r_k + a_k - a_{k+1}]$. Let $g_k(t) = h(t) \cdot \chi_{\Delta_k}(t)$, $k \in \mathbb{N}$. Define functions $p_k(t)$ by the induction:

$$p_1(t) = g_1(t - b_1)\chi_{[0;1]}(t),$$
$$p_k(t) = (p_{k-1}(t)(1 - \chi_{\Delta_k}(t - b_k)) + g_k(t - b_k)) \cdot \chi_{[0;1]}(t), \quad k > 1.$$

It is clear that $h(t)$ is decreasing and therefore $p_k(t) \leq p_{k+1}(t)$, for all $t \in [0;1]$ and all $k \in \mathbb{N}$. Also for all $k \in \mathbb{N}$ we have

$$\int_0^1 p_k(t)dt \leq \int_0^1 h(t)dt \leq \int_0^1 \ln(e/t)dt = 2. \tag{3.4}$$

Now define $q(\cdot)$ function by

$$q(t) = \lim_{k\to+\infty} p_k(t), \quad t \in [0;1].$$

By (3.4) we get that the function $q(\cdot)$ is almost everywhere finite. By the construction it is clear that $q^*(t) \leq h(t) \leq p^*(t)$. Now by the well known result (see [1, Chapter 2, Theorem 7.5]) there exists measure preserving transformation $\omega : [0;1] \to [0;1]$ such that $q(t) = q^*(\omega(t))$. Now define $\bar{p}(\cdot)$ by $\bar{p}(t) = p^*(\omega(t))$. Since $q^*(t) \leq p^*(t)$ it is obvious that $q^*(\omega(t)) \leq p^*(\omega(t))$, then for all $t \in (0;1)$ we get the following inequality

$$q(t) \leq \bar{p}(t). \tag{3.5}$$

Given $I := (a;b) \subset (0;1)$ let estimate $||\chi_I||_{\bar{p}(\cdot)}$. Since $c > 1$ by (3.5) the following inequality is obvious

$$\int_I c^{\bar{p}(t)}dt \geq \int_I c^{q(t)}dt.$$

Please cite this article in press as: T. Kopaliani, Sh. Zviadadze, Note on the variable exponent Lebesgue function spaces close to L^∞, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.02.029

By the construction of $q(\cdot)$ there exists number k_0 such that $A_{k_0} \subset I$. We have

$$\int\limits_I c^{q(t)}dt \geq \int\limits_{A_{k_0}} c^{q(t)}dt \geq \int\limits_{A_{k_0}} c^{p_{k_0}(t)}dt = \int\limits_{A_{k_0}} c^{g_{k_0}(t-b_{k_0})}dt =$$

$$= \int\limits_{r_{k_0}}^{r_{k_0}+a_{k_0}-a_{k_0+1}} c^{h(t-b_{k_0})\cdot\chi_{\Delta_{k_0}}(t-b_{k_0})}dt =$$

$$= \int\limits_{a_{k_0+1}}^{a_{k_0}} c^{h(t)}dt = 1.$$

Now by the definition of the norm in variable Lebesgue space and by the above estimations we get that for all intervals $(a;b)$ we have $||\chi_{(a;b)}||_{\bar{p}(\cdot)} \geq 1/c$. By the Theorem 2.2 we get the proof of sufficiency of the Theorem 1.1.

Note that method of construction of the function $\bar{p}(\cdot)$ is different than the method of construction of the function with analogous property from [5]. Particularly in [5] the corresponding exponent $p(\cdot)$ is defined as

$$p(x) = 2 + \sum_{k=1}^{\infty} \ln\left(\frac{1}{x-r_k}\right)\chi_{[r_k,r_k+\delta_k)}(x), \tag{3.6}$$

where $\{r_k\}$ any dense sequence in $[0;1]$, and $\delta_k \downarrow 0$ sequence is such that

$$\sum_{k=1}^{+\infty}\int\limits_0^{\delta_k} \ln(1/t)dt < +\infty.$$

Note that

$$p(x) \geq \ln\left(\frac{1}{x-r_1}\right)\chi_{[r_1;r_1+\delta_1)}(x),$$

and consequently

$$p^*(t) \geq \ln(1/t), \quad t \in (0;\delta_1).$$

Acknowledgment

The authors are very grateful to the referee for the careful reading of the paper and helpful comments and remarks.

References

[1] C. Bennet, R. Sharpley, Interpolation of Operators, Pure Appl. Math., vol. 129, Academic Press, 1988.
[2] D. Cruz-Uribe, A. Fiorenza, Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Birkhäuser, Basel, 2013.
[3] L. Diening, P. Hästö, P. Harjulehto, M. Råužička, Lebesgue and Sobolev Spaces with Variable Exponents, Springer Lecture Notes, vol. 2017, Springer-Verlag, Berlin, 2011.
[4] D. Edmunds, D. Evans, Hardy Operators, Function Spaces and Embeddings, Springer, Berlin, Heidelberg, 2004.
[5] D. Edmunds, A. Gogatishvili, T. Kopaliani, Construction of function spaces close to L^∞ with associate space close to L^1, J. Fourier Anal. Appl. (2017), https://doi.org/10.1007/s00041-017-9574-2.
[6] D.E. Edmunds, J. Lang, A. Nekvinda, On $L^{p(x)}$ norms, Proc. R. Soc. Lond. Ser. A 455 (1999) 219–225.
[7] T. Kopaliani, On unconditional bases in certain Banach function spaces, Anal. Math. 30 (3) (2004) 193–205.

[8] T. Kopaliani, The singularity property of Banach function spaces and unconditional convergence in $L^1[0,1]$, Positivity 10 (3) (2006) 467–474.

[9] S.G. Krein, Yu.I. Petunin, E.M. Semenov, Interpolation of Linear Operators, American Mathematical Society, Providence, 1982.

[10] Q. Lai, L. Pick, The Hardy operator, L^∞ and BMO, J. Lond. Math. Soc. 48 (1993) 167–177.

[11] J. Lang, A. Nekvinda, A difference between continuous and absolutely continuous norms in Banach function spaces, Czechoslovak Math. J. 47 (2) (1997) 221–232.

Please cite this article in press as: T. Kopaliani, Sh. Zviadadze, Note on the variable exponent Lebesgue function spaces close to L^∞, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.02.029