Loading monotonicity of weighted premiums, and total positivity properties of weight functions

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A B S T R A C T

We consider the construction of insurance premiums that are monotonically increasing with respect to a loading parameter. By introducing weight functions that are totally positive of higher order, we derive higher monotonicity properties of generalized weighted premiums; in particular, we deduce for weight functions that are totally positive of order three a monotonicity property of the variance-to-mean ratio, or index of dispersion, of the loss variable. We derive the higher order total positivity properties of some ratios that arise in actuarial and insurance analysis of combined risks. Further, we examine seven classes of weight functions that have appeared in the literature and we ascertain the higher order total positivity properties of those functions.

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1. Introduction

Consider the problem of estimating the premiums that an insurance operation is to charge its clients in order to underwrite their risks. On the one hand, the insurer is limited by competition as to how much it may charge to underwrite a given risk. On the other hand the insurer, so as to remain solvent, necessarily must charge premiums that are suitably large in order to cover its insured risks and its operating expenses.

To formulate this problem probabilistically, suppose that we have a probability triplet $(\mathbf{\Omega}, \mathcal{A}, \mathbf{P})$, consisting of a sample space $\mathbf{\Omega}$, a sigma-algebra $\mathcal{A}$ of subsets of $\mathbf{\Omega}$, and a probability measure $\mathbf{P}(\cdot)$ on $\mathcal{A}$. We suppose that there corresponds to $\mathbf{P}(\cdot)$ a random variable $X : \mathbf{\Omega} \to \mathbb{R}_+$, called the *loss variable*, that arises when the insurer underwrites a randomly chosen risk. We assume that X is nonnegative and refer to its mean, $\mathbb{E}[X] = \int_{\mathbf{\Omega}} X(\omega) \mathbf{P}(\mathrm{d}\omega)$, as the *net premium*. Noting that the net premium $\mathbb{E}[X]$ will cover only the average insured risk, the insurer, in order to remain profitable, necessarily must charge an amount $H[X]$ that is *loaded*, meaning that $H[X] \geq \mathbb{E}[X]$.

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We refer to Furman and Zitikis [5] for an extensive review of the construction of loaded premiums, and to Sendov, Wang, and Zitikis [17] for further results on the same topic. We also cite [6–8,11] for motivating accounts and related analyses of actuarial and insurance problems that involve loading parameters.

A method for constructing weighted premiums begins with the insurer choosing a nonnegative *weight function* $w(\lambda, x)$ that depends on a *loading parameter* $\lambda > 0$. We are especially interested in the weighted premium,

$$H[\lambda, X] = \frac{\mathbb{E}[X \, w(\lambda, X)]}{\mathbb{E}[w(\lambda, X)]},$$

where it is assumed that, for each $\lambda > 0$, the function $x \mapsto w(\lambda, x)$ is Borel-measurable. Suppose that the weight function $(\lambda, x) \mapsto w(\lambda, x)$ is *totally positive of order* 2, i.e.,

$$w(\lambda_1, x_1) \, w(\lambda_2, x_2) \geq w(\lambda_1, x_2) \, w(\lambda_2, x_1),$$

whenever $\lambda_1 > \lambda_2$ and $x_1 > x_2$; then Sendov et al. [17, Theorem 2.1] proved that $H[\lambda, X]$ is non-decreasing in λ, thereby relating the study of weighted premiums with the theory of total positivity. The implication for insurance pricing is that if $w(\lambda, x)$ is totally positive of order two then a riskier venture, with risk represented by the parameter λ, will not be assigned a lower weighted net premium. We refer to Furman and Zitikis [5] who introduced the concept of a *weighted premium* in research on the construction of insurance premiums.

Noting the general theory of total positivity (Karlin [12]), we wish to determine the behavior of the weighted premium $H[\lambda, X]$ for weights $w(\lambda, x)$ that are totally positive of order higher than two. In this paper, we study $H[\lambda, X]$, and some of its generalizations, when the weight function $w(\lambda, x)$ is totally positive of any given order.

Our results may be described as follows. In Section 2, we introduce the theory of total positivity, providing a self-contained introduction to results needed in the sequel.

We consider in Section 3 classes of *generalized weighted premiums*, as defined in [8], extending $H[\lambda, X]$. We establish monotonicity properties of the generalized weighted premiums, recovering as a special case the previously cited result of Sendov et al. [17, Theorem 2.1], and we deduce for weight functions that are totally positive of order three a monotonicity property of the variance-to-mean ratio (or index of dispersion) of the loss variable X. In Section 4, we derive some total positivity properties of $\mathcal{R}_c$ and $\mathcal{C}_c$, two actuarial ratios that were defined and studied in [6,7] in the analysis of combined risks.

In Section 5, we consider seven classes of weight functions treated previously by [17]. We ascertain the higher order total positivity properties of these weight functions, proving that five of them are strictly totally positive of order infinity, one is totally positive of order infinity, and one is not totally positive of order three.

Finally, in Section 6, we summarize with concluding remarks on the implications of working with weighted premiums that are totally positive of higher order.

2. Total positivity

We begin by recalling from [12] the concepts of total positivity, strict total positivity, and sign regularity. For $k \in \mathbb{N}$, a weight function $w : \mathbb{R}^2 \to \mathbb{R}$ is *totally positive of order* k, denoted TP_k, if for all $\lambda_1 > \cdots > \lambda_k$, $x_1 > \cdots > x_k$, and for all $r = 1, \ldots, k$, the $r \times r$ determinant,

$$\det\left(w(\lambda_i, x_j)\right) := \begin{vmatrix} w(\lambda_1, x_1) & \cdots & w(\lambda_1, x_r) \\ \vdots & \ddots & \vdots \\ w(\lambda_r, x_1) & \cdots & w(\lambda_r, x_r) \end{vmatrix} \geq 0.$$

The function $w(\lambda, x)$ is *totally positive of order infinity*, denoted TP_∞, if $w(x, \lambda)$ is TP_k for all $k \geq 1$. Similarly, $w(\lambda, x)$ is *strictly totally positive of order k*, denoted STP_k if the $r \times r$ determinant $\det\big(w(\lambda_i, x_j)\big)$ is strictly positive for all $\lambda_1 > \cdots > \lambda_k$, $x_1 > \cdots > x_k$, and all $r = 1, \ldots, k$. Further, $w(\lambda, x)$ is *strictly totally positive of order infinity*, denoted STP_∞, if $w(\lambda, x)$ is STP_k for all $k \geq 1$.

The function $w(\lambda, x)$ is said to be *reverse-rule of order k*, denoted RR_k, if for all $\lambda_1 > \cdots > \lambda_k$ and $x_1 > \cdots > x_k$, $(-1)^{r(r-1)/2} \det\big(w(\lambda_i, x_j)\big)$ is nonnegative for all $r = 1, \ldots, k$; if this holds for all $k \geq 1$ then $w(\lambda, x)$ is called *reverse-rule of order infinity*, denoted RR_∞. If $(-1)^{r(r-1)/2} \det\big(w(\lambda_i, x_j)\big)$ is strictly positive for all $\lambda_1 > \cdots > \lambda_k$, $x_1 > \cdots > x_k$, $r = 1, \ldots, k$ then $w(\lambda, x)$ is said to be *strictly reverse-rule of order k* (SRR_k); and $w(\lambda, x)$ is called *strictly reverse-rule of order infinity* (SRR_∞) if it is SRR_k for all $k \geq 1$.

Throughout the remainder of the paper, we will assume that all integrals or sums converge absolutely. Whenever it is necessary to provide explicit conditions under which such convergence holds then we will provide the details.

The *Binet-Cauchy formula* often is stated in terms of calculating the minors of a matrix product, AB, from the minors of A and B [12, p. 1]. We will need a continuous and a discrete generalization of this formula: Let ν be a Borel-finite measure on a totally ordered measure space $\mathfrak{X}$. Also, for $r \in \mathbb{N}$, let $\phi_1, \ldots, \phi_r$ and $\psi_1, \ldots, \psi_r$ be complex-valued functions on $\mathfrak{X}$. The Binet-Cauchy formula is that the $r \times r$ determinant with (i, j)th entry $\int_{\mathfrak{X}} \phi_i(x) \psi_j(x) \, d\nu(x)$ satisfies the identity

$$\det \left(\int_{\mathfrak{X}} \phi_i(x) \psi_j(x) \, d\nu(x) \right) = \int \cdots \int_{x_1 > \cdots > x_r} \det \big(\phi_i(x_j) \big) \det \big(\psi_i(x_j) \big) \prod_{j=1}^{r} d\nu(x_j). \tag{2.1}$$

For the case in which $\mathfrak{X} = \mathbb{N}_0$, the set of nonnegative integers, and ν is a discrete measure on $\mathbb{N}_0$ with weights $\nu(m)$, $m = 0, 1, 2, \ldots$, the Binet-Cauchy formula is the statement that

$$\det \left(\sum_{m=0}^{\infty} \phi_i(m) \psi_j(m) \nu(m) \right) = \sum_{m_1 > \cdots > m_r \geq 0} \det \big(\phi_i(m_j) \big) \det \big(\psi_i(m_j) \big) \prod_{j=1}^{r} \nu(m_j). \tag{2.2}$$

The continuous version of the *Basic Composition Formula* is that if the weight functions $w_1(\lambda, x)$ and $w_2(\lambda, x)$ are TP_k on $\mathbb{R}^2$, and if ν is a sigma-finite measure on $\mathbb{R}$, then the weight function

$$w(\lambda, x) = \int_{\mathbb{R}} w_1(\lambda, t) w_2(t, x) \, d\nu(t) \tag{2.3}$$

also is TP_k on $\mathbb{R}^2$.

The discrete version of the Basic Composition Formula, analogous to (2.2), is that if $w_1(\lambda, x)$ and $w_2(\lambda, x)$ are TP_k on $\mathbb{N}_0 \times \mathbb{N}_0$, and ν is a discrete measure on $\mathbb{N}_0$ with nonnegative weights $\nu(m)$, $m = 0, 1, 2, \ldots$, then the function

$$w(\lambda, x) = \sum_{m=0}^{\infty} w_1(\lambda, m) w_2(m, x) \nu(m) \tag{2.4}$$

also is TP_k on $\mathbb{N}_0 \times \mathbb{N}_0$.

We remark that a crucial difference between total positivity of order two and total positivity of higher orders is that if a positive function $w(\lambda, x)$ is TP_2 then the function $1/w(\lambda, x)$ is RR_2. However this result does not generally extend to TP_k functions for $k > 2$. This explains why some of the weight functions considered in Section 5 have relatively straightforward TP_2 or RR_2 properties, while their higher-order total positivity properties are more difficult to establish. We refer to Carlson and Gustafson [2, Eq. (1.5)] for further remarks on this point.

3. Monotonicity properties of generalized weighted premiums

Let X be a nonnegative random variable with probability density function g. Sendov et al. [17] derived a monotonicity property of the weighted premium function,

$$H[\lambda, X] = \frac{\mathbb{E}[w(\lambda, X)\, X]}{\mathbb{E}[w(\lambda, X)]},$$

where the expectations are taken with respect to the distribution of X; it was proved in [17, Theorem 2.1] that if $w(\lambda, x)$ is TP$_2$ then the function $\lambda \mapsto H[\lambda, X]$ is non-decreasing. We shall generalize this property in two ways. Following [8], we consider for a *utility function* $f : \mathbb{R}_+ \to \mathbb{R}_+$, the *generalized weighted premium*,

$$H[\lambda, f(X)] = \frac{\mathbb{E}[w(\lambda, X)f(X)]}{\mathbb{E}[w(\lambda, X)]},$$

whenever these expectations exist. Let Y be the random variable that has the weighted probability density function,

$$\frac{w(\lambda, y)}{\mathbb{E}[w(\lambda, X)]}\, g(y) \tag{3.1}$$

$y \geq 0$, where, as defined earlier, g is the density function of X. Then $H[\lambda, f(X)]$ can also be viewed as the expectation $\mathbb{E}_Y f(Y)$, where the expectation is with respect to the distribution of Y. We will establish the monotonicity of $H[\lambda, f(X)]$ for the case in which f is monotonically increasing.

Second, for the case in which the weight function $w(\lambda, x)$ is TP$_k$ or STP$_k$, we obtain generalizations of the monotonicity property arising from the case in which $k = 2$.

Theorem 3.1. *Suppose that the weight function $w(\lambda, x)$ is TP$_k$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function, and $\lambda_1 > \cdots > \lambda_k$. Then, all minors of the $k \times k$ determinant*

$$\det\left(H\left[\lambda_i, \left(f(X)\right)^{k-j}\right]\right) \tag{3.2}$$

are nonnegative. Further, if $w(\lambda, x)$ is STP$_k$ and the set of points of increase of f contains an open set then all minors of the matrix (3.2) are positive.

Proof. Consider the $m \times m$ minor of (3.2) corresponding to rows $r_1, \ldots, r_m$ and columns $c_1, \ldots, c_m$, where $r_1 < \cdots < r_m$ and $c_1 < \cdots < c_m$. Applying the Binet-Cauchy formula (2.1) with $\mathfrak{X} = \mathbb{R}$, $\phi_i(x) = w(\lambda_{c_i}, x)$ and $\psi_i(x) = \left(f(x)\right)^{k-r_i}$, $i = 1, \ldots, m$, and $\mathrm{d}\nu(x) = g(x)\,\mathrm{d}x$, we obtain

$$\det\left(\mathbb{E}[w(\lambda_{c_i}, X)\left(f(X)\right)^{k-r_j}]\right) = \int \cdots \int\limits_{x_1 > \cdots > x_m} \det\left(w(\lambda_{c_i}, x_j)\right) \det\left(\left(f(x_j)\right)^{k-r_i}\right) \prod_{j=1}^m g(x_j)\,\mathrm{d}x_j. \tag{3.3}$$

Since $w(\lambda, x)$ is TP$_r$ and $\lambda_{c_1} > \cdots > \lambda_{c_m}$ then $\det\left(w(\lambda_{c_i}, x_j)\right)$ is nonnegative on the orthant $\{(x_1, \ldots, x_m) : x_1 > \cdots > x_m\}$.

As for the second determinant in the integrand in (3.3), let $\theta_i = k - r_i - m + i$, $i = 1, \ldots, m$, and set $\theta = (\theta_1, \ldots, \theta_m)$. Since $1 \leq r_1 < \cdots < r_m \leq k$ then $k - m \geq \theta_1 \geq \cdots \geq \theta_m \geq 0$. The determinant

$$\det\left(t_j^{k-r_i}\right) \equiv \det\left(t_j^{\theta_i + m - i}\right)$$

is well-known; see Macdonald [13, p. 40]. In particular, this determinant is divisible by the product $\prod_{1 \leq i < j \leq m}(t_i - t_j)$, and the ratio of these two polynomials defines the *Schur function*,

$$\chi_\theta(t_1, \ldots, t_m) = \frac{\det\left(t_j^{\theta_i + m - i}\right)}{\prod_{1 \leq i < j \leq m}(t_i - t_j)}. \tag{3.4}$$

It is straightforward to verify that $\chi_\theta(t_1, \ldots, t_m)$ is a homogeneous polynomial of degree $\theta_1 + \cdots + \theta_m$. It is also well-known that the coefficients appearing in the monomial expansion of $\chi_\theta(t_1, \ldots, t_m)$ are nonnegative integers [13, p. 75]. Therefore, $\chi_\theta(t_1, \ldots, t_m) > 0$ for $t_1, \ldots, t_m > 0$. Writing (3.4) in the form

$$\det\left(t_j^{k-r_i}\right) = \prod_{1 \leq i < j \leq m}(t_i - t_j) \cdot \chi_\theta(t_1, \ldots, t_m),$$

it follows that $\det\left(t_j^{k-r_i}\right) > 0$ for all $t_1, \ldots, t_m > 0$. Consequently, by substituting $t_i = f(x_i)$, we obtain

$$\det\left(\left(f(x_j)\right)^{k-r_i}\right) = \prod_{1 \leq i < j \leq m}\left(f(x_i) - f(x_j)\right) \cdot \chi_\theta\left(f(x_1), \ldots, f(x_m)\right), \tag{3.5}$$

and since f is increasing then it follows that the determinant in (3.5) is nonnegative for $x_1 > \cdots > x_m$.

Therefore, the integrand in (3.3) is nonnegative for $\lambda_{c_1} > \cdots > \lambda_{c_m}$ and $x_1 > \cdots > x_m$, so it follows that $\det\left(\mathbb{E}\left[w(\lambda_{c_i}, X)\left(f(X)\right)^{k-r_j}\right]\right) \geq 0$. Since $m, r_1, \ldots, r_m$ and $c_1, \ldots, c_m$ were chosen arbitrarily then we deduce that all minors of the $k \times k$ determinant $\det\left(\mathbb{E}\left[w(\lambda_i, X)\left(f(X)\right)^{k-j}\right]\right)$ are nonnegative.

If $w(\lambda, x)$ is STP_k then $\det\left(w(\lambda_{c_i}, x_j)\right) > 0$ for all $\lambda_{c_1} > \cdots > \lambda_{c_m}$ and $x_1 > \cdots > x_m$. If also the set of points of increase of f contains an open set then the determinant (3.5) is positive on an open set in the orthant $\{(x_1, \ldots, x_m) : x_1 > \cdots > x_m\}$. Then, the integrand in (3.3) is positive on an open set, so it follows that $\det\left(\mathbb{E}\left[w(\lambda_{c_i}, X)\left(f(X)\right)^{k-r_j}\right]\right) > 0$.

For $j = 1, \ldots, r$, we divide by $\mathbb{E}\left[w(\lambda_{c_i}, X)\right]$ the jth column of the determinant $\det\left(\mathbb{E}\left[w(\lambda_{c_i}, X) \cdot \left(f(X)\right)^{k-r_j}\right]\right)$. Since

$$\frac{\mathbb{E}\left[w(\lambda_{c_i}, X)\left(f(X)\right)^{k-r_j}\right]}{\mathbb{E}\left[w(\lambda_{c_i}, X)\right]} = H\left[\lambda_{c_i}, \left(f(X)\right)^{k-r_j}\right]$$

then we find that $\det\left(H\left[\lambda_{c_i}, \left(f(X)\right)^{k-r_j}\right]\right) \geq 0$ for all $\lambda_{c_1} > \cdots > \lambda_{c_m}$. As before, it follows that all minors of $\det\left(H\left[\lambda_i, \left(f(X)\right)^{k-j}\right]\right)$ are nonnegative.

Finally, for the case in which $w(\lambda, x)$ is STP_k and f is strictly increasing on an open set, we deduce analogously that $\det\left(H\left[\lambda_{c_i}, \left(f(X)\right)^{k-r_j}\right]\right) > 0$ for all $\lambda_{c_1} > \cdots > \lambda_{c_m}$. Therefore, all minors of $\det\left(H\left[\lambda_i, \left(f(X)\right)^{k-j}\right]\right)$ are positive. $\square$

Remark 3.2. (1) Consider the case in which $k = 2$. As $H[\lambda, 1] \equiv 1$, Theorem 3.1 provides that if f is increasing, $w(\lambda, x)$ is TP_2, and if $\lambda_1 > \lambda_2$ then

$$\begin{vmatrix} H[\lambda_1, f(X)] & H[\lambda_1, 1] \\ H[\lambda_2, f(X)] & H[\lambda_2, 1] \end{vmatrix} = H[\lambda_1, f(X)] - H[\lambda_2, f(X)] \geq 0;$$

that is, the function $\lambda \mapsto H[\lambda, f(X)]$ is non-decreasing. For the case in which $f(x) = x$, we recover the result of Sendov et al. [17, Theorem 2.1].

Let μ_λ denote $H[\lambda, f(X)]$; equivalently, μ_λ is the mean of $f(Y)$ with respect to the weighted distribution (3.1). Then the hypothesis that $w(\lambda, x)$ is TP_2 leads to the conclusion that μ_λ is increasing in λ.

(2) Suppose that $k = 3$; then Theorem 3.1 provides that if f is increasing, $w(\lambda, x)$ is TP$_3$, and if $\lambda_1 > \lambda_2 > \lambda_3$ then all minors of the determinant

$$
\begin{vmatrix}
H[\lambda_1, (f(X))^2] & H[\lambda_1, f(X)] & 1 \\
H[\lambda_2, (f(X))^2] & H[\lambda_2, f(X)] & 1 \\
H[\lambda_3, (f(X))^2] & H[\lambda_3, f(X)] & 1
\end{vmatrix}
$$

are nonnegative. In particular, the 2×2 minor,

$$
\begin{vmatrix}
H[\lambda_1, (f(X))^2] & H[\lambda_1, f(X)] \\
H[\lambda_2, (f(X))^2] & H[\lambda_2, f(X)]
\end{vmatrix}
= H[\lambda_1, (f(X))^2]H[\lambda_2, f(X)] - H[\lambda_1, f(X)]H[\lambda_2, (f(X))^2] \tag{3.6}
$$

is nonnegative.

Suppose that the weight function $w(\lambda, x)$ is differentiable in λ. Also, suppose that its partial derivative, $\partial w(\lambda, x)/\partial \lambda$, is integrable and that

$$
H_1[\lambda, f(X)] := \frac{\partial}{\partial \lambda} H[\lambda, f(X)]
$$

exists. Dividing (3.6) by $\lambda_1 - \lambda_2$ and then letting $\lambda_1, \lambda_2 \to \lambda$, we obtain

$$
\begin{aligned}
0 \leq \lim_{\lambda_1, \lambda_2 \to \lambda} & \frac{H[\lambda_1, (f(X))^2]H[\lambda, f(X)] - H[\lambda_1, f(X)]H[\lambda, (f(X))^2]}{\lambda_1 - \lambda_2} \\
&= H_1[\lambda, (f(X))^2]H[\lambda, f(X)] - H_1[\lambda, f(X)]H[\lambda, (f(X))^2] \\
&= H[\lambda, f(X)]H[\lambda, (f(X))^2] \frac{\partial}{\partial \lambda} \log \frac{H[\lambda, (f(X))^2]}{H[\lambda, f(X)]};
\end{aligned}
$$

equivalently,

$$
\frac{\partial}{\partial \lambda} \log \frac{H[\lambda, (f(X))^2]}{H[\lambda, f(X)]} \geq 0.
$$

Hence, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and $w(\lambda, x)$ is TP$_3$ then the function $\lambda \mapsto H[\lambda, (f(X))^2]/H[\lambda, f(X)]$ is increasing.

To interpret this result within a statistical context, define

$$
\sigma_\lambda^2 := H[\lambda, (f(X))^2] - (H[\lambda, f(X)])^2,
$$

representing the variance of $f(Y)$ with respect to the weighted distribution (3.1). Then,

$$
\frac{H[\lambda, (f(X))^2]}{H[\lambda, f(X)]} = \frac{\sigma_\lambda^2 + \mu_\lambda^2}{\mu_\lambda} = \frac{\sigma_\lambda^2}{\mu_\lambda} + \mu_\lambda.
$$

The ratio $\sigma_\lambda^2/\mu_\lambda$ is known classically as the *variance-to-mean ratio* or *index of dispersion* [4, p. 72], and we denote it by VMR$_\lambda$. The variance-to-mean ratio is a normalized measure of the extent to which the possible values of X are dispersed, so that smaller values of VMR$_\lambda$ correspond to more concentrated clustering of the possible values of X. Thus, the assumption that $w(\lambda, x)$ is TP$_3$ leads to the conclusions that μ_λ and VMR$_\lambda + \mu_\lambda$ are increasing functions of λ. In the context of premium calculations, the variance-to-mean ratio was studied earlier by [10].

We now see that as k, the order of total positivity of the weight function $w(\lambda, x)$, increases, we are able to deduce correspondingly more intricate aspects of the monotonicity properties of $H[\lambda, f(X)]$ as a function

of λ. An implication of the above remark is that if an insurer expects greater variance-to-mean ratios for increasing values of the loading parameter λ then it would be advisable to calculate premiums using weight functions that are STP_k with $k \geq 3$.

For $k = 2$, a consequence of the proof of Theorem 3.1 is that it provides in (3.3) an explicit representation for the difference $H[\lambda_1, f(X)] - H[\lambda_2, f(X)]$ as the integral of a nonnegative function, viz.,

$$
\mathbb{E}[w(\lambda_1, X)]\,\mathbb{E}[w(\lambda_2, X)]\,\big(H[\lambda_1, f(X)] - H[\lambda_2, f(X)]\big)
$$
$$
= \int\!\!\int_{x_1 > x_2} (f(x_1) - f(x_2)) \begin{vmatrix} w(\lambda_1, x_1) & w(\lambda_1, x_2) \\ w(\lambda_2, x_1) & w(\lambda_2, x_2) \end{vmatrix} g(x_1)g(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2. \quad (3.7)
$$

Hence, the nonnegativity of the difference, $H[\lambda_1, f(X)] - H[\lambda_2, f(X)]$, is obtained immediately. Further, as the following result shows, the integral representation (3.7) combined with an estimate on the variation of $f(x)$ leads to an upper bound on $H[\lambda_1, f(X)] - H[\lambda_2, f(X)]$.

Corollary 3.3. *Suppose that* $(\lambda, x) \mapsto w(\lambda, x)$ *is* TP_2 *and* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is a non-decreasing function that satisfies a uniform Lipschitz condition of order* 1, *viz.,*

$$
|f(x_1) - f(x_2)| \leq |x_1 - x_2|
$$

for all x_1 *and* x_2. *Then, for all* $\lambda_1 > \lambda_2$,

$$
H[\lambda_1, f(X)] - H[\lambda_2, f(X)] \leq H[\lambda_1, X] - H[\lambda_2, X]. \quad (3.8)
$$

Proof. Applying to (3.7) the Lipschitz condition on f, we obtain

$$
\mathbb{E}[w(\lambda_1, X)]\,\mathbb{E}[w(\lambda_2, X)]\,\big(H[\lambda_1, f(X)] - H[\lambda_2, f(X)]\big) \quad (3.9)
$$
$$
\leq \int\!\!\int_{x_1 > x_2} (x_1 - x_2) \begin{vmatrix} w(\lambda_1, x_1) & w(\lambda_1, x_2) \\ w(\lambda_2, x_1) & w(\lambda_2, x_2) \end{vmatrix} g(x_1)g(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2
$$
$$
\equiv \int\!\!\int_{x_1 > x_2} \det\left(x_i^{2-j}\right) \cdot \det\left(w(\lambda_i, x_j)\right) g(x_1)g(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2. \quad (3.10)
$$

Applying the Binet-Cauchy formula (2.1), we deduce that (3.10) equals

$$
\det\left(\mathbb{E}[w(\lambda_i, X)X^{2-j}]\right) = \det\left(\mathbb{E}[w(\lambda_i, X)]H[\lambda_i, X^{2-j}]\right)
$$
$$
= \mathbb{E}[w(\lambda_1, X)]\,\mathbb{E}[w(\lambda_2, X)]\,\big(H[\lambda_1, X] - H[\lambda_2, X]\big). \quad (3.11)
$$

On comparing (3.9) and (3.11), and clearing the common terms on each side of that inequality, we obtain (3.8). $\square$

Remark 3.4. Furman and Zitikis [5, Section 4] provide examples of utility functions that are of the form $f(x) = \int_0^x h(x)\,\mathrm{d}x$, where $h(x) \geq 0$ for all x. Suppose that h is uniformly bounded with $h(x) \leq 1$ for all x; examples of such h are the cumulative distribution functions of nonnegative random variables. Then for $x_1, x_2 \in \mathbb{R}$, it follows from the triangle inequality that

$$
|f(x_1) - f(x_2)| = \left| \int_{x_2}^{x_1} h(x)\,\mathrm{d}x \right| \leq |x_1 - x_2|,
$$

so f satisfies a uniform Lipschitz condition of order 1. Hence, the class of utility functions that satisfy the Lipschitz condition is at least as large as the class of cumulative distribution functions.

In general, the bound in Corollary 3.3 provides under a specified degree-of-variation on f an upper limit on the increase in the premium $H[\lambda, f(X)]$ resulting from an increase in λ, the loading parameter. This enables an insurer to assess the extent to which it is charging suitable additional amounts for perceived increases in risk as measured by higher values of the loading parameter.

4. Total positivity properties of some actuarial ratios

Concepts of total positivity of higher order are germane to other insurance-related problems. For $u > 0$ and $v \geq 0$, the well-known *upper incomplete gamma function* is defined as

$$\Gamma(u, v) = \int_v^\infty x^{u-1} \, \mathrm{e}^{-x} \, \mathrm{d}x.$$

Further, for $c > 0$, define the ratio,

$$\mathcal{R}_c(u, v) = \frac{\Gamma(c + u, v)}{\Gamma(u, v)};$$

this function was shown by Furman and Zitikis [6,7] to arise in the study of losses from collections of insurable risks, and their Proposition 2.1 proved that $\mathcal{R}_c(u, v)$ is strictly increasing in u for each fixed v and c. Extending this observation, we obtain the following total positivity properties of the function $\mathcal{R}_c$.

Proposition 4.1. *(i) For fixed $v \geq 0$, the function $(c, u) \mapsto \mathcal{R}_c(u, v)$, $c > 0$, $u > 0$, is STP_∞.*
  (ii) For fixed $u > 0$, the function $(c, v) \mapsto \mathcal{R}_c(u, v)$, $c > 0$, $v \geq 0$, is STP_∞.
  (iii) For fixed $c > 0$, the function $(u, v) \mapsto \mathcal{R}_c(u, v)$, $u > 0$, $v \geq 0$, is SRR_2.

Proof. For $r \in \mathbb{N}$, and for $c_1 > \cdots > c_r > 0$ and $u_1 > \cdots > u_r > 0$, consider the $r \times r$ determinant,

$$\det \big(\Gamma(u_j, v) \, \mathcal{R}_{c_i}(u_j, v) \big) = \det \big(\Gamma(c_i + u_j, v) \big)$$

$$= \det \left(\int_v^\infty x^{c_i + u_j - 1} \, \mathrm{e}^{-x} \, \mathrm{d}x \right).$$

Applying the continuous version of the Binet-Cauchy formula, (2.1), with $\phi_i(x) = x^{c_i}$, $\psi_j(x) = x^{u_j}$, and $\mathrm{d}\nu(x) = x^{-1} \mathrm{e}^{-x} \, \mathrm{d}x$, we find that

$$\det \big(\Gamma(u_j, v) \, \mathcal{R}_{c_i}(u_j, v) \big) = \int \cdots \int_{x_1 > \cdots > x_r > v} \det \big(x_i^{c_j} \big) \det \big(x_i^{u_j} \big) \prod_{j=1}^r x_j^{-1} \mathrm{e}^{-x_j} \, \mathrm{d}x_j. \qquad (4.1)$$

As shown below in (5.1), the determinant $\det \big(x_i^{k_j} \big)$ is positive for $x_1 > \cdots > x_r > 0$ and $k_1 > \cdots > k_r \geq 0$. Therefore the integrand on the right-hand side of (4.1) is positive on an open subset of $\mathbb{R}^r$, and so the integral is positive. Hence the determinant on the left-hand side of (4.1) is positive, and by extracting the factors $\Gamma(u_j, v)$ from that determinant, we obtain

$$\det \big(\mathcal{R}_{c_i}(u_j, v) \big) > 0. \qquad (4.2)$$

Since r was chosen arbitrarily then it follows that the function $(c, u) \mapsto \mathcal{R}_c(u, v)$, $c > 0$, $u > 0$, is STP_∞. This completes the proof of (i).

To prove (ii), let $\mathbf{1}(x > v)$ denote the indicator function of the interval (v, ∞); then,

$$\Gamma(u, v)\mathcal{R}_c(u, v) = \int_0^\infty x^c \, \mathbf{1}(x > v) \, x^{u-1} e^{-x} \, \mathrm{d}x.$$

For $c_1 > \cdots > c_r > 0$ and $v_1 > \cdots > v_r \geq 0$, it follows from the Binet-Cauchy formula (2.1) that

$$\det\left(\Gamma(u, v_j)\, \mathcal{R}_{c_i}(u, v_j)\right) = \int \cdots \int_{x_1 > \cdots > x_r > 0} \det\left(x_i^{c_j}\right) \det\left(\mathbf{1}(x_i > v_j)\right) \prod_{j=1}^r x_j^{u-1} e^{-x_j} \, \mathrm{d}x_j. \tag{4.3}$$

As noted before, $\det\left(x_i^{c_j}\right) > 0$ for $x_1 > \cdots > x_r > 0$ and $c_1 > \cdots > c_r > 0$. Further, we note in (5.2) that for $v_1 > \cdots > v_r \geq 0$, $\det\left(\mathbf{1}(x_i > v_j)\right) \geq 0$ for all $x_1 > \cdots > x_r > 0$ and also is strictly positive on an open set in $\mathbb{R}^r$. Therefore, the integrand in (4.3) is positive on an open set in $\mathbb{R}^r$, so we deduce that the function $(c, v) \mapsto \mathcal{R}_c(u, v)$, $c > 0$, $v \geq 0$, is STP_∞.

To establish (iii), we apply the "$2m$-function theorem" of Rinott and Saks [15]. For $x > 0$, define the eight functions,

$$\begin{aligned}
f_1(x) &= x^{c+u_1} \, \mathbf{1}(x > v_1), & g_1(x) &= x^{c+u_1} \, \mathbf{1}(x > v_2) \\
f_2(x) &= x^{c+u_2} \, \mathbf{1}(x > v_2), & g_2(x) &= x^{c+u_2} \, \mathbf{1}(x > v_1) \\
f_3(x) &= x^{u_1} \, \mathbf{1}(x > v_2), & g_3(x) &= x^{u_1} \, \mathbf{1}(x > v_1) \\
f_4(x) &= x^{u_2} \, \mathbf{1}(x > v_1), & g_4(x) &= x^{u_2} \, \mathbf{1}(x > v_2)
\end{aligned}$$

and, as before, let $\mathrm{d}\nu(x) = x^{-1} e^{-x} \, \mathrm{d}x$.

We now verify that these functions satisfy the hypotheses of Theorem 1.1 of [15], viz., for $x_1 > x_2 > x_3 > x_4$,

$$\prod_{j=1}^4 f_j(x_j) \leq \prod_{j=1}^4 g_j(x_j). \tag{4.4}$$

By (5.2),

$$\mathbf{1}(z_1 > w_2)\, \mathbf{1}(z_2 > w_1) \leq \mathbf{1}(z_1 > w_1)\, \mathbf{1}(z_2 > w_2)$$

whenever $z_1 > z_2$ and $w_1 > w_2$. Applying this result repeatedly, and noting that $v_1 > v_2$, we obtain

$$\begin{aligned}
\mathbf{1}(x_1 > v_1)\, \mathbf{1}(x_2 > v_2)\, \mathbf{1}(x_3 > v_2)\, \mathbf{1}(x_4 > v_1) &\leq \mathbf{1}(x_1 > v_1)\, \mathbf{1}(x_2 > v_2)\, \mathbf{1}(x_3 > v_1)\, \mathbf{1}(x_4 > v_2) \\
&\leq \mathbf{1}(x_1 > v_1)\, \mathbf{1}(x_2 > v_1)\, \mathbf{1}(x_3 > v_2)\, \mathbf{1}(x_4 > v_2) \\
&\leq \mathbf{1}(x_1 > v_2)\, \mathbf{1}(x_2 > v_1)\, \mathbf{1}(x_3 > v_1)\, \mathbf{1}(x_4 > v_2).
\end{aligned}$$

Multiplying each side of this inequality by $x_1^{c+u_1} x_2^{c+u_2} x_3^{u_1} x_4^{u_2}$, we obtain (4.4); moreover, the inequality is strict on an open set in $(0, \infty)^4$. It is also trivial that the measure $\mathrm{d}\nu(x)$ is an *FKG measure* [15, p. 270], so we obtain

$$\Gamma(c + u_1, v_1)\Gamma(c + u_2, v_2)\Gamma(u_1, v_2)\Gamma(u_2, v_1) = \prod_{j=1}^{4} \int_{0}^{\infty} f_j(x) \, d\nu(x)$$

$$< \prod_{j=1}^{4} \int_{0}^{\infty} g_j(x) \, d\nu(x)$$

$$= \Gamma(c + u_1, v_2)\Gamma(c + u_2, v_1)\Gamma(u_1, v_1)\Gamma(u_2, v_2).$$

Dividing both sides of this inequality by $\Gamma(u_1, v_1)\Gamma(u_2, v_2)\Gamma(u_1, v_2)\Gamma(u_2, v_1)$, we obtain $\det\big(\mathcal{R}_c(u_i, v_j)\big) < 0$ for $u_1 > u_2$ and $v_1 > v_2$. Hence, $\mathcal{R}_c(u, v)$ is SRR_2 in (u, v). $\square$

Remark 4.2. (i) As a special case of (4.2), suppose that $r = 2$, $c_1 = c$, and $c_2 = 0$; since $\mathcal{R}_0(u, v) = 1$ then (4.2) reduces to the monotonicity result of Furman and Zitikis [6]. More generally, Proposition 4.1 can be applied to obtain inequalities for the higher moments of sums of risks similar to the way in which higher moment inequalities are described in Remark 3.2.

(ii) We remark that the function $(u, v) \mapsto \mathcal{R}_c(u, v)$, $u > 0$, $v \geq 0$ is not RR_3. For $c = 3.5$, $(u_1, u_2, u_3) = (4, 3, 2)$, and $(v_1, v_2, v_3) = (6, 5, 4)$, we calculate that $\det\big(\mathcal{R}_c(u_i, v_j)\big) = 7.04$ which, since it is positive, violates the RR_3 condition. Here and throughout, all numerical computations were carried out in high precision, with accuracy to over 100 significant digits.

We will also provide some results on a more complex ratio defined by Furman and Zitikis [7]. Define, for $u > 0$ and $v \geq 0$ the function

$$Q(u, v) = \frac{\Gamma(u, v)}{\Gamma(u)} = \int_{v}^{\infty} \frac{1}{\Gamma(u)} x^{u-1} e^{-x} \, dx. \tag{4.5}$$

For fixed $u > 0$, the function $v \mapsto Q(u, v)$, $v \geq 0$, is strictly decreasing, so we denote by $v \mapsto Q^{-1}(u, v)$, $0 \leq v \leq 1$, the corresponding inverse function. Furman and Zitikis [7] defined the function,

$$\mathcal{C}_c(u, v) = Q\big(c + u, Q^{-1}(u, v)\big) = \frac{\Gamma\big(c + u, Q^{-1}(u, v)\big)}{\Gamma(c + u)},$$

$c > 0$, $u > 0$, $0 \leq v \leq 1$; further, they proved that, for fixed c and v, the function $u \mapsto \mathcal{C}_c(u, v)$ is decreasing. In light of Proposition 4.1 we will investigate the total positivity properties of $\mathcal{C}_c(u, v)$, obtaining the following result.

Proposition 4.3. (i) For fixed $v \in [0, 1]$, the function $(c, u) \mapsto \Gamma(c + u)\mathcal{C}_c(u, v)$, $c > 0$, $u > 0$, is STP_2.
(ii) For fixed $u > 0$, the function $(c, v) \mapsto \mathcal{C}_c(u, v)$, $c > 0$, $0 \leq v \leq 1$, is SRR_∞.

Proof. We have

$$\Gamma(c + u)\mathcal{C}_c(u, v) = \Gamma\big(c + u, Q^{-1}(u, v)\big)$$

$$= \int_{Q^{-1}(u,v)}^{\infty} x^{c+u-1} e^{-x} \, dx$$

$$= \int_{0}^{\infty} x^{c+u-1} \mathbf{1}\big(x > Q^{-1}(u, v)\big) e^{-x} \, dx.$$

For $c_1 > c_2 > 0$ and $u_1 > u_2 > 0$, it follows from the Binet-Cauchy formula that

$$\det\left(\Gamma(c_i + u_j)\, \mathcal{C}_{c_i + u_j}(u_j, v)\right) = \int \cdots \int_{x_1 > x_2 > 0} \det(x_i^{c_j})\, \det\left(x_i^{u_j} \mathbf{1}\left(x_i > Q^{-1}(u_j, v)\right)\right) \prod_{j=1}^{2} x_j^{-1} e^{-x}\, \mathrm{d}x_j. \quad (4.6)$$

For 2×2 matrices $A = (a_{ij})$ and $B = (b_{ij})$, there holds the identity,

$$\det(a_{ij} b_{ij}) = b_{11} b_{22} \det(A) + a_{12} a_{21} \det(B). \quad (4.7)$$

We set $a_{ij} = x_i^{u_j}$; then $\det(A) = \det(x_i^{c_j}) > 0$ and $a_{12} a_{21} > 0$ for $x_1 > x_2 > 0$ and $c_1 > c_2 > 0$. Also, let $b_{ij} = \mathbf{1}\left(x_i > Q^{-1}(u_j, v)\right)$; then we need to determine the sign of

$$\det(B) = \det\left(\mathbf{1}\left(x_i > Q^{-1}(u_j, v)\right)\right) \quad (4.8)$$

for $x_1 > x_2 > 0$ and $u_1 > u_2 > 0$.

We claim that if $u_1 > u_2 > 0$ then $Q^{-1}(u_1, v) > Q^{-1}(u_2, v)$ for all $v > 0$. By applying $Q(u_1, \cdot)$ to both sides of this inequality, and noting that $v \mapsto Q(u_1, v)$ is decreasing, we see that the claim is equivalent to $v < Q\left(u_1, Q^{-1}(u_2, v)\right)$; and by replacing v further by $Q(u_2, v)$, the claim is now seen to be equivalent to $Q(u_2, v) < Q(u_1, v)$, $v > 0$. So, consider

$$\Gamma(u_1)\Gamma(u_2)\left[Q(u_1, v) - Q(u_2, v)\right] = \Gamma(u_2)\Gamma(u_1, v) - \Gamma(u_1)\Gamma(u_2, v)$$

$$= \int_{x_2=0}^{\infty} \int_{x_1=v}^{\infty} (x_1^{u_1} x_2^{u_2} - x_1^{u_2} x_2^{u_1}) \prod_{j=1}^{2} x_j^{-1} e^{-x_j}\, \mathrm{d}x_j$$

$$= \int_{x_2=0}^{\infty} \int_{x_1=v}^{\infty} \det(x_i^{u_j}) \prod_{j=1}^{2} x_j^{-1} e^{-x_j}\, \mathrm{d}x_j.$$

Decomposing the interval $(0, \infty)$ into $(0, v) \cup [v, \infty)$, we obtain

$$\Gamma(u_1)\Gamma(u_2)\left[Q(u_1, v) - Q(u_2, v)\right] = \int_{v}^{\infty} \int_{v}^{\infty} \det(x_i^{u_j}) \prod_{j=1}^{2} x_j^{-1} e^{-x_j}\, \mathrm{d}x_j + \int_{0}^{v} \int_{v}^{\infty} \det(x_i^{u_j}) \prod_{j=1}^{2} x_j^{-1} e^{-x_j}\, \mathrm{d}x_j.$$

By anti-symmetry,

$$\int_{v}^{\infty} \int_{v}^{\infty} \det(x_i^{u_j}) \prod_{j=1}^{2} x_j^{-1} e^{-x_j}\, \mathrm{d}x_j = 0,$$

therefore

$$\Gamma(u_1)\Gamma(u_2)\left[Q(u_1, v) - Q(u_2, v)\right] = \int_{0}^{v} \int_{v}^{\infty} \det(x_i^{u_j}) \prod_{j=1}^{2} x_j^{-1} e^{-x_j}\, \mathrm{d}x_j$$

$$\equiv \int \int_{x_1 > v > x_2 > 0} \det(x_i^{u_j}) \prod_{j=1}^{2} x_j^{-1} e^{-x_j}\, \mathrm{d}x_j.$$

Since $\det(x_i^{u_j}) > 0$ for $x_1 > x_2 > 0$ and $u_1 > u_2 > 0$ then we obtain $Q(u_1, v) > Q(u_2, v)$, equivalently, $Q^{-1}(u_1, v) > Q^{-1}(u_2, v)$.

Since $u_1 > u_2$ implies $Q^{-1}(u_1, v) > Q^{-1}(u_2, v)$ then it follows from (4.8) and (5.2) that for $x_1 > x_2$ and $u_1 > u_2$,

$$\det(B) = \begin{cases} 1, & x_1 > Q^{-1}(u_1, v) > x_2 > Q^{-1}(u_2, v) \\ 0, & \text{otherwise} \end{cases}.$$

Therefore for $u_1 > u_2 > 0$, $\det(B)$, b_{11}, and b_{22}, as functions of (x_1, x_2), are nonnegative everywhere and are positive on an open subset of the orthant $\{x_1 > x_2 > 0\}$. Applying (4.7), we find that

$$\det\left(x_i^{u_j}\mathbf{1}\left(x_i > Q^{-1}(u_j, v)\right)\right) > 0$$

on an open subset of the orthant $\{x_1 > x_2 > 0\}$, consequently, the same applies to the integrand in (4.6), so the integral is positive. Therefore, the function $(c, u) \mapsto \Gamma\left(c + u, Q^{-1}(u, v)\right)$, $c > 0$, $u > 0$, is STP$_2$. The proof of (i) now is complete.

To prove (ii), let $r \in \mathbb{N}$, $c_1 > \cdots > c_r > 0$, and $v_1 > \cdots > v_r \geq 0$; then,

$$\left[\prod_{i=1}^{r}\Gamma(c_i + u)\right]\det\left(\mathcal{C}_{c_i}(u, v_j)\right) = \det\left(\Gamma(c_i + u)\,\mathcal{C}_{c_i}(u, v_j)\right)$$

$$= \det\left(\Gamma\left(c_i + u, Q^{-1}(u, v_j)\right)\right)$$

$$= \det\left(\int_{Q^{-1}(u, v_j)}^{\infty} x^{c_i + u - 1}\,e^{-x}\,\mathrm{d}x\right).$$

Applying the Binet-Cauchy formula, we obtain

$$\left[\prod_{i=1}^{r}\Gamma(c_i + u)\right]\det\left(\mathcal{C}_{c_i}(u, v_j)\right) = \int\cdots\int_{x_1 > \cdots > x_r > 0} \det(x_i^{c_j})\,\det\left(\mathbf{1}\left(x_i > Q^{-1}(u, v_j)\right)\right)\prod_{j=1}^{r} x_j^{u-1}e^{-x}\,\mathrm{d}x_j. \quad (4.9)$$

As before, $\det(x_i^{c_j}) > 0$ for $x_1 > \cdots > x_r > 0$. To derive the sign of the remaining determinant, note that $Q^{-1}(u, v_r) > Q^{-1}(u, v_{r-1}) > \cdots > Q^{-1}(u, v_r)$ for $v_1 > \cdots > v_r$. Therefore, for $x_1 > \cdots > x_r$ and $v_1 > \cdots > v_r$,

$$\det\left(\mathbf{1}\left(x_i > Q^{-1}(u, v_j)\right)\right)$$

$$= (-1)^{r(r-1)/2}\det\left(\mathbf{1}\left(x_i > Q^{-1}(u, v_{r-j+1})\right)\right)$$

$$= \begin{cases} (-1)^{r(r-1)/2}, & x_1 > Q^{-1}(u, v_r) > x_2 > Q^{-1}(u, v_{r-1}) > \cdots > x_r > Q^{-1}(u, v_1) \\ 0, & \text{otherwise} \end{cases}.$$

It now follows that the sign of (4.9) is $(-1)^{r(r-1)/2}$. Since r was chosen arbitrarily, it follows that the function $\mathcal{C}_c(u, v)$ is SRR$_\infty$ in (c, v). $\square$

Remark 4.4. As we noted before, Furman and Zitikis [7] proved that the function $u \mapsto \mathcal{C}_c(u, v)$ is decreasing, a result which raises the issue of whether the function $(c, u) \mapsto \mathcal{C}_c(u, v)$, $c > 0$, $u > 0$ is RR$_2$.

Let F_u denote the cumulative distribution function of a gamma-distributed random variable having index parameter u, i.e., a random variable whose probability density function is the integrand in (4.5). Then, $\mathcal{C}_c(u, v) = 1 - F_{c+u}\big(Q^{-1}(u, v)\big)$. Since $Q(u, v) = 1 - F_u(v)$ then $Q^{-1}(u, v) = F_u^{-1}(1 - v)$, so we obtain

$$\mathcal{C}_c(u, v) = 1 - F_{c+u}\big(F_u^{-1}(1 - v)\big).$$

We performed extensive calculations using this identity and determined that the function $(c, u) \mapsto \mathcal{C}_c(u, v)$ is not RR$_2$ as many of its 2×2 determinants are positive. Further, this function is not TP$_2$, although it seems to fail barely to be so; for $v = 0.211$, $(c_1, c_2) = (4.047, 1.210)$, and $(u_1, u_2) = (3.203, 0.189)$, we obtained $\det\big(\mathcal{C}_{c_i}(u_j, v)\big) = -0.026$, and this negative value of the determinant was similar in magnitude to all other negative values that we found.

We also carried out calculations regarding the total positivity properties of the function $(u, v) \mapsto \mathcal{C}_c(u, v)$ and found substantial evidence that this function is both STP$_2$ and STP$_3$. As we have not been able to establish such results analytically, we pose them as open problems.

5. Seven classes of weight functions

In this section, we determine the total positivity properties of some classes of weight functions treated by Sendov et al. [17, Section 3].

Example 5.1. Let $w_1(\lambda, x) = e^{\lambda x}$. The corresponding weighted premium, $H[\lambda, X]$, is called the *Esscher premium*; see [17] and references given therein. It is well-known that the weight function w_1 is STP$_\infty$ [12, p. 15].

Indeed, by a result of Gross and Richards [9, p. 233], for each $r \geq 2$, the $r \times r$ determinant, $\det\big(w_1(\lambda_i, x_j)\big)$ has an integral representation,

$$\frac{\det\big(w_1(\lambda_i, x_j)\big)}{\prod_{1 \leq i < j \leq r}(\lambda_i - \lambda_j)(x_i - x_j)} = \int_U \Phi(\Lambda, X, u)\, d\nu(u),$$

where U is a certain set of $r \times r$ matrices, $\Lambda = (\lambda_1, \ldots, \lambda_r)$, $X = (x_1, \ldots, x_r)$, $\Phi(\Lambda, X, u)$ is a strictly positive function, and ν is a probability measure on U. This integral formula yields immediately the positivity of the determinant $\det\big(w_1(\lambda_i, x_j)\big)$ for $\lambda_1 > \cdots > \lambda_r$ and $x_1 > \cdots > x_r$. Hence w_1 is STP$_r$ for all r and therefore also STP$_\infty$.

More generally, if $F : \mathbb{R} \to \mathbb{R}$ is strictly increasing then the weight function $\tilde{w}_1(\lambda, x) = \exp\big(\lambda F(x)\big)$ is STP$_\infty$, and the corresponding premium is known as the *Aumann-Shapley premium* [8]. With regard to the closing comments in Remark 3.2, we would advise an insurer to base its premium calculations on an STP$_\infty$ weight function if the possible values of the loss variable X are greatly dispersed, i.e., if X has an extremely large variance-to-mean ratio.

A consequence of the STP$_\infty$ property of w_1 is that the weight function $w_1(\log \lambda, x) = \lambda^x$, $(\lambda, x) \in \mathbb{R}_+ \times \mathbb{R}$, is STP$_\infty$. That is, for any $r \in \mathbb{N}$, the $r \times r$ determinant,

$$\det\big(\lambda_i^{x_j}\big) > 0 \tag{5.1}$$

for $\lambda_1 > \cdots > \lambda_r > 0$ and $x_1 > \cdots > x_r$. This result holds because the transformation $\lambda \to \log \lambda$ is strictly increasing and therefore preserves the total positivity properties of $w_1(\lambda, x)$. In the actuarial literature, the weight function $(\lambda, x) \mapsto \lambda^x$, $\lambda, x > 0$, gives rise to a weighted premium known as the *size-biased premium* [8].

Example 5.2. Let $w_2(\lambda, x) = \mathbf{1}(x > \lambda)$, the weight function corresponding to the *conditional tail expectation* (CTE) premium, $H[\lambda, X] = \mathbb{E}(X|X > \lambda)$. It is well-known that the weight function w_2 is TP_∞ [12, p. 16]. Indeed, for any $r = 1, 2, \ldots$, and for $\lambda_1 > \cdots > \lambda_r$ and $x_1 > \cdots > x_r$,

$$\det\big(w_2(\lambda_i, x_j)\big) = \begin{cases} 1, & \text{if } x_1 > \lambda_1 > x_2 > \lambda_2 > \cdots > x_r > \lambda_r \\ 0, & \text{otherwise} \end{cases} \tag{5.2}$$

which proves that the determinant is nonnegative.

Example 5.3. Let $w_3(\lambda, x) = 1 - e^{-x/\lambda}$. The corresponding weighted premium $H[\lambda, X]$ is called the *Kamps premium* (see [17] and the references therein). The weight function w_3 is STP_∞, as we now prove.

It is straightforward to verify that

$$w_3(\lambda, x) = \int\limits_0^x \lambda^{-1} e^{-t/\lambda} \, dt$$

$$\equiv \int\limits_0^\infty w(\lambda, t) \, w_2(t, x) \, dt,$$

where $w(\lambda, t) = \lambda^{-1} e^{-t/\lambda}$, and $w_2(t, x) = \mathbf{1}(x > t)$ is the weight function given in Example 5.2. Applying the Binet-Cauchy formula (2.1), we obtain for each $r \geq 2$,

$$\det\big(w_3(\lambda_i, x_j)\big) = \int\limits_{t_1 > \cdots > t_r} \cdots \int \det\big(w(\lambda_i, t_j)\big) \, \det\big(w_2(t_i, x_j)\big) \, dt_1 \cdots dt_r. \tag{5.3}$$

Suppose that $\lambda_1 > \cdots > \lambda_r$ and $x_1 > \cdots > x_r$. By Example 5.1, the function $w(\lambda, t)$ is STP_∞, so $\det\big(w(t_i, x_j)\big)$ is positive on the orthant $\{(t_1, \ldots, t_r) : t_1 > \cdots > t_r\}$. Also, by (5.2), the determinant $\det\big(w_2(t_i, x_j)\big)$ is positive on an open neighborhood in the same orthant. Therefore, the integrand in (5.3) is positive on an open set, hence the integral is positive for any choice of $(\lambda_1, \ldots, \lambda_r)$ and $(x_1, \ldots, x_r)$. Therefore, $w_3(\lambda, x)$ is STP_r for all r, hence it is STP_∞.

These results for $w_3(\lambda, x)$ also extend to a more general class of weight functions. For each nonnegative integer k, define the weight function

$$w_{3,k}(\lambda, x) = k! \left[1 - e^{-x/\lambda} \sum_{j=0}^k \frac{(x/\lambda)^j}{j!} \right].$$

For $k = 0$, $w_{3,k}(\lambda, x)$ reduces to $w_3(\lambda, x)$, the weight function corresponding to Kamps' premium. By repeated integration-by-parts, we obtain

$$w_{3,k}(\lambda, x) = \lambda^{-(k+1)} \int\limits_0^x t^k e^{-t/\lambda} \, dt$$

$$= \int\limits_0^\infty w(\lambda, t) \, w_2(t, x) \, dt,$$

where $w(\lambda, t) = \lambda^{-(k+1)} e^{-t/\lambda}$ and $w_2(t, x)$ is the weight function in Example 5.2. Finally, we proceed using arguments similar to the case of w_3: Since $w(\lambda, t)$ and $w_2(\lambda, t)$ are TP_∞ then, by applying the Basic Composition Formula (2.3) and the Binet-Cauchy formula (2.1), we deduce that $w_{3,k}(\lambda, x)$ is STP_∞.

Example 5.4. The fourth weight function considered in [17, Section 3] is

$$\widetilde{w}_4(\lambda, x) = \exp\left(\frac{(1+x)^\lambda - 1}{\lambda}\right) - x,$$

$\lambda, x > 0$. We replace x by $e^x - 1$, a transformation that is strictly increasing and therefore preserves any total positivity properties of $\widetilde{w}_4(\lambda, x)$. Then we are to determine the total positivity properties of

$$w_4(\lambda, x) := \widetilde{w}_4(\lambda, e^x - 1) = \exp\left(f(\lambda, x)\right) - e^x + 1,$$

$\lambda, x > 0$, where

$$f(\lambda, x) = \frac{e^{\lambda x} - 1}{\lambda}.$$

We also define $f(0, x)$ by right-continuity:

$$f(0, x) := \lim_{\lambda \to 0+} f(\lambda, x) = x.$$

Then,

$$w_4(\lambda, x) = \exp\left(f(\lambda, x)\right) - \exp\left(f(0, x)\right) + 1. \tag{5.4}$$

For $r \in \mathbb{N}$, let $\lambda_1 > \cdots > \lambda_r > 0$, $x_1 > \cdots > x_r > 0$, and consider the $r \times r$ determinant,

$$\det\left(w_4(\lambda_i, x_j)\right) = \begin{vmatrix} w_4(\lambda_1, x_1) & \cdots & w_4(\lambda_1, x_r) \\ w_4(\lambda_2, x_1) & \cdots & w_4(\lambda_2, x_r) \\ \vdots & \vdots & \vdots \\ w_4(\lambda_{r-1}, x_1) & \cdots & w_4(\lambda_{r-1}, x_r) \\ w_4(\lambda_r, x_1) & \cdots & w_4(\lambda_r, x_r) \end{vmatrix}.$$

For $i = 1, \ldots, r - 1$, we subtract row $i + 1$ from row i, obtaining

$$\det\left(w_4(\lambda_i, x_j)\right) = D_1 + D_2,$$

where

$$D_1 = \begin{vmatrix} w_4(\lambda_1, x_1) - w_4(\lambda_2, x_1) & \cdots & w_4(\lambda_1, x_r) - w_4(\lambda_2, x_r) \\ w_4(\lambda_2, x_1) - w_4(\lambda_3, x_1) & \cdots & w_4(\lambda_2, x_r) - w_4(\lambda_3, x_1) \\ \vdots & \vdots & \vdots \\ w_4(\lambda_{r-1}, x_1) - w_4(\lambda_r, x_1) & \cdots & w_4(\lambda_{r-1}, x_r) - w_4(\lambda_r, x_1) \\ w_4(\lambda_r, x_1) - w_4(0, x_1) & \cdots & w_4(\lambda_r, x_r) - w_4(0, x_1) \end{vmatrix}$$

and

$$D_2 = \begin{vmatrix} w_4(\lambda_1, x_1) - w_4(\lambda_2, x_1) & \cdots & w_4(\lambda_1, x_r) - w_4(\lambda_2, x_r) \\ w_4(\lambda_2, x_1) - w_4(\lambda_3, x_1) & \cdots & w_4(\lambda_2, x_r) - w_4(\lambda_3, x_1) \\ \vdots & \vdots & \vdots \\ w_4(\lambda_{r-1}, x_1) - w_4(\lambda_r, x_1) & \cdots & w_4(\lambda_{r-1}, x_r) - w_4(\lambda_r, x_1) \\ w_4(0, x_1) & \cdots & w_4(0, x_1) \end{vmatrix}. \tag{5.5}$$

Define

$$w_{41}(\lambda, x) = \frac{\partial}{\partial \lambda} \omega_4(\lambda, x) \equiv \frac{\partial}{\partial \lambda} \exp\big(f(\lambda, x)\big),$$

and set $\lambda_{r+1} \equiv 0$. By Taylor's theorem, there exists $\rho_i \in (\lambda_{i+1}, \lambda_i)$ such that

$$w_4(\lambda_i, x) - w_4(\lambda_{i+1}, x) = (\lambda_i - \lambda_{i+1}) w_{41}(\rho_i, x), \tag{5.6}$$

$i = 1, \ldots, r$. Therefore,

$$\begin{aligned} D_1 &= \det\big((\lambda_i - \lambda_{i+1}) w_{41}(\rho_i, x_j)\big) \\ &\equiv \prod_{i=1}^{r} (\lambda_i - \lambda_{i+1}) \cdot \det\big(w_{41}(\rho_i, x_j)\big), \end{aligned}$$

where $\lambda_1 > \rho_1 > \lambda_2 > \rho_2 > \cdots > \lambda_r > \rho_r > 0$. So, to prove that D_1 is positive, it suffices to show that w_{41} is STP_r, and we begin by observing from (5.4) that

$$\begin{aligned} \det\big(w_{41}(\lambda_i, x_j)\big) &= \det\Big(\frac{\partial}{\partial \lambda_i} \exp\big(f(\lambda_i, x_j)\big)\Big) \\ &= \frac{\partial^r}{\partial \lambda_1 \cdots \partial \lambda_r} \det\Big(\exp\big(f(\lambda_i, x_j)\big)\Big). \end{aligned} \tag{5.7}$$

We now recall the *Bell* (or *exponential*) *polynomials* B_k, $k = 0, 1, 2, \ldots$, defined through the generating function,

$$\exp\big(u(e^t - 1)\big) = \sum_{k=0}^{\infty} B_k(u) \frac{t^k}{k!}. \tag{5.8}$$

We refer to Comtet [3, p. 133 ff.] and Roman [16, pp. 63-67] for further details on these polynomials. For $k \geq 1$, $B_k(u)$ is monic and of degree k; moreover,

$$B_k(u) = \sum_{m=1}^{k} S(k, m) u^m, \tag{5.9}$$

where the coefficients $S(k, m)$ are the *Stirling numbers of the second kind*, viz., the number of partitions of a set of size k into m non-empty subsets [3, p. 50]. In particular, $S(k, 1) = S(k, k) = 1$, and $S(k, m) = 0$ if $m > k$.

An alternative representation for the Bell polynomials arises from the observation that the left-hand side of (5.8) is, for $u > 0$, the moment-generating function of U, a Poisson-distributed random variable with mean parameter u; therefore,

$$B_k(u) = E(U^k) = \sum_{m=0}^{\infty} \frac{e^{-u} u^m}{m!} m^k. \tag{5.10}$$

We now apply to (5.10) the discrete Binet-Cauchy formula (2.2) with $\phi_i(m) = e^{-u_i} u_i^m$ and $\psi_k(m) = m^k$, each of which is STP_∞ by (5.1), and weights $\nu(m) = 1/m!$. Written explicitly, we have, for $k_1 > \cdots > k_r \geq 0$ and $u_1 > \cdots > u_r > 0$,

$$\det\big(B_{k_i}(u_j)\big) = e^{-(u_1 + \cdots + u_r)} \sum_{m_1 > \cdots > m_r \geq 0} \frac{1}{m_1! \cdots m_r!} \det(u_i^{m_j}) \det(m_i^{k_j}).$$

Then the positivity of $\det \left(B_{k_i}(u_j) \right)$ follows from the positivity of each determinant inside the summation. Since r was chosen arbitrarily then it follows that $B_k(u)$ is STP$_\infty$ in (k, u).

Define

$$\widetilde{B}_k(\lambda) := \lambda^k B_k(\lambda^{-1}) = \sum_{m=0}^{k-1} S(k, k-m)\lambda^m. \tag{5.11}$$

Then, by (5.8) and (5.9),

$$\exp\left(f(\lambda, x)\right) = \sum_{k=0}^{\infty} B_k(\lambda^{-1})\frac{(\lambda x)^k}{k!} = \sum_{k=0}^{\infty} \widetilde{B}_k(\lambda)\frac{x^k}{k!}. \tag{5.12}$$

Applying to (5.12) the Binet-Cauchy formula (2.1), we obtain for $\lambda_1 > \cdots > \lambda_r > 0$ and $x_1 > \cdots > x_r > 0$,

$$\det\left(\exp\left(f(\lambda_i, x_j)\right)\right) = \det\left(\sum_{k=0}^{\infty} \frac{1}{k!}\widetilde{B}_k(\lambda_i)\,x_j^k\right)$$

$$= \sum_{k_1 > \cdots > k_r \geq 0} \frac{1}{k_1!\cdots k_r!}\,\det\left(\widetilde{B}_{k_j}(\lambda_i)\right)\,\det\left(x_i^{k_j}\right). \tag{5.13}$$

By (5.1), $\det\left(x_i^{k_j}\right) > 0$ for $x_1 > \cdots > x_r$ and $k_1 > \cdots > k_r$.

We note two consequences of (5.11). First, since $S(k, 1) = 1$ then the polynomial $\widetilde{B}_k$ is monic and of degree $k - 1$. Second, since $S(k, m) = 0$ for $m > k$ and $S(k, k) = 1$ then the polynomials $\{\widetilde{B}_k(\lambda) : k = 0, 1, 2, \ldots\}$ satisfy a linear system of equations in terms of $\{\lambda^m : m = 0, 1, 2, \ldots\}$, with a triangular matrix of coefficients having the (k, m)th entry equal to $S(k, m)$, $1 \leq m \leq k$. Writing out these equations for rows $k_r, k_{r-1}, \ldots, k_1$, in that order, and for columns $1, 2, \ldots, r$ results in a matrix equation,

$$\widetilde{\boldsymbol{B}} = \boldsymbol{S}\boldsymbol{\Lambda}, \tag{5.14}$$

where the $r \times r$ matrix $\widetilde{\boldsymbol{B}}$ has (i, j)th entry $\widetilde{B}_{k_{r-i+1}}(\lambda_j)$, $i, j = 1, \ldots, r$; $\boldsymbol{S}$ is $r \times k_1$ with (i, j)th entry $S(k_{r-i+1}, k_{r-i+1} - j + 1)$, $i = 1, \ldots, r$, $j = 1, \ldots, k_1$; and $\boldsymbol{\Lambda}$ is $k_1 \times r$ with (i, j)th entry λ_j^{i-1}, $i = 1, \ldots, k_1$, $j = 1, \ldots, r$.

Each $r \times r$ minor of $\boldsymbol{\Lambda}$, being of the form $\det\left(\lambda_j^{l_i}\right)$ with $\lambda_1 > \cdots > \lambda_r$ and $l_1 < \cdots < l_r$, is non-zero and has sign $(-1)^{r(r-1)/2}$ as $r(r - 1)/2$ row interchanges are needed to order the λ_i and l_i similarly.

To determine the sign of the minors of $\boldsymbol{S}$, we apply the results of Brenti [1, Section 5] (cf., Mongelli [14]). According to those results, the infinite matrix $S(k, m)$ is totally positive, i.e., all minors of the matrix $S(k, m)$, where k and m are similarly ordered, are nonnegative. In the ith row of $\boldsymbol{S}$, the columns are indexed by the decreasing sequence $k_{r-i+1} - j + 1$, $j = 1, \ldots, k_1$; and in the jth column, the rows are indexed by the increasing sequence k_{r-i+1}, $i = 1, \ldots, r$; therefore, it follows that each non-zero $r \times r$ minor of $\boldsymbol{S}$ also has sign $(-1)^{r(r-1)/2}$. Further, if $k_1, \ldots, k_r$ are consecutive integers then the resulting matrix is lower triangular with non-zero diagonal entries, so the corresponding minor of $\boldsymbol{S}$ is non-zero.

By the classical Binet-Cauchy formula, $\det(\widetilde{\boldsymbol{B}})$ equals a sum of products of $r \times r$ minors of $\boldsymbol{S}$ and $\boldsymbol{\Lambda}$ [12, p. 1]. By the preceding discussion, each such product is nonnegative; this establishes the positivity of each $r \times r$ minor of $\widetilde{\boldsymbol{B}}$, proving that it is at least TP$_r$. Further, the sum of all such products of minors is positive since some minors of $\boldsymbol{S}$, and all minors of $\boldsymbol{\Lambda}$, are non-zero. Therefore, $\widetilde{\boldsymbol{B}}$ is STP$_r$; and since r is arbitrary then it is STP$_\infty$.

To complete the proof that $w_{41}(\lambda, x)$ is STP$_r$, we need to show that (5.7) is positive. By the same argument as at (5.14), *infra*, we find that

$$\frac{\partial^r}{\partial \lambda_1 \cdots \partial \lambda_r} \det \left(\widetilde{B}_{k_j}(\lambda_i) \right) \tag{5.15}$$

is a sum of products of minors of $\boldsymbol{S}$ with derivatives of minors of $\boldsymbol{\Lambda}$. However, the derivatives of the minors of $\boldsymbol{\Lambda}$ are the form

$$\frac{\partial^r}{\partial \lambda_1 \cdots \partial \lambda_r} \det \left(\lambda_j^{l_i} \right) \equiv \det \left(\frac{\partial}{\partial \lambda_i} \lambda_j^{l_i} \right) = \frac{l_1 \cdots l_r}{\lambda_1 \cdots \lambda_r} \det \left(\lambda_j^{l_i} \right), \tag{5.16}$$

which is of the same sign, viz., $(-1)^{r(r-1)/2}$, as each $r \times r$ minor of $\boldsymbol{\Lambda}$. Therefore, the derivatives (5.15) are nonnegative, and some are positive. By differentiating the series (5.13), it follows that $w_{41}(\lambda, x)$ is STP_r, and hence is STP_∞.

For future reference, we note that in addition to (5.7) being positive for $\lambda_1 > \cdots > \lambda_r$ and $x_1 > \cdots > x_r$, there also holds

$$\det \left(\frac{\partial^2}{\partial x_j \partial \lambda_i} \exp \left(f(\lambda_i, x_j) \right) \right) > 0 \tag{5.17}$$

under the same conditions. To prove this, we observe that the above determinant equals

$$\frac{\partial^r}{\partial x_1 \cdots \partial x_r} \frac{\partial^r}{\partial \lambda_1 \cdots \partial \lambda_r} \det \left(\exp \left(f(\lambda_i, x_j) \right) \right),$$

so we can expand this determinant using (5.13). As shown before, the resulting terms in $(\lambda_1, \ldots, \lambda_r)$ are nonnegative. Also, the resulting terms in $(x_1, \ldots, x_r)$ are of the form (5.16) (with each λ_i replaced by x_i), and hence also are nonnegative. Moreover, it is straightforward to see that some of these terms are non-zero. Therefore, (5.17) is positive for $\lambda_1 > \cdots > \lambda_r$ and $x_1 > \cdots > x_r$.

Turning to the determinant D_2 in (5.5), we note first that

$$w_4(0, x) := \lim_{\lambda \to 0+} w_4(0, x) = 1.$$

Therefore,

$$D_2 = \begin{vmatrix} w_4(\lambda_1, x_1) - w_4(\lambda_2, x_1) & \cdots & w_4(\lambda_1, x_r) - w_4(\lambda_2, x_r) \\ w_4(\lambda_2, x_1) - w_4(\lambda_3, x_1) & \cdots & w_4(\lambda_2, x_r) - w_4(\lambda_3, x_1) \\ \vdots & \vdots & \vdots \\ w_4(\lambda_{r-1}, x_1) - w_4(\lambda_r, x_1) & \cdots & w_4(\lambda_{r-1}, x_r) - w_4(\lambda_r, x_1) \\ 1 & \cdots & 1 \end{vmatrix}.$$

We again apply Taylor's theorem, as in (5.6), to each entry in rows $1, \ldots, r-1$, obtaining

$$D_2 = \prod_{i=1}^{r-1} (\lambda_i - \lambda_{i+1}) \cdot D_3$$

where

$$D_3 = \begin{vmatrix} w_{41}(\rho_1, x_1) & \cdots & w_{41}(\rho_1, x_r) \\ w_{41}(\rho_2, x_1) & \cdots & w_{41}(\rho_2, x_r) \\ \vdots & \vdots & \vdots \\ w_{41}(\rho_{r-1}, x_1) & \cdots & w_{41}(\rho_{r-1}, x_r) \\ 1 & \cdots & 1 \end{vmatrix}.$$

Carrying out elementary column operations, subtracting column $j+1$ from column j, for $j = 1, \ldots, r-1$, we obtain

$$D_3 = \begin{vmatrix} w_{41}(\rho_1, x_1) - w_{41}(\rho_1, x_2) & \cdots & w_{41}(\rho_1, x_{r-1}) - w_{41}(\rho_1, x_r) & w_{41}(\rho_1, x_r) \\ \vdots & \vdots & \vdots & \vdots \\ w_{41}(\rho_{r-1}, x_1) - w_{41}(\rho_{r-1}, x_2) & \cdots & w_{41}(\rho_{r-1}, x_{r-1}) - w_{41}(\rho_{r-1}, x_r) & w_{41}(\rho_{r-1}, x_r) \\ 0 & \cdots & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} w_{41}(\rho_1, x_1) - w_{41}(\rho_1, x_2) & \cdots & w_{41}(\rho_1, x_{r-1}) - w_{41}(\rho_1, x_r) \\ \vdots & \vdots & \vdots \\ w_{41}(\rho_{r-1}, x_1) - w_{41}(\rho_{r-1}, x_2) & \cdots & w_{41}(\rho_{r-1}, x_{r-1}) - w_{41}(\rho_{r-1}, x_r) \end{vmatrix}.$$

Define

$$w_{411}(\lambda, x) := \frac{\partial}{\partial x} w_{41}(\lambda, x) = \frac{\partial^2}{\partial \lambda \partial x} \exp\left(f(\lambda, x)\right).$$

Applying Taylor's theorem again, we find that there exists $y_j \in (x_{j+1}, x_j)$ such that

$$w_{41}(\rho_i, x_j) - w_{41}(\rho_i, x_{j+1}) = (x_j - x_{j+1}) w_{411}(\rho_i, y_j),$$

$i, j = 1, \ldots, r-1$; therefore,

$$D_3 = \det\left((x_j - x_{j+1}) w_{411}(\rho_i, y_j)\right) = \prod_{j=1}^{r-1} (x_j - x_{j+1}) \cdot \det\left(w_{411}(\rho_i, y_j)\right).$$

Noting that

$$\det\left(w_{411}(\rho_i, y_j)\right) = \det\left(\frac{\partial^2}{\partial \rho_i \partial y_j} \exp\left(f(\rho_i, y_j)\right)\right)$$

is of the form (5.17), it follows that D_3, and therefore D_2 is a product of terms, each of which is positive for $x_1 > \cdots > x_r$, $y_1 > \cdots > y_{r-1}$, and $\rho_1 > \cdots > \rho_{r-1}$; therefore $D_2 > 0$. Consequently, $w_4(\lambda, x)$, and hence $\widetilde{w}_4(\lambda, x)$, are STP$_r$, and since r was chosen arbitrarily then they both are STP$_\infty$.

Example 5.5. Let

$$w_5(\lambda, x) = \frac{(1+\lambda)^x - 1}{\lambda x},$$

$\lambda, x > 0$. We show that $w_5(\lambda, x)$ is STP$_\infty$.

We observe that

$$w_5(\lambda, x) = \lambda^{-1} \int_0^\lambda (1+t)^{x-1} \, dt$$

$$= \lambda^{-1} \int_0^\infty \mathbf{1}(\lambda > t(\,(1+t)^{x-1} \, dt.$$

Recall that the function $w_2(\lambda, t) = \mathbf{1}(\lambda > t)$ is TP_∞; moreover, the corresponding $r \times r$ determinants are positive on an open set in $\mathbb{R}_+^r$. Also, the function $w(t, x) = (1 + t)^{x-1}$ is STP_∞. Therefore, by the Basic Composition Formula (2.3), $w_5(\lambda, x)$ is STP_∞.

Example 5.6. Let

$$w_6(\lambda, x) = \frac{\lambda x}{\log(1 + \lambda x)},$$

$\lambda, x > 0$. We shall prove that w_6 is STP_∞ on the region $\{(\lambda, x) : \lambda > 0, x > 0, \lambda x < 1\}$.

First, we note that for all $\lambda, x > 0$,

$$w_6(\lambda, x) = \int_0^1 (1 + \lambda x)^t \, dt. \tag{5.18}$$

For $\lambda x < 1$, we expand the integrand, obtaining

$$(1 + \lambda x)^t = \sum_{k=0}^\infty \frac{(-1)^k}{k!} (-t)_k \, \lambda^k x^k,$$

where $(a)_k = a(a+1)(a+2) \cdots (a+k-1)$ is the classical rising factorial. Substituting this series into the integral at (5.18) and integrating term-by-term, we obtain

$$w_6(\lambda, x) = \sum_{k=0}^\infty \theta_k \, \lambda^k x^k, \tag{5.19}$$

where

$$\theta_k = \frac{(-1)^k}{k!} \int_0^1 (-t)_k \, dt$$

$$= \frac{1}{k!} \int_0^1 t(1 - t)(2 - t) \cdots (k - 1 - t) \, dt.$$

This representation shows immediately that $\theta_k > 0$ for all $k \geq 0$.

By applying to (5.19) the discrete version (2.4) of the Basic Composition Formula, with $w_1(\lambda, k) = \lambda^k$, $w_2(k, x) = x^k$, and $\nu(k) = \theta_k$, we deduce that w_6 is STP_r for all r; hence, $w_6(\lambda, x)$ is STP_∞ on the region $\{(\lambda, x) : \lambda > 0, x > 0, \lambda x < 1\}$. By the discrete Binet-Cauchy formula (2.2), we also obtain the representation,

$$\det\left(w_6(\lambda, x)\right) = \sum_{k_1 > \cdots > k_r \geq 0} \theta_{k_1} \cdots \theta_{k_r} \det(\lambda_i^{k_j}) \det(x_i^{k_j}).$$

As regards the total positivity properties of $w_6(\lambda, x)$ on $\mathbb{R}_+^2$, we calculated that for $(x_1, x_2, x_3) = (20000, 0.3, 0.1)$ and $(\lambda_1, \lambda_2, \lambda_3) = (3, 0.4, 0.1)$,

$$\det\left(w_6(\lambda_i, x_j)\right)_{3 \times 3} = -5.17488\ldots < 0.$$

Therefore, $w_6(\lambda, x)$ is not TP_3 on $\mathbb{R}_+^2$.

Example 5.7. Let

$$w_7(\lambda, x) = \frac{\log(1 + \lambda + x)}{\lambda + x} \frac{x}{\log(1 + x)},$$

$\lambda, x > 0$. It is straightforward to verify that

$$\frac{\log(1 + \lambda + x)}{\lambda + x} = \int_0^\infty (1 + t)^{-1}(1 + \lambda + x + t)^{-1} \, dt. \tag{5.20}$$

We remark that this integral representation arose in work of Carlson and Gustafson [2] on the total positivity properties of mean value kernels; in the notation of Carlson and Gustafson, the function in (5.20) is denoted by $R_{-1}(1, 1; 1 + \lambda + x, 1)$. Writing

$$(1 + \lambda + x + t)^{-1} = \int_0^\infty e^{-(1+\lambda+x+t)u} \, du,$$

substituting this formula into (5.20), and applying Fubini's theorem to justify an interchange of the order of integration, we obtain

$$\frac{\log(1 + \lambda + x)}{\lambda + x} = \int_0^\infty (1 + t)^{-1} \int_0^\infty e^{-(1+\lambda+x+t)u} \, du \, dt$$

$$= \int_0^\infty e^{-\lambda u} e^{-ux} \, d\nu(u),$$

where the positive measure ν is given explicitly by

$$d\nu(u) = \Big[\int_0^\infty e^{-tu}(1 + t)^{-1} \, dt \Big] e^{-u} \, du,$$

$u > 0$. Consequently, we have obtained an integral representation,

$$w_7(\lambda, x) = \frac{x}{\log(1 + x)} \int_0^\infty e^{-\lambda u} e^{-ux} \, d\nu(u).$$

Applying the Binet-Cauchy formula (2.1), we obtain

$$\det \left(w_7(\lambda_i, x_j) \right) = \Big[\prod_{j=1}^r \frac{x_j}{\log(1 + x_j)} \Big] \int \cdots \int_{u_1 > \cdots > u_r} \det \left(e^{-\lambda_i u_j} \right) \det \left(e^{-u_i x_j} \right) d\mu(u_1) \cdots d\nu(u_r)$$

for $\lambda_1 > \cdots > \lambda_r$ and $x_1 > \cdots > x_r$. Since the sign of each determinant in the integrand equals $(-1)^{r(r-1)/2}$ then their product is positive everywhere on the range of integration. Therefore, $w_7(\lambda, x)$ is STP$_r$ for all $r \geq 1$, hence it is STP$_\infty$.

6. Conclusions

In this paper, we have explored the implications for the loading monotonicity problem of the use of higher-order totally positive weight functions for constructing weighted premiums. In doing this, we applied results from the areas of total positivity [12] and symmetric functions [13]. As a consequence, we obtained monotonicity properties of generalized weighted premiums and an upper bound under a Lipschitz hypothesis for the increase in the weighted premium in response to an increase in the loading parameter. Further, we obtain the total positivity properties of two actuarial ratios that arise in research on combined insurance risks.

We also examined the higher order total positivity properties of a class of kernels that have appeared in the actuarial literature. We established the highest order of total positivity of each of these kernels, thereby adding to the collection of examples of strictly totally positive kernels.

We related the use of weight functions that are totally positive of higher order to the degree of randomness of insured risks, and we advise insurers to relate the order of total positivity of the chosen weight function to the index of dispersion of the loss variable. These results indicate that a broad list of TP_∞, and even STP_∞, weight functions is needed to develop weighted premiums for the purpose of underwriting insurable risks of any degree of randomness.

Acknowledgments

References

[1] F. Brenti, Combinatorics and total positivity, J. Combin. Theory Ser. A 71 (1995) 175–218.
[2] B.C. Carlson, J.L. Gustafson, Total positivity of mean values and hypergeometric functions, SIAM J. Math. Anal. 14 (1983) 389–395.
[3] L. Comtet, Advanced Combinatorics: The Art of Finite and Infinite Expansions, Reidel, Boston, MA, 1974.
[4] D.R. Cox, P.A.W. Lewis, The Statistical Analysis of Series of Events, Methuen, London, 1966.
[5] E. Furman, R. Zitikis, Weighted premium calculation principles, Insurance Math. Econom. 42 (2008) 459–465.
[6] E. Furman, R. Zitikis, Monotonicity of ratios involving incomplete gamma functions with actuarial applications, JIPAM. J. Inequal. Pure Appl. Math. (2008) 61.
[7] E. Furman, R. Zitikis, A monotonicity property of the composition of regularized and inverted-regularized gamma functions with applications, J. Math. Anal. Appl. 348 (2008) 971–976.
[8] E. Furman, R. Zitikis, Weighted pricing functionals with applications to insurance: an overview, N. Am. Actuar. J. 13 (2009) 483–496.
[9] K.I. Gross, D.St.P. Richards, Total positivity, spherical series, and hypergeometric functions of matrix argument, J. Approx. Theory 59 (1989) 224–226.
[10] W.-R. Heilmann, Decision theoretic foundations of credibility theory, Insurance Math. Econom. 8 (1989) 77–95.
[11] B.L. Jones, R. Zitikis, Risk measures, distortion parameters, and their empirical estimation, Insurance Math. Econom. 41 (2007) 279–297.
[12] S. Karlin, Total Positivity, Stanford University Press, Palo Alto, CA, 1968.
[13] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second edition, Oxford University Press, New York, 1995.
[14] P. Mongelli, Total positivity properties of Jacobi-Stirling numbers, Adv. in Appl. Math. 48 (2012) 354–364.
[15] Y. Rinott, M. Saks, Correlation inequalities and a conjecture for permanents, Combinatorica 13 (1993) 269–277.
[16] S. Roman, The Umbral Calculus, Academic Press, New York, 1984.
[17] H.S. Sendov, Y. Wang, R. Zitikis, Log-supermodularity of weight functions, ordering weighted losses, and the loading monotonicity of weighted premiums, Insurance Math. Econom. 48 (2011) 257–264.