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$L^1 - L^1$ estimates for the strongly damped plate equation

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ABSTRACT

In this paper, we derive $L^1 - L^1$ long time estimates for the strongly damped plate equation

$$u_{tt} + \Delta^2 u + \Delta^2 u_t = 0 \quad x \in \mathbb{R}^n,\, t \in \mathbb{R}_+, \quad u(0,x) = u_0(x),\ u_t(0,x) = u_1(x).$$

In particular, we prove that

$$\|u(t,\cdot)\|_{L^1} \leq C\,(1+t)^{\frac{n}{4}} \big(\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}} \|u_1\|_{L^1}\big),$$

for any $t \geq 0$, in space dimension $n \geq 5$.

1. Introduction

In this paper we consider the forward Cauchy problem for the following linear strongly damped plate equation

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta^2 u_t = 0, & x \in \mathbb{R}^n,\, t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x). \end{cases} \tag{1}$$

Fourth-order evolution partial differential equations as in (1) arise in problems of solid mechanics as, for example, in the theory of thin plates and beams. Also, in particular formulations of problems related with the

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Navier-Stokes equations (see [29]) appear elliptic equations of fourth-order. Models to study the vibrations of thin plates ($n = 2$) given by the full von Kármán system have been studied by several authors, in particular, see [2,15,26].

The action of damping dissipates the energy of problem (1), due to

$$\mathcal{E}'(t) = -\|\Delta u_t(t,\cdot)\|_{L^2}^2, \qquad \mathcal{E}(t) = \frac{1}{2}\|u_t(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\|\Delta u(t,\cdot)\|_{L^2}^2,$$

and, more precisely, one can easily show that

$$\mathcal{E}(t) \leq C\,(1+t)^{-\frac{n}{4}}\big(E(0) + \|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2\big),$$

for any $t \geq 0$, for a constant $C > 0$, independent of the initial data, in any space dimension $n \geq 2$, provided that the initial data are in the energy space $H^2 \times L^2$ and in L^1. However, the action of the damping also influences the $L^p - L^q$ long time estimates for the solution to (1), where $1 \leq p \leq q \leq \infty$. The purpose of this paper is to investigate this influence, in particular in the more difficult setting of $L^1 - L^1$ estimates.

The dissipation of the energy, as a consequence of the damping, comes into play for more general evolution equations. The first example of dissipative evolution equation is the wave equation with weak damping

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x). \end{cases} \tag{2}$$

Long time decay estimates and energy estimates for the solution to (2), with initial data in the energy space and in L^1, have been obtained by A. Matsumura [17]. In particular,

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^2} \leq C\,(1+t)^{-\frac{n}{4}-k-\frac{|\alpha|}{2}}\,\big(\|u_0\|_{H^{k+|\alpha|}} + \|u_1\|_{H^{k+|\alpha|-1}} + \|u_0\|_{L^1} + \|u_1\|_{L^1}\big),$$

for $k + |\alpha| \geq 0$. The asymptotic profile of the solution to (2) is described by the solution to the problem for a heat equation

$$\begin{cases} v_t - \Delta v = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ v(0,x) = v_0(x), \end{cases} \tag{3}$$

with initial data $v_0 = u_0 + u_1$, see [16,21,31]. Namely,

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^2} = o(\|v(t,\cdot)\|_{L^2}),$$

under the moment condition $\int_{\mathbb{R}^n} v_0(x)dx \neq 0$. This phenomenon, sometimes called *diffusion phenomenon,* clarifies how the dissipative action of the damping in (2) influences the long time $L^p - L^q$ decay estimates. Indeed,

$$\|v(t,\cdot)\|_{L^p} \approx t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}\,\|v(1,\cdot)\|_{L^p}, \qquad 1 \leq p \leq \infty,$$

where v is the solution to (3). The dissipative action of the damping remains valid for more general evolution equations [14], say

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + u_t = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{4}$$

for instance, for the damped plate equation, obtained for $\sigma = 2$. The long time decay estimates allow to study several types of nonlinear problems related to a damped evolution equation. In particular, the critical exponent for global small data solution of the damped wave equation with power nonlinearity $|u|^p$ is the same critical exponent of the corresponding semilinear heat equation, as proved in space dimension $n = 1, 2$ in [17] and in space dimension $n \geq 3$ by G. Todorova and B. Yordanov [30].

The main difficulty in the study of nonlinear problems associated to the linear equation in (2), or in (4), is related to the regularity of the solution. Indeed, the solution operator for (2) has a regularity similar to regularity of the solution operator for the wave equation, as one can see from the study of $L^p - L^q$ estimates, $1 \leq p \leq q \leq 2$, for (2) (see [19]).

In particular, if $u_0 = 0$ and $u_1 \in L^1$, the solution $u(t,\cdot)$ to (2), or to the wave equation without damping, is not in L^1, in space dimension $n \geq 4$, for any $t > 0$. Namely, the fundamental solution to (2) is not a L^1-bounded operator.

Going back to the plate equation,

$$\begin{cases} u_{tt} + \Delta^2 u = 0, \quad x \in \mathbb{R}^n,\, t \in \mathbb{R}_+, \\ u(0,x) = 0, \\ u_t(0,x) = u_1(x), \end{cases} \tag{5}$$

with $u_1 \in L^p$, we have that the solution u is in L^p if $n|1/p - 1/2| < 1$, and is not in L^p if $n|1/p - 1/2| > 1$, see [22]. A similar regularity issue appears for the plate equation with weak damping and for the solution to (4), with $\sigma \neq 1$. For the wave equation, the regularity restriction is relaxed to $(n-1)|1/p - 1/2| \leq 1$.

In order to guarantee that the solution $u(t,\cdot)$ to (4) is in L^p, for some $p \in [1,2)$, it is possible to assume that the initial data are in suitable Sobolev spaces $W^{m,p}$, where $m = m(n,p,\sigma) \geq 0$ represents a *loss of regularity*. Incidentally, we remark that it is easier to guarantee that the solution remains in L^p, when $p \in (2,\infty]$, since it is possible to use higher order energies (the Cauchy problem is well-posed in H^k) together with Sobolev embeddings $H^k \hookrightarrow L^p$, with $p(n-2k) \leq 2n$ (see [17] for (2)).

The situation is completely different in the case of a strong damping, as showed by Y. Shibata [27] for the wave equation with viscoelastic damping

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t = 0, \quad x \in \mathbb{R}^n,\, t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x). \end{cases} \tag{6}$$

Assuming initial data in L^1, the solution $u(t,\cdot)$ to (6) remains in L^1, for any $t > 0$. The motivation behind this phenomenon is that now the diffusive structure of the equation cancels the influence of oscillations on the regularity of the fundamental solution. On the other hand the diffusive structure created by the strong damping interacts with the wave structure of the problem, in the description of the asymptotic profile of the fundamental solution to (6). This latter is now given by the convolution of the Gauss kernel and the fundamental solution of the wave equation. The precise asymptotic profile to (6) has been first obtained in [10].

It turns out that the interaction between oscillations and damping, create a *loss of decay rate* for the solution to (6) in $L^p - L^q$ estimates, with $1 \leq p \leq q < 2$ (we always use the wording *loss of decay rate* even if a decay only appears if $p < q$ and/or if derivatives of the solution are considered). This loss is completely analogous to the *loss of regularity* created by the interaction between oscillations and damping for the solution to (2). In particular [27, Theorem 2.1,4], the solution to (6) verifies the estimate

$$\|u(t,\cdot)\|_{L^1} \leq C\,(1+t)^{\frac{n}{4}}\big(\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}}\|u_1\|_{L^1}\big), \tag{7}$$

in space dimension $n \geq 2$, even, and the better estimate

$$\|u(t,\cdot)\|_{L^1} \leq C\,(1+t)^{\frac{n-1}{4}}\big(\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}}\|u_1\|_{L^1}\big), \tag{8}$$

in space dimension $n \geq 3$, odd. We stress out that the quantity $(1+t)^{\frac{n}{4}}$ appearing in (7) is increasing in time, so it represents a loss with respect to the corresponding $L^2 - L^2$ estimate:

$$\|u(t,\cdot)\|_{L^2} \leq C\,\big(\|u_0\|_{L^2} + (1+t)^{\frac{1}{2}}\|u_1\|_{L^2}\big),$$

where no loss appears. The fact that in odd space dimension, i.e. in (8), the loss is reduced, can be explained by the Huygens' principle for waves, which is more effective in odd space dimension.

The main purpose of this paper is to prove the following.

Theorem 1. *Let $n \geq 1$. Assume that $u_0, u_1 \in L^1$. Then the solution to (1) verifies estimate*

$$\|u(t,\cdot)\|_{L^1} \leq C\,(1+t)^{\frac{n}{4}}\big(\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}}\|u_1\|_{L^1}\big),$$

provided that $n \geq 5$ or $u_1 = 0$, and estimate

$$\|u(t,\cdot)\|_{L^1} \leq C\,(1+t)\big(\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}}\,\log(e+t)\|u_1\|_{L^1}\big),$$

if $n = 4$, for any $t \geq 0$ and for some $C > 0$, independent of the data.

We remark that, in space dimension $n \geq 5$, the estimate in Theorem 1 is the same estimate obtained for the wave equation with viscoelastic damping by Y. Shibata, i.e., (7). However, we cannot expect improvements in odd space dimension for the plate equation, due to the lack of Huygens' principle for the plate.

The technique employed in [27] to prove (7) and (8) for (6) heavily relies on the very special structure of the fundamental solution to the wave equation. For this reason, in this paper we employ a different technique to estimate the Fourier multiplier associated to problem (1) at low frequencies, where oscillations interact with the damping (see later, Section 2). On the other hand, at high frequencies, where oscillations are neglected, the L^1 well-posedness of the problem may be easily derived, with straightforward calculations.

To complete the picture and clarify the roles played by the damping, we mention a few other results.

T. Narazaki and M. Reissig [20] derived $L^1 - L^1$ estimates for the wave equation with structural damping

$$\begin{cases} u_{tt} + \Delta u + (-\Delta)^{\theta} u_t = 0, & x \in \mathbb{R}^n,\, t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{9}$$

with $\theta \in (0,1)$. In particular, for $\theta \in (1/2, 1)$ they found estimates analogous to the limit case $\theta = 1$ described in (6), including the reduction of the loss in odd space dimension:

$$\|u(t,\cdot)\|_{L^1} \leq C\,(1+t)^{[\frac{n}{2}](1-\frac{1}{2\theta})}\big(\|u_0\|_{L^1} + (1+t)^{\frac{1}{2\sigma}}\|u_1\|_{L^1}\big). \tag{10}$$

On the other hand, $L^p - L^q$ estimates for (9) in the case $\theta \in (0, 1/2)$ are analogous to the estimates for the limit case $\theta = 0$ described in (2). In particular a loss of regularity appears [5]. The diffusion phenomenon for this case has been investigated in [4].

Not surprisingly, under the simultaneous action of a weak and a strong damping, there is no loss of decay, neither a loss of regularity. For the wave equation with double damping [11]

$$
\begin{cases}
u_{tt} + \Delta u + (1-\Delta)u_t = 0, \quad x \in \mathbb{R}^n,\, t \in \mathbb{R}_+,\\
u(0,x) = u_0(x),\\
u_t(0,x) = u_1(x),
\end{cases} \tag{11}
$$

$L^1 - L^1$ estimates

$$
\|u(t,\cdot)\|_{L^1} \leq C\left(\|u_0\|_{L^1} + \|u_1\|_{L^1}\right),
$$

have been derived and applied to semilinear problems in [3]. The reasoning for this result is that oscillations are completely canceled by the simultaneous action of the two damping terms. An interesting model for which oscillations are not canceled, but their influence can be neglected is the limit case $\theta = 1/2$ in (9), for which (10) remains valid:

$$
\|u(t,\cdot)\|_{L^1} \leq C\left(\|u_0\|_{L^1} + (1+t)\|u_1\|_{L^1}\right).
$$

We address the interested reader to [12,13,18,24] for additional long time estimates on σ-evolution models with strong or structural damping, and to [1,8,9] for more general equations with damping.

2. Decomposition of the solution at low and high frequencies

Applying the Fourier transform with respect to the space variable, we see that

$$
\hat{u}(t,\xi) = \mathscr{F}[u(t,\cdot)](\xi) = \int_{\mathbb{R}^n} e^{-ix\xi}\, u(t,x)\, dx
$$

solves the Cauchy problem for the damped harmonic oscillator

$$
\begin{cases}
\hat{u}_{tt} + |\xi|^4 \hat{u} + |\xi|^4 \hat{u}_t = 0, \quad t \in \mathbb{R}_+,\\
\hat{u}(0,\xi) = \hat{u}_0(\xi),\\
\hat{u}_t(0,\xi) = \hat{u}_1(\xi),
\end{cases} \tag{12}
$$

for any $\xi \in \mathbb{R}^n$. The overdamping region, where oscillations are neglected by the action of the damping, corresponds to $|\xi| \geq \sqrt{2}$, whereas we have damped oscillations for $|\xi| < \sqrt{2}$.

In particular, damped oscillations appear in the low frequencies region (the region containing small values of $|\xi|$), and oscillations are neglected in the high frequencies region (the region containing large values of $|\xi|$). The same property appears, more in general, for σ-evolution equations with strong damping:

$$
\begin{cases}
u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\sigma u_t = 0, \quad x \in \mathbb{R}^n,\, t \in \mathbb{R}_+,\\
u(0,x) = u_0(x),\\
u_t(0,x) = u_1(x).
\end{cases} \tag{13}
$$

On the other hand, in the case of a weak damping for σ-evolution equations, the situation is the opposite: damped oscillations appear in the high frequencies region, and oscillations are neglected in the low frequencies region.

Since the regularity of the solution is determined by its high frequencies part and its asymptotic profile is determined by its low frequencies part, this phenomenon explains the different properties of the solution in the two cases. A classification between these two cases has been discussed and provided for evolution equations with time-dependent coefficients and structural damping in [6].

Due to the different behavior of the solution to (1) at low and high frequencies, it is more convenient to give independent results for the solution localized at low frequencies, and for the solution localized at high frequencies. We notice that, due to the fact that problem (1) is linear, the localization of the solution is equivalent to study problem (1) with localized data. To localize the solution to (1) at low and high frequencies, we fix $\varphi_0 \in C_c^\infty(\mathbb{R}^n)$ with $\varphi_0(\xi) = 1$ in a neighborhood of the origin, we set $\varphi_\infty = 1 - \varphi_0$, and we put

$$E_0(t)(u_0, u_1)(x) = \mathscr{F}^{-1}[\varphi_0 \hat{u}(t, \cdot)](x), \tag{14}$$

$$E_\infty(t)(u_0, u_1)(x) = \mathscr{F}^{-1}[\varphi_\infty \hat{u}(t, \cdot)](x), \tag{15}$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform with respect to the space variable, and u is the solution to (1).

The Fourier multipliers associated with the solution to (1) are smooth in $\mathbb{R}^n \setminus \{0\}$. Therefore, our analysis will focus on the singularities of the multipliers as $\xi \to 0$ and on their behavior as $|\xi| \to \infty$. The characteristic equation to (12) is

$$\lambda^2 + |\xi|^4 \lambda + |\xi|^4 = 0. \tag{16}$$

If $|\xi| > \sqrt{2}$, equation (16) has real-valued, non positive, roots:

$$\lambda_\pm(\xi) = \frac{-|\xi|^4 \pm |\xi|^2 \sqrt{|\xi|^4 - 4}}{2}. \tag{17}$$

If $|\xi| < \sqrt{2}$, the roots of (16) are complex-valued with negative real part:

$$\lambda_\pm(\xi) = \frac{-|\xi|^4 \pm i|\xi|^2 \sqrt{4 - |\xi|^4}}{2}. \tag{18}$$

If we fix φ_0, and φ_∞ in $C^\infty(\mathbb{R}^n)$ such that

$$\varphi_0(\xi) = \begin{cases} 1 & \text{if } |\xi| \le \frac{3}{4}, \\ 0 & \text{if } |\xi| \ge 1, \end{cases} \qquad \varphi_\infty(\xi) = \begin{cases} 1 & \text{if } |\xi| \ge 4, \\ 0 & \text{if } |\xi| \le 3, \end{cases} \tag{19}$$

and we define $\varphi_1 \in C^\infty(\mathbb{R}^n)$ by

$$\varphi_1(\xi) = 1 - (\varphi_0(\xi) + \varphi_\infty(\xi)),$$

then the solution to (1) at "intermediate" frequencies

$$E_1(t)(u_0, u_1) = \mathscr{F}^{-1}[\varphi_1 \hat{u}(t, \cdot)](x)$$

trivially verifies the estimate

$$\|\partial_x^\alpha \partial_t^k E_1(t)(u_0, u_1)\|_{L^q} \le C\, e^{-ct}\, \|(u_0, u_1)\|_{L^p},$$

for any $1 \le p \le q \le \infty$, $k + |\alpha| \ge 0$, and $t \ge 0$, for some $C, c > 0$, independent of the data. Indeed, φ_1 is compactly supported in $\mathbb{R}^n \setminus \{0\}$ and the real parts of the roots $\lambda_\pm(\xi)$ are negative in $\mathbb{R}^n \setminus \{0\}$.

Therefore, it is not restrictive to assume that $\varphi_0, \varphi_\infty$ are as in (19), in the definition of E_0 and E_∞, when we prove our statements for (14) and (15).

We first present our estimates for the solution to (1) at low frequencies $E_0(t)(u_0, u_1)$.

Theorem 2. *Let $n \geq 5$. Assume that $u_0, u_1 \in L^1$. Then the solution to (1) at low frequencies $E_0(t)(u_0, u_1)$ verifies the following $L^1 - L^1$ estimate*

$$\|E_0(t)(u_0, u_1)\|_{L^1} \leq C\,(1+t)^{\frac{n}{4}}\big(\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}}\|u_1\|_{L^1}\big), \tag{20}$$

for any $t \geq 0$, for some $C > 0$, independent of the data. In space dimension $n = 4$, estimate (20) remains valid with a log-loss, namely,

$$\|E_0(t)(u_0, u_1)\|_{L^1} \leq C\,(1+t)\big(\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}}\,\log(e+t)\|u_1\|_{L^1}\big). \tag{21}$$

In space dimension $n = 1, 2, 3$, estimate (20) remains valid if $u_1 = 0$.

If we consider estimates at the "energy level", i.e., at least one time derivative or two spatial derivatives, the restriction on the space dimension disappears, as it happens in Theorem 2 for $u_1 = 0$. More in general, we have the following.

Theorem 3. *Let $n \geq 1$, $\alpha \in \mathbb{N}^n$ and $j \in \mathbb{N}$. Assume that $u_0, u_1 \in L^1$, and that $|\alpha| \geq 2$ or $j \geq 1$. Then the solution to (1) at low frequencies $E_0(t)(u_0, u_1)$ verifies the estimate*

$$\|\partial_t^j \partial_x^\alpha E_0(t)(u_0, u_1)\|_{L^1} \leq C\,(1+t)^{\frac{n}{4}-\frac{j}{2}-\frac{|\alpha|}{4}}\big(\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}}\|u_1\|_{L^1}\big), \tag{22}$$

for any $t \geq 0$, for some $C > 0$, independent of the data.

Our estimates are supplemented by the following regularity result for the solution to (1) at high frequencies $E_\infty(t)(u_0, u_1)$.

Theorem 4. *Let $n \geq 1$, $\alpha \in \mathbb{N}^n$ and $j, k \in \mathbb{N}$ with $0 \leq k \leq j$. Then the solution to (1) at high frequencies $E_\infty(t)(u_0, u_1)$ verifies the estimate*

$$\|\partial_t^j \partial_x^\alpha E_\infty(t)(u_0, u_1)\|_{L_p} \leq C\,e^{-ct}\big(\|u_0\|_{W_p^{|\alpha|}} + \|u_1\|_{W_p^{(|\alpha|-3)_+}} + t^{-(j-k)}\|(u_0, u_1)\|_{W_p^{4k+(|\alpha|-3)_+}}\big), \tag{23}$$

for $p = 1, \infty$, for any $t > 0$, for some constant $C, c > 0$, independent of the data.

In Theorem 4 and in the following, we denote

$$W_p^m = \{u \in L^p :\ \partial_x^\alpha u \in L^p,\ |\alpha| \leq m\},$$

the usual Sobolev space on $\mathbb{R}^n$.

Remark 2.1. We notice that estimate (23) is singular at $t = 0$ if one takes $k < j$. However, taking $k < j$, it is possible to estimate the L^p norm of time derivatives of the solution to (1) for any $t > 0$, assuming less regular initial data. That is, after some time, the time derivatives of the solution gain additional regularity. This phenomenon of *smoothing effect* is related to the presence of the strong damping, but it also appears in presence of structural damping for evolution models, effective or not, see [4,5,7] and when a strong and a weak damping are considered at the same time [3].

We immediately obtain that the proof of Theorem 1 follows as a consequence of Theorems 2 and 4. Similarly, combining Theorem 3 and 4, one may derive estimates for the derivatives of the solution to (1). For instance, one has the following.

Corollary 2.1. *Let $n \geq 1$ and $\alpha \in \mathbb{N}^n$, $j \in \mathbb{N}$. Moreover, let $|\alpha| \geq 2$ if $j = 0$ and u_1 be not trivial. Assume that $u_0 \in W_1^{|\alpha|}$ and $u_1 \in W_1^{(|\alpha|-3)^+}$. Then the solution to (1) verifies the following $L^1 - L^1$ estimate*

$$\|\partial_t^j \partial_x^\alpha u(t,\cdot)\|_{L^1} \leq C(1+t)^{\frac{n}{4}-\frac{j}{2}-\frac{|\alpha|}{4}} \left(\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}}\|u_1\|_{L^1}\right) + Ce^{-ct} \left(\|u_0\|_{W_1^{|\alpha|}} + t^{-j}\,\|(u_0,u_1)\|_{W_1^{(|\alpha|-3)^+}}\right),$$

for any $t > 0$, for some $C, c > 0$, independent of the data.

Remark 2.2. Let us consider problem (13) for the strongly damped σ-evolution equation, with $\sigma \in \mathbb{N} \setminus \{0\}$, or even $\sigma \in (0,\infty)$. One may easily follow the proof of Theorem 2 and 3 and show that the solution at low frequencies verifies estimate (20) in space dimension $n > 2\sigma$, estimate (21) if $n = 2\sigma$, and estimate (22) in space dimension $n \geq 1$, provided that $j \geq 1$ or $|\alpha| \geq \sigma$. If the space dimension n is even, estimates (20) and (21) for $\sigma = 1$ have been obtained in [27] with a different technique, which cannot be used for $\sigma \neq 1$. In fact, the technique employed in [27] allowed to improve the estimates in odd space dimension $n \geq 3$, as a consequence of (strong) Huygen's principle, namely, the factor $(1+t)^{\frac{n}{4}}$ could be replaced by $(1+t)^{\frac{n-1}{4}}$. However, in general we cannot expect an improvement in odd space dimension, if $\sigma \neq 1$, due to the lack of the classical Huygens' principle.

2.1. Additional $L^p - L^q$ estimates for the solution

In this section, we collect some additional $L^p - L^q$ estimates for the solution to (1).

We may derive $L^p - L^p$ estimates for E_0, with $p \in (1,2)$, as a consequence of Theorem 2 and the following result, which easily follows from Plancherel's theorem.

Lemma 2.1. *Let $n \geq 1$. Assume that $u_0, u_1 \in L^2$. Then the solution to (1) at low frequencies $E_0(t)(u_0,u_1)$ verifies the following estimate*

$$\|E_0(t)(u_0,u_1)\|_{L^2} \leq C\left(\|u_0\|_{L^2} + (1+t)\|u_1\|_{L^2}\right), \tag{24}$$

for any $t \geq 0$, for some $C > 0$, independent of the data.

As a corollary of Theorem 2 and Lemma 2.1, we immediately obtain the following.

Corollary 2.2. *Let $n \geq 5$ and assume that $u_0, u_1 \in L^p$ for some $p \in (1,2)$. Then the solution to (1) at low frequencies $E_0(t)(u_0,u_1)$ verifies the following $L^p - L^p$ estimate*

$$\|E_0(t)(u_0,u_1)\|_{L^p} \leq C\,(1+t)^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\left(\|u_0\|_{L^p} + (1+t)^{1-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\|u_1\|_{L^p}\right), \tag{25}$$

for any $t \geq 0$, for some $C > 0$, independent of the data. In space dimension $n = 4$, estimate (25) remains valid with a log-loss. In space dimension $n = 1, 2, 3$, estimate (25) remains valid if $u_1 = 0$.

At low frequencies, it is possible to derive additional decay rate for $L^p - L^q$ estimates, where $1 \leq p < q \leq \infty$. For the sake of brevity we restrict to the easier case $1 \leq p \leq 2 \leq q \leq \infty$ (studied for (6) in [25]). Indeed, in this case, the estimate may be easily obtained by using the Hausdorff-Young inequality.

Lemma 2.2. *Let $n \geq 1$ and $1 \leq p \leq 2 \leq q \leq \infty$, with $p \neq q$, be such that $n(1/p - 1/q) > 2$. Assume that $u_0, u_1 \in L^p$. Then the solution to (1) at low frequencies $E_0(t)(u_0,u_1)$ verifies the following decay estimate*

$$\|E_0(t)(u_0,u_1)\|_{L^q} \leq C\,(1+t)^{-\frac{n}{4}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^p} + (1+t)^{\frac{1}{2}}\|u_1\|_{L^p}\right), \tag{26}$$

for any $t \geq 0$, for some $C > 0$, independent of the data. If $n(1/p - 1/q) = 2$, estimate (26) remains valid with a log-loss, namely,

$$\|E_0(t)(u_0, u_1)\|_{L^q} \leq C\,(1+t)^{-\frac{1}{2}}\,\|u_0\|_{L^p} + C\,\log(e+t)\,\|u_1\|_{L^p}. \tag{27}$$

If $n(1/p - 1/q) < 2$, the solution to (1) at low frequencies $E_0(t)(u_0, u_1)$ verifies the following estimate

$$\|E_0(t)(u_0, u_1)\|_{L^q} \leq C\,(1+t)^{-\frac{n}{4}\left(\frac{1}{p}-\frac{1}{q}\right)}\,\|u_0\|_{L^p} + C\,(1+t)^{1-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|u_1\|_{L^p}, \tag{28}$$

for any $t \geq 0$, for some $C > 0$, independent of the data.

In particular, as a consequence of Theorem 2 and Lemma 2.2 with $p = 1$, we obtain, by interpolation, the following corollary, for $q \in (1, 2)$.

Corollary 2.3. *Let $n \geq 5$ and $q \in (1, 2)$. Assume that $u_0, u_1 \in L^1$. Then the solution to (1) at low frequencies $E_0(t)(u_0, u_1)$ verifies the following estimate*

$$\|E_0(t)(u_0, u_1)\|_{L^q} \leq C\,(1+t)^{\frac{n}{4}\left(\frac{3}{q}-2\right)}\left(\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}}\|u_1\|_{L^1}\right), \tag{29}$$

for any $t \geq 0$, for some $C > 0$, independent of the data. If $n = 4$, estimate (29) remains valid with a log-loss. In space dimension $n = 1, 2, 3$, estimate (29) remains valid if $u_1 = 0$.

As we did for E_0 in Corollaries 2.2 and 2.3, we may derive $L^p - L^p$ and $L^1 - L^p$ estimates for the derivatives of E_0, with $p \in (1, 2)$, as a consequence of Theorem 3 and the following result, which easily follows from the Hausdorff-Young inequality.

Lemma 2.3. *Let $n \geq 1$, $\alpha \in \mathbb{N}^n$ and $j \in \mathbb{N}$. Let $1 \leq p \leq 2 \leq q \leq \infty$. Assume that $u_0, u_1 \in L^p$. Then the solution to (1) at low frequencies $E_0(t)(u_0, u_1)$ verifies the estimate*

$$\|\partial_t^j \partial_x^\alpha E_0(t)(u_0, u_1)\|_{L^q} \leq C\,(1+t)^{-\frac{n}{4}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j}{2}-\frac{|\alpha|}{4}}\left(\|u_0\|_{L^p} + (1+t)^{\frac{1}{2}}\|u_1\|_{L^p}\right), \tag{30}$$

for any $t \geq 0$, for some $C > 0$, independent of the data.

Finally, we may derive $L^p - L^p$ estimates at high frequencies for the solution to (1), with $p \in (1, \infty)$, by relying on the Mikhlin-Hörmander multiplier theorem.

Proposition 2.1. *Let $n \geq 1$, $\alpha \in \mathbb{N}^n$ and $j \in \mathbb{N}$. Let $N \in \mathbb{N}$ be such that $N \leq 4j$, and $p \in (1, \infty)$. Then the solution to (1) at high frequencies $E_\infty(t)(u_0, u_1)$ verifies the estimate*

$$\|\partial_t^j \partial_x^\alpha E_\infty(t)(u_0, u_1)\|_{L_p} \leq C\,e^{-ct}\left(\|u_0\|_{W_p^{|\alpha|}} + \|u_1\|_{W_p^{(|\alpha|-4)_+}} + t^{-N/4}\|(u_0, u_1)\|_{W_p^{(4j+|\alpha|-4-N)_+}}\right), \tag{31}$$

for any $t > 0$, for some constant $C, c > 0$, independent of the data.

3. Proof of Theorems 2 and 3 and other low frequencies estimates

Let

$$K(t, x) = \mathscr{F}^{-1}\left[\frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)}\varphi_0(\xi)\right](x). \tag{32}$$

Then it holds

$$E_0(t,\cdot)(u_0,u_1) = (\partial_t + \Delta^2)K(t,\cdot) * u_0 + K(t,\cdot) * u_1, \tag{33}$$

where $*$ denotes the convolution with respect to the space variable x. In view of (33), to prove our results it is sufficient to obtain estimates for $K(t,\cdot) * u_1$ and its derivatives.

First of all, we give a straight-forward regularity result.

Proposition 3.1. *Let $T \geq 1$. Then, for any $t \in [0,T]$ and $1 \leq p \leq q \leq \infty$, it holds*

$$\|\partial_t^\ell \partial_x^\alpha K(t,\cdot) * u_1\|_{L_q(\mathbb{R}^n)} \leq C(T)\|u_1\|_{L_p(\mathbb{R}^n)},$$

for some $C(T)$, independent of u_1.

Proof. To prove Proposition 3.1, it is sufficient to notice that $\hat{K} \in \mathcal{C}^\infty([0,T],\mathcal{S})$, so that $K \in \mathcal{C}^\infty([0,T],\mathcal{S})$ as well. Here $\mathcal{S}$ denotes the Schwartz space. Indeed, $\hat{K} \in \mathcal{C}^\infty([0,T],\mathcal{C}_c^\infty)$, and this concludes the proof.

In view of Proposition 3.1, we may assume $t \geq 1$ with no loss of generality, in this section.

For the sake of brevity, we use the following notation.

Notation 1. We write $f \lesssim g$ when there exists a constant $C > 0$ such that $f \leq Cg$, and $f \approx g$ when $g \lesssim f \lesssim g$.

3.1. Proofs of Lemmas 2.1, 2.2, 2.3

For the ease of reading, before proving Theorems 2 and 3, we will first give the easier proofs of Lemmas 2.1, 2.2, 2.3, which follows as a consequence of Plancherel's theorem and the Hausdorff-Young inequality. Indeed, for any $1 \leq p \leq 2 \leq q \leq \infty$, by Hölder inequality, it holds

$$\|\partial_t^\ell \partial_x^\alpha K(t,\cdot) * u_1\|_{L^q} \lesssim \|(i\xi)^\alpha \partial_t^\ell \hat{K}(t,\cdot)\hat{u}_1\|_{L^{q'}} \lesssim \|(i\xi)^\alpha \partial_t^\ell \hat{K}(t,\cdot)\|_{L^r} \|\hat{u}_1\|_{L^{p'}} \lesssim \|(i\xi)^\alpha \partial_t^\ell \hat{K}(t,\cdot)\|_{L^r} \|u_1\|_{L^p},$$

where $q' = q/(q-1)$ and r verifies $1/r = 1/q' - 1/p' = 1/p - 1/q$.

By formula (32) and (18), we get that

$$|(i\xi)^\alpha \partial_t^\ell \hat{K}(t,\xi)| \lesssim |\xi|^{|\alpha|+2\ell} e^{-|\xi|^4 t} \frac{\sin(|\xi|^2 t)}{|\xi|^2} \varphi_0(\xi).$$

We notice that we may estimate

$$\frac{\sin(|\xi|^2 t)}{|\xi|^2} \leq t^\theta |\xi|^{2\theta-2}, \tag{34}$$

for any $\theta \in [0,1]$, and

$$|\xi|^a e^{-|\xi|^4 t} \lesssim t^{-\frac{a}{4}},$$

for any $a > 0$. If $r = \infty$, that is, $p = q$, then

$$\sup_\xi |(i\xi)^\alpha \partial_t^\ell \hat{K}(t,\xi)| \lesssim \begin{cases} t^{-\frac{|\alpha|+2\ell-2}{4}} & \text{if } \ell \geq 1 \text{ or } |\alpha| \geq 2, \\ t^{1-\frac{|\alpha|}{2}} & \text{if } \ell = 0 \text{ and } |\alpha| = 0,1. \end{cases}$$

Namely, we set $\theta = 0$ in the first case, and $\theta = 1 - |\alpha|/2$ in the second one.

If $r \in [1, \infty)$, that is, $p < q$, then we may use the change of variable $\eta = t^{\frac{1}{4}}\xi$ to estimate

$$\|(i\cdot)^\alpha \partial_t^\ell \hat{K}(t,\cdot)\|_{L^r} \lesssim t^{-\frac{n}{4r}-\frac{|\alpha|+2\ell-2}{4}} \Big(\int_{\mathbb{R}^n} |\eta|^{r(|\alpha|+2\ell-2)}\, e^{-r|\eta|/2}\, d\eta\Big)^{\frac{1}{r}}.$$

The integral in the right-hand side is finite if $|\alpha| + 2\ell > 2 - n/r$. Assume that this latter does not hold, that is, $\ell = 0$ and $|\alpha| + n/r \leq 2$. We split the integral into two parts, $\{t|\xi|^2 \leq 1\}$ and $\{|\xi|^2 \leq 1 \leq t|\xi|^2\}$, and we employ (34) with $\theta = 1$ in the first one and with $\theta = 0$ in the second one, obtaining:

$$\|(i\cdot)^\alpha \hat{K}(t,\cdot)\|_{L^r} \lesssim t\Big(\int_{|\xi|\leq t^{-\frac{1}{2}}} |\xi|^{r|\alpha|} d\xi\Big)^{\frac{1}{r}} + \Big(\int_{t^{-\frac{1}{2}}\leq|\xi|\leq 1} |\xi|^{r(|\alpha|-2)} d\xi\Big)^{\frac{1}{r}} \lesssim \begin{cases} t^{1-\frac{n}{2r}-\frac{|\alpha|}{2}} & \text{if } |\alpha| + n/r < 2, \\ \log(e+t) & \text{if } |\alpha| + n/r = 2. \end{cases}$$

This concludes the proof of Lemmas 2.1, 2.2, 2.3.

3.2. Preliminary results

In the following, for any $p \in [1, 2]$ we denote by $M_p = M_p(\mathbb{R}^n)$ the set of Fourier multipliers on $L^p = L^p(\mathbb{R}^n)$. That is, the set of tempered distributions $\hat{a}$, such that

$$\|\hat{a}\|_{M_p} = \sup_{\|f\|_{L^p}=1} \|\mathscr{F}^{-1}(\hat{a}\hat{f})\|_{L^p} < \infty.$$

In particular, it holds $M_2 = L^\infty$ and, by Young inequality, $\mathscr{F}(L^1) \hookrightarrow M_p$, with

$$\|\hat{a}\|_{M_p} \leq \|\mathscr{F}^{-1}(\hat{a})\|_{L^1}.$$

On the other hand, by Riesz-Thorin interpolation theorem, we know that

$$\|\hat{a}\|_{M_p} \lesssim \|\hat{a}\|_{M_1}^{1-\theta} \|\hat{a}\|_{M_2}^{\theta}, \quad \theta = 2\left(1 - \frac{1}{p}\right),$$

for any $p \in (1, 2)$. We already know that $K(t,\cdot) \in \mathcal{S}$, so that $\hat{K}(t,\cdot) \in M_1$. To estimate $\|\hat{K}(t,\cdot)\|_{M_1}$ with respect to t, we will use the following.

Theorem 5 *(Bernstein, see [23,28]). Let $n \geq 1$ and $N > \frac{n}{2}$. Assume that $\hat{a} \in H^N$. Then $\mathscr{F}^{-1}(\hat{a}) \in L^1$ and there exists a constant $C > 0$, such that*

$$\|\mathscr{F}^{-1}(\hat{a})\|_{L^1} \leq C \|\hat{a}\|_{L_2}^{1-\frac{n}{2N}} \|D^N \hat{a}\|_{L^2}^{\frac{n}{2N}},$$

where

$$\|D^N \hat{a}\|_{L^2} = \sum_{|\beta|=N} \|\partial_\xi^\beta \hat{a}\|_{L^2}.$$

For the sake of completeness, we give the proof of the previous result.

Proof. Let $a = \mathscr{F}^{-1}\hat{a}$. If $a = 0$, the statement is trivial. Otherwise, let

$$r = \|\hat{a}\|_{L^2}^{-\frac{1}{N}} \|D^N \hat{a}\|_{L^2}^{\frac{1}{N}}.$$

Then, by Hölder inequality,

$$\|a\|_{L^1} = \int\limits_{|x|\le r} |a(x)|\,dx + \int\limits_{|x|\ge r} |x|^{-N}\,|x|^N\,|a(x)|\,dx \lesssim r^{n/2}\,\|a\|_{L^2} + r^{-N+n/2}\,\||\cdot|^N a\|_{L^2}$$
$$\approx r^{n/2}\,\|\hat{a}\|_{L^2} + r^{-N+n/2}\,\|D^N\hat{a}\|_{L^2},$$

so that the proof follows.

To apply Theorem 5, we need to estimate the derivatives of $\hat{K}(t,\xi)$ with respect to ξ, whereas in Section 3.1 no derivative with respect to ξ was involved. Therefore, we will investigate the asymptotic profile of $\hat{K}(t,\xi)$ as $\xi \to 0$.

First of all, we notice that the solutions to (16) verify

$$\Re\lambda_\pm = -|\xi|^4, \quad \Im\lambda_\pm = \pm|\xi|^2(1 + |\xi|^4\,g(|\xi|^4)),$$

where $g = O(1)$; explicitly,

$$g(s) = \frac{\sqrt{4-s}-2}{s} = -\frac{1}{2}\int_0^1 \frac{1}{\sqrt{4-\theta s}}\,d\theta.$$

We also put $f(\xi) = \sqrt{4-|\xi|^4}$. By Taylor's formula we have the following asymptotic expansion.

Lemma 3.1. *Let $K(t,x)$ be as in (32). Then it holds*

$$\hat{K}(t,\xi) = \sum_{j=0}^{Q} \hat{a}_j(\xi,t) + e^{-|\xi|^4 t/2} R_Q(\xi,t), \tag{35}$$

where

$$\hat{a}_j(\xi,t) = A_j(\xi)t^j[\partial_t^j e^{-|\xi|^4 t/2}]\left[\partial_t^j \frac{\sin|\xi|^2 t}{|\xi|^2}\right],$$
$$A_j(\xi) = \frac{1}{j!}[-g(|\xi|^4)]^j\,\frac{2\varphi_0(\xi)}{f(\xi)} \in C_c^\infty(\mathbb{R}^n),$$
$$R_Q(\xi,t) = \int_0^1 (1-\theta)^Q[e^{i|\xi|^2 t(2+\theta|\xi|^4 g(|\xi|^4))/2} - (-1)^{Q+1}e^{-i|\xi|^2 t(2+\theta|\xi|^4 g(|\xi|^4))/2}]d\theta$$
$$\times\frac{1}{Q!}\left[\frac{ig(|\xi|^4)}{2}\right]^{Q+1}|\xi|^{6Q+4}t^{Q+1}\frac{\varphi_0(\xi)}{if(\xi)}.$$

Proof. We may write

$$e^{\lambda_\pm(\xi)t} = e^{-|\xi|^4 t/2}e^{\pm i|\xi|^2 f(\xi)t/2} = e^{-|\xi|^4 t/2}e^{\pm i(2|\xi|^2+|\xi|^6 g(|\xi|^4))t/2}.$$

By Taylor expansion formula, we have

$$e^{\pm i|\xi|^6 g(|\xi|^4)t/2} = \sum_{j=0}^{Q}\frac{1}{j!}\left(\frac{\pm i|\xi|^6 g(|\xi|^4)t}{2}\right)^j + \frac{1}{Q!}\left(\frac{\pm i|\xi|^6 g(|\xi|^4)t}{2}\right)^{Q+1}\int_0^1 (1-\theta)^Q e^{\pm i\theta|\xi|^6 g(|\xi|^4)t/2}d\theta.$$

Due to

$$\big(e^{i|\xi|^2t}-(-1)^j e^{-i|\xi|^2t}\big)\Big(\frac{i|\xi|^6 g(|\xi|^4)t}{2}\Big)^j = 2i[\partial_t^j \sin(|\xi|^2 t)]\left(\frac{|\xi|^4 g(|\xi|^4)t}{2}\right)^j,$$

noticing that $\lambda_+(\xi)-\lambda_-(\xi)=i|\xi|^2 f(\xi)$, we conclude the proof of (35).

Taking advantage of the expansion in Lemma 3.1, it is sufficient to estimate derivatives with respect to ξ of the two terms $e^{-t|\xi|^4}$ and $|\xi|^{-2}\sin(t|\xi|^2)$. This latter term creates the oscillations which influence the L^1 estimate of $K(t,\cdot)$.

For the ease of reading, we will prepare a series of straight-forward estimates for the derivatives of $|\xi|^{-2}\sin(t|\xi|^2)$.

Lemma 3.2. *It holds*

$$|\xi|^{-2}|\sin(t|\xi|^2)| \lesssim t(1+t|\xi|^2)^{-1}. \tag{36}$$

The proof is trivial, but we present it to clarify the scheme used in the following results.

Proof. Having in mind (34), it is clear that estimate (36) follows by the estimate $|\sin\rho|\le C|\rho|(1+|\rho|)^{-1}$, that is, $|\sin\rho|\le|\rho|$ is used for small values of $|\rho|$ and $|\sin\rho|\le 1$ for large ones.

Lemma 3.3. *For any $|\beta|=N\ge 1$, it holds*

$$|\partial_\xi^\beta \cos(t|\xi|^2)| \lesssim t^2|\xi|^4(1+t|\xi|^2)^{N-2}\,|\xi|^{-N}. \tag{37}$$

Proof. First we prove (37) for $N=1$. Due to

$$\partial_{\xi_j}\cos(t|\xi|^2)=2t\xi_j\,\sin(t|\xi|^2),$$

then estimate (37) follows by the estimate $|\sin\rho|\le C|\rho|(1+|\rho|)^{-1}$, that is, $|\sin\rho|\le|\rho|$ is used for small values of $|\rho|$ and $|\sin\rho|\le 1$ for large ones. Now let $N\ge 2$. Setting $\ell=\lceil N/2\rceil$, the smallest integer number which verifies $\ell\ge N/2$, we may estimate

$$|\partial_\xi^\beta\cos(t|\xi|^2)| \le \sum_{k\ge\ell} t^k\,|\xi|^{2k-N}\,|\cos^{(k)}(t|\xi|^2)| \le Ct^2|\xi|^{4-N}\sum_{k=0}^{N-2} t^k|\xi|^{2k} \le C_1\,t^2|\xi|^{4-N}\,(1+t|\xi|^2)^{N-2},$$

where it has been sufficient to use $\ell\ge 1$, together with $|\cos^{(k)}\rho|\le C|\rho|(1+|\rho|)^{-1}$ only for $k=1$ and $|\cos^{(k)}\rho|\le 1$ for $k\ge 2$ (independently if k is even or odd). This concludes the proof of (37).

Lemma 3.4. *For any $\beta\ne 0$, it holds*

$$|\partial_\xi^\beta|\xi|^{-2}\sin(t|\xi|^2)| \lesssim t^3|\xi|^4(1+t|\xi|^2)^{N-3}\,|\xi|^{-N}, \quad |\beta|=N\ge 1. \tag{38}$$

Proof. The proof follows from straightforward calculations. In particular, we notice that

$$\partial_\xi^\beta|\xi|^{-2}\sin(t|\xi|^2) = t\,\partial_\xi^\beta\,\mathrm{sinc}\,(t|\xi|^2),$$

where $\mathrm{sinc}\,\rho=\rho^{-1}\sin\rho$, and that $|\mathrm{sinc}^{(k)}(\rho)|\le C\min\{|\rho|,|\rho|^{-1}\}$ if k is odd, and $|\mathrm{sinc}^{(k)}\rho|\le C\min\{1,|\rho|^{-1}\}$ if k is even. In particular,

$$
\begin{aligned}
|\operatorname{sinc}'(\rho)| &\lesssim |\rho|(1+|\rho|)^{-2},\\
|\operatorname{sinc}''(\rho)| &\lesssim (1+|\rho|)^{-1}
\end{aligned}
$$

whereas it is sufficient to use $|\operatorname{sinc}^{(k)}(\rho)| \leq |\rho|^{-1}$, for any $k \geq 3$. If $N = 1, 2$, we immediately derive (38) from the previous estimates, and

$$
\begin{aligned}
\partial_{\xi_j} \operatorname{sinc}(t|\xi|^2) &= 2t\xi_j \operatorname{sinc}'(t|\xi|^2),\\
\partial_{\xi_j}\partial_{\xi_k} \operatorname{sinc}(t|\xi|^2) &= 4t^2\xi_j\xi_k \operatorname{sinc}''(t|\xi|^2) + \delta_{jk} 2t \operatorname{sinc}(t|\xi|^2).
\end{aligned}
$$

Now let $N \geq 3$. Setting $\ell = \lceil N/2 \rceil$, the smallest integer number which verifies $\ell \geq N/2$, we get

$$
|\partial_\xi^\beta \operatorname{sinc}(t|\xi|^2)| \leq \sum_{k\geq \ell} t^k\,|\xi|^{2k-N}\,|\operatorname{sinc}^{(k)}(t|\xi|^2)| \leq Ct^2|\xi|^{4-N}\sum_{k=0}^{N-3} t^k|\xi|^{2k} \leq C_1\,t^2|\xi|^{4-N}(1+t|\xi|^2)^{N-3},
$$

where it has been sufficient to use $\ell \geq 1$, together with the previous estimates for the derivatives of sinc.

Having in mind that $\partial_t|\xi|^{-2}\sin(t|\xi|^2) = \cos(t|\xi|^2)$, we need only to estimate $\partial_\xi^\beta\partial_t^\ell\cos(t|\xi|^2)$ for $|\beta| \geq 1$ and $\ell \geq 1$.

Lemma 3.5. *For any $\ell \geq 1$ and $\beta \neq 0$, it holds:*

$$
|\partial_\xi^\beta\partial_t^\ell\cos(t|\xi|^2)| \lesssim |\xi|^{2\ell}(1+t|\xi|^2)^N\,|\xi|^{-N}, \quad |\beta| = N \geq 1. \tag{39}
$$

Moreover, if ℓ is odd, it also holds

$$
|\partial_\xi^\beta\partial_t^\ell\cos(t|\xi|^2)| \lesssim t|\xi|^{2+2\ell}\,(1+t|\xi|^2)^{N-1}\,|\xi|^{-N}, \quad |\beta| = N \geq 1. \tag{40}
$$

Proof. To prove (39) and (40), we first notice that

$$
\partial_t^\ell\cos(t|\xi|^2) = \begin{cases} (-1)^{\frac{\ell+1}{2}}\,t|\xi|^{2\ell+2}\operatorname{sinc}(t|\xi|^2) & \text{if } \ell \text{ is odd},\\ (-1)^{\frac{\ell}{2}}|\xi|^{2\ell}\cos(t|\xi|^2) & \text{if } \ell \text{ is even}, \end{cases}
$$

and then we apply (36), (37) and (38).

Lemma 3.6. *For any $\ell \geq 1$ and $\beta \neq 0$, it holds:*

$$
|\partial_\xi^\beta\partial_t^\ell e^{-t|\xi|^4/2}| \lesssim |\xi|^{4\ell}(1+t|\xi|^4)^N\,|\xi|^{-N}\,e^{-t|\xi|^4/2}, \quad |\beta| = N \geq 1. \tag{41}
$$

Moreover, if $\ell = 0$, then it holds

$$
|\partial_\xi^\beta e^{-t|\xi|^4/2}| \lesssim t|\xi|^4(1+t|\xi|^4)^{N-1}\,|\xi|^{-N}\,e^{-t|\xi|^4/2}, \quad |\beta| = N \geq 1. \tag{42}
$$

We first apply Theorem 5 to $\hat{a}_j$, with $j = 1, \ldots, Q$.

Lemma 3.7. *Let $n \geq 1$ and $j \geq 1$. Then*

$$
\|(i\cdot)^\alpha\,\partial_t^\ell\hat{a}_j(t,\cdot)\|_{M_1} \lesssim t^{\frac{n}{4}-\frac{|\alpha|}{4}-\frac{\ell}{2}}, \tag{43}
$$

uniformly with respect to $j \geq 1$, for any $t \geq 1$.

Proof. Due to $A_j \in \mathcal{C}_c^\infty \subset M_1$ with $\|A_j\|_{M_1}$ independent of t, it is sufficient to estimate

$$\partial_\xi^\beta((i\xi)^\alpha \partial_t^\ell(t^j(\partial_t^j e^{-t|\xi|^4/2})(\partial_t^{j-1}\cos(t|\xi|^2)))).$$

Thanks to (37) and (39), we may estimate

$$|\partial_\xi^\beta \partial_t^{j+k-1}\cos(t|\xi|^2)| \lesssim |\xi|^{2(j+k-1)-|\beta|}\,(1+t|\xi|^2)^{|\beta|},$$

for any $j \geq 1$, $k \geq 0$ and $|\beta| \geq 0$. Therefore, for $|\beta| = N$, recalling that $|\xi|^4 \leq |\xi|^2$ in $\operatorname{supp}\varphi_0$ (due to $|\xi| \leq 1$), and using (41), we may estimate

$$\begin{aligned}|\partial_\xi^\beta((i\xi)^\alpha \partial_t^\ell(t^j\,(\partial_t^j e^{-t|\xi|^4/2})(\partial_t^{j-1}\cos(t|\xi|^2))))| &\lesssim t^j\,|\xi|^{6j+|\alpha|-2-N}\,(t^{-1}+|\xi|^2)^\ell\,(1+t|\xi|^2)^N\,e^{-t|\xi|^4/2},\\ &= t^{j-\ell}\,|\xi|^{6j+|\alpha|-2-N}\,(1+t|\xi|^2)^{N+\ell}\,e^{-t|\xi|^4/2},\\ &\lesssim t^{1-\ell}\,|\xi|^{4+|\alpha|-N}\,(1+t|\xi|^2)^{N+\ell}\,e^{-t|\xi|^4/3},\end{aligned}$$

uniformly with respect to $j \geq 1$, where we used $(t|\xi|^6)^{j-1}e^{-t|\xi|^4/6} \lesssim |\xi|^{2(j-1)} \leq 1$. We remark that the term

$$(t^{-1}+|\xi|^2)^\ell = t^{-\ell}(1+t|\xi|^2)^\ell$$

in the previous estimate takes into account of the possibility that each time derivative of ∂_t^ℓ is applied, in Leibniz's rule, either to t^j or to $(\partial_t^j e^{-t|\xi|^4})(\partial_t^{j-1}\cos(t|\xi|^2))$.

Setting first $N = 0$, and then $N = (n+1)/2$ if n is odd and $N = (n+2)/2$ if n is even, and using the change of variables $\eta = t^{\frac14}\xi$ as in Section 3.1, we derive

$$\begin{aligned}&\|\partial_\xi^\beta((i\xi)^\alpha \partial_t^\ell(t^j\,(\partial_t^j e^{-t|\xi|^4/2})(\partial_t^{j-1}\cos(t|\xi|^2))))\|_{L^2}\\ &\lesssim t^{1-\ell}\Big(\int_{\mathbb{R}^n}|\xi|^{2(4+|\alpha|-N)}\,(1+t|\xi|^2)^{2(N+\ell)}\,e^{-2t|\xi|^4/3}\,d\xi\Big)^{\frac12}\\ &= t^{-\frac{n}{8}+\frac{N}{4}-\frac{|\alpha|}{4}-\ell}\Big(\int_{\mathbb{R}^n}|\eta|^{2(4+|\alpha|-N)}\,(1+\sqrt{t}|\eta|^2)^{2(N+\ell)}\,e^{-2|\eta|^4/3}\,d\eta\Big)^{\frac12}\\ &\lesssim t^{-\frac{n}{8}+\frac{3N}{4}-\frac{|\alpha|}{4}-\frac{\ell}{2}}\Big(\int_{\mathbb{R}^n}|\eta|^{2(4+|\alpha|-N)}\,(1+|\eta|^2)^{2(N+\ell)}\,e^{-2|\eta|^4/3}\,d\eta\Big)^{\frac12}\\ &\approx t^{-\frac{n}{8}+\frac{3N}{4}-\frac{|\alpha|}{4}-\frac{\ell}{2}}.\end{aligned}$$

In the last estimate we used $2(4+|\alpha|-N) \geq 8-2N > -n$ to have a convergent integral.

Applying Theorem 5, we derive (43), uniformly, with respect to $j \geq 1$.

In a completely similar way, we may estimate the Fourier multiplier $\hat{b}_{Q+1}(t,\xi) = e^{-|\xi|^4 t/2}R_Q(\xi,t)$.

Lemma 3.8. *Let $n \geq 1$ and $Q \geq 0$. Then*

$$\|(i\cdot)^\alpha\,\partial_t^\ell e^{-|\cdot|^4 t/2}R_Q(t,\cdot)\|_{M_1} \lesssim t^{\frac{n}{4}-\frac{|\alpha|}{4}-\frac{\ell}{2}}, \tag{44}$$

uniformly with respect to $Q \geq 0$, for any $t \geq 1$.

To apply Theorem 5 to $\hat{a}_0$, we shall refine the proof of Lemma 3.7, since this multiplier is more singular at $\xi = 0$ than $\hat{a}_j$ for $j \geq 1$. Some differences appear in the proof of Theorems 2 and 3, due to the difficulties to deal with the singularity of $\hat{a}_0$.

3.3. Proof of Theorem 3

We will first prove Theorem 3.

Proof of Theorem 3. In view of Proposition 3.1, Theorem 5 and Lemmas 3.1, 3.7 and 3.8, the proof of Theorem 3 follows if we prove that

$$\|(i\cdot)^\alpha\,\partial_t^\ell \hat{a}_0(t,\cdot)\|_{M_1} \lesssim t^{\frac{n}{4}+\frac{1}{2}-\frac{|\alpha|}{4}-\frac{\ell}{2}}, \tag{45}$$

provided that $\ell \geq 1$ or $|\alpha| \geq 2$, for $t \geq 1$.

Due to $A_0 \in \mathcal{C}_c^\infty \subset M_1$ with $\|A_0\|_{M_1}$ independent of t, it is sufficient to estimate

$$\partial_\xi^\beta((i\xi)^\alpha\,\partial_t^\ell(e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2))).$$

First, we set $\beta = 0$. Then we may easily estimate

$$|(i\xi)^\alpha\partial_t^\ell(e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2))| \lesssim |\xi|^{2\ell+|\alpha|-2}\,e^{-t|\xi|^4/2}$$

so that, by the change of variable $\eta = t^{\frac{1}{4}}\xi$, we derive

$$\begin{aligned}\|(i\xi)^\alpha\partial_t^\ell(e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2))\|_{L^2} &\lesssim \Big(\int_{\mathbb{R}^n}|\xi|^{2(2\ell+|\alpha|-2)}\,e^{-t|\xi|^4}\,d\xi\Big)^{\frac{1}{2}}\\ &= t^{-\frac{n}{8}+\frac{1}{2}-\frac{|\alpha|}{4}-\frac{\ell}{2}}\Big(\int_{\mathbb{R}^n}|\eta|^{2(2\ell+|\alpha|-2)}\,e^{-|\eta|^4}\,d\eta\Big)^{\frac{1}{2}}\\ &\approx t^{-\frac{n}{8}+\frac{1}{2}-\frac{|\alpha|}{4}-\frac{\ell}{2}}.\end{aligned}$$

Indeed, the integral is convergent as a consequence of $2\ell+|\alpha| \geq 2$. Now, let $|\beta| = N$, with $N = (n+1)/2$ if n is odd and $N = (n+2)/2$ if n is even.

Assume first that $\ell = 0$. For $|\beta| = N$, recalling that $|\xi|^4 \leq |\xi|^2$ in $\operatorname{supp}\varphi_0$ (due to $|\xi| \leq 1$), by (36) and (38), we may estimate

$$|\partial_\xi^\beta((i\xi)^\alpha e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2))| \lesssim t|\xi|^{|\alpha|-N}(1+t|\xi|^2)^{N-1}\,e^{-t|\xi|^4/2},$$

so that, by the change of variable $\eta = t^{\frac{1}{4}}\xi$, we derive

$$\begin{aligned}&\|\partial_\xi^\beta((i\xi)^\alpha e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2))\|_{L^2}\\ &\quad\lesssim t\Big(\int_{\mathbb{R}^n}|\xi|^{2(|\alpha|-N)}\,(1+t|\xi|^2)^{2(N-1)}\,e^{-t|\xi|^4}\,d\xi\Big)^{\frac{1}{2}}\\ &\quad= t^{1-\frac{n}{8}+\frac{N}{4}-\frac{|\alpha|}{4}}\Big(\int_{\mathbb{R}^n}|\eta|^{2(|\alpha|-N)}\,(1+\sqrt{t}|\eta|^2)^{2(N-1)}\,e^{-|\eta|^4}\,d\eta\Big)^{\frac{1}{2}}\\ &\quad\approx t^{\frac{1}{2}-\frac{n}{8}+\frac{3N}{4}-\frac{|\alpha|}{4}}.\end{aligned}$$

Indeed, the integral is convergent as a consequence of $2|\alpha|-2N \geq 4-2N > -n$. Now, let $\ell \geq 1$. Noticing that

$$\partial_t^\ell(e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2)) = \frac{\sin(t|\xi|^2)}{|\xi|^2}\partial_t^\ell e^{-t|\xi|^4/2} + \sum_{k=1}^{\ell}\binom{\ell}{k}(\partial_t^{\ell-k}e^{-t|\xi|^4/2})(\partial_t^{k-1}\cos(t|\xi|^2)),$$

we may apply (36) and (38) to estimate the first term, and we may apply (39) and (41) to estimate the second one. Therefore, we get

$$\begin{aligned}
&|\partial_\xi^\beta((i\xi)^\alpha\partial_t^\ell(e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2)))| \\
&\quad\lesssim \big(t\,|\xi|^{4\ell+|\alpha|-N}(1+t|\xi|^2)^{N-1} + |\xi|^{2(\ell-1)+|\alpha|-N}(1+t|\xi|^2)^N\big)\,e^{-t|\xi|^4/2} \\
&\quad\lesssim |\xi|^{2\ell+|\alpha|-2-N}(1+t|\xi|^2)^N\,e^{-t|\xi|^4/2},
\end{aligned} \tag{46}$$

so that, by the change of variable $\eta = t^{\frac14}\xi$, we derive

$$\begin{aligned}
\|\partial_\xi^\beta\partial_t^\ell((i\xi)^\alpha e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2))\|_{L^2} &\lesssim \Big(\int_{\mathbb{R}^n}|\xi|^{2(2\ell+|\alpha|-2-N)}\,(1+t|\xi|^2)^{2N}\,e^{-t|\xi|^4}\,d\xi\Big)^{\frac12} \\
&= t^{\frac12-\frac{n}{8}+\frac{N}{4}-\frac{\ell}{2}-\frac{|\alpha|}{4}}\Big(\int_{\mathbb{R}^n}|\eta|^{2(2\ell+|\alpha|-2-N)}\,(1+\sqrt{t}|\eta|^2)^{2N}\,e^{-|\eta|^4}\,d\eta\Big)^{\frac12} \\
&\approx t^{\frac12-\frac{n}{8}+\frac{3N}{4}-\frac{\ell}{2}-\frac{|\alpha|}{4}},
\end{aligned}$$

provided that the latter integral is convergent, that is,

$$2(2\ell+|\alpha|-2-N) > -n.$$

The previous inequality fails if, and only if, $\ell = 1$, and either $\alpha = 0$, or $|\alpha| = 1$ with n even. In these cases, we shall refine our approach. We notice that, as a consequence of $\alpha = 0$ if $n = 1$, and $|\alpha| \le 1$ otherwise, it holds $|\beta| = N \ge |\alpha| + 1$. That is, when we consider

$$\begin{aligned}
&\partial_\xi^\beta\partial_t((i\xi)^\alpha\,e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2)) \\
&\quad= \partial_\xi^\beta\Big((i\xi)^\alpha\,\frac{\sin(t|\xi|^2)}{|\xi|^2}\partial_t e^{-t|\xi|^4/2} + (i\xi)^\alpha\,e^{-t|\xi|^4/2}\cos(t|\xi|^2)\Big) \\
&\quad= -\frac12\partial_\xi^\beta\Big((i\xi)^\alpha\,|\xi|^4\,\frac{\sin(t|\xi|^2)}{|\xi|^2}e^{-t|\xi|^4/2}\Big) + \partial_\xi^\beta\big((i\xi)^\alpha\,e^{-t|\xi|^4/2}\cos(t|\xi|^2)\big)
\end{aligned}$$

at least one derivative of the derivation ∂_ξ^β is not applied to $(i\xi)^\alpha$ in Leibniz's rule for product derivation of the second term in the sum, as a consequence of $|\beta| \ge |\alpha| + 1$. Namely,

$$\partial_\xi^\beta\big((i\xi)^\alpha\,e^{-t|\xi|^4/2}\cos(t|\xi|^2)\big) = \sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_2\neq 0}}\binom{\beta}{\beta_2}(\partial_\xi^{\beta_1}(i\xi)^\alpha)\,\partial_\xi^{\beta_2}(e^{-t|\xi|^4/2}\,\cos(t|\xi|^2)),$$

since we may assume $|\beta_1| \le |\alpha|$. Due to the fact that we may apply (42) and (37) to $\partial_\xi^{\beta_2}(e^{-t|\xi|^4/2}\,\cos(t|\xi|^2))$, we may refine 46 with the following:

$$\begin{aligned}
&|\partial_\xi^\beta((i\xi)^\alpha\partial_t(e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2)))| \\
&\quad\lesssim \big(t\,|\xi|^{4+|\alpha|-N}(1+t|\xi|^2)^{N-1} + t|\xi|^{2+|\alpha|-N}(1+t|\xi|^2)^{N-1}\big)\,e^{-t|\xi|^4/2} \\
&\quad\lesssim t|\xi|^{2+|\alpha|-N}(1+t|\xi|^2)^{N-1}\,e^{-t|\xi|^4/2}.
\end{aligned} \tag{47}$$

Comparing the estimate in (46) for $\ell = 1$, with the estimate in (47), we see that $(1+t|\xi|^2)^N$ has been replaced by $t|\xi|^2(1+t|\xi|^2)^{N-1}$, and this additional power $|\xi|^2$ allow us to weaken enough the singularity of the multiplier at $\xi = 0$. Indeed, after the change of variable $\eta = t^{\frac{1}{4}}\xi$, now we get

$$\begin{aligned}\|\partial_\xi^\beta \partial_t((i\xi)^\alpha e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2))\|_{L^2} &\lesssim t\Big(\int_{\mathbb{R}^n} |\xi|^{2(2+|\alpha|-N)}\,(1+t|\xi|^2)^{2(N-1)}\,e^{-t|\xi|^4}\,d\xi\Big)^{\frac{1}{2}}\\ &= t^{\frac{1}{2}-\frac{n}{8}+\frac{N}{4}-\frac{|\alpha|}{4}}\Big(\int_{\mathbb{R}^n} |\eta|^{2(2+|\alpha|-N)}\,(1+\sqrt{t}|\eta|^2)^{2(N-1)}\,e^{-|\eta|^4}\,d\eta\Big)^{\frac{1}{2}}\\ &\approx t^{-\frac{n}{8}+\frac{3N}{4}-\frac{|\alpha|}{4}},\end{aligned}$$

since integral is convergent, due to

$$2(2+|\alpha|-N) > -n.$$

We are now ready to apply Theorem 5. Thanks to

$$\begin{aligned}\|(i\cdot)^\alpha\,\partial_t^\ell((i\xi)^\alpha e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2))\|_{L^2} &\lesssim t^{-\frac{n}{8}+\frac{1}{2}-\frac{|\alpha|}{4}-\frac{\ell}{2}},\\ \|\partial_\xi^\beta \partial_t^\ell((i\xi)^\alpha e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2))\|_{L^2} &\lesssim t^{-\frac{n}{8}+\frac{1}{2}+\frac{3N}{4}-\frac{|\alpha|}{4}-\frac{\ell}{2}}, \quad |\beta| = N,\end{aligned}$$

we obtain (45). This concludes the proof of Theorem 3.

3.4. Proof of Theorem 2

We will now prove Theorem 2.

Proof of Theorem 2. By virtue of (33), the proof of Theorem 2 with $u_1 = 0$ is a consequence of Theorem 3 with $(\ell, |\alpha|) = (1,0), (0,2)$. Therefore, it remains to prove the statement for $u_0 = 0$.

In view of Proposition 3.1, Theorem 5 and Lemmas 3.1, 3.7 and 3.8, the proof of Theorem 2 follows if we prove that (45) holds, that is,

$$\|\hat{a}_0(t,\cdot)\|_{M_1} \lesssim t^{\frac{n+2}{4}}, \tag{48}$$

if $n \geq 5$, and

$$\|\hat{a}_0(t,\cdot)\|_{M_1} \lesssim t^{\frac{3}{2}}\log(e+t), \tag{49}$$

if $n = 4$, for $t \geq 1$. Due to $A_0 \in \mathcal{C}_c^\infty \subset M_1$ with $\|A_0\|_{M_1}$ independent of t, it is sufficient to estimate

$$\partial_\xi^\beta(e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2)).$$

First, let $\beta = 0$. If $n \geq 5$, by the change of variable $\eta = t^{\frac{1}{4}}\xi$, we derive

$$\begin{aligned}\|e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2)\|_{L^2} &\lesssim \Big(\int_{\mathbb{R}^n} |\xi|^{-4}\,e^{-t|\xi|^4}\,d\xi\Big)^{\frac{1}{2}}\\ &= t^{-\frac{n}{8}+\frac{1}{2}}\Big(\int_{\mathbb{R}^n} |\eta|^{-4}\,e^{-|\eta|^4}\,d\eta\Big)^{\frac{1}{2}} \approx t^{-\frac{n}{8}+\frac{1}{2}}.\end{aligned}$$

If $n = 4$, we split the integral into two parts, $\{t|\xi|^2 \le 1\}$ and $\{|\xi|^2 \le 1 \le t|\xi|^2\}$, and we employ (34) with $\theta = 1$ in the first one and with $\theta = 0$ in the second one, obtaining:

$$\|e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2)\|_{L^2} \lesssim t\Big(\int\limits_{|\xi|\le t^{-\frac{1}{2}}} 1\,d\xi\Big)^{\frac{1}{2}} + \Big(\int\limits_{t^{-\frac{1}{2}}\le|\xi|\le 1} |\xi|^{-4}d\xi\Big)^{\frac{1}{2}} \lesssim \log(e+t).$$

Now let $|\beta| = N$ with $N = (n+1)/2$ if n is odd and $N = (n+2)/2$ if n is even. Noticing that

$$\begin{aligned}\partial_\xi^\beta(e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2)) &= \sum_{\beta_1+\beta_2=\beta}\binom{\beta}{\beta_1}(\partial_\xi^{\beta_1}e^{-t|\xi|^4/2})\,(\partial_\xi^{\beta_2}|\xi|^{-2}\sin(t|\xi|^2))\\ &= (\partial_\xi^\beta e^{-t|\xi|^4/2})\,|\xi|^{-2}\sin(t|\xi|^2) + \sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_2\neq 0}}\binom{\beta}{\beta_1}(\partial_\xi^{\beta_1}e^{-t|\xi|^4/2})\,(\partial_\xi^{\beta_2}|\xi|^{-2}\sin(t|\xi|^2)),\end{aligned}$$

we may employ (42) and (36) to estimate the first term, and (42) and (38) to estimate the other ones. We derive:

$$\begin{aligned}|\partial_\xi^\beta(e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2))| &\lesssim t^2|\xi|^{4-N}(1+t|\xi|^4)^{N-1}(1+t|\xi|^2)^{-1} + t^3|\xi|^{4-N}(1+t|\xi|^2)^{N-3}\\ &\lesssim t^2|\xi|^{2-N}(1+t|\xi|^2)^{N-2},\end{aligned}$$

where we used that $|\xi|^4 \le |\xi|^2$ in $\operatorname{supp}\varphi_0$ (due to $|\xi| \le 1$). Therefore (after the change of variable $\eta = t^{\frac{1}{4}}\xi$), we obtain

$$\|\partial_\xi^\beta(e^{-t|\xi|^4/2}|\xi|^{-2}\sin(t|\xi|^2))\|_{L^2} \lesssim t^{\frac{3}{2}-\frac{n}{8}+\frac{N}{4}}\Big(\int\limits_{\mathbb{R}^n}|\eta|^{2(2-N)}\,(1+\sqrt{t}|\eta|^2)^{2(N-2)}\,e^{-|\eta|^4}\,d\eta\Big)^{\frac{1}{2}} \approx t^{-\frac{n}{8}+\frac{3N}{4}+\frac{1}{2}}.$$

The integral is convergent, due to $4 - 2N > -n$. Applying Theorem 5, we obtain (45) if $n \ge 5$, and (49) if $n = 4$. This concludes the proof of Theorem 2.

4. Proof of Theorem 4 and Proposition 2.1

In this section, we follow the proof given in [27] with minor modifications, to prove Theorem 4.

Recalling that the roots $\lambda_\pm$ to (16) are given by (17), we define

$$\begin{aligned}(L_\pm v)(x) &= \mathscr{F}^{-1}\left[\frac{e^{\lambda_\pm(\xi)t}\lambda_\mp(\xi)}{\lambda_+(\xi)-\lambda_-(\xi)}\varphi_\infty(\xi)\hat{v}(\xi)\right](x),\\ (M_\pm v)(x) &= \mathscr{F}^{-1}\left[\frac{e^{\lambda_\pm(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}\varphi_\infty(\xi)\hat{v}(\xi)\right](x),\end{aligned}$$

so that

$$E_\infty(t,\cdot)(u_0,u_1) = (L_+ + L_-)u_0 + (M_+ + M_-)u_1. \tag{50}$$

In the following we will assume $|\xi| \ge 3$ and we will study the behavior of $\lambda_\pm$ and their derivatives, as $|\xi| \to \infty$.

By using Taylor's formula in (17), we get

$$\lambda_+(\xi) = -|\xi|^4 + 1 + \mu(\xi), \quad \lambda_-(\xi) = -1 - \mu(\xi),$$

where

$$\mu(\xi)=\frac{2}{|\xi|^4}g\left(\frac{4}{|\xi|^4}\right),\quad g(s)=\int_0^1(1-\theta s)^{-3/2}(1-\theta)d\theta. \tag{51}$$

In particular,

$$|\partial_\xi^\beta\lambda_+(\xi)|\lesssim|\xi|^{4-|\beta|},\qquad |\partial_\xi^\beta\lambda_-(\xi)|\lesssim|\xi|^{-|\beta|},\qquad |\lambda_+(\xi)-\lambda_-(\xi)|\approx|\xi|^4. \tag{52}$$

We are now ready to prove Theorem 4.

Proof of Theorem 4. We will prove the following estimates:

$$\|\partial_t^j\partial_x^\alpha L_+(t)v\|_{L^p(\mathbb{R}^n)}\lesssim e^{-t}t^{-(j-k)}\|v\|_{W_p^{4k+(|\alpha|-3)_+}}, \tag{53}$$

$$\|\partial_t^j\partial_x^\alpha M_+(t)v\|_{L^p(\mathbb{R}^n)}\lesssim e^{-t}t^{-(j-k)}\|v\|_{W_p^{4k+(|\alpha|-3)_+}}, \tag{54}$$

$$\|\partial_t^j\partial_x^\alpha (L_-(t)v-e^{-t}v)\|_{L^p(\mathbb{R}^n)}\lesssim e^{-t/4}\|v\|_{W_p^{(|\alpha|-3)_+}}, \tag{55}$$

$$\|\partial_t^j\partial_x^\alpha M_-(t)v\|_{L^p(\mathbb{R}^n)}\lesssim e^{-t/4}\|v\|_{W_p^{(|\alpha|-3)_+}}, \tag{56}$$

for $p=1,\infty$ and $k\in\mathbb{N}$ with $k\le j$. The proof of Theorem 4 will immediately follow by combining the previous estimates, taking into account that we may estimate

$$\|\partial_t^j\partial_x^\alpha(e^{-t}v)\|_{L^p(\mathbb{R}^n)}\lesssim e^{-t}\|v\|_{W_p^{|\alpha|}},$$

in (55). We fix $j\ge k\ge 0$ and $\alpha_1\le\alpha$, then we consider the multiplier

$$\hat{a}_\ell(t,\xi)=\frac{e^{\lambda_+(\xi)t}\lambda_-(\xi)^\ell\lambda_+(\xi)^j(i\xi)^{\alpha_1}}{(\lambda_+(\xi)-\lambda_-(\xi))(1+|\xi|^2)^{2k}}\,\varphi_\infty(\xi),$$

with $\ell=0,1$, so that

$$\partial_t^j\partial_x^\alpha L_+(t)v=a_0(t,\cdot)*\partial_x^{\alpha-\alpha_1}(1-\Delta)^{2k}v, \tag{57}$$

$$\partial_t^j\partial_x^\alpha M_+(t)v=a_1(t,\cdot)*\partial_x^{\alpha-\alpha_1}(1-\Delta)^{2k}v, \tag{58}$$

where $a_\ell=\mathscr{F}^{-1}(\hat{a}_\ell)$. We choose $\alpha_1=\alpha$, if $|\alpha|\le 3$, whereas we choose some $\alpha_1\le\alpha$ with $|\alpha_1|=3$, if $|\alpha|\ge 4$. We now want to prove that

$$\|a_\ell(t,\cdot)\|_{L^1}\le e^{-t/2}t^{-(j-k)},\qquad \ell=0,1.$$

First, assume that $|x|<1/3$. By using the identity

$$\sum_{j=1}^n\frac{x_j}{i|x|^2}\frac{\partial}{\partial\xi_j}e^{ix\cdot\xi}=e^{ix\cdot\xi}, \tag{59}$$

we may integrate by part $n-1$ times, and get $a_\ell(t,x)=I_1+I_2$, where

$$I_1 = \frac{1}{(2\pi)^n} \sum_{|\beta|=n-1} \left(\frac{ix}{|x|^2}\right)^\beta \int\limits_{3\le|\xi|\le\frac{1}{|x|}} e^{ix\cdot\xi}\partial_\xi^\beta \hat{a}_\ell(t,\xi)d\xi \tag{60}$$

$$I_2 = \frac{1}{(2\pi)^n} \sum_{|\beta|=n-1} \left(\frac{ix}{|x|^2}\right)^\beta \int\limits_{|\xi|\ge\frac{1}{|x|}} e^{ix\cdot\xi}\partial_\xi^\beta \hat{a}_\ell(t,\xi)d\xi. \tag{61}$$

By using (52) and

$$|\xi|^{4(j-k)}\, e^{\lambda_+ t} \lesssim t^{-(j-k)}\, e^{-t},$$

for any $k \le j$ (and $|\xi| \ge 3$), we immediately obtain

$$\begin{aligned} |I_1| &\lesssim e^{-t}t^{-(j-k)}|x|^{-(n-1)} \int\limits_{3\le|\xi|\le\frac{1}{|x|}} |\xi|^{|\alpha_1|-n-3}d\xi \\ &\approx e^{-t}t^{-(j-k)} \begin{cases} |x|^{-(n-1)}\ln(|x|^{-1}), & \text{if } |\alpha_1| = 3 \\ |x|^{-(n-1)}, & \text{otherwise.} \end{cases} \end{aligned} \tag{62}$$

For I_2, integrating one more time, we obtain

$$\begin{aligned} I_2 = {} & \frac{1}{(2\pi)^n} \sum_{|\beta|=n-1} \left(\frac{ix}{|x|^2}\right)^\beta \sum_{j=1}^n \frac{-ix_j}{|x|^2} \int\limits_{|\xi|=\frac{1}{|x|}} e^{ix\cdot\xi}\partial_\xi^\beta \hat{a}_\ell(t,\xi)dS \\ & + \frac{1}{(2\pi)^n} \sum_{|\beta|=n-1} \left(\frac{ix}{|x|^2}\right)^\beta \sum_{j=1}^n \frac{ix_j}{|x|^2} \int\limits_{|\xi|\ge\frac{1}{|x|}} e^{ix\cdot\xi}\partial_{\xi_j}\partial_\xi^\beta \hat{a}_\ell(t,\xi)d\xi, \end{aligned}$$

so that

$$\begin{aligned} |I_2| &\lesssim e^{-t}t^{-(j-k)}|x|^{-n}\Big(\int\limits_{|\xi|=\frac{1}{|x|}} |\xi|^{|\alpha_1|-n-3}dS + \int\limits_{|\xi|\ge\frac{1}{|x|}} |\xi|^{|\alpha_1|-4-n}d\xi\Big) \lesssim e^{-t}t^{-(j-k)}|x|^{-n+4-|\alpha_1|} \\ &\le e^{-t}t^{-(j-k)}|x|^{-(n-1)}, \end{aligned} \tag{63}$$

as a consequence of $|x| < 1/3$ and $|\alpha_1| \le 3$.

Now let $|x| \ge 1/3$. Using (59) and the integration by parts $n+1$ times, we get

$$|a_\ell(t,x)| \lesssim e^{-t}t^{-(j-k)}|x|^{-(n+1)} \int\limits_{|\xi|\ge 3} |\xi|^{|\alpha_1|-n-5}d\xi \approx e^{-t}t^{-(j-k)}|x|^{-(n+1)}. \tag{64}$$

This concludes the proof of (53) and (54). To prove (56), we proceed as before, but we set

$$\hat{a}_2(t,\xi) = \frac{e^{\lambda_-(\xi)t}\lambda_-(\xi)^j(i\xi)^{\alpha_1}}{(\lambda_+(\xi)-\lambda_-(\xi))}\,\varphi_\infty(\xi).$$

By using (52) and $e^{\lambda_-(\xi)t} \le e^{-t/4}$, we now derive $\|a_2(t,\cdot)\|_{L^1} \le e^{-t/4}$. This concludes the proof of (56). Let us notice that

$$L_-(t)v = \mathcal{L}(t,\cdot)v + \partial_t M_-(t)v,$$

where

$$\big(\mathcal{L}_-(t)v\big)(x) = \mathscr{F}^{-1}\left[e^{\lambda_-(\xi)t}\,\varphi_\infty(\xi)\hat{v}(\xi)\right](x).$$

We now take advantage of the asymptotic expansion of $\lambda_-(\xi)$ to gain some regularity from the cancelations of $\mathcal{L}_-(t)v - e^{-t}v$. Due to the fact that $\varphi_\infty - 1$ is a $\mathcal{C}_c^\infty(\mathbb{R}^n)$ function, we trivially obtain

$$\|\partial_t^j\partial_x^{\alpha_1}(e^{-t}\mathscr{F}^{-1}[\varphi_\infty(\xi)-1])\|_{L_1(\mathbb{R}^n)} \lesssim e^{-t},$$

so that, by using (51), it remains to estimate

$$\hat{a}_3(t,\xi) = (i\xi)^{\alpha_1}\varphi_\infty(\xi)\partial_t^j\big(e^{-t}(e^{-\mu(\xi)t}-1)\big) = (i\xi)^{\alpha_1}\varphi_\infty(\xi)\mu(\xi)\partial_t^j\Big(-te^{-t}\int_0^1 e^{-\theta t\mu(\xi)}d\theta\Big)$$

in M_1. Due to

$$\Big|\partial_\xi^\beta\big(\mu(\xi)^{j+1}(i\xi)^{\alpha_1}\int_0^1 e^{-\theta t\mu(\xi)}\theta^j d\theta\big)\Big| \lesssim |\xi|^{-4(j+1)+|\alpha_1|-|\beta|} \lesssim |\xi|^{-4+|\alpha_1|-|\beta|},$$

uniformly with respect to j, we may proceed as we did for a_0, a_1, a_2, obtaining $\|a_3(t,\cdot)\|_{L^1} \le e^{-t/4}$.

This concludes the proof of (55).

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. The proof of Proposition 2.1 follows by a straight-forward application of the Mikhlin-Hörmander multiplier theorem to $L_\pm v, M_\pm v$, which we omit. Namely, it follows once we prove that

$$\begin{aligned}
&\|\partial_t^j\partial_x^\alpha L_+(t)v\|_{L^p(\mathbb{R}^n)} \lesssim e^{-t/2}t^{-N/4}\|v\|_{W_p^{(4j+|\alpha|-4-N)^+}(\mathbb{R}^n)}, \quad \forall N\ge 0,\\
&\|\partial_t^j\partial_x^\alpha M_+(t)v\|_{L^p(\mathbb{R}^n)} \lesssim e^{-t/2}t^{-N/4}\|v\|_{W_p^{(4j+|\alpha|-4-N)^+}(\mathbb{R}^n)}, \quad \forall N\ge 0,\\
&\|\partial_t^j\partial_x^\alpha L_-(t)v\|_{L^p(\mathbb{R}^n)} \lesssim e^{-t/2}\|v\|_{W_p^{|\alpha|}(\mathbb{R}^n)},\\
&\|\partial_t^j\partial_x^\alpha M_-(t)v\|_{L^p(\mathbb{R}^n)} \lesssim e^{-t/2}\|v\|_{W_p^{(|\alpha|-4)^+}(\mathbb{R}^n)},
\end{aligned}$$

for $p\in(1,\infty)$.

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