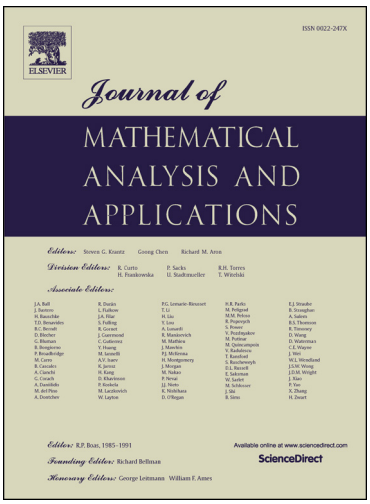
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CONVERGENCE OF SEQUENCES OF SCHRÖDINGER MEANS

PER SJÖLIN AND JAN-OLOV STRÖMBERG

Abstract. We study convergence almost everywhere of sequences of Schrödinger means. We also replace sequences by uncountable sets.

1. Introduction

For $f \in L^2(\mathbb{R}^n), n \geq 1$ and $a > 0$ we set

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-i\xi\cdot x} f(x)\, dx, \xi \in \mathbb{R}^n,$$

and

$$S_t f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x} \mathrm{e}^{it|\xi|^a} \hat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n, t \geq 0.$$

For $a = 2$ and f belonging to the Schwartz class $\mathscr{S}(\mathbb{R}^n)$ we set $u(x,t) = S_t f(x)$. It then follows that $u(x,0) = f(x)$ and u satisfies the Schrödinger equation $i\partial u/\partial t = \Delta u$.

We introduce Sobolev spaces $H_s = H_s(\mathbb{R}^n)$ by setting

$$H_s = \{f \in \mathscr{S}'; \|f\|_{H_s} < \infty\}, s \in \mathbb{R},$$

where

$$\|f\|_{H_s} = \left(\int_{\mathbb{R}^n} (1 + [\xi[^2)^s |\hat{f}(\xi)|^2\, d\xi\right)^{1/2}.$$

In the case $a = 2$ and $n = 1$ it is well-known (se Carleson [3] and Dahlberg and Kenig [5]) that

$$\lim_{t\to 0} S_t f(x) = f(x) \tag{1}$$

almost everywhere if $f \in H_{1/4}$. Also it is known that $H_{1/4}$ cannot be replaced by H_s if $s < 1/4$.

In the case $a = 2$ and $n > 2$ Sjölin [11] and Vega [16] proved independently that (1) holds almost everywhere if $f \in H_s(\mathbb{R}^n), s > 1/2$. This result was improved by Bourgain [1] who proved that $f \in H_s(\mathbb{R}^n), s > 1/2 - 1/4n$, is sufficient for convergence almost everywhere. On the other hand Bourgain [2] has proved that $s \geq n/2(n+1)$ is necessary for convergence for $a = 2$ and $n \geq 2$.

In the case $n = 2$ and $a = 2$, Du, Guth and Li [6] proved that the condition $s > 1/3$ is sufficient. Recently Du and Zhang [7] proved that the condition $s > n/2(n+1)$ is sufficient for $a = 2$ and $n \geq 3$.

In the case $a > 1, n = 1$, (1) holds almost everywhere if $f \in H_{1/4}$ and $H_{1/4}$ cannot be replaced by H_s if $s < 1/4$. In the case $a > 1, n = 2$, it is known that (1) holds almost everywhere if $f \in H_{1/2}$ and in the case $a > 1, n \geq 3$ convergence has been proved for

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Journal Pre-proof

$f \in H_s$ with $s > 1/2$. For the results in the case $a > 1$ see Sjölin [11, 12] and Vega [16, 17].

If $f \in L^2(\mathbb{R}^n)$ then $S_t f \to f$ in L^2 as $t \to 0$. It follows that there exists a sequence $(t_k)_1^\infty$ satisfying

$$1 > t_1 > t_2 > t_3 > \cdots > 0 \text{ and } \lim_{k\to\infty} t_k = 0. \tag{2}$$

such that

$$\lim_{k\to\infty} S_{t_k} f(x) = f(x) \tag{3}$$

almost everywhere.

In Sjölin [13] we studied the problem of deciding for which sequences $(t_k)_1^\infty$ one has (3) almost everywhere if $f \in H_s$. The following result was obtained in [13].

Theorem A *Assume that $n \geq 1$ and $a > 1$ and $s > 0$. We assume that (2) holds and that $\sum_{k=1}^\infty t_k^{2s/a} < \infty$ and $f \in H_s(\mathbb{R}^n)$. Then*

$$\lim_{k\to\infty} S_{t_k} f(x) = f(x)$$

for almost every $x \in \mathbb{R}^n$.

We shall here continue the study of conditions on sequences $(t_k)_1^\infty$ which imply that (3) holds almost everywhere. We shall also replace the set $\{t_k; k = 1, 2, 3, \dots\}$ with sets E which are not countable, for instance the Cantor set . Our first theorem is an extension of Therorem A in which we replace the spaces H_s with Bessel potential spaces L_s^p. We need some more notations.

Let $1 < p \leq 2$ and $s > 0$. Set $k_s(\xi) = (1 + |\xi|^2)^{-s/2}$ for $\xi \in \mathbb{R}^n$.
Let the operator $\mathscr{J}_s$ be defined by

$$\mathscr{J}_s f = \mathscr{F}^{-1}(k_s \hat{f}), f \in L^2 \cap L^p,$$

where $\mathscr{F}$ denotes the Fourier transformation, i.e. $\mathscr{F} f = \hat{f}$. Then $\mathscr{J}_s$ can be extended to a bounded operator on L^p, that is $k_s \in M_p$, where M_p denotes the space of Fourier multipliers on L_p (see Stein [14], p.132).

We introduce the Bessel potential space L_s^p by setting $L_s^p = \{\mathscr{J}_s g; g \in L^p\}$, $s > 0$.
We let I denote an interval defined in the following way. In the case $n = 1, s < a/2$, and in the case $n \geq 2$, we have $I = [p_0, 2]$, where $p_0 = 2/(1 + 2s/na)$. In the remaining case $n = 1, s \geq a/2$, we have $I = (1, 2]$.
For $f \in L_s^p, p \in I$, and $a > 1$, and $0 < s < a$, we shall define $S_t f$ so that

$$(S_t f)\hat{}(\xi) = \mathrm{e}^{it|\xi|^a} \hat{f}(\xi)$$

and then have the following theorem.

Theorem 1. *Assume $a > 1$, $0 < s < a$, and $f \in L_s^p$, where $p \in I$. Let the sequence $(t_k)_1^\infty$ satisfy (2), and assume also that $\sum_{t=1}^\infty t_k^{ps/a} < \infty$. Then*

$$\lim_{k\to\infty} S_{t_k} f(x) = f(x)$$

almost everywhere.

Journal Pre-proof

Journal Pre-proof

In the proof of Theorem 1 we shall use the following theorem on Fourier multipliers.

Theorem 2. *Let $a > 1, 0 < s < a$, and assume also that $0 < \delta < 1$. Set*

$$m(\xi) = \frac{e^{i\delta|\xi|^a} - 1}{(1 + |\xi|^2)^{s/2}}, \quad \xi \in \mathbb{R}^n.$$

Then $m \in M_p$ and

$$\|m\|_{M_p} \leq C_p \delta^{s/a} \text{ for } p \in I,$$

where C_p does not depend on δ.

We remark that in Sjölin [13] we used Theorem 2 in the special case $p = 2$. Now let the sequence $(t_k)_1^\infty$ satisfy (2) and set

$$A_j = \{t_k; 2^{-j-1} < t_k \leq 2^{-j}\} \text{ for } j = 1, 2, 3, \dots.$$

Let $\#A$ denote the number of elements in a set A. We have the following theorem.

Theorem 3. *Assume that $n \geq 1, a > 1$, and $0 < s \leq 1/2$ and $b \leq 2s/(a - s)$. Assume also that*

$$\#A_j \leq C 2^{bj} \text{ for } j = 1, 2, 3, \dots \tag{4}$$

and that $f \in H_s$. Then

$$\lim_{k\to\infty} S_{t_k} f(x) = f(x)$$

almost everywhere.

Theorem 3 has the following two corollaries.

Corollary 1. *Assume that $(t_k)_1^\infty$ satisfies (2) and that $n \geq 1, a > 1, 0 < s \leq 1/2$, and that $\sum_{t=1}^\infty t_k^\gamma < \infty$, where $\gamma = 2s/(a - s)$. If also $f \in H_s$ then (3) holds almost everywhere.*

We remark that Corollary 1 gives an improvement of Theorem A.

Corollary 2. *Assume that $(t_k)_1^\infty$ statisfies (2), and that $n \geq 1$, $a > 1$, $1 < p < 2$, $r > 0$, and*

$$s = \frac{n}{2} + r - \frac{n}{p}.$$

If $f \in L_r^p$ and $s > 1/2$ then (3), holds almost everywhere.
If $0 < s \leq 1/2$ set $\gamma = 2s/(a - s)$. If also $\sum_{t=1}^\infty t_k^\gamma < \infty$, and $f \in L_r^p$ then (3) holds almost everywhere.

Now let E denote a bounded set in $\mathbb{R}$. For $r > 0$ we let $N_E(r)$ denote the minimal number N of intervals $I_l, l = 1, 2, \dots, N$, of length r, such that $E \subset \bigcup_1^N I_l$.
For $f \in \mathscr{S}$ we introduce the maximal function

$$S^* f(x) = \sup_{t\in E} |S_t f(x)|, \quad x \in \mathbb{R}^n.$$

We shall prove the following estimate.

Journal Pre-proof

Journal Pre-proof

Theorem 4. *Assume $n \geq 1, a > 0$, and $s > 0$. If $f \in \mathscr{S}$ then one has*

$$\int |S^* f(x)|^2 \, dx \leq C \left(\sum_{m=0}^{\infty} N_E(2^{-m}) \, 2^{-2ms/a} \right) \|f\|_{H_s}^2 .$$

The following corollary follows directly

Corollary 3. *Assume that $n \geq 1, a > 0, s > 0$, $f \in \mathscr{S}$, and*

$$\sum_{m=0}^{\infty} N_E(2^{-m}) \, 2^{-2ms/a} < \infty. \tag{5}$$

Then one has

$$\left(\int |S^* f(x)|^2 \, dx \right)^{1/2} \leq C \|f\|_{H_s}.$$

Now let $E = \{t_k, k = 1.2, 3, \dots\}$ where the sequence $(t_k)_1^{\infty}$ satisfies (2). We define $S^* f$ as above so that

$$S^* f(x) = \sup_k |S_{t_k} f(x)| , \quad f \in \mathscr{S}.$$

We then have the following corollary.

Corollary 4. *We let $n \geq 1$, $a > 0$, $s > 0$, and assume that*

$$\sum_{m=0}^{\infty} N_E(2^{-m}) \, 2^{-2ms/a} < \infty,$$

and $f \in H_s$. Then (3) holds almost everywhere.

Now assume $0 < \kappa < 1$ and that let m_κ denote κ-dimensional Hausdorff measure on $\mathbb{R}$ (see Mattila [8], p.55). Let $E \subset \mathbb{R}$ be a Borel set with Hausdorff dimension κ and $0 < m_\kappa(E) < \infty$. Assume also that $0 \in E$.
We shall use a precise definition of $S_t f(x)$ for $f \in L^2(\mathbb{R}^n)$ and $(x, t) \in \mathbb{R}^n \times E$. Let Q denote the unit cube $[-\frac{1}{2}, \frac{1}{2}]^n$ in $\mathbb{R}^n$. Set

$$f_N(x, t) = (2\pi)^{-n} \int_{NQ} \mathrm{e}^{i\xi \cdot x} \mathrm{e}^{it|\xi|^a} \hat{f}(\xi) \, d\xi, \text{ for } (x, t) \in \mathbb{R}^n \times E$$

and $N = 1, 2, 3, \dots$. It follows from well-known estimates (See Sjölin [10]) that there exists a set $F \subset \mathbb{R}^n \times E$ with $m \times m_\kappa((\mathbb{R}^n \times E) \setminus F) = 0$ such that

$$\lim_{N \to \infty} f_N(x, t)$$

exists for every $(x, t) \in F$. Here m denotes Lebesque measure. We set $S_t f(x)$ equal to this limit for $(x, t) \in F$ and $S_t f(x)$ will then be a measurable function on $\mathbb{R}^n \times E$ with respect to the measure $m \times m_\kappa$
Then one has the following convergence result

Theorem 5. *Let $n \geq 1, a > 0$, and assume that $s > 0$ and*

$$\sum_{m=0}^{\infty} N_E(2^{-m}) \, 2^{-2ms/a} < \infty \tag{6}$$

Journal Pre-proof

Journal Pre-proof

and $f \in H_s$. Then for almost every $x \in \mathbb{R}^n$ we can modify $S_t f(x)$ on a m_κ - nullset so that

$$\lim_{\substack{t\to 0\\ t\in E}} S_t f(x) = f(x).$$

Note that if $0 < a < 2s$ then (6) holds when E is the interval $[0, 1]$. Thus one of the consequences of the above results is the following well-known fact (see Cowling [4]).

Corollary 5. *If $0 < a < 2s$ and $f \in H_s$ then (1) holds.*

We also have

Corollary 6. *In Theorem 3 the conditions $a > 1$ and $b \le 2s/(a - s)$ can be replaced by the conditions $a \ge 2s$ and $1/b > (a - 2s)/2s$.*

and

Corollary 7. *Assume that $(t_k)_1^\infty$ satisfies (2), and that $n \ge 1, a \ge 2s, 0 < s \le 1/2$, and that $\sum_{t=1}^\infty t_k^\gamma < \infty$, where $1/\gamma > (a - 2s)/2s$. If also $f \in H_s$ then (3) holds almost everywhere.*

We remark that Corollary 7 gives an improvement of Theorem A and Corollary 1.

We shall now study the case where E is a Cantor set. Assume $0 < \lambda < 1/2$. We set $I_{0,1} = [0, 1]$, $I_{1,1} = [0, \lambda]$ and $I_{1,2} = [1 - \lambda, 1]$. Having defined $I_{k-1,1}, \dots, I_{k-1,2^{k-1}}$, we define $I_{k,1}, \dots, I_{k,2^k}$ by taking away from the middle of each interval $I_{k-1,j}$ an interval of length $(1 - 2\lambda)l(I_{k-1,j}) = (1 - 2\lambda)\lambda^{k-1}$, where $l(I)$ denotes the length of an interval I. We then define Cantor sets by setting

$$C(\lambda) = \bigcap_{k=0}^{\infty} \bigcup_{j=1}^{2^k} I_{k,j}.$$

It can be proved that $C(\lambda)$ has Hausdorff dimension

$$\kappa = \log 2/\log(1/\lambda)$$

and that $m_\kappa(C(\lambda)) = 1$ (See [8], p. 60-62). We have the following result, where $S_t f(x)$ is defined as in Theorem 5 with $E = C(\lambda)$.

Theorem 6. *Assume $n \ge 1, a > 0$, and $0 < \lambda < 1/2$. Also assume $s > a\kappa/2$ and $f \in H_s$. Then we can for almost every x modify $S_t f(x)$ on m_κ-nullset so that*

$$\lim_{\substack{t\to 0\\ t\in C(\lambda)}} S_t f(x) = f(x).$$

Remark. In the proofs of Corollary 4 and Theorem 5 we first in the main part of the proof obtain a maximal estimate for smooth functions and then prove a convergence result for functions in H_s. In the passage from the maximal estimate for smooth functions to the convergence result we use an approach which was mentioned to one of the authors by P. Sjögren in a conversation, 2009.

Journal Pre-proof

Journal Pre-proof

In Secton 2 we shall prove Theorems 1 and 2, and Section 3 contains the proof of Theorem 3. In section 4 we prove Theorem 4, and in Section 5 the proofs of Theorems 5 and 6 are given.
We shall finally construct a counter-example which gives the following theorem.

Theorem 7. *Assume $t_k = 1/(\log k)$ for $k = 2, 3, 4, \dots$, and set*

$$S^* f(x) = \sup_k |S_{t_k} f(x)|, x \in \mathbb{R}^n,$$

for $f \in L^2(\mathbb{R}^n)$. Then S^* is not a bounded operator on $L^2(R^n)$ in the case $n = 1, a > 1$, and also in the case $n \geq 2, a = 2$.

2. Proofs of Theorems 1 and 2

For $m \in L^\infty(\mathbb{R}^n)$ and $1 < p < \infty$ we set

$$T_m f = \mathscr{F}^{-1}(m\hat{f}), \quad f \in L^p \cup L^2.$$

We say that m is a Fourier multiplier for L^p if T_m can be extended to a bounded operator on L^p, and we let M_p denote the class of multipliers on L^p. We set $\|m\|_{M_p}$ equal to the norm of T_m as an operator on L^p.

Now let $1 < p \leq 2$ and $0 < s < a$. For $f \in \mathscr{S}$ and with $\hat{f}(\xi) = (1 + |\xi|^2)^{-s/2}\hat{g}(\xi)$ one obtains

$$\mathbf{S}_t f(x) = \left(\mathscr{F}^{-1}\left(\mu(\xi)\hat{g}(\xi)\right)\right)(x) = T_\mu g(x),$$

where

$$\mu(\xi) = \frac{\mathrm{e}^{it|\xi|^a}}{(1 + |\xi|^2)^{s/2}}.$$

We shall prove that $\mu \in M_p$ for $p \in I$, where I is an interval defined in the introduction. We need som well-known results.

Lemma 1. *Assume that $m \in M_p$ for some p which $1 < p < \infty$. Let b be a positive number and let $k(\xi) = m(b\xi)$ for $\xi \in \mathbb{R}^n$. Then $k \in M_p$ and $\|k\|_{M_p} = \|m\|_{M_P}$.*

We shall also use the following multiplier theorem (see Stein ([14], p. 96).

Theorem B: *Assume that m is a bounded function on $\mathbb{R}^n \setminus \{0\}$ and that*

$$|D^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}$$

for $\xi \neq 0$ and $|\alpha| \leq k$, where k is an integer and $k > n/2$. Then $m \in M_p$ for $1 < p < \infty$.

We shall also need the following result (see Miyachi [9], p 283)

Theorem C: *Assume $\psi \in C^\infty(\mathbb{R}^n)$ and that ψ vanishes in a neighbourhood of the origin and is equal to 1 outside a compact set. Set*

$$m_{a,s}(\xi) = \psi(\xi)|\xi|^{-s} e^{i|\xi|^a}, \quad \xi \in \mathbb{R}^n,$$

where $a > 1$ and $0 < s < a$. Then $m_{a,s} \in M_p$ if $1 < p < \infty$ and $|1/p - 1/2| \leq s/na$.

Remark. In Miyachi's formulation of this result the function ψ is replaced by a function ψ_1 with the properties that $\psi_1 \in C^\infty, 0 \leq \psi_1 \leq 1, \psi_1(\xi) = 0$ for $|\xi| \leq 1$, and $\psi_1(\xi) = 1$ for $|\xi| \geq 2$. However, the two formulations are equivalent since the function

Journal Pre-proof

$(\psi - \psi_1)|\xi|^{-s}\mathrm{e}^{i|\xi|^a}$ belongs to C_0^∞.

It follows from Theorem C that $m_{a,s} \in M_p$ if $p \in I$.
We shall then give the proof of the above statement about the function μ.

Lemma 2. *Assume $a > 1$ and $0 < s < a$ and also $t > 0$. Set*

$$\mu(\xi) = e^{it|\xi|^a}(1 + |\xi|^2)^{-s/2}, \quad \xi \in \mathbb{R}^n.$$

Then $\mu \in M_p$ for $p \in I$.

Proof of Lemma 2. We first take ψ as in Theorem C and also set $\varphi = 1 - \psi$. One then has

$$\mu(\xi) = \varphi(\xi)\mathrm{e}^{it|\xi|^a}(1 + |\xi|^2)^{-s/2} + \psi(\xi)\mathrm{e}^{it|\xi|^a}(1 + |\xi|^2)^{-s/2} = \mu_1(\xi) + \mu_2(\xi).$$

We write $\mu_2 = \mu_3\mu_4$, where

$$\mu_3(\xi) = \psi(\xi)\frac{\mathrm{e}^{it|\xi|^a}}{|\xi|^s}$$

and

$$\mu_4(\xi) = \frac{|\xi|^s}{(1 + |\xi|^2)^{s/2}}.$$

We have

$$\mu_3(t^{-1/a}\eta) \,=\, \psi(t^{-1/a}\eta)\,\frac{\mathrm{e}^{i|\eta|^a}}{|t^{-1/a}\eta|^s} = \psi(t^{-1/a}\eta)\,t^{s/a}\,\frac{\mathrm{e}^{i|\eta|^a}}{|\eta|^s}\,.$$

We let $p \in I$ and it then follows from the Remark after Theorem C that $\mu_3 \in M_p$. Also $\mu_4 \in M_p$ since $I \subset (1, \infty)$ (see Stein [14], p. 133).
Finally

$$\mu_1(\xi) = \varphi(\xi)\,\frac{\mathrm{e}^{it|\xi|^a}}{(1 + |\xi|^2)^{s/2}}$$

and it is easy to see that μ_1 satisfies the conditions in Theorem B. We conclude that $\mu_1 \in M_p$ and thus also $\mu \in M_p$.

□

For $f \in L^p_s$, $p \in I$, and $a > 1$, and $0 < s < a$, we define $S_t f$ by setting $S_t f = T_\mu g$. It is then easy to see that

$$(S_t f)\hat{}\,(\xi) \,=\, \mathrm{e}^{it|\xi|^a}\hat{f}(\xi).$$

Observe that according to the Hausdorff-Young theorem $\hat{f} \in L^q$ where $1/p + 1/q = 1$.

We shall then give the proof of Theorem 2. We shall write $A \lesssim B$ if there is a constant K such that $A \le KB$.

Proof of Theorem 2. We set $C = \delta^{-1/a}$ and then have $C^{-s} = \delta^{s/a}$. It follows that

$$m(C\xi) = \frac{\mathrm{e}^{i|\xi|^a} - 1}{(1 + C^2|\xi|^2)^{s/2}} = m_1(\xi) + m_2(\xi) - m_3(\xi),$$

where

$$m_1(\xi) = \varphi(\xi)\,\frac{\mathrm{e}^{i|\xi|^a} - 1}{(1 + C^2|\xi|^2)^{s/2}},$$

Journal Pre-proof

Journal Pre-proof

$$m_2(\xi) = \psi(\xi)\,\frac{\mathrm{e}^{i|\xi|^a}}{(1+C^2|\xi|^2)^{s/2}}$$

and

$$m_3(\xi) = \psi(\xi)\,\frac{1}{(1+C^2|\xi|^2)^{s/2}}.$$

Here φ and ψ are defined as in the proof of Lemma 2, and we may assume that φ and ψ are radial functions.
We have

$$m_2(\xi) = m_4(\xi)\, m_5(\xi),$$

where

$$m_4(\xi) = \psi(\xi)\,\frac{\mathrm{e}^{i|\xi|^a}}{(C^2|\xi|^2)^{s/2}} = \delta^{s/a}\,\psi(\xi)\,\frac{\mathrm{e}^{i|\xi|^a}}{|\xi|^s}$$

and

$$m_5(\xi) = \frac{(C^2|\xi|^2)^{s/2}}{(1+C^2|\xi|^2)^{s/2}}.$$

It follows from Theorem C that $m_4 \in M_p$ and $\|m\|_{M_p} \lesssim \delta^{s/a}$ for $p \in I$. Also m_5 has the same multiplier norm as the function $|\xi|^s(1+|\xi|^2)^{-s/2}$. We conclude that $\|m_2\|_{M_p} \lesssim \delta^{s/a}$ for $p \in I$.

We want to show that

$$|D^\alpha m_1(\xi)| \lesssim C^{-s}|\xi|^{-|\alpha|} \text{ for } \xi \in \mathbb{R}^n \setminus \{0\}$$

for all multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, where α_i are non-negative integers. Invoking Theorem B we conclude that

$$\|m_1\|_{M_p} \lesssim C^{-s} = \delta^{s/a}$$

for $1 < p < \infty$.

First we set

$$m_{10}(x) = \varphi_0(x)\frac{\mathrm{e}^{ix^{a/2}} - 1}{(1+C^2x)^{s/2}},$$

where we define φ_0 by taking $\varphi_0(x) = \varphi(\xi)$ if $x = |\xi|^2$ and we then have $m_1(\xi) = m_{10}(|\xi|^2)$.

We get for $x > 0$

$$D^j\frac{1}{(1+C^2x))^{s/2}} = \frac{C_jC^{2j}}{(1+C^2x))^{s/2+j}}.$$

Hence we have

$$|D^j\frac{1}{(1+C^2x)^{s/2}}| \lesssim x^{-j}\,C^{-s}x^{-s/2}. \tag{7}$$

on support of φ_0. One also has $|\mathrm{e}^{ix^{a/2}} - 1| \le x^{a/2}$ and $D^j(\mathrm{e}^{ix^{a/2}} - 1)$ are linear combinations of functions $\mathrm{e}^{ix^{a/2}}x^{ka/2-j}$ for $j \ge 1$, where k is an integer $1 \le k \le j$. Hence

$$|D^j(\mathrm{e}^{ix^{a/2}} - 1)| \lesssim x^{a/2-j}, \quad j = 0, 1, 2, \dots, \tag{8}$$

Journal Pre-proof

Journal Pre-proof

for $x \in \operatorname{supp}\varphi$.
A combination of (7) and (8) then gives

$$|D_j m_{10}(x)| \lesssim x^{-j} C^{-s} x^{a/2\,-\,s/2}$$

Let α and β denote n-dimenisonal muti-index. By induction over $j = 0, 1, 2, \dots$, and $|\alpha| = j$ we can write $D^\alpha m_1(\xi)$ as a finite linear combination of functions of the form

$$D^k m_{10}(|\xi|^2)\xi^\beta$$

with $j/2 \le k \le j$ and $|\beta| = 2k - j$. We conclude that

$$|D^\alpha m_1(\xi)| \lesssim \max_{|\alpha|/2 \le k \le |\alpha|} |\xi|^{-2k} C^{-s} |\xi|^{a-s} |\xi|^{2k-j} = C^{-s} |\xi|^{-|\alpha|} \lesssim \delta^{s/a} |\xi|^{-|\alpha|}.$$

It remains to study m_3. Define $m_{30}(x)$ analogously to the definition of $m_{10}(x)$ on $\operatorname{supp}\varphi_0$, such that $m_{30}(x) = m_3(\xi)$ when $x = |\xi|^2$, we have

$$m_{30}(x) = \psi_0(x)\,\frac{1}{(1 + C^2 x)^{s/2}}$$

and invoking (7)

$$|D^j (1 + C^2 x)^{-s/2}| \lesssim C^{-s}\, x^{-j}$$

on $\operatorname{supp}\psi_0$. Also $|D^j \psi_0(x)| \lesssim x^{-j}$ on $\operatorname{supp}\psi_0$.
We conclude that

$$|D^j (m_{30}(x)| \lesssim C^{-s}\, x^{-j}$$

and arguing as above we obtain

$$|D^\alpha m_3(\xi)| \lesssim \max_{|\alpha||/2 \le k \le |\alpha|} |\xi|^{-2k} C^{-s} |\xi|^{2k-j} = C^{-s} |\xi|^{-|\alpha|} \lesssim \delta^{s/a} |\xi|^{-|\alpha|}$$

for $\xi \in$ supp m_3 and $j = 0, 1, 2, \dots$. Invoking Theorem B we conclude that $\|m_3\|_{M_p} \lesssim \delta^{s/a}$ for $1 < p < \infty$. This completes the proof of Theorem 2 □

We shall finally give the proof of Theorem 1.

Proof of Theorem 1. We set

$$\mu_0(\xi) = \frac{\mathrm{e}^{it_k|\xi|^a}}{(1 + |\xi|^2)s/2}$$

$$m(\xi) = \frac{\mathrm{e}^{it_k|\xi|^a} - 1}{(1 + |\xi|^2)^{s/2}}$$

and also have

$$k_s(\xi) = (1 + |\xi|^2)^{-s/2}.$$

It follows that

$$T_{\mu_0} g - \mathscr{J}_s g = T_m g$$

for $g \in \mathscr{S}$.
We have $f \in L^p_s$ where $p \in I$ and it follows that $f = \mathscr{J}_s g$ for some $g \in L^p$. We choose a sequence $(g_j)_1^\infty$ such that $g_j \in \mathscr{S}$ and $g_j \to g$ in L^p as $j \to \infty$.
One then has

$$T_{\mu_0} g_j - \mathscr{J}_s g_j = T_m g_j$$

Journal Pre-proof

for every j. Letting j tend to ∞ we obtain

$$T_{\mu_0} g - \mathscr{J}_s g = T_m g$$

since the three operators T_{μ_0}, $\mathscr{J}_s$ and T_m are all bounded on L^p. It follows that

$$S_{t_k} f - f = T_m g.$$

Here we have used Lemma 2 and Theorem 2.
We now set $h_k = S_{t_k} f - f$ and hence $h_k = T_m g$. It follows from Theorem 2 that

$$\|h_k\|_p \lesssim t_k^{s/a} \|g\|_p$$

and we conclude that

$$\sum_{k=1}^{\infty} \int |h_k|^p \, dx \le \left(\sum_{k=1}^{\infty} t_k^{ps/a} \right) \int |g|^p \, dx < \infty.$$

Applying the theorem on monotone convergence on then obtain

$$\int \left(\sum_{1}^{\infty} |h_k|^p \right) dx < \infty$$

and hence $\sum_1^\infty |h_k|^p$ is convergent almost everywhere. It follows that $\lim_{k\to\infty} |h_k| = 0$ alomst everywhere and we conclude that

$$\lim_{k\to\infty} S_{t_k} f(x) = f(x)$$

almost everywhere. This completes the proof of Theorem 1. □

3. Proof of Theorem 3 and its corollaries

We first give the proof of Theorem 3.

Proof of Theorem 3. We may assume $b = 2s/(a-s)$. Fix j. By adding points to A_j we can get an increasing sequence $(v_k)_{k=0}^N$ and $\tilde{A}_j = \{v_k; k =, 0, \dots, N\}$ such that $v_0 = 0, v_N = 2^{-j}, \#\tilde{A}_j \le C2^{bj}$, and $v_k - v_{k-1} \le C2^{-j}2^{-bj}$.
We split the operator S_{v_k} into a low frequency part and a high frequency part

$$S_{v_k} f(x) = S_{v_k,\mathrm{low}_j} f(x) + S_{v_k,\mathrm{high}_j} f(x)$$

where

$$S_{k,\mathrm{low}_j} f(x)0 = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x} \mathrm{e}^{iv_k|\xi|^a} \chi_{E_j} \hat{f}(\xi) \, d\xi,$$

and

$$S_{k,\mathrm{high}_j} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x} \mathrm{e}^{iv_k|\xi|^a} \chi_{E_j^c} \hat{f}(\xi) \, d\xi,$$

with $E_j = \{\xi \in \mathbb{R}^n; |\xi| \le 2^{b_1 j}\}$ and $b_1 = b/\,2s$.
We shall prove that

$$\sum_j 2^{bj} \sum_{\substack{v_k \in \tilde{A}_j \\ k>0}} \|S_{k,\mathrm{low}_j} f - S_{k-1,\mathrm{low}_j} f\|_2^2 \le C\|f\|_{H_s} 2 \tag{9}$$

Journal Pre-proof

Journal Pre-proof

and

$$\sum_j \sum_{v_k\in\tilde{A}_j} \|S_{k,\mathrm{high}_j} f\|_2^2 \le C\|f\|_{H_s}^2. \tag{10}$$

We first assume that (9) and (10) hold. Using the Schwarz inequality we then have

$$\sup_{v_k\in\tilde{A}_j} |S_{k,\mathrm{low}_j} f(x) - f(x)|^2 \le \left(|S_{0,\mathrm{high}_j} f(x)| + \sum_{\substack{v_k\in\tilde{A}_j\\ k>0}} |S_{k,\mathrm{low}_j} f(x) - S_{k-1,\mathrm{low}_j} f(x)| \right)^2$$

$$\le 2|S_{0,\mathrm{high}_j} f(x)|^2 + C2^{bj} \sum_{\substack{v_k\in\tilde{A}_j\\ k>0}} |S_{k,\mathrm{low}_j} f(x) - S_{k-1,\mathrm{low}_j} f(x)|^2$$

and invoking (9) and (10)

$$\sum_j \sup_{v_k\in\tilde{A}_j} |S_{k,\mathrm{low}_j} f(x) - f(x)|^2 \le 2\sum_j |S_{0,\mathrm{high}_j} f(x)|^2 + C\sum_j 2^{bj} \sum_{\substack{v_k\in\tilde{A}_j\\ k>0}} |S_{k,\mathrm{low}_j} f(x) - S_{k-1,\mathrm{low}_j} f(x)|^2$$

and

$$\int \sum_j \sup_{v_k\in A_j} |S_{k,\mathrm{low}_j} f(x) - f(x)|^2\, dx \le C\|f\|_{H_s}^2. \tag{11}$$

Using (10) we also obtain

$$\begin{aligned} &\textstyle\int \sum_j \sup_{v_k\in A_j} |S_{k,\mathrm{high}_j} f(x)|^2\, dx \\ &\quad\le \textstyle\int \sum_j \sup_{v_k\in\tilde{A}_j} |S_{k,\mathrm{high}_j} f(x)|^2\, dx \le C\|f\|_{H_s}^2. \end{aligned} \tag{12}$$

The theorem follows from (11) and (12).
We shall now prove (9) an first observe that

$$S_{k,\mathrm{low}_j} f(x) - S_{k-1,\mathrm{low}_j} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x} \left(\mathrm{e}^{iv_k|\xi|^a} - \mathrm{e}^{iv_{k-1}|\xi|^a} \right) \chi_{E_j} \hat{f}(\xi)\, d\xi,$$

Applying Plancherel's theorem we obtain

$$\begin{aligned} &\|S_{k,\mathrm{low}_j} f - S_{k-1,\mathrm{low}_j} f\|_2^2 = C\textstyle\int_{E_j} |\mathrm{e}^{iv_k|\xi|^a} - \mathrm{e}^{iv_{k-1}|\xi|^a}|^2 |\hat{f}(\xi)|^2\, d\xi \\ &\quad\le C\textstyle\int_{E_j} |v_k - v_{k-1}|^2 |\xi|^{2a} |\hat{f}(\xi)|^2\, d\xi \le C2^{-2j}2^{-2bj} \textstyle\int_{E_j} |\xi|^{2a} |\hat{f}(\xi)|^2\, d\xi \end{aligned}$$

and

$$\begin{aligned} &\textstyle\sum_j 2^{bj} \sum_{\substack{v_k\in\tilde{A}_j\\ k>0}} \|S_{k,\mathrm{low}_j} f - S_{k-1,\mathrm{low}_j} f\|_2^2 \\ &\quad\le C\textstyle\sum_j 2^{-2j} \left(2^{-bj} \sum_{\substack{v_k\in\tilde{A}_j\\ k>0}} 1 \right) \int_{E_j} |\xi|^{2a} |\hat{f}(\xi)|^2\, d\xi \\ &\quad\le C\textstyle\int \left(\sum_{2^{b_1 j}\ge|\xi|} 2^{-2j} \right) |\xi|^{2a} |\hat{f}(\xi)|^2\, d\xi. \end{aligned}$$

The inequality $2^{b_1 j} \ge |\xi|$ implies $2^j \ge |\xi|^{1/b_1}$ and thus we get

$$\sum_{2^{b_1 j}\ge|\xi|} 2^{-2j} \le C|\xi|^{-2/b_1}.$$

Journal Pre-proof

Hence the left hand side of (9) is majorized by

$$C\int |\xi|^{2a-2/b_1}|\hat f(\xi)|^2\,d\xi.$$

We have $b=2s/(a-s)$ and $b_1=1/(a-s)$ and $2a-2/b_1=2a-2(a-s)=2s$ and the inequality (9) follows.

To prove (10) we first observe that Plancherel's theorem implies

$$\|S_{k,\mathrm{high}_j}f\|_2^2\le C\int_{|\xi|\ge 2^{b_1 j}}|\hat f(\xi)|^2|,d\xi.$$

and hence

$$\begin{array}{c}\sum_j\sum_{v_k\in\tilde A_j}\|S_{k,\mathrm{high}_j}f\|_2^2\le\sum_j 2^{bj}\int_{|\xi|\ge 2^{b_1 j}}|\hat f(\xi)|^2\,d\xi\\ =\int\left(\sum_{2^{b_1 j}\le|\xi|}2^{bj}\right)|\hat f(\xi)|^2\,d\xi\le C\int|\xi|^{b/b_1}|\hat f(\xi)|^2\,d\xi.\end{array}.$$

Since $b=2s/(a-s)$ and $b_1=1/(a-s)$ we obtain $b/b_1=2s$ and (10) follows.
Thus the proof of Theorem 3 is complete. □

We shall then prove the two corollaries to Theorem 3.

Proof of Corollary 1. Since $\sum_1^\infty t_k^\gamma$ is convergent we obtain

$$\left(\#\{k;t_k>2^{-j-1}\}\right)2^{(-j-1)\gamma}\le\sum_{t_k>2^{-j-1}}t^\gamma\le C$$

an $\#A_j\le C2^{j\gamma}$ for $j=1,2,3,\dots$. Since $\gamma=2s/(a-s)$ the corollary follows from Theorem 3. □

Proof of Corollary 2 . Assume that $f\in L^p_r$, where $1<p<2$, amd $r>0$. Also let $s=n/2+r-n/p$. Then there exists $g\in L^p$ such that $f=\mathscr{J}_r(g)=\mathscr{J}_s(\mathscr{J}_{r-s}g)$ and we have

$$\frac12=\frac1p-\frac{r-s}{n}.$$

It follows from the Hardy-Littlewood-Sobolev theorem that $\mathscr{J}_{r-s}g\in L^2$ and hence $f\in H_s$ (see Stein [14]. p. 119). The corollary then follows from Theorem 3. □

4. Proofs of Theorem 4 and its corollaries

In Sections 4 and 5 we assume $n\ge1$ and $a>0$. We remark that (1) holds almost everywhere if $f\in H_s$ and $n=1, 0<a<1$, and $s>a/4$ or $n\ge1, a=1$ and $s>1/2$) (se Walther [18],[19]).
Before proving Theorem 4 we need some preliminary estimates. We set $B(x_0;r)=\{x;|x-x_0|\le r\}$. Using the estimate

$$|\mathrm{e}^{it|\xi|^a}-e^{iu|\xi|^a}|\le|t-u|\,|\xi|^a$$

and with $A\ge1$ and $\mathrm{supp}\hat f\subset B(0;A)$ we obtain by Schwarz inequality

$$\begin{aligned}\|S_tf-S_uf\|_\infty&\le\textstyle\int_{|\xi|\le A}|t-u|\,|\xi|^a\,|\hat f(\xi)|\,d\xi\\ &\le|t-u|\left(\textstyle\int_{|\xi|\le A}|\xi|^{2a}\,d\xi\right)^{1/2}\left(\textstyle\int|\hat f(\xi)|^2\,d\xi\right)^{1/2}\\ &\le C|t-u|\left(\textstyle\int_0^A r^{2a+n-1}\,dr\right)^{1/2}\|f\|_2\le C|t-u|\,A^{a+n/2}\|f\|_2\end{aligned}\qquad(13)$$

Journal Pre-proof

Now assume $T = \{t_j; j = 0, 1, 2, \dots, N\}$ where $t_j \in \mathbb{R}$ and $t_{j-1} < t_j$. We shall prove that that if $\operatorname{supp}\hat{f} \subset B(0; A)$ then

$$\int \max_{t,u\in T} |S_t f(x) - S_u f(x)|^2 \, dx \le C \max_{t,u\in T} |t-u|^2 A^{2a} \|f\|_2^2. \tag{14}$$

Using the Schwarz inequality we obtain

$$\begin{aligned}
&\max_{t,u\in T} |S_t f(x) - S_u f(x)| \le \textstyle\sum_1^N |S_{t_i} f(x) - S_{t_{i-1}} f(x)| \\
&\quad \le \textstyle\sum_1^N |t_i - t_{i-1}|^{-1/2} |S_{t_i} f(x) - S_{t_{i-1}} f(x)| \, |t_i - t_{i-1}|^{1/2} \\
&\quad \le \Big(\textstyle\sum_1^N |t_i - t_{i-1}|^{-1} |S_{t_i} f(x) - S_{t_{i-1}} f(x)|^2\Big)^{1/2} \Big(\textstyle\sum_1^N |t_i - t_{i-1}|\Big)^{1/2}
\end{aligned}$$

where the last sum equals $\max_{t,u\in T} |t-u|$, and the Plancherel theorem gives

$$\begin{aligned}
&\textstyle\int \max_{t,u\in T} |S_t f(x) - S_u f(x)|^2 \, dx \le (\max_{t,u\in T} |t-u|) \sum_1^N |t_i - t_{i-1}|^{-1} \int |S_{t_i} f(x) - S_{t_{i-1}} f(x)|^2 \, dx \\
&\quad \le (\max_{t,u\in T} |t-u|) \textstyle\sum_1^N |t_i - t_{i-1}|^{-1} \int |t_i - t_{i-1}|^2 |\xi|^{2a} |\hat{f}(\xi)|^2 \, d\xi \\
&\quad \le (\max_{t,u\in T} |t-u|)^2 \textstyle\int |\xi|^{2a} |\hat{f}(\xi)|^2 \, d\xi \le C \max_{t,u\in T} |t-u|^2 A^{2a} \|f\|_2^2
\end{aligned}$$

Hence (14) is proved.

We shall then prove the following lemma

Lemma 3. *Let I denote an interval of length r Then*

$$\int \sup_{t,u\in I} |S_t f(x) - S_u f(x)|^2 \, dx \le C r^2 A^{2a} \|f\|_2^2 \tag{15}$$

if $f \in L^2(\mathbb{R}^n)$ and supp $\hat{f} \subset B(0; A)$.

Proof of Lemma 3. Asumme $I = [b, b + r]$ and let N be a positive integer. Set $t_i = b + ir/N, i = 0, 1, 2, \dots, N$, and $T = \{t_i; i = 0, 1, 2, \dots, N\}$. We have

$$S_t f(x) - S_u f(x) = S_{t_i} f(x) - S_{t_j} f(x) + S_t f(x) - S_{t_i} f(x) - (S_u f(x) - S_{t_j} f(x)),$$

where we choose t_i close to t and t_j close to u. Invoking (13) we obtain

$$|S_t f(x) - S_{t_i} f(x)| \le C|t - t_i| \, A^{a+n/2} \|f\|_2 \le C \frac{r}{N} A^{a+n/2} \|f\|_2 = C_f \frac{r}{N},$$

and

$$|S_u f(x) - S_{t_j} f(x)| \le C|u - t_j| \, A^{a+n/2} \|f\|_2 \le C \frac{r}{N} A^{a+n/2} \|f\|_2 = C_f \frac{r}{N}.$$

where C_f depends on f. It follows that

$$|S_t f(x) - S_u f(x)| \le \max_{i,j} |S_{t_i} f(x) - S_{t_j} f(x)| + C_f \frac{r}{N}.$$

Setting $F_N(x) = \max_{i,j} |S_{t_i} f(x) - S_{t_j} f(x)|$ we obtain

$$|S_t f(x) - S_u f(x)| \le F_N(x) + C_f \frac{r}{N}$$

Letting $N \to \infty$ we obtain

$$|S_t f(x) - S_u f(x)| \le \varlimsup_{N\to\infty} F_N(x).$$

Journal Pre-proof

Journal Pre-proof

An application of Fatou's lemma and the inequality (14) then gives

$$\int \sup_{t,u\in I} |S_t f(x) - S_u f(x)|^2\, dx \le \int \varliminf_{N\to\infty} F_N(x)^2\, dx$$
$$\le \varliminf_{N\to\infty} \int F_N(x)^2\, dx \le Cr^2A^{2a}\|f\|_2^2$$

and the lemma follows. □

Let I and f have the properties in the above lemma. Then

$$\int \sup_{t\in I} |S_t f(x) - f(x)|^2\, dx \le C\left(r^2A^{2a}+1\right)\|f\|_2^2. \tag{16}$$

To prove (16) we take $u_0 \in I$ and observe that

$$\sup_{t\in I} |S_t f(x) - f(x)| \le \sup_{t\in I} |S_t f(x) - S_{u_0} f(x)| + |S_{u_0} f(x)| + |f(x)|$$

and (16) follows from Lemma 3 and Plancherel theorem.

We shall then prove the following lemma

Lemma 4. *Let f have the same properties as in Lemma 3. Assume $r > 0$ and set $I_l = [t_l - r/2, t_l + r/2], l = 1, 2, \dots, N$. Assume that E is a set and $E \subset \bigcup_1^N I_l$. Then*

$$\int \sup_{t\in E} |S_t f(x) - f(x)|^2\, dx \le CN\left(r^2A^{2a}+1\right)\|f\|_2^2. \tag{17}$$

Poof of Lemma 4. The lemma follows from the inequality (16) and the inequality

$$\sup_{t\in E} |S_t f(x) - f(x)|^2 \le \sum_{l=1}^{N} \sup_{t\in I_l} |S_t f(x) - f(x)|^2$$

□

Now assume $f \in \mathscr{S}$ and write

$$f = \sum_{k=0}^{\infty} f_k,$$

where $\hat{f}_0$ is supported in $B(0;1)$ and $\hat{f}_k$ has support in $\{\xi; 2^{k-1} \le |\xi| \le 2^k\}$ for $k = 1, 2, 3, \dots$. We shall prove the following lemma

Lemma 5. *Let $f \in \mathscr{S}$ and $s > 0$ and and let E be a bounded set in $\mathbb{R}$. Then*

$$\int \sup_{t\in E} |S_t f(x) - f(x)|^2\, dx \le C\|f\|_{H_s}^2 \left(\sum_{k=0}^{\infty} N_E(2^{-ka}) 2^{-2ks}\right),$$

where $N_E(r)$ for $r > 0$ denotes the minimal number N of intervals $I_l, l = 1, 2, \dots, N$, of length r such that $E \subset \sum_1^N I_l$.

Proof of Lemma 5. With real numbers $g_k > 0, k = 0, 1, 2, \dots$, we have

$$\sup_{t\in E} |S_t f(x) - f(x)| \le \sum_{k=0}^{\infty} \sup_{t\in E} |S_t f_k((x) - f_k(x)|$$
$$= \sum_{k=0}^{\infty} g_k^{-1/2} \sup_{t\in E} |S_t f({}_k x) - f_k(x)| g_k^{1/2}$$
$$\le \left(\sum_{k=0}^{\infty} g_k^{-1} \sup_{t\in E} |S_t f_k(x) - f_k(x)|^2\right)^{1/2} \left(\sum_{k=0}^{\infty} g_k\right)^{1/2}$$

Journal Pre-proof

Journal Pre-proof

and invoking Lemma 4 with $r = 2^{-ka}$ and $A = 2^k$ we obtain

$$\int \sup_{t\in E} |S_t f(x) - f(x)|^2\, dx \le \left(\sum_{k=0}^{\infty} g_k\right)\left(\sum_{k=0}^{\infty} g_k^{-1} C N_E(2^{-ka})(2^{-2ak}2^{2ak}+1)\|f_k\|_2^2\right)$$

Choosing $g_k = N_E(2^{-ka})2^{-2ks}$ one obtains

$$\begin{aligned}\textstyle\int \sup_{t\in E} |S_t f(x) - f(x)|^2\, dx &\le C\left(\textstyle\sum_{k=0}^{\infty} g_k\right)\left(\textstyle\sum_0^\infty 2^{2ks}\|f_k\|_2^2\right)\\ &\le C\left(\textstyle\sum_0^\infty N_E(2^{-ka})2^{-2ks}\right)\|f\|_{H_s}^2\end{aligned}$$

and the proof of the lemma is complete. □

We shall prove Theorem 4.

Proof of Theorem 4. Let m take the values $0, 1, 2, \ldots$. If

$$2^{-m-1} < 2^{-ka} \le 2^{-m} \tag{18}$$

for some integer $k \ge 0$ then

$$N_E(2^{-ka}) \le C N_E(2^{-m})$$

and since $a > 0$ there is for any fixed m only a bounded number of values of k for which (18) holds. It follows that

$$N_E(2^{-ka})2^{-2ks} \le C N_E(2^{-m})2^{-2ms/a}.$$

Combining this inequality with the estimate

$$\sup_{t\in E} |S_t f(x)| \le \sup_{t\in E} |S_t f(x) - f(x)| + |f(x)|$$

one obtains the theorem from Lemma 5 □

Corollary 3 follows directly from Theorem 4 and we shall then prove Corollary 4.

Proof of Corollary 4. Set $E_0 = E \cup \{0\}$ and

$$S_0^* f(x) = \sup_{E_0} |S_t f(x)|,\ x \in \mathbb{R}^n.$$

It then follows from Corollary 3 that for $f \in \mathscr{S}$ one has

$$\|S_0^* f\|_2 \le C\|f\|_{H_s}.$$

It follows that for every cube I in $\mathbb{R}^n$ one has

$$\int_I S_0^* f(x)\, dx \le C_I \|f\|_{H_s},\ f \in \mathscr{S}.$$

Now fix $f \in H_s$ and a cube I. Then there exists a sequence $(f_j)_1^\infty$ such that $f_j \in C_0^\infty$ and

$$\|f_j - f\|_{H_s} < 2^{-j},\ j = 1, 2, 3, \ldots.$$

One then has $\|f_j - f_{j+1}\|_{H_s} < 2 \cdot 2^{-j}$ and

$$\int_I \sup_{t\in E_0} |S_t f_j(x) - S_t f_{j+1}(x)|\, dx \le C 2^{-j}.$$

Journal Pre-proof

Journal Pre-proof

Hence

$$\sum_1^\infty \sup_{t\in E_0} |S_t f_j(x) - S_t f_{j+1}(x)| < \infty \tag{19}$$

for almost every $x \in I$.
Then choose x so that (19) holds. It follow that $S_t f_j(x) \to u_x(t)$, as $j \to \infty$, uniformly in $t \in E_0$, where u_x is a continuous function on E_0.
It is also clear that $S_t f_j \to S_t f$ in L^2 as $j \to \infty$, for every $t \in E_0$. Since E_0 is countable we can find a subsequence $(f_{j_l})_1^\infty$ such that for almost every x $S_t f_{j_l} \to S_t f(x)$ for all $t \in E_0$.
It follows that for almost every $x \in I$ one has $S_t f(x) = u_x(t)$ for all $t \in E_0$. Since

$$\lim_{\substack{t\to 0\\ t\in E}} u_x(t) = u_x(0)$$

almost everywhere one also has

$$\lim_{\substack{t\to 0\\ t\in E}} S_t f(x) = f(x)$$

for almost every $x \in I$. Since I is arbitrary it follows that (3) holds almost everywhere in $\mathbb{R}^n$. □

5. Proofs of Theorems 5 and 6 and Corollaries 6 and 7

We shall first give the proof of Theorem 5

Proof of Theorem 5. It follows from Corollary 3 that

$$\|S^* f\|_2 \le C\|f\|_{H_s}, \quad f \in \mathscr{S},$$

where

$$S^* f(x) = \sup_{t\in E} |S_t f(x)|, x \in \mathbb{R}^n, f \in \mathscr{S}.$$

Now take $f \in H_s$.
Let I denote a cube in R^n. It follows that $\int_I S^* f(x)\, dx \le C_I \|f\|_{H_s}$ for $f \in C_0^\infty$.
We choose a sequence $(f_j)_1^\infty$ such that $f_j \in C_0^\infty$ and

$$\|f_j - f\|_{H_s} < 2^{-j},\ j = 1, 2, 3, \dots.$$

One then has $\|f_j - f_{j+1}\|_{H_s} < C2^{-j}$ and

$$\int \sup_{t\in E} |S_t f_j(x) - S_t f_{j+1}(x)|\, dx \le C2^{-j}.$$

It follows that

$$\sum_1^\infty \sup_{t\in E} |S_t f_j(x) - S_t f_{j+1}(x)| < \infty$$

for almost every $x \in I$. Now choose x such that the above inequality holds. We conclude that $S_t f_j(x) \to u_x(t)$, as $j \to \infty$, uniformly in $t \in E$, where u_x is a continuous function on E.
On the other hand $S_t f_j \to S_t f$ in $L^2(\mathbb{R}^n \times E; m \times m_\kappa)$ as $j \to \infty$. Hence there is a subsequence $(f_{j_l})_1^\infty$ such that $S_t f_{j_l}(x) \to S_t f(x)$ almost everywhere in $\mathbb{R}^n \times E$ with

Journal Pre-proof

respect to $m \times m_\kappa$. It follows that for almost every $x \in I$ one has $S_t f(x) = u_x(t)$ for almost all $t \in E$ with respect to m_κ. We have

$$\lim_{\substack{t\to 0\\ t\in E}} u_x(t) = f(x)$$

for almost every $x \in I$ and it follows that for almost every $x \in I$ we can modify $S_t f(x)$ on a m_κ-nullset so that

$$\lim_{\substack{t\to 0\\ t\in E}} S_t f(x) = f(x).$$

This completes the proof of Theorem 5. □

For the proof of Corollary 6 we need the following lemma

Lemma 6. *Let A_j be defined as in Theorem 3 satisfying*

$$\#A_j \le C2^{bj} \text{ for } j = 0, 1, 2, \dots$$

for some $b > 0$. Let $E = \bigcup_1^\infty A_j$ and N_E be as above then

$$N_E(2^{-m}) \le C2^{bm/(b+1)}$$

Proof of Lemma 6. Fix a k. We have

$$\#(\left(\bigcup_1^k A_j\right) \le C\sum_{j=1}^k 2^{bj} \le C2^{bk}$$

and $\bigcup_{j=k+1}^\infty A_j \subset \{t; 0 \le t \le 2^{-k-1}\}$, which can be covered by 2^{m-k+1} intervals of length 2^{-m}. Thus

$$N_E(2^{-m}) \le 2^{m-k+1} + C2^{bk}$$

Choose k such that $k \le (m+1)/(b+1) < k+1$ We get $2^{b+1} \cdot 2^{(b+1)k} > 2^{m+1}$ and $2^{bk} \le C2^{mb/(b+1)}$. We conclude that

$$N_E(2^{-m}) \le C2^{bk} \le C2^{bm/(b+1)}.$$

This ends the proof of the Lemma 6 □

We can now prove Corollary 6 by using Lemma 6 and Corollary 4

Proof of Corollary 6. With $1/b > (a-2s)/2s$ as in Corollary 6 we get

$$b/(b+1) = \frac{1}{(1+1/b)} < 1/\left(1 + \frac{a-2s}{2s}\right) = 2s/a,$$

and we get

$$\sum_1^\infty N_E(2^{-m})2^{-2ms/a} \le C\sum_1^\infty 2^{bm/(b+1)}2^{-2ms/a} \le C\sum_1^\infty 2^{m(b/(b+1)-2s/a)} < \infty$$

since $b/(b+1) - 2s/a < 0$.
By Corollary 4 the Corollary 6 will follow. □

The Corollary 7 will now follow by similar arguments as in the proof Corollary 1.
Finally we shall give the proof of Theorem 6.

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Proof of Theorem 6. We shall use Theorem 5 with

$$\kappa = \log 2/(\log 1/\lambda).$$

For $k = 0, 1, 2, 3, \dots$, $C(\lambda)$ can be covered by 2^k intervals of length λ^k
Let m be a positive integer. Choose k such that $\lambda^{k+1} < 2^{-m} \le \lambda^k$. It follows that $N_E(2^{-m}) \le 2^{k+1}$ and that

$$(1/\lambda)^k \le 2^m$$

and

$$k \le m\frac{\log 2}{\log(1/\lambda)} = \kappa m.$$

Hence

$$\sum_{m=1}^{\infty} N_E(2^{-m})2^{-2sm/a} \le C\sum_{m=1}^{\infty} 2^{\kappa m}2^{-2sm/a} < \infty,$$

if $\kappa - 2s/a < 0$, i.e. $s > a\kappa/2$. Theorem 6 follows from an application of Theorem 5. □

6. Proof of Theorem 7

We first assume $n = 1$ and $a > 1$. We choose a function $\varphi \in C_0^\infty(\mathbb{R})$ with the property that $\varphi(\xi) = 1$ for $|\xi| = a^{-1/(a-1)}$ and also $\varphi \ge 0$. We also assume that there exists a constant $A > 1$ such that supp $\varphi \subset \{\xi \in \mathbb{R}; 1/A \le |\xi| \le A\}$. We then define a function f_ν by setting $\hat{f}_\nu(\xi) = \varphi\,(2^{-\nu}\xi)$ where $\nu = 1, 2, 3, \dots$. One then has

$$\|f_\nu\|_2 = c\|\hat{f}_\nu\|_2 = c\left(\int |\varphi(2^{-\nu}\xi)|^2\,d\xi\right)^{1/2} = c\left(\int |\varphi(\eta)|^2\,d\eta\,2^\nu\right)^{1/2} = c2^{\nu/2},$$

where c denotes positive constants. Setting $\eta = 2^{-\nu}\xi$ we also obtain

$$S_tf_\nu(x) = c\int \mathrm{e}^{i\xi x}\mathrm{e}^{it|\xi|^a}\varphi(2^{-\nu}\xi)\,d\xi = c2^\nu\int \mathrm{e}^{i2^\nu\eta x}\mathrm{e}^{it2^{\nu a}|\eta|^a}\varphi(\eta)\,d\eta = c2^\nu\int \mathrm{e}^{iF(\xi)}\varphi(\xi)\,d\xi,$$

where $F(\xi) = t2^{\nu a}|\xi|^a + 2^\nu x\xi$.
We then assume $C2^{-\nu} \le x \le 1$ where C denotes a large positive constant. It is clear that $F = G + H$, where

$$G(\xi) = 2^\nu x|\xi|^a + 2^\nu x\xi$$

and

$$H(\xi) = t2^{\nu a}|\xi|^a - 2^\nu x|\xi|^a.$$

We shall first study the integral

$$\int \mathrm{e}^{iG(\xi)}\varphi(\xi)\,d\xi = \int \mathrm{e}^{i2^\nu xK(\xi)}\varphi(\xi)\,d\xi,$$

where $K(\xi) = |\xi|^a + \xi$ for $\xi \in \mathbb{R}$.
For $\xi > 0$ we have $K'(\xi) = a\xi^{a-1} + 1$ and for $\xi < 0$ one has $K'(\xi) = 1 - a|\xi|^{a-1}$. It follows that $K'(\xi) = 0$ for $\xi = -a^{-1/(a-1)}$. Also $K''(\xi) \ne 0$ for $\xi \in$ supp φ. We now apply the method of stationary phase (se Stein [15], p. 334). One obtains

$$\left|\int \mathrm{e}^{iG}\varphi\,d\xi\right| \gtrsim (2^\nu x)^{-1/2} = 2^{-\nu/2}x^{-1/2.}$$

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Hence

$$\left|\int \mathrm{e}^{iF}\varphi\,d\xi\right| = \left|\int \mathrm{e}^{i(G+H)}\varphi\,d\xi\right| = \left|\int \mathrm{e}^{iG}\varphi\,d\xi + \int (\mathrm{e}^{iG+H} - \mathrm{e}^{iG})\varphi\,d\xi\right|$$
$$\gtrsim 2^{-\nu/2}x^{-1/2} - O\left(\int \left|\mathrm{e}^{iH}-1\right|\varphi\,d\xi\right) \geq 2^{-\nu/2}x^{-1/2} - O\left(\int |H|\varphi\,d\xi\right), \tag{20}$$

and we need an estimate of H. One obtains

$$|H(\xi)| = |t2^{\nu a} - 2^{\nu}x||\xi|^a \lesssim |t2^{\nu a} - 2^{\nu}x|$$

on supp φ. We then choose k such that

$$t_{k+1} < \frac{2^{\nu}x}{2^{\nu a}} \leq t_k$$

where we assume that ν is large. It follows that

$$t_k \leq 2\frac{2^{\nu}x}{2^{\nu a}} \leq 2\frac{2^{\nu}}{2^{\nu a}} = 2\cdot 2^{\nu(1-a)}$$

and hence

$$\log k \geq \frac{1}{2}2^{\nu(a-1)} \geq 2^{\nu\epsilon}$$

where $\epsilon > 0$. It is then easy to see that

$$k \geq \mathrm{e}^{2^{\nu\epsilon}}$$

and

$$t_k - t_{k+1} \leq \frac{1}{k} \leq \mathrm{e}^{-2^{\nu\epsilon}}$$

which implies that

$$\left|t_k - \frac{2^{\nu}x}{2^{\nu a}}\right| \leq t_k - t_{k+1} \leq \mathrm{e}^{-2^{\nu\epsilon}}$$

We conclude that

$$|t_k 2^{\nu a} - 2^{\nu}x| \leq 2^{\nu a}\mathrm{e}^{-2^{\nu\epsilon}}\mathrm{e}^{-100\nu}$$

for ν large.
Setting $t = t_k$, invoking the inequality (20), and using the fact that $x \leq 1$, one obtains

$$\left|\int \mathrm{e}^{iF}\varphi\,d\xi\right| \gtrsim 2^{-v/2}x^{-1/2} - O\left(\mathrm{e}^{-100\nu}\right) \gtrsim 2^{-v/2}x^{-1/2}.$$

It follows that

$$\int |S^*f(x)|^2\,dx \gtrsim \int_{C2^{-\nu}}^1 2^{\nu}\frac{1}{x}\,dx \gtrsim 2^{\nu}\nu$$

for ν large.
We have $\|f_\nu\|_2 = c2^{\nu/2}$ and we have proved that $\|S^*f_\nu\|_2 \gtrsim 2^{\nu/2}\nu^{1/2}$ and it follows that S^* is not a bounded operator on $L^2(\mathbb{R})$.

We shall then study the case $n \geq 2$ and $a = 2$. We let $\varphi \in C_0^\infty(\mathbb{R})$ be the same function as in the case $n = 1$. Also let $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ and assume that $\|\psi\|_2 > 0$.
For $x \in \mathbb{R}^n$ we write $x = (x_1, x')$, where $x' = (x_2, x_2, \ldots, x_n)$. We define f_ν by setting $\hat{f}\nu(\xi) = \varphi(2^{-\nu}\xi_1)\psi(\xi')$ for $v = 1, 2, 3, \ldots$.

Journal Pre-proof

Journal Pre-proof

It is then easy to see that $\|f_\nu\|_2 = c2^{\nu/2}$ for some constant c.
We also have

$$\begin{aligned} S_t f_\nu(x) &= c\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} \mathrm{e}^{i(\xi_1 x_1+\xi'\cdot x')}\mathrm{e}^{it(\xi_1^2+|\xi'|^2)}\varphi(2^{-\nu}\xi_1)\psi(\xi')\,d\xi_1 d\xi' \\ &= \int_{\mathbb{R}} \mathrm{e}^{i(\xi_1 x_1+t\xi_1^2)}\varphi(2^{-\nu}\xi_1)\,d\xi_1 \int_{\mathbb{R}^{n-1}} \mathrm{e}^{i(\xi'\cdot x'+t|\xi'|^2)}\psi(\xi')\,d\xi', \end{aligned}$$

where c denotes a constant. Setting $\eta_1 = 2^{-\nu}\xi_1$ we obtain

$$S_t f_\nu(x) = c2^\nu \left(\int_{\mathbb{R}} \mathrm{e}^{i(t2^{2\nu}\eta_1^2+2^\nu\eta_1 x_1)}\varphi(\eta_1)\,d\eta_1\right)\left(\int_{\mathbb{R}^{n-1}} \mathrm{e}^{i(\xi'\cdot x'+t|\xi'|^2)}\psi(\xi')\,d\xi'\right).$$

We then choose t_k as an approximation for $\frac{2^\nu x_1}{2^{\nu a}}$ as in the one-dimensional case and set $t(x_1) = t_k$. It follows that

$$S_{t(x_1)} f_\nu(x) = c2^\nu I(x_1)\, J(x_1, x')$$

where

$$I(x_1) = \int_{\mathbb{R}} \mathrm{e}^{i(t(x_1)2^{2\nu}\eta_1^2+2^\nu\eta_1 x_1)}\varphi(\eta_1)\,d\eta_1$$

and

$$J(x_1, x') = \int_{\mathbb{R}^{n-1}} \mathrm{e}^{i(\xi'\cdot x'+t(x_1)|\xi'|^2)}\psi(\xi')\,d\xi'.$$

Above we proved that $|I(x_1)| \gtrsim 2^{-\nu/2}x_1^{-1/2}$ for $C2^{-\nu} \le x_1 \le 1$. We also have

$$S^* f_\nu(x) \gtrsim 2^\nu |I(x_1)|\,|J(x_1, x')|.$$

It follows that

$$\int_{\mathbb{R}^{n-1}} (S^* f_\nu(x))^2\,dx' \gtrsim 2^{2\nu}|I(x_1)|^2 \int_{\mathbb{R}^{n-1}} |J(x_1, x')|^2\,dx',$$

and invoking Plancherel's theorem we obtain

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} (S^* f_\nu(x))^2\,dx' &\gtrsim 2^{2\nu}|I(x_1)|^2 \int_{\mathbb{R}^{n-1}} |\psi(\xi')|^2\,d\xi' \\ &= c2^{2\nu}|I(x_1)|^2 \gtrsim 2^{2\nu}2^{-\nu}x_1^{-1} = 2^\nu x_1^{-1} \end{aligned}$$

for $C2^{-\nu} \le x_1 \le 1$.
We conclude that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} (S^* f_\nu(x))^2\,dx_1\,dx' \gtrsim 2^\nu \int_{C2^{-\nu}}^1 1/x_1\,dx_1 \gtrsim 2^\nu\nu$$

and

$$\|S^* f_\nu\|_2 \gtrsim 2^{\nu/2}\nu^{1/2}.$$

Since $\|f_\nu\|_2 = c2^{\nu/2}$ it follows that S^* is not a bounded operator on $L^2(\mathbb{R}^n)$.

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Journal Pre-proof

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