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Long range scattering for the nonlinear Schrödinger equation with higher order anisotropic dispersion in two dimensions

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ABSTRACT

This paper is a continuation of our previous study [13] on the long time behavior of solution to the nonlinear Schrödinger equation with higher order anisotropic dispersion (4NLS). We prove the long range scattering for (4NLS) with the quadratic nonlinearity in two dimensions. More precisely, for a given asymptotic profile u_+, we construct a solution to (4NLS) which converges to u_+ as $t \to \infty$, where u_+ is given by the leading term of the solution to the linearized equation of (4NLS) with a logarithmic phase correction.

1. Introduction

This paper is a continuation of our previous study [13] on the long time behavior of solution to the nonlinear Schrödinger equation with higher order anisotropic dispersion:

$$i\partial_t u + \frac{1}{2}\Delta u - \frac{1}{4}\partial_{x_1}^4 u = \lambda |u|^{p-1}u, \qquad t > 0, x \in \mathbb{R}^d, \tag{1.1}$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is an unknown function, $\lambda \in \mathbb{R}\backslash\{0\}$ and $p > 1$ are constants. Equation (1.1) arises in nonlinear optics to model the propagation of ultrashort laser pulses in a medium with anomalous time-dispersion in the presence of fourth-order time-dispersion (see [3,6,16] and the references therein). It also arises in models of propagation in fiber arrays (see [1,5]). The readers can consult [4] for the well-posedness of (1.1), and existence/non-existence and qualitative properties results of solitary wave solutions for (1.1).

In this paper, we consider the scattering problem for (1.1). Since the solution to the linearized equation of (1.1) decays like $O(t^{-d/2})$ in $L^\infty(\mathbb{R}^d)$ as $t \to \infty$ (see Ben-Artzi, Koch and Saut [2]), we expect that if $p > 1 + 2/d$, then the (small) solution to (1.1) will scatter to the solution to the linearized equation and if

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$p \leqslant 1+2/d$, then the solution to (1.1) will not scatter. The homogeneous fourth order nonlinear Schrödinger type equation

$$i\partial_t u + \frac{1}{2}\Delta^2 u = \lambda |u|^{p-1}u, \qquad t > 0, x \in \mathbb{R}^d \tag{1.2}$$

has been studied by many authors from the point of view of the scattering. See [13] for a review of the known results on the scattering and blow-up problem for (1.2). Compared to the homogeneous equation (1.2), there are few results on the long time behavior of solution for (1.1). For the one dimensional cubic case, the second author [14] proved that for a given asymptotic profile, there exists a solution u to (1.1) which converges to the given asymptotic profile as $t \to \infty$, where the asymptotic profile is given by the leading term of the solution to the linearized equation with a logarithmic phase correction. Furthermore, Hayashi and Naumkin [10] proved that for any small initial data, there exists a global solution to (1.1) with $d = 1, p = 3$ which behaves like a solution to the linearized equation with a logarithmic phase correction. Recently, the authors [13] have shown the unique existence of solution u to (1.1) which scatters to the free solution for $2 < p < 3$ if $d = 2$ and $9/5 < p < 7/3$ if $d = 3$. In this paper, refining the asymptotic formula [13, Proposition 2.1] as $t \to \infty$ for the solution to the linearized equation of (1.1), we prove the long range scattering for (1.1) with the quadratic nonlinearity in two dimensions.

Let us consider the final state problem:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u - \frac{1}{4}\partial_{x_1}^4 u = \lambda |u|u, & t > 0, x \in \mathbb{R}^2, \\ \lim\limits_{t\to+\infty} (u(t) - u_+(t)) = 0, & \text{in } L^2(\mathbb{R}^2), \end{cases} \tag{1.3}$$

where $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ is an unknown function and $u_+ : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ is a "modified" asymptotic profile given by

$$\begin{aligned} u_+(t,x) &= \frac{t^{-1}}{\sqrt{3\mu_1^2+1}} \hat{\psi}_+(\mu) e^{\frac{3}{4}it\mu_1^4 + \frac{1}{2}it|\mu|^2 + iS_+(t,\mu) - i\frac{\pi}{2}}, \\ S_+(t,\xi) &= -\lambda \frac{|\hat{\psi}_+(\xi)|}{\sqrt{3\xi_1^2+1}} \log t, \end{aligned} \tag{1.4}$$

where $\mu = (\mu_1, \mu_2) \in \mathbb{R} \times \mathbb{R}$ is a stationary point for the oscillatory integral (2.2) associated with the linearized equation of (1.3), i.e.,

$$\begin{aligned} \mu_1 &= \frac{1}{2^{\frac{1}{3}}} \left\{ \left(\frac{x_1}{t} + \sqrt{\left(\frac{x_1}{t}\right)^2 + \frac{4}{27}} \right) \right\}^{1/3} + \left\{ \left(\frac{x_1}{t} - \sqrt{\left(\frac{x_1}{t}\right)^2 + \frac{4}{27}} \right) \right\}^{1/3}, \\ \mu_2 &= \frac{x_2}{t}. \end{aligned} \tag{1.5}$$

Our main result in this paper is as follows:

Theorem 1.1 *(Long range scattering). There exists $\varepsilon > 0$ with the following properties: for any $\psi_+ \in H^{0,2}(\mathbb{R}^2)$ with $\|\psi_+\|_{H^{0,2}} < \varepsilon$ (see (1.9) for the definition of $H^{0,2}$), there exists a unique global solution $u \in C(\mathbb{R}; L_x^2(\mathbb{R}^2)) \cap \langle\partial_{x_1}\rangle^{-1/4} L_{loc}^4(\mathbb{R}; L_x^4(\mathbb{R}^2))$ to (1.3) satisfying*

$$\|u(t) - u_+(t)\|_{L_x^2} \leqslant Ct^{-\alpha}$$

for $t \geqslant 3$, where $1/2 < \alpha < 3/4$ and u_+ is given by (1.4).

We give an outline of the proof of Theorem 1.1. To prove Theorem 1.1, we employ the argument by Ozawa [12], Hayashi and Naumkin [8,9]. We first construct a solution u to the final state problem

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u - \frac{1}{4}\partial_{x_1}^4 u = \lambda|u|u, & t > 0, x \in \mathbb{R}^2, \\ \lim_{t\to+\infty}(u(t) - W(t)\mathcal{F}^{-1}w) = 0, & \text{in } L^2(\mathbb{R}^2), \end{cases} \tag{1.6}$$

where $w(t,\xi) = \hat{\psi}_+(\xi)e^{iS_+(t,\xi)}$ and $\{W(t)\}_{t\in\mathbb{R}}$ is a unitary group generated by the operator $(1/2)i\Delta - (1/4)i\partial_{x_1}^4$. To prove this, we first rewrite (1.6) as the integral equation

$$\begin{aligned} &u(t) - W(t)\mathcal{F}^{-1}w \\ &= i\lambda \int_t^{+\infty} W(t-\tau)[|u|u - |W(t)\mathcal{F}^{-1}w|W(t)\mathcal{F}^{-1}w](\tau)d\tau \\ &\quad -i\int_t^{+\infty} W(t-\tau)R(\tau)d\tau, \end{aligned} \tag{1.7}$$

where

$$R = W(t)\mathcal{F}^{-1}\left[\frac{\lambda t^{-1}}{\sqrt{3\xi_1^2+1}}|\hat{\psi}_+|\hat{\psi}_+(\xi)e^{iS_+(t,\xi)}\right] - \lambda|W(t)\mathcal{F}^{-1}w|W(t)\mathcal{F}^{-1}w.$$

Next, we apply the contraction mapping principle to the integral equation (1.7) in a suitable function space. In this step the large time asymptotic formula (Proposition 2.1) and the Strichartz estimate (Lemma 2.2) for solution to the linear equation (2.1) play an important role. Finally, we show that the solutions of (1.6) converge to u_+ in L^2 as $t \to \infty$.

Remark 1.2. The proof of the large time asymptotic formula for solution to the linear equation (2.1) (Proposition 2.1) depends heavily on the fact that the fourth-order dispersion is one dimensional. So far, we do not know whether the our arguments are applicable for the scattering problem of the isotropic fourth order nonlinear Schrödinger equation (1.2) for $d \geqslant 2$.

By using the argument by Glassey [7] we can prove the non-existence of asymptotically free solution for (1.1) with $p \leqslant 1 + 2/d$.

Theorem 1.3 *(Nonexistence of asymptotically free solution). Let $d \geqslant 2$ and $1 < p \leqslant 1 + 2/d$. Let $u \in C(\mathbb{R}, L^2(\mathbb{R}^d))$ be a solution to (1.1) with $u(0,x) = u_0 \in L^2(\mathbb{R}^d)$. Assume that there exists a function $\psi_+ \in H^{0,s} \cap \langle\partial_{x_1}\rangle^{p-1}L^{2/p}$ with $s > (4-d)/2 + (d-1)p/2$ such that*

$$\|u(t) - W(t)\psi_+\|_{L_x^2} \to 0, \tag{1.8}$$

as $t \to \infty$, where $\{W(t)\}_{t\in\mathbb{R}}$ is a unitary group generated by the operator $(1/2)i\Delta - (1/4)i\partial_{x_1}^4$. Then $u \equiv 0$.

Remark 1.4. The assumption for the regularity of the final state ψ_+ stems from the asymptotic formula for the linear fourth order Schrödinger equation (2.1) which is proved by our previous paper [13, Proposition 2.1].

We introduce several notations and function spaces which are used throughout this paper. For $\psi \in \mathcal{S}'(\mathbb{R}^d)$, $\hat{\psi}(\xi) = \mathcal{F}[\psi](\xi)$ denote the Fourier transform of ψ. Let $\langle\xi\rangle = \sqrt{|\xi|^2+1}$. The differential operator

$\langle\nabla\rangle^s = (1-\Delta)^{s/2}$ denotes the Bessel potential of order $-s$. We define $\langle\partial_{x_1}\rangle^s = \mathcal{F}^{-1}\langle\xi_1\rangle^s\mathcal{F}^{-1}$ for $s \in \mathbb{R}$. For $1 \leqslant q, r \leqslant \infty$, $L^q(t,\infty; L^r_x(\mathbb{R}^d))$ is defined as follows:

$$L^q(t,\infty; L^r_x(\mathbb{R}^d)) = \{u \in \mathcal{S}'(\mathbb{R}^{1+d}); \|u\|_{L^q(t,\infty;L^r_x)} < \infty\},$$

$$\|u\|_{L^q(t,\infty;L^r_x)} = \left(\int_t^\infty \|u(\tau)\|^q_{L^r_x} d\tau\right)^{1/q}.$$

We will use the Sobolev spaces

$$H^s(\mathbb{R}^d) = \{\phi \in \mathcal{S}'(\mathbb{R}^d); \|\phi\|_{H^s} = \|\langle\nabla\rangle^s\phi\|_{L^2} < \infty\}$$

and the weighted Sobolev spaces

$$H^{m,s}(\mathbb{R}^d) = \{\phi \in \mathcal{S}'(\mathbb{R}^d); \|\phi\|_{H^{m,s}} = \|\langle x\rangle^s\langle\nabla\rangle^m\phi\|_{L^2} < \infty\}. \tag{1.9}$$

We denote various constants by C and so forth. They may differ from line to line, when this does not cause any confusion.

The plan of the present paper is as follows. In Section 2, we prove several linear estimates for the fourth order Schrödinger type equation (2.1). In Section 3, we prove Theorem 1.1 by applying the contraction mapping principle to the integral equation (1.7). Finally in Section 4, we give the proof of Theorem 1.3.

2. Linear estimates

In this section, we derive several linear estimates that will be crucial for the proof of Theorem 1.1, for the fourth order Schrödinger type equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u - \frac{1}{4}\partial_{x_1}^4 u = 0, & t > 0, x \in \mathbb{R}^2, \\ u(0,x) = \psi(x), & x \in \mathbb{R}^2. \end{cases} \tag{2.1}$$

The solution to (2.1) can be rewritten as

$$u(t,x) = [W(t)\psi](x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\xi - \frac{i}{2}t|\xi|^2 - \frac{i}{4}t\xi_1^4}\hat{\psi}(\xi)d\xi. \tag{2.2}$$

The following proposition is a refinement of [13, Proposition 2.1] for $d = 2$ and $p = 2$.

Proposition 2.1. *We have*

$$[W(t)\psi](x) = \frac{t^{-1}}{\sqrt{3\mu_1^2+1}}\hat{\psi}(\mu)e^{\frac{3}{4}it\mu_1^4 + \frac{1}{2}it|\mu|^2 - i\frac{\pi}{2}} + R(t,x)$$

for $t \geqslant 2$, where $\mu = (\mu_1, \mu_2)$ is given by (1.5) and R satisfies

$$\|R(t)\|_{L^2_x} \leqslant Ct^{-\beta}\|\psi\|_{H^{0,2}_x},$$

for $0 < \beta < 3/4$.

Proof of Proposition 2.1. We easily see

$$[W(t)\psi](x) = \int\limits_{\mathbb{R}^2} K(t, x-y)\psi(y)dy,$$

where

$$K(t,z) = \left(\frac{1}{2\pi}\right)^2 \int\limits_{\mathbb{R}^2} e^{iz\xi - \frac{i}{2}t|\xi|^2 - \frac{i}{4}t\xi_1^4} d\xi.$$

By the Fresnel integral formula

$$\frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{iz_2\xi_2 - \frac{i}{2}t\xi_2^2} d\xi_2 = t^{-\frac{1}{2}} e^{\frac{iz_2^2}{2t} - i\frac{\pi}{4}},$$

we have

$$K(t,z) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} t^{-\frac{1}{2}} e^{\frac{iz_2^2}{2t} - i\frac{\pi}{4}} \int\limits_{\mathbb{R}} e^{iz_1\xi_1 - \frac{i}{2}t\xi_1^2 - \frac{i}{4}t\xi_1^4} d\xi_1.$$

Therefore, we find

$$u(t,x) = \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{\frac{i}{2}t\mu_2^2 - i\frac{\pi}{4}} \int\limits_{\mathbb{R}} e^{ix_1\xi_1 - \frac{i}{2}t\xi_1^2 - \frac{i}{4}t\xi_1^4} \mathcal{F}[e^{\frac{iy_2^2}{2t}}\psi](\xi_1, \mu_2) d\xi_1.$$

We split u into the following two pieces:

$$\begin{aligned} &u(t,x) \\ &= \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{\frac{i}{2}t\mu_2^2 - i\frac{1}{4}\pi} \int\limits_{\mathbb{R}} e^{ix_1\xi_1 - \frac{i}{2}t\xi_1^2 - \frac{i}{4}t\xi_1^4} \mathcal{F}[\psi](\xi_1, \mu_2) d\xi_1 \\ &\quad + \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{\frac{i}{2}t\mu_2^2 - i\frac{1}{4}\pi} \int\limits_{\mathbb{R}} e^{ix_1\xi_1 - \frac{i}{2}t\xi_1^2 - \frac{i}{4}t\xi_1^4} \mathcal{F}[(e^{\frac{iy_2^2}{2t}} - 1)\psi](\xi_1, \mu_2) d\xi_1 \\ &=: L(t,x) + R(t,x). \end{aligned} \tag{2.3}$$

To evaluate L, we split L into

$$\begin{aligned} L(t,x) \;=\; &\frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{\frac{i}{2}t\mu_2^2 - i\frac{1}{4}\pi} \mathcal{F}[\psi](\mu) \int\limits_{\mathbb{R}} e^{ix_1\xi_1 - \frac{i}{2}t\xi_1^2 - \frac{i}{4}t\xi_1^4} d\xi_1 \\ &+ \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{\frac{i}{2}t\mu_2^2 - i\frac{1}{4}\pi} \partial_{\xi_1}\mathcal{F}[\psi](\mu) \int\limits_{\mathbb{R}} e^{ix_1\xi_1 - \frac{i}{2}t\xi_1^2 - \frac{i}{4}t\xi_1^4} (\xi_1 - \mu_1) d\xi_1 \\ &+ \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{\frac{i}{2}t\mu_2^2 - i\frac{1}{4}\pi} \int\limits_{\mathbb{R}} e^{ix_1\xi_1 - \frac{i}{2}t\xi_1^2 - \frac{i}{4}t\xi_1^4} \\ &\qquad \times (\mathcal{F}[\psi](\xi_1, \mu_2) - \mathcal{F}[\psi](\mu_1, \mu_2) - \partial_{\xi_1}\mathcal{F}[\psi](\mu)(\xi_1 - \mu_1)) d\xi_1 \\ =:\; &L_1(t,x) + L_2(t,x) + L_3(t,x). \end{aligned} \tag{2.4}$$

We rewrite L_1 as follows:

$$L_1(t,x) = \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{\frac{3}{4}it\mu_1^4 + \frac{i}{2}t|\mu|^2 - i\frac{1}{4}\pi} \mathcal{F}[\psi](\mu) \int_{\mathbb{R}} e^{-itS(\mu_1,\xi_1)} d\xi_1,$$

where $S(\mu_1, \xi_1)$ is defined by

$$S(\mu_1,\xi_1) = \frac{1}{4}\xi_1^4 + \frac{1}{2}\xi_1^2 - (\mu_1^3 + \mu_1)\xi_1 + \frac{3}{4}\mu_1^4 + \frac{1}{2}\mu_1^2.$$

Let

$$\eta_1 = \mu_1 + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{3\mu_1^2+1}} (\xi_1 - \mu_1)\sqrt{\xi_1^2 + 2\mu_1\xi_1 + 3\mu_1^2 + 2}.$$

Then, L_1 can be rewritten as follows:

$$\begin{aligned}
&L_1(t,x) \\
&= \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} \mathcal{F}[\psi](\mu) e^{\frac{3}{4}it\mu_1^4 + \frac{i}{2}t|\mu|^2 - i\frac{1}{4}\pi} \\
&\quad \times \left\{ \int_{\mathbb{R}} e^{-itS(\mu_1,\xi_1)} \frac{d\eta_1}{d\xi_1} d\xi_1 - \left.\frac{d^2\eta_1}{d\xi_1^2}\right|_{\xi_1=\mu_1} \int_{\mathbb{R}} e^{-itS(\mu_1\xi_1)} (\xi_1 - \mu_1) d\xi_1 \right. \\
&\qquad \left. + \int_{\mathbb{R}} e^{-itS(\mu_1\xi_1)} (1 - \frac{d\eta_1}{d\xi_1} + \left.\frac{d^2\eta_1}{d\xi_1^2}\right|_{\xi_1=\mu_1} (\xi_1 - \mu_1)) d\xi_1 \right\} \\
&=: t^{-\frac{1}{2}} \mathcal{F}[\psi](\mu) e^{\frac{3}{4}it\mu_1^4 + \frac{i}{2}t|\mu|^2 - i\frac{1}{4}\pi} (L_{1,1}(t,x) + L_{1,2}(t,x) + L_{1,3}(t,x)).
\end{aligned} \tag{2.5}$$

For $L_{1,1}$, changing the variable $\xi_1 \mapsto \eta_1$, we have

$$L_{1,1}(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}it(3\mu_1^2+1)(\eta_1-\mu_1)^2} d\eta_1.$$

In addition, changing the variable $\zeta_1 = (1/\sqrt{2})t^{1/2}\sqrt{3\mu_1^2+1}(\eta_1 - \mu_1)$ $(\eta_1 \mapsto \zeta_1)$ and using the Fresnel integral formula, we obtain

$$L_{1,1}(t,x) = \sqrt{\frac{2}{\pi}} \frac{t^{-1/2}}{\sqrt{3\mu_1^2+1}} \int_{\mathbb{R}} e^{-i\zeta^2} d\zeta = \frac{t^{-1/2}}{\sqrt{3\mu_1^2+1}} e^{-i\frac{\pi}{4}}. \tag{2.6}$$

Next we evaluate $L_{1,2}$. Integrating by parts via the identity

$$e^{-itS(\mu_1,\xi_1)}(\xi_1 - \mu_1) = it^{-1} G(\mu_1,\xi_1) \partial_{\xi_1} e^{-itS(\mu_1,\xi_1)} \tag{2.7}$$

with

$$G(\mu_1,\xi_1) = \frac{1}{\xi_1^2 + \mu_1\xi_1 + \mu_1^2 + 1},$$

we have

$$\int_{\mathbb{R}} e^{-itS(\mu_1,\xi_1)}(\xi_1-\mu_1)d\xi_1 = -it^{-1}\int_{\mathbb{R}} e^{-itS(\mu_1,\xi_1)}\partial_{\xi_1}G(\mu_1,\xi_1)d\xi_1.$$

Furthermore, integrating by parts via the identity

$$e^{-itS(\mu_1,\xi_1)} = H(t,\mu_1,\xi_1)\partial_{\xi_1}\{(\xi_1-\mu_1)e^{-itS(\mu_1,\xi_1)}\} \tag{2.8}$$

with

$$H(t,\mu_1,\xi_1) = \frac{1}{1-it(\xi_1-\mu_1)^2(\xi_1^2+\mu_1\xi_1+\mu_1^2+1)},$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-itS(\mu_1,\xi_1)}(\xi_1-\mu_1)d\xi_1 \\
&= it^{-1}\int_{\mathbb{R}} e^{-itS(\mu_1,\xi_1)}(\xi_1-\mu_1)\partial_{\xi_1}\left\{H(t,\mu_1,\xi_1)\partial_{\xi_1}G(\mu_1,\xi_1)\right\}d\xi_1 \\
&= it^{-1}\int_{\mathbb{R}} e^{-itS(\mu_1,\xi_1)}(\xi_1-\mu_1)H(t,\mu_1,\xi_1)\partial_{\xi_1}^2G(\mu_1,\xi_1)d\xi_1 \\
&\quad + it^{-1}\int_{\mathbb{R}} e^{-itS(\mu_1,\xi_1)}(\xi_1-\mu_1)\partial_{\xi_1}H(t,\mu_1,\xi_1)\partial_{\xi_1}G(\mu_1,\xi_1)d\xi_1.
\end{aligned}$$

Using the inequalities

$$\left|\partial_{\xi_1}^j G(\mu_1,\xi_1)\right| \leqslant C\langle\mu_1\rangle^{-j-2}, \tag{2.9}$$

$$\left|\partial_{\xi_1}^j H(t,\mu_1,\xi_1)\right| \leqslant C\frac{|\xi_1-\mu_1|^{-j}}{1+t(\xi_1-\mu_1)^2(\xi_1^2+\mu_1\xi_1+\mu_1^2+1)} \tag{2.10}$$

for $j=0,1,2$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}} e^{-itS(\mu_1,\xi_1)}(\xi_1-\mu_1)d\xi_1\right| \\
&\leqslant Ct^{-1}\langle\mu_1\rangle^{-4}\int_{\mathbb{R}}\frac{|\xi_1-\mu_1|+\langle\mu_1\rangle}{1+t(\xi_1-\mu_1)^2(\xi_1^2+\mu_1\xi_1+\mu_1^2+1)}d\xi_1.
\end{aligned}$$

By using the inequalities

$$\xi_1^2+\mu_1\xi_1+\mu_1^2+1 \geqslant \begin{cases} \frac{1}{2}\langle\mu_1\rangle^2, & \text{if } |\xi_1-\mu_1|\leqslant\langle\mu_1\rangle, \\ \frac{1}{4}(\xi_1-\mu_1)^2, & \text{if } |\xi_1-\mu_1|\geqslant\langle\mu_1\rangle, \end{cases}$$

we see

$$\left|\int_{\mathbb{R}} e^{-itS(\mu_1,\xi_1)}(\xi_1-\mu_1)d\xi_1\right|$$

$$
\begin{aligned}
&\leqslant Ct^{-1}\langle\mu_1\rangle^{-3}\int\limits_{|\xi_1-\mu_1|\leqslant\langle\mu_1\rangle}\frac{1}{1+t\langle\mu_1\rangle^2(\xi_1-\mu_1)^2}d\xi_1\\
&\quad+Ct^{-1}\langle\mu_1\rangle^{-4}\int\limits_{|\xi_1-\mu_1|\geqslant\langle\mu_1\rangle}\frac{|\xi_1-\mu_1|}{1+t(\xi_1-\mu_1)^4}d\xi_1\\
&\leqslant Ct^{-1-\gamma}\langle\mu_1\rangle^{-2\gamma-3}\int\limits_{|\xi_1-\mu_1|\leqslant\langle\mu_1\rangle}|\xi_1-\mu_1|^{-2\gamma}d\xi_1\\
&\quad+Ct^{-2}\langle\mu_1\rangle^{-4}\int\limits_{|\xi_1-\mu_1|\geqslant\langle\mu_1\rangle}|\xi_1-\mu_1|^{-3}d\xi_1\\
&\leqslant C(t^{-1-\gamma}\langle\mu_1\rangle^{-4\gamma-2}+t^{-2}\langle\mu_1\rangle^{-6})\\
&\leqslant Ct^{-1-\gamma}\langle\mu_1\rangle^{-4\gamma-2},
\end{aligned}
\tag{2.11}
$$

where $0<\gamma<1/2$. Hence

$$
|L_{1,2}(t,x)|\leqslant Ct^{-1-\gamma}\left|\left.\frac{d^2\eta_1}{d\xi^2}\right|_{\xi_1=\mu_1}\right|\langle\mu_1\rangle^{-4\gamma-2}\leqslant Ct^{-1-\gamma}\langle\mu_1\rangle^{-4\gamma-3}. \tag{2.12}
$$

For $L_{1,3}$, integrating by parts via the identity (2.8), we have

$$
\begin{aligned}
L_{1,3}(t,x)=&-\frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}}(\xi_1-\mu_1)e^{-itS(\mu_1,\xi_1)}\\
&\times\partial_{\xi_1}\left\{H(t,\mu_1,\xi_1)(1-\frac{d\eta_1}{d\xi_1}+\left.\frac{d^2\eta_1}{d\xi_1^2}\right|_{\xi_1=\mu_1}(\xi_1-\mu_1))\right\}d\xi_1.
\end{aligned}
$$

Furthermore, integrating by parts via the identity (2.7), we have

$$
\begin{aligned}
&L_{1,3}(t,x)\\
&=\frac{it^{-1}}{\sqrt{2\pi}}\int\limits_{\mathbb{R}}e^{-itS(\mu_1,\xi_1)}\\
&\qquad\times\partial_{\xi_1}\left[G(\mu_1,\xi_1)\partial_{\xi_1}\left\{H(t,\mu_1,\xi_1)(1-\frac{d\eta_1}{d\xi_1}+\left.\frac{d^2\eta_1}{d\xi_1^2}\right|_{\xi_1=\mu_1}(\xi_1-\mu_1))\right\}\right]d\xi_1\\
&=\frac{it^{-1}}{\sqrt{2\pi}}\int\limits_{\mathbb{R}}e^{-itS(\mu_1,\xi_1)}F_1(\mu_1,\xi_1)\frac{d^3\eta_1}{d\xi_1^3}d\xi_1\\
&\quad+\frac{it^{-1}}{\sqrt{2\pi}}\int\limits_{\mathbb{R}}e^{-itS(\mu_1,\xi_1)}F_2(\mu_1,\xi_1)\left(\frac{d^2\eta_1}{d\xi_1^2}-\left.\frac{d^2\eta_1}{d\xi_1^2}\right|_{\xi_1=\mu_1}\right)d\xi_1\\
&\quad+\frac{it^{-1}}{\sqrt{2\pi}}\int\limits_{\mathbb{R}}e^{-itS(\mu_1,\xi_1)}F_3(\mu_1,\xi_1)\left(1-\frac{d\eta_1}{d\xi_1}+\left.\frac{d^2\eta_1}{d\xi_1^2}\right|_{\xi_1=\mu_1}(\xi_1-\mu_1)\right)d\xi_1,
\end{aligned}
\tag{2.13}
$$

where

$$\begin{aligned}
F_1(\mu_1,\xi_1) &= -G(\mu_1,\xi_1)H(t,\mu_1,\xi_1),\\
F_2(\mu_1,\xi_1) &= -2G(\mu_1,\xi_1)\partial_{\xi_1}H(t,\mu_1,\xi_1) - \partial_{\xi_1}G(\mu_1,\xi_1)H(t,\mu_1,\xi_1),\\
F_3(\mu_1,\xi_1) &= G(\mu_1,\xi_1)\partial_{\xi_1}^2 H(t,\mu_1,\xi_1) + \partial_{\xi_1}G(\mu_1,\xi_1)\partial_{\xi_1}H(t,\mu_1,\xi_1).
\end{aligned}$$

Since $\left|\dfrac{d^3\eta_1}{d\xi_1^3}\right| \leqslant C\langle\mu_1\rangle^{-2}$, we see that

$$\left|\frac{d^2\eta_1}{d\xi_1^2} - \frac{d^2\eta_1}{d\xi_1^2}\bigg|_{\xi_1=\mu_1}\right| \leqslant \sup_{\xi_1\in\mathbb{R}}\left|\frac{d^3\eta_1}{d\xi_1^3}\right| |\xi_1-\mu_1| \leqslant C\langle\mu_1\rangle^{-2}|\xi_1-\mu_1|,$$

$$\begin{aligned}
&\left|1 - \frac{d\eta_1}{d\xi_1} + \frac{d^2\eta_1}{d\xi_1^2}\bigg|_{\xi_1=\mu_1}(\xi_1-\mu_1)\right|\\
&= \left|\frac{d\eta_1}{d\xi_1}\bigg|_{\xi_1=\mu_1} - \frac{d\eta_1}{d\xi_1} + \frac{d^2\eta_1}{d\xi_1^2}\bigg|_{\xi_1=\mu_1}(\xi_1-\mu_1)\right|\\
&\leqslant \frac{1}{2}\sup_{\xi_1\in\mathbb{R}}\left|\frac{d^3\eta_1}{d\xi_1^3}\right| |\xi_1-\mu_1|^2 \leqslant C\langle\mu_1\rangle^{-2}|\xi_1-\mu_1|^2.
\end{aligned}$$

Combining (2.9) and (2.10) with the above three inequalities, we have

$$|L_{1,3}(t,x)| \leqslant Ct^{-1}\langle\mu_1\rangle^{-5}\int_{\mathbb{R}} \frac{|\xi_1-\mu_1| + \langle\mu_1\rangle}{1+t(\xi_1-\mu_1)^2(\xi_1^2+\mu_1\xi_1+\mu_1^2+1)}d\xi_1.$$

Hence, by an argument similar to (2.11), we have

$$|L_{1,3}(t,x)| \leqslant Ct^{-1-\gamma}\langle\mu_1\rangle^{-4\gamma-3}, \tag{2.14}$$

where $0<\gamma<1/2$. By (2.5), (2.6), (2.12) and (2.14), we have

$$L_1(t,x) = \frac{t^{-1}}{\sqrt{3\mu_1^2+1}}\mathcal{F}[\psi](\mu)e^{\frac{3}{4}it\mu_1^4+\frac{i}{2}t|\mu|^2-i\frac{\pi}{2}} + R_1(t,x), \tag{2.15}$$

where R_1 satisfies

$$|R_1(t,x)| \leqslant Ct^{-\frac{3}{2}-\gamma}\langle\mu_1\rangle^{-4\gamma-3}|\mathcal{F}[\psi](\mu)|$$

with $0<\gamma<1/2$. Hence the Plancherel identity yield

$$\begin{aligned}
\|R_1(t)\|_{L_x^2} &\leqslant Ct^{-\frac{1}{2}-\gamma}\|\langle\mu_1\rangle^{-4\gamma-2}\mathcal{F}[\psi](\mu)\|_{L_\mu^2}\\
&\leqslant Ct^{-\frac{1}{2}-\gamma}\|\mathcal{F}[\psi](\mu)\|_{L_\mu^2}\\
&= Ct^{-\frac{1}{2}-\gamma}\|\psi\|_{L_x^2}.
\end{aligned} \tag{2.16}$$

Next we evaluate L_2. By (2.11), we obtain

$$|L_2(t,x)| \leqslant Ct^{-\frac{3}{2}-\gamma}\langle\mu_1\rangle^{-4\gamma-2}|\partial_{\xi_1}\mathcal{F}[\psi](\mu)|.$$

By an argument similar to that in (2.16), we have

$$\|L_2(t)\|_{L^2_x} \leqslant Ct^{-\frac{1}{2}-\gamma}\|\psi\|_{H^{0,1}}. \tag{2.17}$$

Next, we evaluate L_3. We write

$$L_3(t,x) =: t^{-\frac{1}{2}} e^{\frac{3}{4}it\mu_1^4+\frac{i}{2}t|\mu|^2-i\frac{1}{4}\pi}\tilde{L}_3(t,x).$$

The same argument as that in (2.13) yields that $\tilde{L}_3$ is equal to the right hand side of (2.13) by replacing $1-\frac{d\eta_1}{d\xi_1}+\left.\frac{d^2\eta_1}{d\xi_1^2}\right|_{\xi_1=\mu_1}(\xi_1-\mu_1)$ by $\mathcal{F}[\psi](\xi_1,\mu_2)-\mathcal{F}[\psi](\mu_1,\mu_2)-\partial_{\xi_1}\mathcal{F}[\psi](\mu)(\xi_1-\mu_1)$. Since

$$\begin{aligned}
&|\partial_{\xi_1}\mathcal{F}[\psi](\xi_1,\mu_2)-\partial_{\xi_1}\mathcal{F}[\psi](\mu)| \leqslant \|\partial_{\xi_1}^2\mathcal{F}[\psi](\cdot,\mu_2)\|_{L^2_{\xi_1}}|\xi_1-\mu_1|^{\frac{1}{2}},\\
&|\mathcal{F}[\psi](\xi_1,\mu_2)-\mathcal{F}[\psi](\mu_1,\mu_2)-\partial_{\xi_1}\mathcal{F}[\psi](\mu)(\xi_1-\mu_1)|\\
&\qquad\qquad \leqslant \frac{2}{3}\|\partial_{\xi_1}^2\mathcal{F}[\psi](\cdot,\mu_2)\|_{L^2_{\xi_1}}|\xi_1-\mu_1|^{\frac{3}{2}},
\end{aligned}$$

we have

$$\begin{aligned}
&|\tilde{L}_3(t,x)|\\
&\leqslant Ct^{-1}\|\partial_{\xi_1}^2\mathcal{F}[\psi](\cdot,\mu_2)\|_{L^2_{\xi_1}}\\
&\quad\times\left\{\langle\mu_1\rangle^{-2}\left(\int_{\mathbb{R}}\frac{1}{\{1+t(\xi_1-\mu_1)^2(\xi_1^2+\mu_1\xi_1+\mu_1^2+1)\}^2}d\xi_1\right)^{\frac{1}{2}}\right.\\
&\qquad\left.+\langle\mu_1\rangle^{-3}\int_{\mathbb{R}}\frac{|\xi_1-\mu_1|^{\frac{1}{2}}+\langle\mu_1\rangle|\xi_1-\mu_1|^{-\frac{1}{2}}}{1+t(\xi_1-\mu_1)^2(\xi_1^2+\mu_1\xi_1+\mu_1^2+1)}d\xi_1\right\}.
\end{aligned}$$

By an argument similar to that in (2.11), we obtain

$$|\tilde{L}_3(t,x)| \leqslant Ct^{-1-\gamma}\langle\mu_1\rangle^{-4\gamma-\frac{3}{2}}\|\partial_{\xi_1}^2\mathcal{F}[\psi](\cdot,\mu_2)\|_{L^2_{\xi_1}},$$

where $0<\gamma<1/4$. Hence

$$\begin{aligned}
\|L_3(t)\|_{L^2_{x_1}} &\leqslant Ct^{-\frac{3}{2}-\gamma}\|\langle\mu_1\rangle^{-4\gamma-\frac{3}{2}}\|_{L^2_{x_1}}\|\partial_{\xi_1}^2\mathcal{F}[\psi](\cdot,\mu_2)\|_{L^2_{\xi_1}}\\
&\leqslant Ct^{-1-\gamma}\|\langle\mu_1\rangle^{-4\gamma-\frac{1}{2}}\|_{L^2_{\mu_1}}\|\partial_{\xi_1}^2\mathcal{F}[\psi](\cdot,\mu_2)\|_{L^2_{\xi_1}}\\
&\leqslant Ct^{-1-\gamma}\|\partial_{\xi_1}^2\mathcal{F}[\psi](\cdot,\mu_2).\|_{L^2_{\xi_1}}
\end{aligned}$$

Combining the above inequality and the Plancherel identity, we have

$$\begin{aligned}
\|L_3(t)\|_{L^2_x} &\leqslant Ct^{-1-\gamma}\|\|\partial_{\xi_1}^2\mathcal{F}[\psi](\cdot,\mu_2)\|_{L^2_{\mu_1}}\|_{L^2_{x_2}}\\
&\leqslant Ct^{-\frac{1}{2}-\gamma}\|\psi\|_{H^{0,2}}.
\end{aligned} \tag{2.18}$$

Finally let us evaluate R. R can be rewritten as

$$R = t^{-\frac{1}{2}}e^{\frac{i}{2}t|\mu_2|^2-i\frac{1}{4}\pi}W_{4LS}(t)\mathcal{F}_{x_2\mapsto\xi_2}[(e^{\frac{iy_2^2}{2t}}-1)\psi](x_1,\mu_2),$$

where $\{W_{4LS}(t)\}_{t\in\mathbb{R}}$ is a unitary group generated by the linear operator $(i/2)\partial_{x_1}^2-(i/4)\partial_{x_1}^4$:

$$W_{4LS}(t)\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix_1\xi_1 - \frac{i}{2}t\xi_1^2 - \frac{i}{4}t\xi_1^4} \mathcal{F}_{x_1\mapsto\xi_1}[\phi](\xi_1)d\xi_1.$$

Then, we obtain

$$\|R(t)\|_{L^2_{x_1}} = t^{-\frac{1}{2}} \|\mathcal{F}_{x_2\mapsto\xi_2}[(e^{\frac{iy_2^2}{2t}} - 1)\psi](x_1, \mu_2)\|_{L^2_{x_1}}.$$

Combining the above identity and the Plancherel identity, we have

$$\begin{aligned}
\|R(t)\|_{L^2_x} &= t^{-\frac{1}{2}} \| \|\mathcal{F}_{x_2\mapsto\xi_2}[(e^{\frac{iy_2^2}{2t}} - 1)\psi](x_1, \mu_2)\|_{L^2_{x_1}} \|_{L^2_{x_2}} \\
&= t^{-\frac{1}{2}} \| \|\mathcal{F}_{x_2\mapsto\xi_2}[(e^{\frac{iy_2^2}{2t}} - 1)\psi](x_1, \mu_2)\|_{L^2_{x_2}} \|_{L^2_{x_1}} \\
&= \| \|\mathcal{F}_{x_2\mapsto\xi_2}[(e^{\frac{iy_2^2}{2t}} - 1)\psi](x_1, \mu_2)\|_{L^2_{\mu_2}} \|_{L^2_{x_1}} \\
&= \|(e^{\frac{ix_2^2}{2t}} - 1)\psi\|_{L^2_x} \\
&\leqslant Ct^{-1}\|\psi\|_{H^{0,2}_x}.
\end{aligned} \tag{2.19}$$

Collecting (2.3), (2.4), (2.15), (2.16), (2.17), (2.18) and (2.19), we obtain the desired result. □

To prove Theorem 1.1, we employ the decay estimate and the Strichartz estimate for the linear fourth order Schrödinger equation (2.1).

Lemma 2.2. *Let $W(t)$ be given by (2.2).*
(i) Let $2 \leqslant p \leqslant \infty$. Then, the inequality

$$\|\langle\partial_{x_1}\rangle^{1-\frac{2}{p}} W(t)\psi\|_{L^p_x} \leqslant Ct^{-d(\frac{1}{2}-\frac{1}{p})}\|\psi\|_{L^{p'}_x}$$

holds.
(ii) Let (q_j, r_j) $(j = 1, 2)$ satisfy $1/q_j + 1/r_j = 1/2$ and $2 \leqslant r_j < \infty$. Then, the inequality

$$\left\| \langle\partial_{x_1}\rangle^{\frac{1}{q_1}} \int_t^{+\infty} W(t-t')F(t')dt' \right\|_{L^{q_1}_t(t,\infty;L^{r_1}_x)} \leqslant C\|\langle\partial_{x_1}\rangle^{-\frac{1}{q_2}} F\|_{L^{q_2'}_t(t,\infty;L^{r_2'}_x)}$$

holds.

Proof of Lemma 2.2. See [11, Theorem 3.1, Theorem 3.2] for instance. □

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. To this end, we show the following lemma for the asymptotic profile.

Lemma 3.1. *Let S_+ be given by (1.4). Then we have for $t \geqslant 3$,*

$$\|\hat{\psi}_+ e^{iS_+(t,\xi)}\|_{H^2_\xi} \leqslant (\log t)^2 P(\|\psi_+\|_{H^{0,2}}),$$

$$\|\frac{1}{\sqrt{3\xi_1^2+1}} |\hat{\psi}_+|\hat{\psi}_+ e^{iS_+(t,\xi)}\|_{H^2_\xi} \leqslant (\log t)^2 P(\|\psi_+\|_{H^{0,2}}),$$

where $P(\|\psi_+\|_{H^{0,2}})$ is a polynomial in $\|\psi_+\|_{H^{0,2}}$ without constant term.

Proof of Lemma 3.1. Since the proof follows from a direct calculations, we omit the detail. □

Let us start the proof of Theorem 1.1. We first rewrite (1.6) as the integral equation. Let $\mathcal{L} = i\partial_t + (1/2)\Delta - (1/4)\partial_{x_1}^4$ and let

$$w(t,\xi) = \hat{\psi}_+(\xi)e^{iS_+(t,\xi)}, \tag{3.1}$$

where S_+ is given by (1.4). From (1.6) and (3.1), we obtain

$$i\partial_t(\mathcal{F}W(-t)u) = \mathcal{F}W(-t)\mathcal{L}u = \lambda\mathcal{F}W(-t)|u|u, \tag{3.2}$$

$$i\partial_t w = \lambda\frac{t^{-1}}{\sqrt{3\xi_1^2+1}}|\hat{\psi}_+(\xi)|\hat{\psi}_+(\xi)e^{iS_+(t,\xi)}. \tag{3.3}$$

Subtracting (3.3) from (3.2), we have

$$\begin{aligned} &i\partial_t(\mathcal{F}W(-t)u - w) \\ &= \lambda\mathcal{F}W(-t)\left[|u|u - W(t)\mathcal{F}^{-1}\left[\frac{t^{-1}}{\sqrt{3\xi_1^2+1}}|\hat{\psi}_+(\xi)|\hat{\psi}_+(\xi)e^{iS_+(t,\xi)}\right]\right]. \end{aligned} \tag{3.4}$$

Proposition 2.1 and Lemma 3.1 yield

$$W(t)\mathcal{F}^{-1}\left[\frac{t^{-1}}{\sqrt{3\xi_1^2+1}}|\hat{\psi}_+(\xi)|\hat{\psi}_+(\xi)e^{iS_+(t,\xi)}\right] = |u_+|u_+ + R_1(t),$$

where u_+ is given by (1.4) and R_1 satisfies

$$\begin{aligned} \|R_1(t)\|_{L_x^2} &\leqslant Ct^{-1-\beta}\|\mathcal{F}^{-1}[\frac{1}{\sqrt{3\xi_1^2+1}}|\hat{\psi}_+(\xi)|\hat{\psi}_+(\xi)e^{iS_+(t,\xi)}]\|_{H_x^{0,2}} \\ &= Ct^{-1-\beta}\|\frac{1}{\sqrt{3\xi_1^2+1}}|\hat{\psi}_+|\hat{\psi}_+e^{iS_+(t,\xi)}\|_{H_\xi^2} \\ &\leqslant Ct^{-1-\beta}(\log t)^2P(\|\psi_+\|_{H_x^{0,2}}), \end{aligned} \tag{3.5}$$

where $0 < \beta < 3/4$. Furthermore, by Proposition 2.1 and Lemma 3.1,

$$\begin{aligned} &W(t)\mathcal{F}^{-1}\left[\frac{t^{-1}}{\sqrt{3\xi_1^2+1}}|\hat{\psi}_+(\xi)|\hat{\psi}_+(\xi)e^{iS_+(t,\xi)}\right] \\ &\qquad = |W(t)\mathcal{F}^{-1}w|W(t)\mathcal{F}^{-1}w + R_1(t) + R_2(t), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \|R_2(t)\|_{L_x^2} &= \||W(t)\mathcal{F}^{-1}w|W(t)\mathcal{F}^{-1}w - |u_+|u_+\|_{L_x^2} \\ &\leqslant (\|W(t)\mathcal{F}^{-1}w\|_{L_x^\infty} + \|u_+\|_{L_x^\infty})\|W(t)\mathcal{F}^{-1}w - u_+\|_{L_x^2} \\ &\leqslant Ct^{-1-\beta}(\log t)^4P(\|\psi_+\|_{H_x^{0,2}}). \end{aligned} \tag{3.7}$$

Substituting (3.6) into (3.4), we obtain

$$\begin{aligned}
&i\partial_t(\mathcal{F}W(-t)u - w) \\
&= \lambda \mathcal{F}W(-t)[|u|u - |W(t)\mathcal{F}^{-1}w|W(t)\mathcal{F}^{-1}w] - \lambda \mathcal{F}W(-t)(R_1 + R_2).
\end{aligned}$$

Integrating the above equation with respect to t variable on (t, ∞), we have

$$\begin{aligned}
&u(t) - W(t)\mathcal{F}^{-1}w \\
&= i\lambda \int_t^{+\infty} W(t-\tau)[|u|u - |W(t)\mathcal{F}^{-1}w|W(t)\mathcal{F}^{-1}w](\tau)d\tau \\
&\quad - i\lambda \int_t^{+\infty} W(t-\tau)(R_1 + R_2)(\tau)d\tau.
\end{aligned} \tag{3.8}$$

To show the existence of u satisfying (3.8), we shall prove that if $\|\psi_+\|_{H^{0,2}}$ is sufficiently small, then the map Φ given by

$$\begin{aligned}
\Phi[u](t) = i\lambda \int_t^{+\infty} W(t-\tau)[|u|u - |W(t)\mathcal{F}^{-1}w|W(t)\mathcal{F}^{-1}w](\tau)d\tau \\
- i\lambda \int_t^{+\infty} W(t-\tau)(R_1 + R_2)(\tau)d\tau
\end{aligned}$$

is a contraction on

$$\begin{aligned}
\mathbf{X}_{\rho,T} = \{u \in C([T,\infty); L^2(\mathbb{R}^2)) \cap \langle\partial_{x_1}\rangle^{-\frac{1}{4}} L^4_{loc}(T,\infty; L^4(\mathbb{R}^2)); \\
\|u - W(t)\mathcal{F}^{-1}w\|_{\mathbf{X}_T} \leqslant \rho\}, \\
\|v\|_{\mathbf{X}_T} = \sup_{t \geqslant T} t^{\alpha}(\|v\|_{L^\infty(t,\infty;L^2_x)} + \|\langle\partial_{x_1}\rangle^{\frac{1}{4}} v\|_{L^4(t,\infty;L^4_x)})
\end{aligned}$$

for some $T \geqslant 3$ and $\rho > 0$.

Let $v(t) = u(t) - W(t)\mathcal{F}^{-1}w$ and $v \in \mathbf{X}_{\rho,T}$. Then the Strichartz estimate (Lemma 2.2) implies

$$\begin{aligned}
&\|\Phi[u] - W(t)\mathcal{F}^{-1}w\|_{L^\infty(t,\infty;L^2_x)} + \|\langle\partial_{x_1}\rangle^{\frac{1}{4}}(\Phi[u] - W(t)\mathcal{F}^{-1}w)\|_{L^4(t,\infty;L^4_x)} \\
&\leqslant C(\||v|v\|_{L^{\frac{4}{3}}(t,\infty;L^{\frac{4}{3}}_x)} + \||W(t)\mathcal{F}^{-1}w|v\|_{L^1(t,\infty;L^2_x)}) \\
&\quad + \|R_1\|_{L^1(t,\infty;L^2_x)}) + \|R_1\|_{L^1(t,\infty;L^2_x)}).
\end{aligned} \tag{3.9}$$

By the Hölder inequality,

$$\begin{aligned}
\||v|v\|_{L^{\frac{4}{3}}(t,\infty;L^{\frac{4}{3}}_x)} &\leqslant C\|\|v\|_{L^2_x}\|v\|_{L^4_x}\|_{L^{\frac{4}{3}}(t,\infty)} \\
&\leqslant C\rho\|t^{-\alpha}\|v\|_{L^4_x}\|_{L^{\frac{4}{3}}(t,\infty)} \\
&\leqslant C\rho\|t^{-\alpha}\|_{L^2(t,\infty)}\|v\|_{L^4(t,\infty;L^4)} \\
&\leqslant C\rho^2 t^{-2\alpha+\frac{1}{2}}, \\
\||W(t)\mathcal{F}^{-1}w|v\|_{L^1(t,\infty;L^2_x)} &\leqslant \|\|W(t)\mathcal{F}^{-1}w\|_{L^\infty_x}\|v\|_{L^2_x}\|_{L^1(t,\infty)} \\
&\leqslant C\rho P(\|\psi_+\|_{H^{0,2}_x})\|t^{-1-\alpha}\|_{L^1(t,\infty)} \\
&\leqslant C\rho P(\|\psi_+\|_{H^{0,2}_x})t^{-\alpha}.
\end{aligned}$$

Substituting the above two inequalities, (3.5), and (3.7) into (3.9), we have

$$\|\Phi[u] - W(t)\mathcal{F}^{-1}w\|_{\mathbf{X}_T} \leqslant C(\rho^2 T^{-\alpha+\frac{1}{2}} + \rho P(\|\psi_+\|_{H_x^{0,2}}) + T^{\alpha-\beta}(\log T)^4).$$

Choosing $1/2 < \alpha < \beta < 3/4$, T large enough, and $\|\psi_+\|_{H_x^{0,2}}$ sufficiently small, we find that Φ is a map onto $\mathbf{X}_{\rho,T}$. In a similar way we can conclude that Φ is a contraction map on $\mathbf{X}_{\rho,T}$. Therefore, by the Banach fixed point theorem we find that Φ has a unique fixed point in $\mathbf{X}_{\rho,T}$ which is the solution to the final state problem (1.6).

From (1.6), we obtain

$$u(t) = W(t-T)u(T) - i\lambda \int_T^t W(\tau)|u|u(\tau)d\tau. \tag{3.10}$$

Since $u(T) \in L_x^2(\mathbb{R}^3)$, combining the argument by [15] with the Strichartz estimate (Lemma 2.2) and L^2 conservation law for (3.10), we can prove that (3.10) has a unique global solution in $C(\mathbb{R}; L_x^2(\mathbb{R}^2)) \cap \langle\partial_{x_1}\rangle^{-1/4} L_{loc}^4(\mathbb{R}; L_x^4(\mathbb{R}^2))$. Therefore the solution u of (1.6) can be extended to all times.

Finally we show that the solution to (1.6) converges to u_+ in L^2 as $t \to \infty$. Since $u - W(t)\mathcal{F}^{-1}w \in \mathbf{X}_{\rho,T}$, Proposition 2.1 and Lemma 3.1 yield

$$\begin{aligned}
&\|u(t) - u_+(t)\|_{L_x^2} \\
&\leqslant \|u(t) - W(t)\mathcal{F}^{-1}w\|_{L_x^2} + \|W(t)\mathcal{F}^{-1}w - u_+\|_{L_x^2} \\
&\leqslant Ct^{-\alpha} + Ct^{-\beta}(\log t)^2 \\
&\leqslant Ct^{-\alpha},
\end{aligned}$$

where $1/2 < \alpha < \beta < 3/4$. This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.3

In this section we prove Theorem 1.3 via the argument by Glassey [7]. To prove Theorem 1.3, we employ the asymptotic formula [13, Proposition 2.1] as $t \to \infty$ for the solution to

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u - \dfrac{1}{4}\partial_{x_1}^4 u = 0, & t > 0, x \in \mathbb{R}^d, \\ u(0,x) = \psi(x), & x \in \mathbb{R}^d. \end{cases} \tag{4.1}$$

Lemma 4.1. *Let $W(t)\psi$ be a solution to (4.1). Then we have*

$$[W(t)\psi](x) = \frac{t^{-\frac{d}{2}}}{\sqrt{3\mu_1^2+1}} \hat{\psi}(\mu) e^{\frac{3}{4}it\mu_1^4 + \frac{1}{2}it|\mu|^2 - i\frac{d}{4}\pi} + R(t,x)$$

for $t \geqslant 2$, where $\mu = (\mu_1, \mu_\perp)$ is given by

$$\begin{aligned}
\mu_1 &= \left\{\frac{1}{2t}\left(x_1 + \sqrt{x_1^2 + \frac{4}{27}t^2}\right)\right\}^{1/3} + \left\{\frac{1}{2t}\left(x_1 - \sqrt{x_1^2 + \frac{4}{27}t^2}\right)\right\}^{1/3}, \\
\mu_\perp &= \frac{x_\perp}{t}
\end{aligned}$$

and R satisfies

$$\|R(t)\|_{L^p_x} \leqslant Ct^{-d(\frac{1}{2}-\frac{1}{p})-\beta}\|\psi\|_{H^{0,s}_x},$$

for $2 \leqslant p \leqslant \infty$, $1/(4p) < \beta < 1/2$ *and* $s > d/2 - (d-1)/p + 1$.

Proof of Lemma 4.1. See [13, Proposition 2.1]. □

Proof of Theorem 1.3. Let $0 < t_1 < t_2$ and let u be a solution to (1.1) satisfying (1.8). We denote $\langle u, v\rangle = \int u(x)\overline{v(x)}dx$. Then a direct calculation shows

$$\begin{aligned}
&\langle W(-t_1)u(t_1) - W(-t_2)u(t_2), \psi_+\rangle_{L^2_x} \\
&= -i\lambda \int_{t_1}^{t_2} \langle W(-\tau)|u|^{p-1}u(\tau), \psi_+\rangle_{L^2_x} d\tau \\
&= -i\lambda \int_{t_1}^{t_2} \langle (|u|^{p-1}u)(\tau), W(\tau)\psi_+\rangle_{L^2_x} d\tau \\
&= -i\lambda \int_{t_1}^{t_2} \langle (|u^0_+|^{p-1}u^0_+)(\tau), u^0_+(\tau)\rangle_{L^2_x} d\tau \\
&\quad -i\lambda \int_{t_1}^{t_2} \langle (|u|^{p-1}u)(\tau), W(\tau)\psi_+ - u^0_+(\tau)\rangle_{L^2_x} d\tau \\
&\quad -i\lambda \int_{t_1}^{t_2} \langle (|u|^{p-1}u)(\tau) - (|u^0_+|^{p-1}u^0_+)(\tau), u^0_+(\tau)\rangle_{L^2_x} d\tau \\
&=: I_1(t_1,t_2) + I_2(t_1,t_2) + I_3(t_1,t_2),
\end{aligned} \tag{4.2}$$

where u^0_+ is defined by

$$u^0_+(t,x) = \frac{t^{-\frac{d}{2}}}{\sqrt{3\mu_1^2+1}}\hat{\psi}_+(\mu)e^{\frac{3}{4}it\mu_1^4 + \frac{1}{2}it|\mu|^2 - i\frac{d}{4}\pi}.$$

By the definition of u^0_+, we easily see

$$I_1(t_1,t_2) = -i\lambda \left(\int_{\mathbb{R}^d} \frac{|\hat{\psi}_+(\mu)|^{p+1}}{(1+3\mu_1^2)^{\frac{p-1}{2}}} d\mu\right)\left(\int_{t_1}^{t_2} \tau^{-\frac{d}{2}(p-1)} d\tau\right). \tag{4.3}$$

By Lemma 4.1 and the conservation law for L^2 norm of u, we have

$$\begin{aligned}
|I_2(t_1,t_2)| &\leqslant C \int_{t_1}^{t_2} \|u(\tau)\|^p_{L^2_x} \|W(\tau)\psi_+ - u^0_+(\tau)\|_{L_x^{\frac{2}{2-p}}} d\tau \\
&\leqslant C\|u_0\|^p_{L^2_x}\|\psi_+\|_{H^{0,s}_x} \int_{t_1}^{t_2} \tau^{-\frac{d}{2}(p-1)-\beta} d\tau,
\end{aligned} \tag{4.4}$$

where $(2-p)/8 < \beta < 1/2$ and $s > (4-d)/2 + (d-1)p/2$.

By Lemma 4.1 and the conservation law for L^2 norm of u, we have

$$\begin{aligned}|I_3(t_1,t_2)| &\leqslant C\int_{t_1}^{t_2}(\|u(\tau)\|_{L_x^2}^{p-1}+\|u_+^0(\tau)\|_{L_x^2}^{p-1})\|u(\tau)-u_+^0\|_{L_x^2}\|u_+^0\|_{L_x^{\frac{2}{2-p}}}d\tau\\ &\leqslant C(\|u_0\|_{L_x^2}^{p-1}+\|\psi_+\|_{L_x^2}^{p-1})\|\langle\partial_{x_1}\rangle^{-(p-1)}\psi_+\|_{L^{\frac{p}{2}}}\\ &\quad\times\int_{t_1}^{t_2}(\|u(\tau)-W(\tau)\psi_+\|_{L_x^2}+\tau^{-\beta}\|\psi_+\|_{H^{0,s}})\tau^{-\frac{d}{2}(p-1)}d\tau,\end{aligned}\tag{4.5}$$

where $1/8<\beta<1/2$ and $s>3/2$. By (4.2), (4.3), (4.4) and (4.5), we have

$$\begin{aligned}&|\langle W(-t_1)u(t_1)-W(-t_2)u(t_2),\psi_+\rangle_{L_x^2}|\\ &\geqslant|\lambda|\left(\int_{\mathbb{R}^d}\frac{|\hat{\psi}_+(\mu)|^{p+1}}{(1+3\mu_1^2)^{\frac{p-1}{2}}}d\mu\right)\left(\int_{t_1}^{t_2}\tau^{-\frac{d}{2}(p-1)}d\tau\right)\\ &\quad-C\int_{t_1}^{t_2}\tau^{-\frac{d}{2}(p-1)-\beta}d\tau-C\int_{t_1}^{t_2}\|u(\tau)-W(\tau)\psi_+\|_{L_x^2}\tau^{-\frac{d}{2}(p-1)}d\tau.\end{aligned}$$

Hence by the assumption (1.8) on u, we see that there exists $T>0$ such that for $t_2>t_1>T$,

$$\begin{aligned}&\left|\langle W(-t_1)u(t_1)-W(-t_2)u(t_2),\psi_+\rangle_{L_x^2}\right|\\ &\geqslant\frac{|\lambda|}{2}\left(\int_{\mathbb{R}^d}\frac{|\hat{\psi}_+(\mu)|^{p+1}}{(1+3\mu_1^2)^{\frac{p-1}{2}}}d\mu\right)\left(\int_{t_1}^{t_2}\tau^{-\frac{d}{2}(p-1)}d\tau\right).\end{aligned}$$

Hence we have $u\equiv 0$. This completes the proof. $\square$

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