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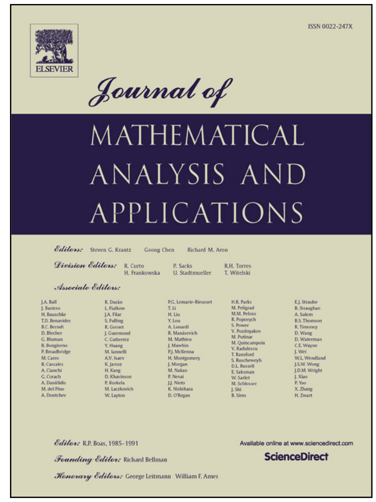

Boundedness and finite-time blow-up in a quasilinear parabolic–elliptic chemotaxis system with logistic source and nonlinear production

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Boundedness and finite-time blow-up in a quasilinear parabolic–elliptic chemotaxis system with logistic source and nonlinear production

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Abstract. This paper deals with the quasilinear parabolic–elliptic chemotaxis system with logistic source and nonlinear production,

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) + \lambda u - \mu u^\kappa, & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \overline{M_f}(t) + f(u), & x \in \Omega,\ t > 0, \end{cases}$$

where $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\overline{M_f}(t) := \frac{1}{|\Omega|}\int_\Omega f(u(x,t))\,dx$, and D, S and f are functions generalizing the prototypes

$$D(u) = (u+1)^{m-1}, \quad S(u) = u(u+1)^{\alpha-1} \quad \text{and} \quad f(u) = u^\ell$$

with $m \in \mathbb{R}$, $\alpha > 0$ and $\ell > 0$. In the case $m = \alpha = \ell = 1$, Fuest (NoDEA Nonlinear Differential Equations Appl.; 2021; 28; 16) obtained conditions for κ such that solutions blow up in finite time. However, in the above system boundedness and finite-time blow-up of solutions have been not yet established. This paper gives boundedness and finite-time blow-up under some conditions for m, α, κ and ℓ.

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1. Introduction

In this paper we consider the following quasilinear parabolic–elliptic chemotaxis system with logistic source and nonlinear production:

$$
(1.1)\qquad \begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(S(u)\nabla v) + \lambda u - \mu u^\kappa, & x\in\Omega,\ t>0,\\ 0 = \Delta v - \overline{M_f}(t) + f(u), & x\in\Omega,\ t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases}
$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$; $\lambda>0$, $\mu>0$ and $\kappa>1$; $D,S\in C^2([0,\infty))$ and $D(0)>0$; $f\in\bigcup_{\beta\in(0,1)}C^{\beta}_{\mathrm{loc}}([0,\infty))\cap C^1((0,\infty))$;

$$
\overline{M_f}(t) := \frac{1}{|\Omega|}\int_\Omega f(u(x,t))\,dx;
$$

ν is the outward normal vector to $\partial\Omega$; $u_0\in\bigcup_{\beta\in(0,1)}C^\beta(\overline{\Omega})$ is nonnegative.

The system (1.1) describes a motion of cellular slime molds with chemotaxis, and the unknown function $u=u(x,t)$ denotes the density of cells and the unknown function $v=v(x,t)$ represents the concentration of the chemical substance at place $x\in\Omega$ and time $t>0$. This system is one of many types of the Keller–Segel system

$$
(1.2)\qquad \begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases}
$$

which was proposed by Keller and Segel [**9**]. A number of variations of the original system (1.2) and related results for blow-up (in the radial setting) and boundedness are introduced in [**1, 6, 11**]:

- We first focus on the quasilinear Keller–Segel system,

$$
\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(S(u)\nabla v), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + f(u), & x\in\Omega,\ t>0, \end{cases}
$$

  where $\tau\in\{0,1\}$. When $f(u)=u$, in the parabolic–parabolic setting ($\tau=1$), Tao and Winkler [**17**] showed that solutions are global and bounded under the conditions that $\frac{S(u)}{D(u)}\leq cu^q$ with $q<\frac{2}{n}$ and $c>0$ and that Ω is a convex domain; Ishida et al. [**7**] removed the convexity of Ω; whereas Winkler [**21**] proved that solutions blow up in either finite or infinite time when $\frac{S(u)}{D(u)}\geq cu^q$ with $q>\frac{2}{n}$ and $c>0$; In the parabolic–elliptic setting ($\tau=0$), Lankeit [**10**] proved that solutions exist globally and are bounded in the case $q<\frac{2}{n}$ and that unbounded solutions are constructed in the case $q>\frac{2}{n}$. When $\tau=1$ and $D(u)=1$, $S(u)=u$ and $f(u)=u^\ell$ with $\ell>0$, Liu and Tao [**13**] established global existence and boundedness under the condition that $0<\ell<\frac{2}{n}$; Moreover, in the case that $D(u)=(u+1)^{m-1}$ and $S(u)=u(1+u)^{\alpha-1}$ with $m\in\mathbb{R}$ and $\alpha\in\mathbb{R}$, it was obtained that solutions are global and bounded under the condition $\alpha-m+\max\left\{\ell,\frac{1}{n}\right\}<\frac{2}{n}$ in [**15**].

- We next review the quasilinear Keller–Segel system with logistic source,

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) + \lambda u - \mu u^\kappa, & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + f(u), & x \in \Omega,\ t > 0, \end{cases}$$

where $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\tau \in \{0, 1\}$. As to this system, blow-up phenomena are suppressed when $\kappa \geq 2$ and $f(u) = u$. Indeed, in the parabolic–parabolic setting ($\tau = 1$), when $D(u) = 1$ and $S(u) = u$, Winkler [**20**] derived that solutions exist globally and are bounded if $\mu > 0$ is so large and $\kappa = 2$; When $D(u) = (u + 1)^{m-1}$ and $S(u) = u(u + 1)^{\alpha-1}$ with $m \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, global existence and boundedness were obtained if $\lambda = \mu = 1$, $\kappa = 2$ and $0 < \alpha - m + 1 < \frac{4}{4+n}$ by Zheng [**28**]. In the parabolic–elliptic setting ($\tau = 0$), when $D(u) = 1$ and $S(u) = u$, Tello and Winker [**18**] showed that solutions exist globally and are bounded in the cases that $\kappa = 2$ and $\mu > \max\left\{0, \frac{n-2}{n}\right\}$ and that $\kappa > 2$ and $\mu > 0$; When $D(u) = u^{m-1}$ and $S(u) = u^\alpha$ for all $u \geq 1$ with $m \geq 1$ and $\alpha > 0$, Zheng [**27**] proved global existence and boundedness in the cases that $\kappa > 1$ and $\alpha + 1 < \max\left\{m + \frac{2}{n}, \kappa\right\}$ and that $\kappa > 1$, $\alpha + 1 = \kappa$ and $\mu > \mu_0$ for some $\mu_0 > 0$. On the other hands, in the parabolic–elliptic setting, it is known that blow-up occurs under the some conditions for $\kappa > 1$ when $f(u) = u$. When $D(u) = 1$ and $S(u) = u$, Winkler [**24**] presented that if $1 < \kappa < \frac{7}{6}$ ($n \in \{3, 4\}$) and $1 < \kappa < 1 + \frac{1}{2(n-1)}$ ($n \geq 5$), then solutions blow up in finite time; Similar blow-up results were obtained in the case that $D(u) = (u+1)^{m-1}$ and $S(u) = u(u + 1)^{\alpha-1}$ with $m \geq 1$ and $\alpha > 0$ (see [**2, 14, 16**]).

- We turn our eyes into the quasilinear parabolic–elliptic chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \overline{M_f}(t) + f(u), & x \in \Omega,\ t > 0. \end{cases}$$

A simplification of this system was introduced by Jöger and Luckhaus [**8**]. When $D(u) = (u + 1)^{m-1}$ with $m \in \mathbb{R}$, $S(u) = u$ and $f(u) = u$, Cieślak and Winkler [**3**] derived global existence and boundedness in the case $2 - m < \frac{2}{n}$ and finite-time blow-up in the case $2 - m > \frac{2}{n}$; When $D(u) = (u + 1)^{m-1}$ and $S(u) = u(u + 1)^{\alpha-1}$ with $m \leq 1$ and $\alpha \in \mathbb{R}$ as well as $f(u) = u$, Winkler and Djie [**25**] proved that solutions are global and bounded if $\alpha - m + 1 < \frac{2}{n}$, whereas finite-time blow-up occurs if $\alpha - m + 1 > \frac{2}{n}$; When $D(u) = 1$, $S(u) = u$ and $f(u) = u^\ell$ with $\ell > 0$, Winkler [**23**] obtained that solutions exist globally and remain bounded in the case $\ell < \frac{2}{n}$ and that there exist solutions which are unbounded in finite time in the case $\ell > \frac{2}{n}$; When $D(u) = (u + 1)^{m-1}$, $S(u) = u$ and $f(u) = u^\ell$ with $m \in \mathbb{R}$ and $\ell > 0$, global existence and boundedness were established if $\ell - m + 1 < \frac{2}{n}$ by Li [**12**]. Moreover, in [**12**] it was asserted that finite-time blow-up occurs under the condition that $\ell - m + 1 > \frac{2}{n}$. However, this condition should be repaired because from assumptions of [**12**, Lemma 3.5] we can obtain the condition that

$$\ell - (m - 1)_+ > \frac{2}{n}, \quad \text{where} \quad (m - 1)_+ := \max\{0, m - 1\}; \tag{1.3}$$

When $D(u) = 1$, $S(u) = u(u + 1)^{\alpha-1}$ and $f(u) = u^\ell$ with $\alpha > 0$ and $\ell > 0$, Wang and Li [**19**] derived the critical value $\alpha + \ell - 1 = \frac{2}{n}$.

- In the system (1.1), when $D(u) = 1$, $S(u) = u$ and $f(u) = u$, Winkler [**22**] showed that if $1 < \kappa < \frac{3}{2} + \frac{1}{2n-2}$ $(n \geq 5)$, then there exists a solution blowing up in finite time; Moreover, a similar blow-up result was obtained in the case that $D(u) = (u+1)^{m-1}$ with $m \geq 1$ in [**2**]; Furthermore, Fuest [**5**] showed that solutions blow up in finite time under the conditions that $1 < \kappa < \min\left\{2, \frac{n}{2}\right\}$ and $\mu > 0$ $(n \geq 3)$ and that $\kappa = 2$ and $\mu \in \left(0, \frac{n-4}{n}\right)$ $(n \geq 5)$; In the two dimensional setting and $\kappa = 2$, global existence and boundedness were established when $\int_\Omega u_0 < 8\pi$, whereas finite-time blow-up occurs when $\int_\Omega u_0 < m_0$ with $m_0 > 8\pi$ in [**4**].

In summary, in [**2, 4, 5, 22**], blow-up results were derived in the chemotaxis system with logistic source and *linear* production. However, boundedness and blow-up results were not obtained in the quasilinear chemotaxis system with logistic source and *nonlinear* production (when $D(u) = 1$ and $S(u) = u$, recently, Yi et al. [**26**] derived the blow-up result under the condition that $\ell + 1 > \kappa\left(1 + \frac{2}{n}\right)$).

Our aim of this paper is to present conditions that solutions of (1.1) are bounded or blow up. Before we state the main results, we give conditions for the functions D, S and f as follows:

$$D \in C^2([0,\infty)) \text{ is positive;} \tag{1.4}$$

$$S \in C^2([0,\infty)) \text{ is nonnegative and nondecreasing;} \tag{1.5}$$

$$f \in \bigcup_{\beta\in(0,1)} C^\beta_{\mathrm{loc}}([0,\infty)) \cap C^1((0,\infty)) \text{ is nonnegative and nondecreasing.} \tag{1.6}$$

We now state the main theorems. The first one asserts boundedness of solutions.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a smooth bounded domain, and let $\delta \in (0,1]$, $m \in \mathbb{R}$, $\alpha > 0$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$. Assume that $u_0 \in \bigcup_{\beta\in(0,1)} C^\beta(\overline{\Omega})$ is nonnegative and D, S and f satisfy (1.4), (1.5) and (1.6) as well as*

$$D(\xi) \geq C_D(\xi+\delta)^{m-1}, \quad S(\xi) \leq C_S\xi(\xi+\delta)^{\alpha-1} \quad \textit{for all } \xi \geq 0 \tag{1.7}$$

and

$$f(\xi) \leq L\xi^\ell \quad \textit{for all } \xi \geq 0 \tag{1.8}$$

with $C_D > 0$, $C_S > 0$ and $L > 0$. If one of the following cases holds:

$$\alpha + \ell < \max\left\{m + \frac{2}{n}, \kappa\right\} \quad \textit{and} \quad \mu > 0, \tag{1.9}$$

$$\alpha + \ell = \kappa \quad \textit{and} \quad \mu > \frac{n(\alpha+\ell-m)-2}{2(\alpha-1)+n(\alpha+\ell-m)}C_SL, \tag{1.10}$$

then there exists an exactly one pair (u,v) of functions

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)), \\ v \in \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega}\times(0,\infty)) \end{cases}$$

which solves (1.1) classically. Moreover, the solution (u,v) is bounded in the sense that there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C \quad \textit{for all } t \in (0,\infty).$$

We next state a result such that solutions blow up in finite time.

Theorem 1.2. *Let* $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \geq 1)$ *be a ball with some* $R > 0$, *and let* $\delta \in (0,1]$, $m \in \mathbb{R}$, $\alpha > 0$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ *and* $\ell > 0$. *Assume that* D, S *and* f *satisfy* (1.4), (1.5) *and* (1.6) *as well as*

$$D(\xi) \leq C_D(\xi+\delta)^{m-1}, \quad S(\xi) \geq C_S\xi(\xi+\delta)^{\alpha-1} \quad \text{for all } \xi \geq 0 \tag{1.11}$$

and

$$f(\xi) \geq L\xi^{\ell} \quad \text{for all } \xi \geq 0 \tag{1.12}$$

with $C_D > 0$, $C_S > 0$ *and* $L > 0$. *Suppose that*

$$\alpha + \ell > \max\left\{m + \frac{2}{n}\kappa, \kappa\right\}, \quad \text{if } m \geq 0, \tag{1.13}$$

$$\text{or} \quad \alpha + \ell > \max\left\{\frac{2}{n}\kappa, \kappa\right\}, \qquad \text{if } m < 0. \tag{1.14}$$

Then for all $M_0 > 0$ *there exist* $\varepsilon_0 \in (0, M_0)$ *and* $r_\star \in (0,R)$ *with the following property: If*

(1.15)
$$u_0 \in \bigcup_{\beta\in(0,1)} C^{\beta}(\overline{\Omega}) \text{ is nonnegative, radially symmetric, nonincreasing with respect to } |x|$$

and

$$\int_\Omega u_0(x)\,dx = M_0 \quad \text{and} \quad \int_{B_{r_\star}(0)} u_0(x)\,dx \geq M_0 - \varepsilon_0, \tag{1.16}$$

then there exist $T^* \in (0,\infty)$ *and an exactly one pair* (u,v) *of functions*

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,T^*)) \cap C^{2,1}(\overline{\Omega}\times(0,T^*)), \\ v \in \bigcap_{q>n} C^0([0,T^*); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega}\times(0,T^*)) \end{cases}$$

which solves (1.1) *classically and blows up in the sense that*

$$\lim_{t\nearrow T^*} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Remark 1.1. As to Theorem 1.1, letting $\kappa \to 1$ implies that the condition (1.9) reduces the condition

$$\alpha + \ell < \max\left\{m + \frac{2}{n}, 1\right\},$$

which is a generalized condition such that solutions remain bounded in [**12, 19, 23**]. Also, as to Theorem 1.2, we see that the condition (1.13) with $m = 1$ and $\kappa \to 1$ is a generalized condition such that solutions blow up in finite time in [**19, 23**].

Remark 1.2. When $\alpha = 1$, letting $\kappa \to 1$ entails from (1.13) and (1.14) that

$$\ell > \max\left\{m - 1 + \frac{2}{n}, 0\right\}, \quad \text{if } m \geq 0, \tag{1.17}$$

$$\ell > \max\left\{-1 + \frac{2}{n}, 0\right\}, \quad \text{if } m < 0. \tag{1.18}$$

For instance, when $m \leq 1 - \frac{2}{n}$, we obtain from (1.3) that $\ell > \frac{2}{n}$, whereas we can observe from (1.17) and (1.18) that $\ell > \left\{\frac{2}{n} - 1, 0\right\}$. Thus the conditions (1.17) and (1.18) improve the condition in [**12**].

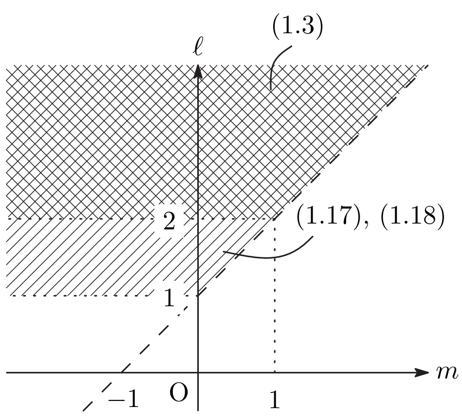

Figure 1: $n = 1$, $\alpha = 1$ and $\kappa \to 1$

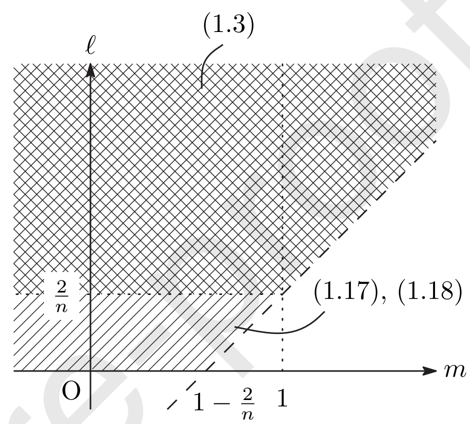

Figure 2: $n \geq 2$, $\alpha = 1$ and $\kappa \to 1$

Moreover, in the case that $m = 1$ and $\alpha = 1$, we can establish that

$$1 + \ell > \max\left\{1 + \frac{2}{n}\kappa, \kappa\right\}. \tag{1.19}$$

Because $\left(1 + \frac{2}{n}\right)\kappa > \max\left\{1 + \frac{2}{n}\kappa, \kappa\right\}$, we can make sure that the condition (1.19) is an improvement on the condition in [**26**].

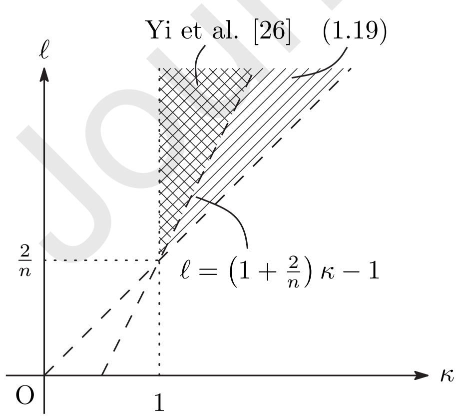

Figure 3: $n \in \{1, 2\}$, $m = 1$ and $\alpha = 1$

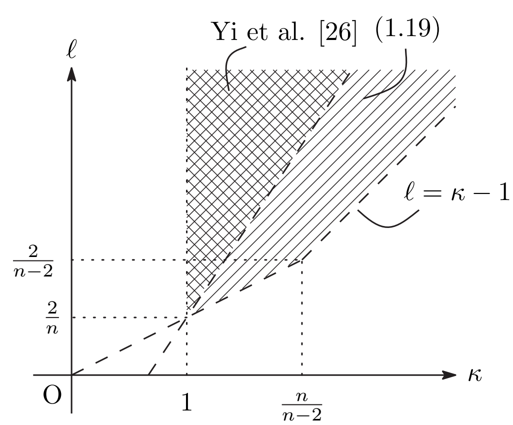

Figure 4: $n \geq 3$, $m = 1$ and $\alpha = 1$

The proofs of Theorems 1.1 and 1.2 are based on those in [**23**]. As to the proof of Theorem 1.1, our purpose is to establish an L^p-estimate for u. In order to obtain an L^p-estimate, we consider three cases. With regard to the proof of Theorem 1.2, we first define the mass accumulation function

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho,t)\,d\rho \quad \text{for } s\in[0,R^n] \text{ and } t\in[0,T_{\max}),$$

where $s := r^n$ for $r \in [0,R]$, and transform the system (1.1) to the parabolic equation

$$\begin{aligned} w_t &= n^2 s^{2-\frac{2}{n}} D(nw_s)w_{ss} - \frac{1}{n} sS(nw_s)\overline{M_f}(t) \\ &\quad + \frac{1}{n} S(nw_s)\int_0^s f(nw_s(\sigma,t))\,d\sigma + \lambda w - n^{\kappa-1}\mu\int_0^s w_s^\kappa(\sigma,t)\,d\sigma. \end{aligned}$$

Next, we introduce the moment-type functional

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0-s)w(s,t)\,ds$$

and the functional

$$\psi(t) := \int_0^{s_0} s^{1-\gamma}(s_0-s)w_s^{\alpha+\ell}(s,t)\,ds$$

with some $s_0 \in (0,R^n)$ and $\gamma \in (-\infty,1)$. By using the above functionals, we will derive nonlinear differential inequalities $\phi' \ge c_1\phi^{\alpha+\ell} - c_2$. In order to attain this inequality, we apply the inequality $\psi \ge c_3\phi^{\alpha+\ell}$ (in [**26**] the inequality $\psi \ge c_4\phi^{\frac{1+\ell}{\kappa}}$ with some $c_4 > 0$ was obtained). Moreover, in the case $m = 0$, due to use the estimate $\log(a+\delta) \le \frac{1}{\varepsilon}a^\varepsilon + c_5$ for all $\varepsilon > 0$ with some $c_5 > 0$, we can improve the condition (1.3) to the conditions (1.17) and (1.18).

This paper is organized as follows. In Section 2 we recall local existence and show Theorem 1.1. In Section 3 we prove Theorem 1.2 and give open problems.

2. Boundedness

In this section we derive global existence and boundedness in (1.1). We first introduce a result on local existence of classical solutions to (1.1). This lemma can be proved by a standard fixed point argument (see e.g., [**25**]).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \ge 1)$ be a smooth bounded domain, and let $\lambda > 0$, $\mu > 0$ and $\kappa > 1$. Assume that $u_0 \in \bigcup_{\beta\in(0,1)} C^\beta(\overline{\Omega})$ is nonnegative and D, S and f fulfill* (1.4), (1.5) *and* (1.6). *Then there exist $T_{\max} \in (0,\infty]$ and a unique classical solution (u,v) of* (1.1) *satisfying*

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,T_{\max})) \cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})), \\ v \in \bigcap_{q>n} C^0([0,T_{\max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega}\times(0,T_{\max})). \end{cases}$$

Moreover, $u \ge 0$ in $\Omega\times(0,T_{\max})$ and

$$\text{if } T_{\max} < \infty, \quad \text{then} \quad \lim_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

If u_0 is radially symmetric, then so are $u(\cdot,t)$ and $v(\cdot,t)$ for all $t \in (0,T_{\max})$.

In the following we assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a smooth bounded domain and $\delta \in (0,1]$, $m \in \mathbb{R}$, $\alpha > 0$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$. Also, we suppose that D, S and f satisfy (1.7) and (1.8). Moreover, let (u, v) be the solution of (1.1) on $[0, T_{\max})$ as in Lemma 2.1. We next recall the following lemma which is obtained from the first equation in (1.1).

Lemma 2.2. *The classical solution u satisfies that*

$$\int_\Omega u(x,t)\,dx \leq M_* := \max\left\{\int_\Omega u_0(x)\,dx, \left(\frac{\lambda}{\mu}|\Omega|^{\kappa-1}\right)^{\frac{1}{\kappa-1}}\right\} \quad \textit{for all } t \in (0, T_{\max}). \tag{2.1}$$

Proof. Integrating the first equation in (1.1) and using Hölder's inequality, we have

$$\frac{d}{dt}\int_\Omega u\,dx \leq \lambda \int_\Omega u\,dx - \mu|\Omega|^{1-\kappa}\left(\int_\Omega u\,dx\right)^\kappa$$

for all $t \in (0, T_{\max})$. By an ODE comparison argument we attain (2.1). □

In order to see global existence and boundedness of solutions, it is sufficient to make sure that for each nonnegative initial data $u_0 \in \bigcup_{\beta \in (0,1)} C^\beta(\overline{\Omega})$ and for any $p > 1$ we can take $C = C(p) > 0$ such that

$$\int_\Omega u^p(x,t)\,dx \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{2.2}$$

In the following subsections we will prove (2.2) in three cases as follows:

- Case 1. $\alpha + \ell < m + \frac{2}{n}$ and $\mu > 0$.
- Case 2. $\alpha + \ell < \kappa$ and $\mu > 0$.
- Case 3. $\alpha + \ell = \kappa$ and $\mu > \frac{n(\alpha+\ell-m)-2}{2(\alpha-1)+n(\alpha+\ell-m)} C_S L$.

2.1. Case 1. $\alpha + \ell < m + \frac{2}{n}$ and $\mu > 0$.

In this subsection we derive (2.2) under the condition that $\alpha + \ell < m + \frac{2}{n}$ and $\mu > 0$.

Lemma 2.3. *Let $\mu > 0$ and assume that $m \in \mathbb{R}$, $\alpha > 0$ and $\ell > 0$ satisfy*

$$\alpha + \ell < m + \frac{2}{n}. \tag{2.3}$$

Then for any $p > \max\left\{1, 2-m, 2-(\alpha+\ell), \frac{n}{2}(1-m) + \left(\frac{n}{2}-1\right)(\alpha+\ell-1)\right\}$ there is $C = C(\Omega, m, \alpha, \lambda, \mu, \kappa, \ell, L, \delta, p, C_D, C_S) > 0$ such that

$$\int_\Omega u^p(x,t)\,dx \leq C \quad \textit{for all } t \in (0, T_{\max}). \tag{2.4}$$

Proof. By virtue of the first equation in (1.1) and $D(u) \geq C_D(u+\delta)^{m-1}$, we have

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx &\leq -p(p-1)C_D\int_\Omega (u+\delta)^{p+m-3}|\nabla u|^2\,dx \\
&\quad + p(p-1)\int_\Omega (u+\delta)^{p-2}S(u)\nabla u\cdot\nabla v\,dx \\
&\quad + p\lambda\int_\Omega u(u+\delta)^{p-1}\,dx - p\mu\int_\Omega u^\kappa(u+\delta)^{p-1}\,dx \\
&= -\frac{4p(p-1)C_D}{(p+m-1)^2}\int_\Omega |\nabla(u+\delta)^{\frac{p+m-1}{2}}|^2\,dx \\
&\quad + p(p-1)\int_\Omega \nabla\left(\int_0^u (\xi+\delta)^{p-2}S(\xi)\,d\xi\right)\cdot\nabla v\,dx \\
&\quad + p\lambda\int_\Omega u(u+\delta)^{p-1}\,dx - p\mu\int_\Omega u^\kappa(u+\delta)^{p-1}\,dx \\
&=: I_1 + I_2 + I_3 + I_4
\end{aligned}
\tag{2.5}
$$

for all $t \in (0, T_{\max})$. Noting from $S(\xi) \leq C_S(\xi+\delta)^\alpha$ and $p > 1-\alpha$ that

$$
\int_0^u (\xi+\delta)^{p-2}S(\xi)\,d\xi \leq C_S\int_0^u (\xi+\delta)^{p+\alpha-2}\,d\xi \leq \frac{C_S}{p+\alpha-1}(u+\delta)^{p+\alpha-1},
$$

from (1.8) and the second equation in (1.1) we can obtain

$$
\begin{aligned}
I_2 &= -p(p-1)\int_\Omega \left(\int_0^u (\xi+\delta)^{p-2}S(\xi)\,d\xi\right)\Delta v\,dx \\
&\leq \frac{p(p-1)C_S}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha-1}f(u)\,dx \\
&\leq \frac{p(p-1)C_SL}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx
\end{aligned}
\tag{2.6}
$$

for all $t \in (0, T_{\max})$. As to I_3 and I_4, since we see from elementary calculations that there exists $\varepsilon > 0$ so small such that

$$
(u+\delta)^\kappa \leq (1+\varepsilon)u^\kappa + C_\varepsilon\delta,
$$

where $C_\varepsilon := \left(\frac{\delta}{1-(1+\varepsilon)^{-\frac{1}{\kappa-1}}}\right)^{\kappa-1} > 0$, we can observe

$$
\begin{aligned}
&I_3 + I_4 \\
&\leq p\lambda\int_\Omega u(u+\delta)^{p-1}\,dx - \frac{p\mu}{1+\varepsilon}\int_\Omega (u+\delta)^{p+\kappa-1}\,dx + \frac{p\mu C_\varepsilon}{1+\varepsilon}\int_\Omega \delta(u+\delta)^{p-1}\,dx \\
&\leq \widetilde{C}_\varepsilon\int_\Omega (u+\delta)^p\,dx - \frac{p\mu}{1+\varepsilon}\int_\Omega (u+\delta)^{p+\kappa-1}\,dx
\end{aligned}
\tag{2.7}
$$

for all $t \in (0, T_{\max})$, where $\widetilde{C}_\varepsilon := \max\left\{p\lambda, \frac{p\mu C_\varepsilon}{1+\varepsilon}\right\} > 0$. From (2.5)–(2.7) it follows that

$$\begin{aligned}(2.8)\quad & \frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \\ &\le -\frac{4p(p-1)C_D}{(p+m-1)^2}\int_\Omega |\nabla (u+\delta)^{\frac{p+m-1}{2}}|^2\,dx + \frac{p(p-1)C_S L}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx \\ &\quad + \widetilde{C}_\varepsilon \int_\Omega (u+\delta)^p\,dx - \frac{p\mu}{1+\varepsilon}\int_\Omega (u+\delta)^{p+\kappa-1}\,dx\end{aligned}$$

for all $t \in (0, T_{\max})$. Here, let

$$\theta := \frac{\frac{p+m-1}{2} - \frac{p+m-1}{2(p+\alpha+\ell-1)}}{\frac{p+m-1}{2} + \frac{1}{n} - \frac{1}{2}}.$$

From $p > \max\left\{1, 2-m-\frac{2}{n}, 2-(\alpha+\ell), \frac{n}{2}(1-m) + \left(\frac{n}{2}-1\right)(\alpha+\ell-1)\right\}$ we see $\theta \in (0,1)$. Thus we can apply the Gagliardo–Nirenberg inequality to find $c_1 = c_1(\Omega, m, \alpha, \ell, p) > 0$ such that

$$\begin{aligned}(2.9)\quad \int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx &= \|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2(p+\alpha+\ell-1)}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}} \\ &\le c_1 \|\nabla (u+\delta)^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}\theta} \|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}(1-\theta)} \\ &\quad + c_1 \|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}}\end{aligned}$$

for all $t \in (0, T_{\max})$. Moreover, thanks to (2.3), we have

$$\frac{2(p+\alpha+\ell-1)}{p+m-1}\theta = \frac{p+\alpha+\ell-2}{\frac{1}{2}\left(p+m-2+\frac{2}{n}\right)} < 2.$$

Hence, noticing from Lemma 2.2 that $\int_\Omega u\,dx \le M_*$, from (2.9) and Young's inequality we can take $c_2 = c_2(\Omega, m, \alpha, \lambda, \mu, \kappa, \ell, L, \delta, p, C_D, C_S) > 0$ such that

$$(2.10)\quad \frac{p(p-1)C_S L}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx \le \frac{2p(p-1)C_D}{(p+m-1)^2}\int_\Omega |\nabla (u+\delta)^{\frac{p+m-1}{2}}|^2\,dx + c_2$$

for all $t \in (0, T_{\max})$. A combination of (2.8) and (2.10) yields that

$$\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \le \widetilde{C}_\varepsilon \int_\Omega (u+\delta)^p\,dx - \frac{p\mu}{2(1+\varepsilon)}\int_\Omega (u+\delta)^{p+\kappa-1}\,dx + c_2$$

for all $t \in (0, T_{\max})$. By Hölder's inequality there exists $c_3 = c_3(\Omega, m, \alpha, \mu, \kappa, \ell, p) > 0$ such that

$$\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \le \widetilde{C}_\varepsilon \int_\Omega (u+\delta)^p\,dx - c_3\left(\int_\Omega (u+\delta)^p\,dx\right)^{\frac{p+\kappa-1}{p}} + c_2$$

for all $t \in (0, T_{\max})$. Noting the fact that $\frac{p+\kappa-1}{p} > 1$, this inequality yields (2.4) by an ODE comparison argument. $\square$

2.2. Case 2. $\alpha+\ell<\kappa$ and $\mu>0$.

In this subsection we show (2.2) under the condition that $\alpha+\ell<\kappa$ and $\mu>0$.

Lemma 2.4. *Let $\mu>0$ and assume that $\alpha>0$, $\kappa>1$ and $\ell>0$ satisfy*

$$\alpha+\ell<\kappa. \tag{2.11}$$

Then for any $p>1$ there exists $C=C(\Omega,\alpha,\lambda,\mu,\kappa,\ell,L,\delta,p,C_S)>0$ such that

$$\int_\Omega u^p(x,t)\,dx \le C \quad \textit{for all } t\in(0,T_{\max}). \tag{2.12}$$

Proof. We know that there exist $\varepsilon>0$ and $\widetilde{C}_\varepsilon>0$ such that (2.8) holds. By virtue of (2.11), we have

$$p+\alpha+\ell-1<p+\kappa-1.$$

Thus, by using Young's inequality, we can find $c_1=c_1(\Omega,\alpha,\mu,\kappa,\ell,L,\delta,p,C_S)>0$ such that

$$\frac{p(p-1)C_SL}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx \le \frac{p\mu}{4(1+\varepsilon)}\int_\Omega (u+\delta)^{p+\kappa-1}\,dx + c_1 \tag{2.13}$$

for all $t\in(0,T_{\max})$. Combining (2.13) with (2.8) and applying Hölder's inequality, we observe that there exists $c_2=c_2(\Omega,\mu,\kappa,p)>0$ such that

$$\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \le \widetilde{C}_\varepsilon\int_\Omega (u+\delta)^p\,dx - c_2\left(\int_\Omega (u+\delta)^p\,dx\right)^{\frac{p+\kappa-1}{p}} + c_1$$

for all $t\in(0,T_{\max})$. Accordingly, we see that (2.12) holds. □

2.3. Case 3. $\alpha+\ell=\kappa$ and $\mu>\frac{n(\alpha+\ell-m)-2}{2(\alpha-1)+n(\alpha+\ell-m)}C_SL$.

In order to prove (2.2) under the condition that $\alpha+\ell=\kappa$ and $\mu>\frac{n(\alpha+\ell-m)-2}{2(\alpha-1)+n(\alpha+\ell-m)}C_SL$, we first derive the L^p-estimate for some $p<1+\frac{\alpha\mu}{(C_SL-\mu)_+}$.

Lemma 2.5. *Let $\mu>0$ and assume that $\alpha>0$, $\kappa>1$ and $\ell>0$ satisfy $\alpha+\ell=\kappa$. Then for any $p\in\left(1,1+\frac{\alpha\mu}{(C_SL-\mu)_+}\right)$ there exists $C=C(\Omega,\alpha,\lambda,\mu,\kappa,L,p,C_S)>0$ such that*

$$\int_\Omega u^p(x,t)\,dx \le C \quad \textit{for all } t\in(0,T_{\max}).$$

Proof. Since the condition $p<1+\frac{\alpha\mu}{(C_SL-\mu)_+}$ implies that

$$\frac{p(p-1)C_SL}{p+\alpha-1}-p\mu<0,$$

we can take $\varepsilon>0$ small enough such that

$$\frac{p(p-1)C_SL}{p+\alpha-1}-\frac{p\mu}{1+\varepsilon}<0.$$

Thus we have that there exists $\widetilde{C}_\varepsilon > 0$ such that

$$\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \le -\frac{4p(p-1)C_D}{(p+m-1)^2}\int_\Omega |\nabla(u+\delta)^{\frac{p+m-1}{2}}|^2\,dx + \widetilde{C}_\varepsilon\int_\Omega (u+\delta)^p\,dx - \left(\frac{p\mu}{1+\varepsilon} - \frac{p(p-1)C_S L}{p+\alpha-1}\right)\int_\Omega (u+\delta)^{p+\kappa-1}\,dx$$

for all $t\in(0,T_{\max})$. By Hölder's inequality, we obtain $c_1 = c_1(\Omega,\alpha,\mu,\kappa,L,p,C_S) > 0$ such that

$$\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \le \widetilde{C}_\varepsilon\int_\Omega (u+\delta)^p\,dx - c_1\left(\int_\Omega (u+\delta)^p\,dx\right)^{\frac{p+\kappa-1}{p}}$$

for all $t\in(0,T_{\max})$, and thereby we can arrive at the conclusion. $\square$

Next we establish the L^p-estimate for any $p>1$.

Lemma 2.6. *Assume that $m\in\mathbb{R}$, $\alpha>0$, $\mu>0$, $\kappa>1$ and $\ell>0$ satisfy*

$$\alpha+\ell=\kappa \quad \text{and} \quad \mu > \frac{n(\alpha+\ell-m)-2}{2(\alpha-1)+n(\alpha+\ell-m)}C_S L. \tag{2.14}$$

Then for any $p>1$ there exists $C = C(\Omega,m,\alpha,\lambda,\mu,\kappa,\ell,L,\delta,p,C_D,C_S) > 0$ such that

$$\int_\Omega u^p(x,t)\,dx \le C \quad \text{for all } t\in(0,T_{\max}). \tag{2.15}$$

Proof. The second condition of (2.14) yields that

$$\left(1+\frac{\alpha\mu}{(C_S L-\mu)_+}\right) - \frac{n}{2}(\alpha+\ell-m) > 0.$$

Therefore we can pick some $p_0 \in \left(\frac{n}{2}(\alpha+\ell-m), 1+\frac{\alpha\mu}{(C_S L-\mu)_+}\right)$. Thanks to Lemma 2.5, we see that there exists $c_1 = c_1(\Omega,\alpha,\lambda,\mu,\kappa,\ell,L,p,C_S) > 0$ such that

$$\int_\Omega u^{p_0}\,dx \le c_1 \tag{2.16}$$

for all $t\in(0,T_{\max})$. Moreover, we choose

$$p > \max\left\{p_0, p_0+1-m, p_0+1-(\alpha+\ell), \frac{n}{2}(1-m)+\left(\frac{n}{2}-1\right)(\alpha+\ell-1)\right\}$$

and take $\varepsilon>0$ and $\widetilde{C}_\varepsilon>0$ such that (2.8) holds. Applying the Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx &= \|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2(p+\alpha+\ell-1)}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}} \\ &\le c_2\|\nabla(u+\delta)^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}\tilde{\theta}}\|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2p_0}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}(1-\tilde{\theta})} \\ &\quad + c_2\|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2p_0}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}}\end{aligned}$$

for all $t \in (0, T_{\max})$ with some $c_2 = c_2(\Omega, m, \alpha, \ell, p) > 0$, where

$$\tilde{\theta} := \frac{\frac{p+m-1}{2p_0} - \frac{p+m-1}{2(p+\alpha+\ell-1)}}{\frac{p+m-1}{2p_0} + \frac{1}{n} - \frac{1}{2}} \in (0,1).$$

Here, we note from $p_0 > \frac{n}{2}(\alpha + \ell - m)$ that

$$\frac{2(p+\alpha+\ell-1)}{p+m-1}\tilde{\theta} - 2 = \frac{\frac{p+\alpha+\ell-1}{p_0} - 1 - \left(\frac{p+m-1}{p_0} + \frac{2}{n} - 1\right)}{\frac{p+m-1}{2p_0} + \frac{1}{n} - \frac{1}{2}} = \frac{\frac{\alpha+\ell-m}{p_0} - \frac{2}{n}}{\frac{p+m-1}{2p_0} + \frac{1}{n} - \frac{1}{2}} < 0.$$

Thus, due to the inequality (2.16) and Young's inequality, we can show that there is $c_3 = c_3(\Omega, m, \alpha, \lambda, \mu, \kappa, \ell, L, \delta, p, C_D, C_S) > 0$ such that

$$\frac{p(p-1)C_S L}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx \le \frac{2p(p-1)C_D}{(p+m-1)^2}\int_\Omega |\nabla(u+\delta)^{\frac{p+m-1}{2}}|^2\,dx + c_3 \tag{2.17}$$

for all $t \in (0, T_{\max})$. From (2.8) and (2.17) we infer that

$$\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \le \widetilde{C}_\varepsilon \int_\Omega (u+\delta)^p\,dx - \frac{p\mu}{1+\varepsilon}\int_\Omega (u+\delta)^{p+\kappa-1}\,dx + c_3$$

for all $t \in (0, T_{\max})$, which implies that (2.15) holds. □

2.4. Proof of Theorem 1.1

In this subsection we complete the proof of boundedness.

Proof of Theorem 1.1. Thanks to (1.9) and (1.10), we can apply Lemmas 2.3, 2.4 and 2.6. Therefore, for any $p > 1$ we can find $c_1 = c_1(\Omega, m, \alpha, \lambda, \mu, \kappa, \ell, L, \delta, p, C_D, C_S) > 0$ such that

$$\int_\Omega u^p\,dx \le c_1$$

for all $t \in (0, T_{\max})$. By the Moser iteration (see [**17**, Lemma A.1]), we obtain

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \infty$$

for all $t \in (0, T_{\max})$, which concludes the proof. □

3. Finite-time blow-up

In this section we will show Theorem 1.2. In the following let $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \ge 1)$ be a ball with some $R > 0$ and let $\lambda > 0$, $\mu > 0$ and $\kappa > 1$. Also, we suppose that D, S and f fulfill (1.4), (1.5) and (1.6), respectively, and u_0 satisfies (1.15). Moreover, introducing $r := |x|$, we denote by $(u, v) = (u(r,t), v(r,t))$ the radially symmetric local solution of (1.1) on $[0, T_{\max})$. Based on [**8**], we define the mass accumulation function w such that

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t)\,d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{\max}). \tag{3.1}$$

This implies that

$$w_s(s,t) = \frac{1}{n}u(s^{\frac{1}{n}},t) \quad \text{and} \quad w_{ss}(s,t) = \frac{1}{n^2}s^{\frac{1}{n}-1}u_r(s^{\frac{1}{n}},t)$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max})$. Thus we have from the first equation in (1.1) that

$$w_t = n^2 s^{2-\frac{2}{n}} D(nw_s)w_{ss} - s^{1-\frac{1}{n}} S(nw_s)v_r + \lambda w - n^{\kappa-1}\mu \int_0^s w_s^\kappa(\sigma,t)\,d\sigma \tag{3.2}$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max})$, and see from the second equation in (1.1) that

$$s^{1-\frac{1}{n}} v_r = \overline{M_f}(t)\frac{s}{n} - \frac{1}{n}\int_0^s f(nw_s(\sigma,t))\,d\sigma \tag{3.3}$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max})$. From (3.2) and (3.3) it follows that

$$\begin{aligned} w_t \ge{}& n^2 s^{2-\frac{2}{n}} D(nw_s)w_{ss} - \frac{1}{n} sS(nw_s)\overline{M_f}(t) \\ &+ \frac{1}{n}S(nw_s)\int_0^s f(nw_s(\sigma,t))\,d\sigma - n^{\kappa-1}\mu\int_0^s w_s^\kappa(\sigma,t)\,d\sigma \end{aligned} \tag{3.4}$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max})$.

In Subsection 3.1 we recall some lemmas in order to obtain inequalities for a derivative of a moment-type functional. In Subsection 3.2 we establish some estimates which lead to differential inequalities for the moment-type functional. The proof of Theorem 3.3 is shown in Subsection 3.3. Finally, we give open problems in Subsection 3.4.

3.1. Preliminaries

We first derive the concavity of w.

Lemma 3.1. *Assume that u_0 satisfies* (1.15). *Then*

$$u_r(r,t) \le 0 \quad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{\max}),$$

that is, for w as in (3.1)

$$w_{ss}(s,t) \le 0 \quad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{\max}).$$

Proof. By an argument similar to that in the proof of [**23**, Lemma 2.2] or [**2**, Lemma 5.1], we can prove this lemma. □

Given $s_0 \in (0, R^n)$ and $\gamma \in (-\infty, 1)$, we set the moment-type functional

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)\,ds \quad \text{for } t \in [0, T_{\max}).$$

Here, we note that $\phi \in C^0([0,T_{\max}))\bigcap C^1((0,T_{\max}))$. Moreover, we introduce the functional

$$\psi(t) := \int_0^{s_0} s^{1-\gamma}(s_0 - s)w_s^{\alpha+\ell}(s,t)\,ds \quad \text{for } t \in (0, T_{\max})$$

and

$$S_\phi := \left\{ t \in (0, T_{\max}) \;\middle|\; \phi(t) \geq \frac{M - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} \right\}.$$

The choices of ϕ, ψ and S_ϕ as well as the underlying overall strategy quite closely follow the approach in [**23**]. However, in our method we do not use the set S_ψ defined in [**23**]. Next we recall the following two lemmas which were shown in [**23**].

Lemma 3.2. *Assume that u_0 satisfies* (1.15) *and let $s_0 \in (0, R^n)$ and $\gamma \in (-\infty, 1)$. Then*

$$w\left(\frac{s_0}{2}, t\right) \geq \frac{1}{\omega_n} \cdot \left(M - \frac{4s_0}{2^\gamma(3-\gamma)} \right) \quad \textit{for all } t \in S_\phi.$$

The following lemma is obtained from Lemmas 3.1 and 3.2 (see [**23**, Lemma 3.2]).

Lemma 3.3. *Assume that u_0 satisfies* (1.15) *and let $s_0 \in \left(0, \frac{R^n}{4}\right]$ and $\gamma \in (-\infty, 1)$. Then*

$$\overline{M_f}(t) \leq f_\gamma + \frac{1}{2s} \int_0^s f(nw_s(\sigma, t))\, d\sigma \quad \textit{for all } s \in (0, s_0) \textit{ and } t \in S_\phi, \tag{3.5}$$

where

$$f_\gamma := f\left(\frac{8n}{2^\gamma(3-\gamma)\omega_n} \right) > 0. \tag{3.6}$$

In order to derive differential inequalities for ϕ we establish an estimate for ϕ'. This method has been developed in [**23**].

Lemma 3.4. *Assume that f fulfills* (1.12) *and u_0 satisfies* (1.15). *Let $s_0 \in \left(0, \frac{R^n}{4}\right]$ and $\gamma \in (-\infty, 1)$ as well as*

$$\gamma < 2 - \frac{2}{n}. \tag{3.7}$$

Then

$$\begin{aligned} \phi'(t) \geq\; & \frac{n^{\ell-1}}{2} L \int_0^{s_0} s^{1-\gamma}(s_0 - s) S(nw_s(s,t)) w_s^\ell(s,t)\, ds \\ & - \frac{f_\gamma}{n} \int_0^{s_0} s^{1-\gamma}(s_0 - s) S(nw_s(s,t))\, ds \\ & + n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s) D(nw_s(s,t)) w_{ss}(s,t)\, ds \\ & - n^{\kappa-1}\mu \int_0^{s_0} s^{-\gamma}(s_0 - s) \left\{ \int_0^s w_s^\kappa(\sigma, t)\, d\sigma \right\} ds \\ =:\; & I_1 + I_2 + I_3 + I_4 \end{aligned} \tag{3.8}$$

for all $t \in S_\phi$, where $f_\gamma > 0$ is defined as (3.6).

Proof. Invoking (3.4) and (3.5), we have

$$w_t \geq n^2 s^{2-\frac{2}{n}} D(nw_s) w_{ss} - \frac{f_\gamma}{n} sS(nw_s) + \frac{1}{2n} S(nw_s) \int_0^s f(nw_s(\sigma,t))\, d\sigma - n^{\kappa-1}\mu \int_0^s w_s^\kappa(\sigma,t)\, d\sigma \tag{3.9}$$

for all $s \in \left(0, \frac{R^n}{4}\right]$ and $t \in S_\phi$. Here, we note from Lemma 3.1 that

$$w_s(\sigma,t) \geq w_s(s,t) \quad (\sigma \leq s).$$

Thanks to this inequality and (1.12), we see that

$$S(sw_s) \int_0^s f(nw_s(\sigma,t))\, d\sigma \geq LS(nw_s) \int_0^s (nw_s(\sigma,t))^\ell\, d\sigma \geq n^\ell LsS(nw_s) w_s^\ell \tag{3.10}$$

for all $s \in \left(0, \frac{R^n}{4}\right]$ and $t \in S_\phi$. By virtue of (3.9) and (3.10), we attain (3.8). □

3.2. Estimates for the four integrals in the inequality (3.8)

In this subsection, in order to derive different inequalities for ϕ we show estimates for the four integrals in (3.8) by using lower bound for ψ. We first provide the estimate for $I_1 + I_2$ in the following lemma.

Lemma 3.5. *Assume that S and f fulfill* (1.11) *and* (1.12)*, and u_0 satisfies* (1.15)*. Let $\gamma \in (-\infty, 1)$. Suppose that $\alpha > 0$ and $\ell > 0$ satisfy*

$$\alpha + \ell > 1. \tag{3.11}$$

Then there exists $C_1 = C_1(\alpha, \ell, L, C_S) > 0$ and $C_2 = C_2(R, \alpha, \ell, L, \gamma) > 0$ such that for any choices of $s_0 \in \left(0, \frac{R^n}{4}\right]$,

$$I_1 + I_2 \geq C_1 \psi(t) - C_2 s_0^{3-\gamma} \tag{3.12}$$

for all $t \in S_\phi$.

Proof. We define the function χ_A as the characteristic function of the set A and put

$$\overline{C} := \left(\frac{4f_\gamma}{L}\right)^{\frac{1}{\ell}} > 0.$$

As to I_2, noticing that S is nondecreasing, we see that

$$\begin{aligned} I_2 &= -\frac{f_\gamma}{n} \int_0^{s_0} \chi_{\{nw_s(\cdot,t) \geq \overline{C}\}} s^{1-\gamma}(s_0 - s) S(nw_s)\, ds \\ &\quad - \frac{f_\gamma}{n} \int_0^{s_0} \chi_{\{nw_s(\cdot,t) < \overline{C}\}} s^{1-\gamma}(s_0 - s) S(nw_s)\, ds \\ &\geq -\frac{f_\gamma}{n} \int_0^{s_0} \chi_{\{nw_s(\cdot,t) \geq \overline{C}\}} s^{1-\gamma}(s_0 - s) S(nw_s)\, ds \\ &\quad - \frac{f_\gamma}{n} S(\overline{C}) \int_0^{s_0} \chi_{\{nw_s(\cdot,t) < \overline{C}\}} s^{1-\gamma}(s_0 - s)\, ds \end{aligned} \tag{3.13}$$

for all $t \in S_\phi$. Moreover, we have that

$$
\begin{aligned}
&-\frac{f_\gamma}{n}\int_0^{s_0}\chi_{\{nw_s(\cdot,t)\geq \overline{C}\}}s^{1-\gamma}(s_0-s)S(nw_s)\,ds \\
&\geq -\frac{f_\gamma}{n}\int_0^{s_0}\chi_{\{nw_s(\cdot,t)\geq \overline{C}\}}s^{1-\gamma}(s_0-s)S(nw_s)\left(\frac{nw_s}{\overline{C}}\right)^\ell ds \\
&\geq -\frac{n^{\ell-1}}{4}L\int_0^{s_0} s^{1-\gamma}(s_0-s)S(nw_s)w_s^\ell\,ds
\end{aligned}
\tag{3.14}
$$

and

$$
-\frac{f_\gamma}{n}S(\overline{C})\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\overline{C}\}}s^{1-\gamma}(s_0-s)\,ds \geq -\frac{f_\gamma S(\overline{C})}{(2-\gamma)(3-\gamma)n}s_0^{3-\gamma}
\tag{3.15}
$$

for all $t \in S_\phi$. In light of (3.13)–(3.15), we observe that

$$
I_1+I_2 \geq \frac{n^{\ell-1}}{4}L\int_0^{s_0} s^{1-\gamma}(s_0-s)S(nw_s)w_s^\ell\,ds - \frac{f_\gamma S(\overline{C})}{(2-\gamma)(3-\gamma)n}s_0^{3-\gamma}
\tag{3.16}
$$

for all $t \in S_\phi$. Recalling (1.11), we can obtain

$$
\int_0^{s_0} s^{1-\gamma}(s_0-s)S(nw_s)w_s^\ell\,ds \geq nC_S\int_0^{s_0} s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell+1}\,ds
\tag{3.17}
$$

for all $t \in S_\phi$. If $\alpha \geq 1$, then it follows from $(nw_s+\delta)^{\alpha-1} \geq (nw_s)^{\alpha-1}$ that

$$
nC_S\int_0^{s_0} s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell+1}\,ds \geq n^\alpha C_S\psi(t)
\tag{3.18}
$$

for all $t \in S_\phi$. Hence, in the case $\alpha \geq 1$ a combination of (3.16), (3.17) and (3.18) yields (3.12). On the other hand, if $\alpha < 1$, then we can show from $w_s^{\ell+1} = \frac{1}{n}w_s^\ell(nw_s+\delta-\delta)$

that

$$
\begin{aligned}
(3.19)\qquad & nC_S\int_0^{s_0} s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell+1}\,ds\\
&= nC_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)\geq\delta\}}s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell+1}\,ds\\
&\quad + nC_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell+1}\,ds\\
&\geq \frac{n^\alpha}{2^{1-\alpha}}C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)\geq\delta\}}s^{1-\gamma}(s_0-s)w_s^{\alpha+\ell}\,ds\\
&\quad + C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha}w_s^{\ell}\,ds\\
&\quad - \delta C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell}\,ds\\
&\geq \frac{n^\alpha}{2^{1-\alpha}}C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)\geq\delta\}}s^{1-\gamma}(s_0-s)w_s^{\alpha+\ell}\,ds\\
&\quad + n^\alpha C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)w_s^{\alpha+\ell}\,ds\\
&\quad - \delta C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell}\,ds
\end{aligned}
$$

for all $t \in S_\phi$. Noting from $\alpha < 1$ that

$$
(nw_s+\delta)^{\alpha-1}w_s^\ell = \left(\frac{nw_s}{nw_s+\delta}\right)^{1-\alpha} n^{\alpha-1}w_s^{\alpha+\ell-1} \leq n^{\alpha-1}w_s^{\alpha+\ell-1},
$$

we establish that

$$
\begin{aligned}
& -\delta C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell}\,ds\\
&\quad \geq -n^{\alpha-1}C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)w_s^{\alpha+\ell-1}\,ds\\
&\quad \geq -n^{\alpha-1}C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)\,ds\\
&\quad \geq -\frac{n^{\alpha-1}C_S}{(2-\gamma)(3-\gamma)}s_0^{3-\gamma}
\end{aligned}
$$

for all $t \in S_\phi$. From this inequality and (3.19) we see that

$$
(3.20)\quad nC_S\int_0^{s_0} s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell+1}\,ds \geq \frac{n^\alpha}{2^{1-\alpha}}C_S\psi(t) - \frac{n^{\alpha-1}C_S}{(2-\gamma)(3-\gamma)}s_0^{3-\gamma}
$$

for all $t \in S_\phi$. Thus, in the case $\alpha < 1$, from (3.16), (3.17) and (3.20) we attain (3.12). $\square$

Next, we show the estimate for I_3. In the case $m \neq 0$ the proof of the following lemma is based on that of [**12**]. However, in the case $m = 0$ we will use a different estimate for $\log(x + 1)$ for any $x \geq 0$ than the one used in the proof of [**12**, Corollary 3.4].

Lemma 3.6. *Assume that D fulfills* (1.11) *and u_0 satisfies* (1.15). *Suppose that $m \in \mathbb{R}$, $\alpha > 0$, $\ell > 0$ and $\gamma \in (-\infty, 1)$ satisfy*

$$\text{(3.21)} \qquad \text{if } m \geq 0, \quad \text{then} \quad \alpha + \ell > m \quad \text{and} \quad 2 - \frac{2}{n} \cdot \frac{\alpha + \ell}{\alpha + \ell - m} > \gamma,$$

$$\text{(3.22)} \qquad \text{if } m < 0, \quad \text{then} \qquad 2 - \frac{2}{n} > \gamma.$$

Then there exist $\varepsilon > 0$ so small, $C_1 = C_1(m, \alpha, \ell, \delta, \gamma, C_D) > 0$, $C_2 = C_2(m, \delta, \gamma, C_D) > 0$, $C_3 = C_3(m, \alpha, \ell, \delta, \gamma, \varepsilon, C_D) > 0$ and $C_4 = C_4(m, \delta, \gamma, \varepsilon, C_D) > 0$ such that for any $s_0 \in \left(0, \frac{R^n}{4}\right]$,

$$\text{(3.23)} \qquad I_3 \geq \begin{cases} -C_1 s_0^{(3-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell} - \frac{2}{n}} \psi^{\frac{m}{\alpha+\ell}}(t) - C_2 s_0^{3-\gamma-\frac{2}{n}} & \text{if } m > 0, \\ -C_3 s_0^{(3-\gamma)\frac{\alpha+\ell-\varepsilon}{\alpha+\ell} - \frac{2}{n}} \psi^{\frac{\varepsilon}{\alpha+\ell}}(t) - C_4 s_0^{3-\gamma-\frac{2}{n}} & \text{if } m = 0, \\ -C_2 s_0^{3-\gamma-\frac{2}{n}} & \text{if } m < 0 \end{cases}$$

for all $t \in S_\phi$.

Remark 3.1. In this lemma, the constants $C_1 > 0$ and $C_2 > 0$ depend on δ. However, in the case $m > 0$, we can take them which are independent of δ.

Proof. We have from (1.11) that

$$\begin{aligned} I_3 &\geq n^2 C_D \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)(nw_s + \delta)^{m-1} w_{ss}\, ds \\ &= nC_D \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s) \frac{d}{ds} \left\{ \int_0^{nw_s} (\xi + \delta)^{m-1}\, d\xi \right\} ds \end{aligned}$$

for all $t \in S_\phi$. Since it follows that

$$\int_0^{nw_s} (\xi + \delta)^{m-1}\, d\xi \leq \begin{cases} \frac{1}{m}(nw_s + \delta)^m & \text{if } m > 0, \\ \log(nw_s + \delta) - \log \delta & \text{if } m = 0, \\ -\frac{1}{m}\delta^m & \text{if } m < 0, \end{cases}$$

we obtain from integrating by parts that

$$\text{(3.24)} \quad I_3 \geq \begin{cases} -\frac{n}{m} C_D \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)(nw_s + \delta)^m\, ds & \text{if } m > 0, \\ -nC_D \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s) \log\left(\frac{nw_s}{\delta} + 1\right) ds & \text{if } m = 0, \\ \frac{n}{m} \delta^m C_D \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)\, ds & \text{if } m < 0 \end{cases}$$

for all $t \in S_\phi$. First, we show the estimate (3.23) in the case $m > 0$. By applying the inequality $(nw_s + \delta)^m \le 2^m((nw_s)^m + \delta^m)$, we know that

$$\begin{aligned}\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)(nw_s+\delta)^m\,ds &\le 2^m n^m \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)w_s^m\,ds \\ &\quad + 2^m\delta^m \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)\,ds \\ &=: J_1 + J_2\end{aligned} \tag{3.25}$$

for all $t \in S_\phi$. Invoking from (3.21) that $\frac{m}{\alpha+\ell} < 1$, we see from Hölder's inequality that

$$\begin{aligned}J_1 &= 2^m n^m \int_0^{s_0} \left[s^{1-\gamma}(s_0-s)w_s^{\alpha+\ell}\right]^{\frac{m}{\alpha+\ell}} \cdot s^{(1-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell}-\frac{2}{n}}(s_0-s)^{\frac{\alpha+\ell-m}{\alpha+\ell}}\,ds \\ &\le 2^m n^m \psi^{\frac{m}{\alpha+\ell}}(t)\cdot\left(\int_0^{s_0} s^{1-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}}(s_0-s)\,ds\right)^{\frac{\alpha+\ell-m}{\alpha+\ell}}\end{aligned}$$

for all $t \in S_\phi$. Moreover, thanks to the condition $2 - \frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m} > \gamma$, we can observe

$$\int_0^{s_0} s^{1-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}}(s_0-s)\,ds = c_1 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}},$$

where

$$c_1 := \frac{1}{\left(2-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}\right)\left(3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}\right)} > 0.$$

Thus we establish that

$$J_1 \le 2^m n^m c_1^{\frac{\alpha+\ell-m}{\alpha+\ell}} s_0^{(3-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell}-\frac{2}{n}} \psi^{\frac{m}{\alpha+\ell}}(t) \tag{3.26}$$

for all $t \in S_\phi$. Also, since $2-\gamma-\frac{2}{n} > 2-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m} > 0$ and $\delta \le 1$, it follows that

$$J_2 = \frac{2^m\delta^m}{\left(2-\gamma-\frac{2}{n}\right)\left(3-\gamma-\frac{2}{n}\right)} s_0^{3-\gamma-\frac{2}{n}} \le \frac{2^m}{\left(2-\gamma-\frac{2}{n}\right)\left(3-\gamma-\frac{2}{n}\right)} s_0^{3-\gamma-\frac{2}{n}}. \tag{3.27}$$

In the case $m > 0$, from (3.24)–(3.27) we can deduce that

$$I_3 \ge -\frac{2^m n^{m+1} C_D}{m}\left(2-\frac{2}{n}-\gamma\right) c_1^{\frac{\alpha+\ell-m}{\alpha+\ell}} s_0^{(3-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell}-\frac{2}{n}} \psi^{\frac{m}{\alpha+\ell}}(t) - \frac{2^m n C_D}{m\left(3-\gamma-\frac{2}{n}\right)} s_0^{3-\gamma-\frac{2}{n}}$$

for all $t \in S_\phi$, which implies (3.23). Next, we confirm that the estimate (3.23) holds in the case $m = 0$. Due to (3.21) with $m = 0$, we can take $\varepsilon > 0$ small enough such that

$$\alpha + \ell > \varepsilon \quad \text{and} \quad 2 - \frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-\varepsilon} > \gamma.$$

Furthermore, we have that

$$\log\left(\frac{nw_s}{\delta}+1\right) \le \frac{1}{\varepsilon}\left(\frac{nw_s}{\delta}+1\right)^\varepsilon - \frac{1}{\varepsilon} = \frac{1}{\varepsilon\delta^\varepsilon}(nw_s+\delta)^\varepsilon - \frac{1}{\varepsilon}.$$

In light of (3.24), we obtain that

$$I_3 \geq -\frac{nC_D}{\varepsilon\delta^\varepsilon}\left(2-\frac{2}{n}-\gamma\right)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)(nw_s+\delta)^\varepsilon\,ds + \frac{nC_D}{\varepsilon}\left(2-\frac{2}{n}-\gamma\right)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)\,ds \tag{3.28}$$

for all $t \in S_\phi$. As in the case $m > 0$, we can verify that

$$\begin{aligned} &-\frac{nC_D}{\varepsilon\delta^\varepsilon}\left(2-\frac{2}{n}-\gamma\right)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)(nw_s+\delta)^\varepsilon\,ds \\ &\geq -\frac{2^\varepsilon n^{\varepsilon+1}C_D}{\varepsilon\delta^\varepsilon}\left(2-\frac{2}{n}-\gamma\right)c_2^{\frac{\alpha+\ell-\varepsilon}{\alpha+\ell}} s_0^{(3-\gamma)\frac{\alpha+\ell-\varepsilon}{\alpha+\ell}-\frac{2}{n}}\psi^{\frac{\varepsilon}{\alpha+\ell}}(t) - \frac{2^\varepsilon nC_D}{\varepsilon\delta^\varepsilon\left(3-\gamma-\frac{2}{n}\right)}s_0^{3-\gamma-\frac{2}{n}} \end{aligned} \tag{3.29}$$

for all $t \in S_\phi$, where

$$c_2 := \frac{1}{\left(2-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-\varepsilon}\right)\left(3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-\varepsilon}\right)} > 0.$$

Accordingly, a combination of (3.28) and (3.29) yields (3.23). Finally, in the case $m < 0$, we can show from (3.24) that

$$\frac{n}{m}\delta^m C_D\left(2-\frac{2}{n}-\gamma\right)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)\,ds = \frac{n\delta^m C_D}{m\left(3-\gamma-\frac{2}{n}\right)}s_0^{3-\gamma-\frac{2}{n}},$$

which concludes the proof. $\square$

In the following lemma we derive the estimate for I_4.

Lemma 3.7. *Assume that u_0 satisfies (1.15). Suppose that $\alpha > 0$, $\kappa > 1$, $\ell > 0$ and $\gamma \in (-\infty, 1)$ fulfill*

$$\alpha+\ell > \kappa \quad \textit{and} \quad 2-\frac{\alpha+\ell}{\kappa} < \gamma < 1. \tag{3.30}$$

Then there exists $C_1 = C_1(\alpha,\mu,\kappa,\ell,\gamma) > 0$ such that for any choices of $s_0 \in \left(0, \frac{R^n}{4}\right]$,

$$I_4 \geq -C_1 s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}}\psi^{\frac{\kappa}{\alpha+\ell}}(t) \tag{3.31}$$

for all $t \in S_\phi$.

Proof. We apply the Fubini theorem to obtain that

$$\begin{aligned} \int_0^{s_0} s^{-\gamma}(s_0-s)\left\{\int_0^s w_s^\kappa(\sigma,t)\,d\sigma\right\}ds &= \int_0^{s_0}\left\{\int_\sigma^{s_0} s^{-\gamma}(s_0-s)\,ds\right\}w_s^\kappa(\sigma,t)\,d\sigma \\ &\leq \frac{1}{1-\gamma}s_0^{1-\gamma}\int_0^{s_0}(s_0-\sigma)w_s^\kappa(\sigma,t)\,d\sigma \end{aligned}$$

for all $t \in S_\phi$. Thus we have that

$$I_4 \ge -\frac{n^{\kappa-1}\mu}{1-\gamma} s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) w_s^\kappa \, ds \tag{3.32}$$

for all $t \in S_\phi$. Owing to the first condition of (3.30), we see from Hölder's inequality that

$$\begin{aligned} \int_0^{s_0} (s_0 - s) w_s^\kappa \, ds &= \int_0^{s_0} \left[s^{1-\gamma} (s_0 - s) w_s^{\alpha+\ell} \right]^{\frac{\kappa}{\alpha+\ell}} \cdot s^{-(1-\gamma)\frac{\kappa}{\alpha+\ell}} (s_0 - s)^{\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \, ds \\ &\le \psi^{\frac{\kappa}{\alpha+\ell}}(t) \cdot \left(\int_0^{s_0} s^{-(1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa}} (s_0 - s) \, ds \right)^{\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \end{aligned} \tag{3.33}$$

for all $t \in S_\phi$. Here, noting from the second condition of (3.30) that

$$1 - (1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa} > 1 - \left(\frac{\alpha+\ell}{\kappa} - 1 \right) \frac{\kappa}{\alpha+\ell-\kappa} = 0,$$

we can verify that

$$\int_0^{s_0} s^{-(1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa}} (s_0 - s) \, ds = c_1 s_0^{2-(1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa}}, \tag{3.34}$$

where

$$c_1 := \frac{1}{\left(1 - (1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa}\right)\left(2 - (1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa}\right)} > 0.$$

Thanks to (3.32)–(3.34), it follows that

$$I_4 \ge -\frac{n^{\kappa-1}\mu}{1-\gamma} c_1^{\frac{\alpha+\ell-\kappa}{\alpha+\ell}} s_0^{1-\gamma+\frac{2(\alpha+\ell-\kappa)}{\alpha+\ell}-(1-\gamma)\frac{\kappa}{\alpha+\ell}} \psi^{\frac{\kappa}{\alpha+\ell}}(t) = -\frac{n^{\kappa-1}\mu}{1-\gamma} c_1^{\frac{\alpha+\ell-\kappa}{\alpha+\ell}} s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \psi^{\frac{\kappa}{\alpha+\ell}}(t)$$

for all $t \in S_\phi$, which implies (3.31). □

In the next lemma we establish the estimate for w which is used later.

Lemma 3.8. *Assume that u_0 satisfies* (1.15). *Suppose that $\alpha > 0$, $\ell > 0$ and $\gamma \in (-\infty, 1)$ fulfill*

$$\alpha + \ell > 1 \quad \text{and} \quad 2 - (\alpha + \ell) < \gamma < 1. \tag{3.35}$$

Then there exists $C_1 = C_1(\alpha, \ell, \gamma) > 0$ such that for any $s_0 \in \left(0, \frac{R^n}{4}\right]$,

$$w(s,t) \le C_1 s^{\frac{\alpha+\ell+\gamma-2}{\alpha+\ell}} (s_0 - s)^{-\frac{1}{\alpha+\ell}} \psi^{\frac{1}{\alpha+\ell}}(t)$$

for all $s \in (0, s_0)$ and $t \in S_\phi$.

Proof. According to the condition $\alpha+\ell>1$, we have from Hölder's inequality that

$$\begin{aligned} w(s,t) &= \int_0^s w_s(\sigma,t)\,d\sigma \\ &= \int_0^s [\sigma^{1-\gamma}(s_0-\sigma)]^{\frac{1}{\alpha+\ell}} w_s(\sigma,t)\cdot[\sigma^{1-\gamma}(s_0-\sigma)]^{-\frac{1}{\alpha+\ell}}\,d\sigma \\ &\le \psi^{\frac{1}{\alpha+\ell}}(t)\cdot\left(\int_0^s \sigma^{-\frac{1-\gamma}{\alpha+\ell-1}}(s_0-\sigma)^{-\frac{1}{\alpha+\ell-1}}\,d\sigma\right)^{\frac{\alpha+\ell-1}{\alpha+\ell}} \end{aligned}$$

for all $s\in(0,s_0)$ and $t\in S_\phi$. Moreover, thanks to the condition $2-(\alpha+\ell)<\gamma<1$, we see that

$$\begin{aligned} \int_0^s \sigma^{-\frac{1-\gamma}{\alpha+\ell-1}}(s_0-\sigma)^{-\frac{1}{\alpha+\ell-1}}\,d\sigma &\le (s_0-s)^{-\frac{1}{\alpha+\ell-1}}\int_0^s \sigma^{-\frac{1-\gamma}{\alpha+\ell-1}}\,d\sigma \\ &= \left(\frac{\alpha+\ell-1}{\alpha+\ell+\gamma-2}\right) s^{\frac{\alpha+\ell+\gamma-2}{\alpha+\ell-1}}(s_0-s)^{-\frac{1}{\alpha+\ell-1}}. \end{aligned}$$

Thus we can obtain that

$$w(s,t)\le\left(\frac{\alpha+\ell-1}{\alpha+\ell+\gamma-2}\right)^{\frac{\alpha+\ell-1}{\alpha+\ell}} s^{\frac{\alpha+\ell+\gamma-2}{\alpha+\ell}}(s_0-s)^{-\frac{1}{\alpha+\ell}}\psi^{\frac{1}{\alpha+\ell}}(t)$$

for all $s\in(0,s_0)$ and $t\in S_\phi$, which concludes the proof. □

From Lemma 3.8 we derive the estimate for ψ.

Lemma 3.9. *Assume that u_0 satisfies* (1.15). *Suppose that $\alpha>0$, $\ell>0$ and $\gamma\in(-\infty,1)$ fulfill*

$$\alpha+\ell>1 \quad \textit{and} \quad 2-(\alpha+\ell)<\gamma<1.$$

Then there exists $C_1=C_1(\alpha,\ell,\gamma)>0$ such that for any choices of $s_0\in\big(0,\frac{R^n}{4}\big]$,

$$\psi(t)\ge C_1 s_0^{-(3-\gamma)(\alpha+\ell-1)}\phi^{\alpha+\ell}(t) \tag{3.36}$$

for all $t\in S_\phi$.

Proof. By an argument similar to that in the proof of [**19**, Lemma 3.7], we can show that (3.36) holds. □

3.3. ODIs for ϕ. Proof of Theorem 1.2

In this subsection we will prove Theorem 1.2. To this end, we first derive the ODIs for the moment-type functional ϕ in the following lemma. The proof is similar to that in [**23**].

Lemma 3.10. *Assume that D, S and f fulfill* (1.11) *and* (1.12). *Suppose that $m \in \mathbb{R}$, $\alpha > 0$, $\kappa > 1$ and $\ell > 0$ satisfy that*

$$\text{if } m \geq 0, \quad \text{then} \quad \alpha + \ell > \max\left\{m + \frac{2}{n}\kappa, \kappa\right\}, \tag{3.37}$$

$$\text{if } m < 0, \quad \text{then} \quad \alpha + \ell > \max\left\{\frac{2}{n}\kappa, \kappa\right\}. \tag{3.38}$$

Then there exist $\varepsilon > 0$ small enough and one can find $\gamma = \gamma(m, \alpha, \kappa, \ell) \in (-\infty, 1)$ and $C = C(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that if u_0 satisfies (1.15) *and $s_0 \in \big(0, \frac{R^n}{4}\big]$, then*

$$\phi'(t) \geq \begin{cases} \frac{1}{C} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) - C s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} & \text{if } m > 0, \\ \frac{1}{C} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) - C s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-\varepsilon}} & \text{if } m = 0, \\ \frac{1}{C} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) - C s_0^{3-\gamma-\frac{2}{n}} & \text{if } m < 0 \end{cases} \tag{3.39}$$

for all $t \in S_\phi$.

Proof. By virtue of (3.37), it follows that if $m \geq 0$, then

$$\begin{aligned} \left(2 - \frac{2}{n} \cdot \frac{\alpha+\ell}{\alpha+\ell-m}\right) - \left(2 - \frac{\alpha+\ell}{\kappa}\right) &= (\alpha+\ell)\left(\frac{1}{\kappa} - \frac{2}{n} \cdot \frac{1}{\alpha+\ell-m}\right) \\ &> (\alpha+\ell)\left(\frac{1}{\kappa} - \frac{2}{n} \cdot \frac{n}{2\kappa}\right) = 0. \end{aligned} \tag{3.40}$$

Thus, in the case $m \geq 0$ we can find $\gamma \in (-\infty, 1)$ such that

$$2 - \frac{\alpha+\ell}{\kappa} < \gamma < 2 - \frac{2}{n} \cdot \frac{\alpha+\ell}{\alpha+\ell-m}. \tag{3.41}$$

Thanks to (3.37) and (3.41), we know that (3.7), (3.11), (3.21), (3.30) and (3.35) hold. In the case $m > 0$, applying Lemmas 3.4–3.7, we see that there exist $c_1 = c_1(\alpha, \ell, L, C_S) > 0$ and $c_2 = c_2(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that

$$\begin{aligned} \phi'(t) \geq\ & c_1 \psi(t) - c_2 s_0^{3-\gamma} - c_2 s_0^{(3-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell} - \frac{2}{n}} \psi^{\frac{m}{\alpha+\ell}}(t) - c_2 s_0^{3-\gamma-\frac{2}{n}} \\ & - c_2 s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \psi^{\frac{\kappa}{\alpha+\ell}}(t) \end{aligned} \tag{3.42}$$

for all $t \in S_\phi$. Noting that $\alpha + \ell > m$ and $\alpha + \ell > \kappa$, from Young's inequality we can take $c_3 = c_3(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ and $c_4 = c_4(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that

$$c_2 s_0^{(3-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell} - \frac{2}{n}} \psi^{\frac{m}{\alpha+\ell}}(t) \leq \frac{c_1}{4} \psi(t) + c_3 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}}$$

and

$$c_2 s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \psi^{\frac{\kappa}{\alpha+\ell}}(t) \le \frac{c_1}{4}\psi(t) + c_4 s_0^{3-\gamma}.$$

In light of (3.42), we obtain that

$$\phi'(t) \ge \frac{c_1}{2}\psi(t) - c_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} \left(s_0^{\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} + \frac{c_3}{c_2} + s_0^{\frac{2}{n}\cdot\frac{m}{\alpha+\ell-m}} + \frac{c_4}{c_2} s_0^{\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} \right)$$

for all $t \in S_\phi$. Since $s_0 \le \frac{R^n}{4}$, there exists $c_5 = c_5(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that

$$\phi'(t) \ge \frac{c_1}{2}\psi(t) - c_5 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}}$$

for all $t \in S_\phi$. Moreover, we have from Lemma 3.9 that there exists $c_6 = c_6(\alpha, \ell, \gamma) > 0$ such that

$$\phi'(t) \ge \frac{c_1 c_6}{2} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) - c_5 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}}$$

for all $t \in S_\phi$, which implies (3.39) in the case $m > 0$. As to the case $m = 0$, due to (3.40), we can pick $\varepsilon > 0$ small enough and $\gamma \in (-\infty, 1)$ such that

$$2 - \frac{\alpha+\ell}{\kappa} < \gamma < 2 - \frac{2}{n} \cdot \frac{\alpha+\ell}{\alpha+\ell-\varepsilon}.$$

Therefore, using Lemmas 3.4–3.7, we establish that there exist $c_7 = c_7(\alpha, \ell, L, C_S) > 0$ and $c_8 = c_8(R, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that

$$\phi'(t) \ge c_7\psi(t) - c_8 s_0^{3-\gamma} - c_8 s_0^{(3-\gamma)\frac{\alpha+\ell-\varepsilon}{\alpha+\ell}-\frac{2}{n}} \psi^{\frac{\varepsilon}{\alpha+\ell}}(t) - c_8 s_0^{3-\gamma-\frac{2}{n}} - c_8 s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \psi^{\frac{\kappa}{\alpha+\ell}}(t)$$

for all $t \in S_\phi$. As in the case $m > 0$, from this inequality we can attain (3.39). Finally, in the case $m < 0$ we see from (3.38) that

$$\left(2 - \frac{2}{n}\right) - \left(2 - \frac{\alpha+\ell}{\kappa}\right) = \frac{\alpha+\ell}{\kappa} - \frac{2}{n} > \frac{1}{\kappa}\cdot\frac{2\kappa}{n} - \frac{2}{n} = 0.$$

Thus we can take $\gamma \in (-\infty, 1)$ satisfying

$$2 - \frac{\alpha+\ell}{\kappa} < \gamma < 2 - \frac{2}{n}.$$

By virtue of Lemmas 3.4–3.7 we know that there exist $c_9 = c_9(\alpha, \ell, L, C_S) > 0$ and $c_{10} = c_{10}(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that

$$\phi'(t) \ge c_9\psi(t) - c_{10} s_0^{3-\gamma} - c_{10} s_0^{3-\gamma-\frac{2}{n}} - c_{10} s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \psi^{\frac{\kappa}{\alpha+\ell}}(t)$$

for all $t \in S_\phi$. By an argument similar to that in the case $m > 0$, from Young's inequality and the relation $s_0 \le \frac{R^n}{4}$ we obtain $c_{11} = c_{11}(R, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that

$$\phi'(t) \ge \frac{c_9}{2}\psi(t) - c_{11} s_0^{3-\gamma-\frac{2}{n}}$$

for all $t \in S_\phi$. Thanks to Lemma 3.9, we can verify that (3.39) holds in the case $m < 0$. □

We are in a position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. We first consider the case $m > 0$. Due to (1.13), we can obtain from Lemma 3.10 that there exist $\gamma \in (-\infty, 1)$, $c_1 = c_1(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ and $c_2 = c_2(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that for each u_0 satisfying (1.15) and $s_0 \le \frac{R^n}{4}$, it follows that

$$\phi'(t) \ge c_1 s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) - c_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} \tag{3.43}$$

for all $t \in S_\phi$. Next we choose $s_0 \le \frac{R^n}{4}$ small enough such that

$$s_0 \le \frac{M_0}{2} \tag{3.44}$$

and

$$s_0^{(\alpha+\ell)\left(1-\frac{2}{n}\cdot\frac{1}{\alpha+\ell-m}\right)} \le \frac{c_1}{2c_2}\left(\frac{M_0}{2(1-\gamma)(2-\gamma)\omega_n}\right)^{\alpha+\ell}. \tag{3.45}$$

Furthermore, we fix $\varepsilon_0 \in \left(0, \frac{s_0}{2}\right)$ so small and take $s_\star \in (0, s_0)$ fulfilling

$$\frac{M_0 - \varepsilon_0}{\omega_n} \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s)\, ds > \frac{M_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma}. \tag{3.46}$$

We define $r_\star := s_\star^{\frac{1}{n}} \in (0, R)$ and suppose that u_0 satisfies (1.15) and (1.16). In order to show $T_{\max} < \infty$, assuming that $T_{\max} = \infty$, we will derive a contradiction. We set

$$\widetilde{S} := \left\{ T \in (0, \infty) \;\middle|\; \phi(t) > \frac{M_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma} \quad \text{for all } t \in [0, T] \right\}. \tag{3.47}$$

Here, we note that $\widetilde{S}$ is not empty. Indeed, since we have that for any $s \in (s_\star, R^n)$

$$w(s, 0) \ge w(s_\star, 0) = \frac{1}{\omega_n} \int_{B_{r_\star}(0)} u_0\, dx \ge \frac{M_0 - \varepsilon_0}{\omega_n},$$

we see from (3.46) that

$$\begin{aligned}
\phi(0) &\ge \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s) w(s, 0)\, ds \\
&\ge \frac{M_0 - \varepsilon_0}{\omega_n} \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s)\, ds \\
&> \frac{M_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma}.
\end{aligned}$$

Thus we can put $\widetilde{T} := \sup \widetilde{S} \in (0, \infty]$. Moreover, we know that $(0, \widetilde{T}) \subset S_\phi$. Owing to (3.47) and (3.44), we establish that

$$\phi(t) \ge \frac{M_0}{2(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma}$$

for all $t \in (0, \widetilde{T})$. From (3.45) it follows that

$$\frac{\frac{c_1}{2} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t)}{c_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}}} \geq \frac{c_1}{2c_2}\left(\frac{M_0}{2(1-\gamma)(2-\gamma)\omega_n}\right)^{\alpha+\ell} s_0^{-(\alpha+\ell)+\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} \geq 1$$

for all $t \in (0, \widetilde{T})$, which implies from (3.43) that

$$\phi'(t) \geq \frac{c_1}{2} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) \geq 0 \tag{3.48}$$

for all $t \in (0, \widetilde{T})$. This inequality yields that $\widetilde{T} = \infty$. However, from (3.48) and $\alpha+\ell-1 > 0$ we can show that

$$\widetilde{T} \leq \frac{2}{(\alpha+\ell-1)c_1\phi^{\alpha+\ell-1}(0)} s_0^{(3-\gamma)(\alpha+\ell-1)}.$$

As a consequence, we attain that $T_{\max}$ must be finite. In the cases $m = 0$ and $m < 0$, we can prove that $T_{\max} < \infty$ by an argument similar to that in the case $m > 0$. □

3.4. Open problems

In [**5, 19, 23**] the critical values such that solutions remain bounded or blow up in finite time were derived. With regard to the conditions (1.9), (1.13) and (1.14), we see that if $n \geq 3$ and $m \geq 0$ as well as $\frac{n}{n-2}m \leq \kappa$, then

$$\max\left\{m + \frac{2}{n}, \kappa\right\} = \max\left\{m + \frac{2}{n}\kappa, \kappa\right\} = \kappa.$$

Thus we know that the critical value is $\alpha + \ell = \kappa$ in this case. However, in the cases that $n \in \{1, 2\}$ and that $n \geq 3$ and $m \geq 0$ as well as $\frac{n}{n-2}m > \kappa$, the conditions (1.9), (1.13) and (1.14) are not optimal. Moreover, the special cases are as follows:

- In the case that $m = \alpha = 1$, behavior of solutions is an open problem when $\max\left\{\frac{2}{n}, \kappa - 1\right\} \leq \ell \leq \frac{2}{n}\kappa$ (see Figures 5 and 6).

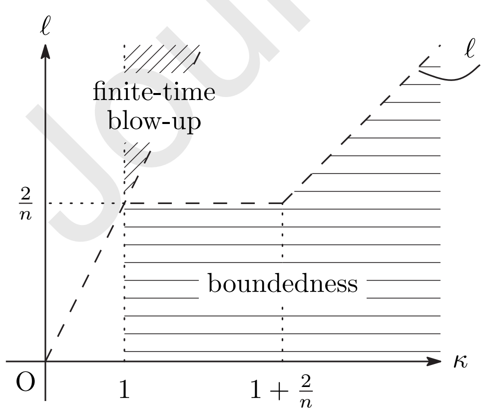

Figure 5: $n \in \{1, 2\}$ and $m = \alpha = 1$

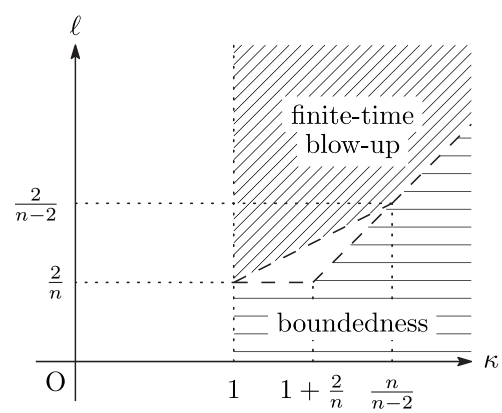

Figure 6: $n \geq 3$ and $m = \alpha = 1$

- In the case that $m = 1$ and $\kappa < \frac{n}{(n-2)_+}$, from Figure 7 we have an open question of whether solutions remain bounded or blow up when $\max\left\{1 + \frac{2}{n}, \kappa\right\} \le \alpha + \ell \le 1 + \frac{2}{n}\kappa$.

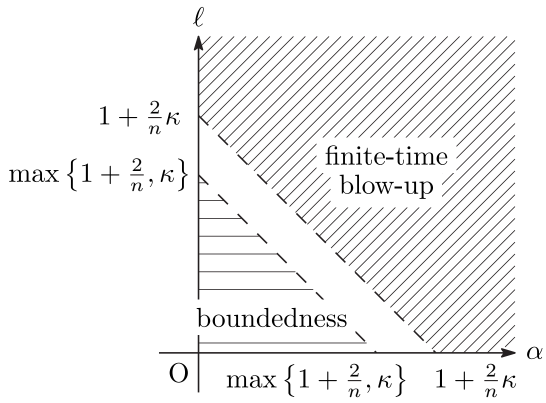

Figure 7: $m = 1$ and $\kappa < \frac{n}{(n-2)_+}$

- In the case that $\alpha = 1$ and $\ell > 0$, there is an open problem for behavior of solutions when $n = 1$ and $\max\{\kappa - 1, m + 1\} \le \ell \le \max\{2\kappa - 1, m + 2\kappa - 1\}$. Also, the same question exists when $n \ge 2$ and $\max\left\{\kappa - 1, m - \left(1 - \frac{2}{n}\right)\right\} \le \ell \le m - \left(1 - \frac{2}{n}\kappa\right)$. Moreover, in the case that $\alpha > 0$ and $\ell = 1$, we obtain regions that ℓ is replaced by α in Figures 8 and 9.

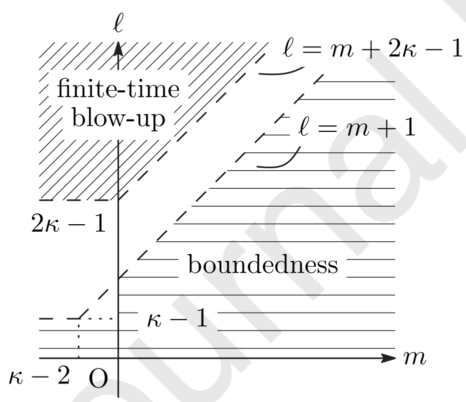

Figure 8: $n = 1$ and $\alpha = 1$

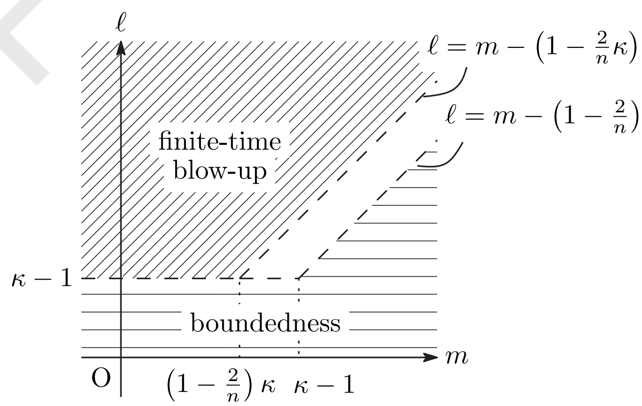

Figure 9: $n \ge 2$ and $\alpha = 1$

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