

Yosida–Hewitt and Lebesgue Decompositions of States on Orthomodular Posets¹

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Orthomodular posets are usually used as event structures of quantum mechanical systems. The states of the systems are described by probability measures (also called states) on it. It is well known that the family of all states on an orthomodular poset is a convex set, compact with respect to the product topology. This suggests using geometrical results to study its structure. In this line, we deal with the problem of the decomposition of states on orthomodular posets with respect to a given face of the state space. For particular choices of this face, we obtain, e.g., Lebesgue-type and Yosida–Hewitt decompositions as special cases. Considering, in particular, the problem of existence and uniqueness of such decompositions, we generalize to this setting numerous results obtained earlier only for orthomodular lattices and orthocomplete orthomodular posets. © 2001 Academic Press

Key Words: face of a convex set; state; probability measure; orthomodular poset; Yosida–Hewitt decomposition; Lebesgue decomposition; filtering set; filtering function; heredity.

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1. INTRODUCTION

Some of the classical decomposition theorems, originally stated for measures on Boolean algebras, have been extended to the case of nondistributive structures—orthomodular posets (OMPs)—giving also interesting applications to functional analysis (think, for example, of the Yosida–Hewitt decomposition theorem [28] stated by Aarnes in [1] for the case of the orthomodular lattice of closed subspaces of a Hilbert space). We investigate this topic from a geometrical point of view: we decompose finitely additive probability measures (states) defined on an OMP with respect to a face of its state space (which is a compact convex subset of a locally convex Hausdorff topological linear space). This approach is sufficiently general to give Lebesgue-type and Yosida–Hewitt decompositions as particular cases. We also deal with some related questions and prove the validity in general OMPs of many results which are known to hold in less general structures.

In extending decomposition results from the case of Boolean algebras to the more general case of OMPs, it is possible to obtain theorems which state the existence of decomposition, but not its uniqueness [7, 8, 19], and in fact in these structures uniqueness fails to hold in general (see Examples 3.13 and 4.3). Only in some particular cases the uniqueness is proved; see, e.g., [27] for the case of valuations, and [2, 3], where it is treated for the case of functions whose kernel is a p -ideal. Hence it is desirable to have some conditions which are sufficient for uniqueness. The paper [16] contributed to the study of the Yosida–Hewitt decomposition by showing that in some sense the lack of uniqueness is “natural” for measures defined on nondistributive structures. The research of the uniqueness of the Yosida–Hewitt decomposition was initiated in [21] and generalized in [9].

In this paper we first study a general type of decomposition, showing that it covers the Lebesgue and Yosida–Hewitt decompositions as particular cases. We then use the concept of filtering states (Definition 7.1) for comparing Yosida–Hewitt decomposition with a type of decomposition introduced by Rüttimann in [21]. The first is known to exist for any OMP; the second does not exist in general, but, if this is the case, then it is unique. When they coincide (this happens exactly when each state admits a Rüttimann decomposition), we obtain the existence and uniqueness of such a decomposition (Theorem 8.1). We then introduce two concepts of “heredity” in OMPs (Definitions 8.6 and 9.5), less general than the previous coincidence, but still strong enough to ensure the uniqueness of Yosida–Hewitt decomposition (Corollary 9.12). We observe that this can be applied to the particular case of Boolean algebras. The result is a modified version of [9, Theorem 2]—it is stated for OMPs (in this sense it is more general) and by using faces (in this sense it is less general).

Examples are important not only to support the reader's intuition, but also to show when the uniqueness of decomposition follows from simple geometric properties of the state space and when it is more involved. We therefore include several of them, providing answers to natural questions arising in this context.

The paper is organized as follows: Section 2 contains all the notions used in the sequel. Section 3 is devoted to the description of D -decomposition (using faces); the results are applied in Section 4 to the D -decomposition of states on OMPs. The general case is followed by several particular cases—Lebesgue decomposition (Section 5) and Yosida–Hewitt decomposition (Section 6). Another decomposition to filtering states and completely additive states (Rüttimann decomposition) is studied in Section 7; its relation to the Yosida–Hewitt decomposition is described in Section 8. Section 9 presents an alternative approach—decompositions using weakly filtering states—and a “graphic” summary of the results (Fig. 4).

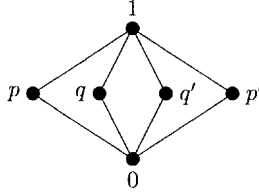
2. BASIC NOTIONS

In this section, we present basic notions and results concerning orthomodular posets. These are generalizations of Boolean algebras which admit the phenomenon of *noncompatibility*: the existence of pairs of elements which can be observed separately, but non-simultaneously. This makes *orthomodular posets* a well-motivated structure for the description of events of quantum mechanical systems where noncompatibility is a characteristic feature.

Consider $(L, \leq, 0, 1, ')$, where L is a set, \leq is a binary relation on L , 0 and 1 are two distinct elements of L , and $'$ is a function from L to L . We say that $(L, \leq, 0, 1, ')$ is an *orthomodular poset* (OMP) if the following conditions are satisfied:

- (i) (L, \leq) is a partially ordered set with a least element 0 and a greatest element 1 ,
- (ii) $' : L \rightarrow L$ is a decreasing function such that $p'' = p$ and $p \wedge p' = 0$ for all $p \in L$,
- (iii) if $p, q \in L$ with p, q *orthogonal* (i.e., $p \leq q'$, in symbols $p \perp q$), then $p \vee q$ exists in L ,
- (iv) if $p, q \in L$ with $p \leq q$, then $q = p \vee (p' \wedge q)$ (*orthomodular law*).

For brevity we will only use L to denote the OMP $(L, \leq, 0, 1, ')$. If, in addition, (L, \leq) is a lattice then L is called an *orthomodular lattice* (OML). A distributive orthomodular lattice is a Boolean algebra. A very important example of a nondistributive orthomodular lattice is the lattice of

FIG. 1. Hasse diagram of the OML MO_2 (Example 2.1).

projections in a Hilbert space or, more generally, in a von Neumann algebra. Numerous examples of (finite or infinite) OMPs can be found, e.g., in [11, 15].

A subset M of L is called *orthogonal* if its elements are pairwise orthogonal. If all orthogonal subsets of L have supremum in L , then L is called *orthocomplete*. For $M \subseteq L$ and $p \in L$, we use the notation $M \leq p$ (resp. $M \perp p$) to say that each element in M is less than or equal to p (resp. orthogonal to p).

EXAMPLE 2.1. The OML MO_2 is described by its Hasse diagram in Fig. 1. Its elements are $0, 1, p, p', q, q'$.

For two OMPs K, L , a mapping $f: K \rightarrow L$ is called a *homomorphism* if

- (1) $f(0) = 0$,
- (2) if $p \in L$, then $f(p') = f(p)'$,
- (3) if $p, q \in L$, $p \perp q$, then $f(p \vee q) = f(p) \vee f(q)$.

If, moreover, f is injective, surjective, and f^{-1} is also a homomorphism, then f is called an *isomorphism*.

The following notion will play an important role in the sequel.

DEFINITION 2.2. A subset I of an OMP L is called *filtering* (join dense) if $\forall p \in L \setminus \{0\} \exists q \in I \setminus \{0\} : q \leq p$.

EXAMPLE 2.3. Let λ be the Lebesgue measure on the algebra L of Lebesgue measurable subsets of $[0, 1]$ and let \tilde{L} be its quotient with respect to the set $\Delta = \{p \in L : \lambda(p) = 0\}$. For any subset F of L , denote by \tilde{F} the corresponding set in \tilde{L} . Note that the set $F_1 = \{\{x\} : x \in [0, 1]\}$ is filtering in L , while \tilde{F}_1 is not filtering in \tilde{L} (in fact \tilde{F}_1 only contains the zero element of \tilde{L}). On the other hand, $F_2 = \{q \in L : \lambda(q) = 2^{-n} \text{ for some } n \in \mathbb{N}\}$ is not filtering in L , while \tilde{F}_2 is filtering in \tilde{L} .

LEMMA 2.4. If M is a maximal orthogonal subset of a filtering set F , then $\vee M = 1$.

Proof. If $p \in L$ is an upper bound of M different from 1, then from the relation $p' \neq 0$ it follows that F contains a nonzero element $q \leq p'$. The system $M \cup \{q\}$ is orthogonal and strictly greater than M . This contradicts the maximality of M . Therefore 1 is the only upper bound for M . ■

Let L be an OMP. For $p \in L$, we denote the interval $L_p = \{x \in L : x \leq p\}$. The interval L_p inherits in a natural way the structure of L in the sense that if \leq_p is the restriction of \leq to L_p and $'^p$ is defined by $x'^p = x' \wedge p$, then the relation \leq_p is an order and $'^p$ is an orthocomplementation which make $(L_p, \leq_p, 0, p, '^p)$ an OMP. We understand L_p as an OMP this way. An interval in an OML is an OML.

The intersection of filtering sets is not filtering in general (in the structure \tilde{L} described in Example 2.3 the filtering set \tilde{F} , where $F = \{q \in L : \lambda(q) \text{ is not rational}\}$ is disjoint from \tilde{F}_2). We can obtain a filtering intersection adding some additional hypothesis.

PROPOSITION 2.5 [6]. *Let P, Q be filtering subsets of an OMP L and let P be an order ideal (i.e., $q \leq p \in P \implies q \in P$). Then $P \cap Q$ is filtering.*

Proof. Let x be a nonzero element in L . There exists $p \in P \cap L_x \setminus \{0\}$. Since p is nonzero, there exists $q \in Q \cap L_p \setminus \{0\}$. As P is an order ideal, the element q also belongs to P and it is a nonzero element of L_x . ■

An element $a \in L$ is called an *atom* if $L_a = \{0, a\}$. We say that L is *atomic* if, for each nonzero $p \in L$, the interval L_p contains an atom of L . From the definition it easily follows that a set which contains a filtering subset is a filtering set itself. Further, a filtering set contains all atoms of L . Moreover, for atomic OMPs we have the equivalence: A subset of an atomic OMP is filtering iff it contains all atoms.

For future use in examples, we shall need the following construction techniques for OMPs:

DEFINITION 2.6. Let $\mathcal{F} = ((K, \leq_K, 0_K, 1_K, '^K))_{K \in \mathcal{K}}$ be a family of OMPs. We take the Cartesian product $L = \prod_{K \in \mathcal{K}} K$ and we endow it with the partial ordering \leq_L and orthocomplementation $'^L$ defined pointwise; i.e., for all $a, b \in L$, $a = (a_K)_{K \in \mathcal{K}}$, $b = (b_K)_{K \in \mathcal{K}}$, we define

$$\begin{aligned} a \leq_L b &\iff \forall K \in \mathcal{K} : a_K \leq_K b_K, \\ a &= b'^L \iff \forall K \in \mathcal{K} : a_K = (b_K)'^K. \end{aligned}$$

The bounds of L are $0_L = (0_K)_{K \in \mathcal{K}}$, $1_L = (1_K)_{K \in \mathcal{K}}$. Then $(L, \leq_L, 0_L, 1_L, '^L)$ is an OMP called the *product* of the family \mathcal{F} . The product of OMLs is an OML.

DEFINITION 2.7. Let \mathcal{F} be a family of OMPs. Take the family \mathcal{G} of the copies of each OMP from \mathcal{F} and consider them disjoint except that their least and greatest elements are identified (and denoted by 0 and 1, respectively). Thus, for each $K, M \in \mathcal{G}$, $K \neq M$, we have $K \cap M = \{0, 1\}$. We take the union $L = \bigcup \mathcal{G}$ and we endow it with the partial ordering \leq_L and orthocomplementation ${}^{\prime L}$ defined by

$$\begin{aligned} a \leq_L b &\iff \exists K \in \mathcal{G} : (a, b \in K, a \leq_K b), \\ a = b^{\prime L} &\iff \exists K \in \mathcal{G} : (a, b \in K, a = b^{\prime K}). \end{aligned}$$

Then $(L, \leq_L, 0, 1, {}^{\prime L})$ is an OMP called the *horizontal sum* of the family \mathcal{F} . The horizontal sum of OMLs is an OML.

EXAMPLE 2.8. Let $A = \{0, 1, p, p'\}$ and $B = \{0, 1, q, q'\}$ be two 4-element Boolean algebras. Their horizontal sum is the OML *MO2* from Example 2.1; their product is the 16-element Boolean algebra with four atoms.

3. GENERAL *D*-DECOMPOSITION

In the sequel, we shall prove theorems on decompositions of states (=probability measures) on OMPs. Our problem was to obtain any state on L as a convex combination of two states which are, in some sense, “far” from each other. Working on states of an OMP L , we deal with a compact subset of $[0, 1]^L$. The required notions can however be defined on arbitrary compact convex sets. We present them in this section.

Assumption 3.1. Throughout this section we assume that V is a locally convex Hausdorff topological linear space and C is a compact convex subset of V .

We say that a convex combination of mutually different points of V is *inner* iff all its coefficients belong to the open interval $(0, 1)$. If D is any subset of V , we denote by $\text{conv } D$ the convex hull of D and by $\text{iconv } D$ the set of all inner convex combinations of D . A σ -convex combination of elements of V is $\sum_{n \in \mathbb{N}} c_n v_n$, where $v_n \in V$ and $c_n \in [0, 1]$ such that $\sum_{n \in \mathbb{N}} c_n = 1$. A subset F of C is a *face* of C if all $\alpha, \beta, \gamma \in C$, with $\gamma \in \text{iconv } \{\alpha, \beta\}$ satisfy the equivalence

$$\gamma \in F \iff \alpha, \beta \in F.$$

The collection of all faces of C ordered by inclusion has a least element \emptyset and a greatest element C . A face is a singleton iff it consists of an extreme point of C . The intersection of a family of faces is again a face. Therefore,

for each subset $D \subseteq C$, there is a least face F containing D ; we call it the face *generated* by D . The case when D is a singleton, $D = \{\mu\}$, is of particular importance. In this case, we speak of a face generated by μ .

Faces generated by a subset D of C have a useful characterization by the relation of *strong absolute continuity*, \ll_S , defined on C as

$$\lambda \ll_S \mu \iff (\lambda = \mu \text{ or } \exists \nu \in C : \mu \in \text{iconv}\{\lambda, \nu\}).$$

It is easy to observe that the relation \ll_S is transitive: Assume that $\alpha \ll_S \beta \ll_S \gamma$. If two of the elements α, β, γ coincide, the result is trivial. Otherwise, there are $\delta, \varepsilon \in C$ such that $\beta \in \text{iconv}\{\alpha, \delta\}$, $\gamma \in \text{iconv}\{\beta, \varepsilon\}$ (see Fig. 2a). Then $\gamma \in \text{iconv}\{\alpha, \delta, \varepsilon\}$, so $\alpha \ll_S \gamma$.

PROPOSITION 3.2. *Let $\mu \in C$. Then the set*

$$F_\mu = \{\lambda \in C : \lambda \ll_S \mu\}$$

is the face generated by μ .

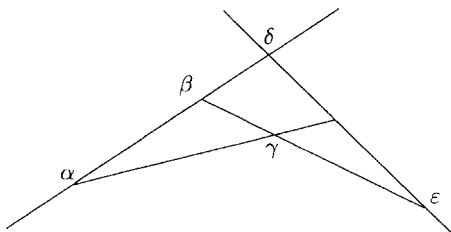
Proof. Let $\alpha \in F_\mu$, i.e., $\alpha \ll_S \mu$. Either $\alpha = \mu$ or we have $\mu \in \text{iconv}\{\alpha, \beta\}$ for some $\beta \in C$. In both cases the face generated by μ contains α , hence also the whole F_μ .

It remains to prove that F_μ is a face of C . Let $\gamma = t\alpha + (1-t)\beta$ for some $\alpha, \beta, \gamma \in C$, $\alpha \neq \beta$ and $t \in (0, 1)$. Then $\alpha \ll_S \gamma$, $\beta \ll_S \gamma$. If $\gamma \in F_\mu$, then $\gamma \ll_S \mu$ and the transitivity of \ll_S gives $\alpha \ll_S \mu$, $\beta \ll_S \mu$, and $\alpha, \beta \in F_\mu$. If $\alpha, \beta \in F_\mu \setminus \{\mu\}$, then there are $\delta, \eta \in C$ and $u, v \in (0, 1)$ such that

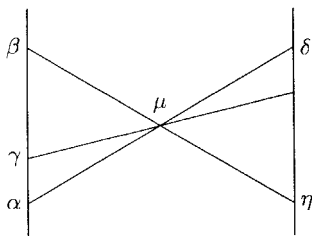
$$\mu = u\alpha + (1-u)\delta = v\beta + (1-v)\eta$$

(see Fig. 2b). For

$$w = \frac{tv}{tv + (1-t)u} \in (0, 1),$$



a



b

FIG. 2. Situations from the proof of Proposition 3.2.

we obtain

$$\begin{aligned}
\mu &= w\mu + (1-w)\mu = w(u\alpha + (1-u)\delta) + (1-w)(v\beta + (1-v)\eta) \\
&= (wu\alpha + (1-w)v\beta) + (w(1-u)\delta + (1-w)(1-v)\eta) \\
&= \frac{uv}{tv + (1-t)u} (t\alpha + (1-t)\beta) + (w(1-u)\delta + (1-w)(1-v)\eta) \\
&= \frac{uv}{tv + (1-t)u} \gamma + (w(1-u)\delta + (1-w)(1-v)\eta).
\end{aligned}$$

As

$$\frac{uv}{tv + (1-t)u} \in (0, 1),$$

we proved that $\gamma \ll_S \mu$, hence $\gamma \in F_\mu$ and F_μ is convex. If $\alpha = \mu$ (resp. $\beta = \mu$) we proceed analogously with $u = 1$ (resp. $v = 1$). The proof is finished. ■

The previous result shows that two points are strongly absolutely continuous with respect to each other if and only if they generate the same face.

For a subset D of C , we define

$$U_D = \{\lambda \in C : (\exists \mu \in D : \lambda \ll_S \mu)\}.$$

If D is a singleton, $D = \{\mu\}$, then U_D coincides with the face F_μ generated by μ . In general, U_D is a subset of the face generated by D , but the reverse inclusion need not hold. In fact, U_D is the least *union of faces* which contains D , and also the union of all faces generated by the elements of D ,

$$U_D = \bigcup_{\mu \in D} F_\mu.$$

Remark 3.3. If D is convex, then U_D is a face.

For a subset D of C , we define its *complementary set* as

$$D^\# = \{\alpha \in C : (\forall \beta \in D : \beta \not\ll_S \alpha)\}.$$

Different definitions of complementary sets can be found in the literature [9, 21]. Their equivalence is a consequence of the following observation:

PROPOSITION 3.4. *Let D be a subset of C . For an element $\alpha \in C$, the following conditions are equivalent:*

- (1) $\alpha \in D^\#$,
- (2) the face generated by α is disjoint from D ,

(3) α belongs to a face disjoint from D ,

(4) $\alpha \notin D$ and if $\alpha \in \text{conv}\{\beta, \gamma\}$ for some $\beta \in D$, $\gamma \in C$, then $\gamma = \alpha$.

Proof. The equivalence of (1) and (2) follows from Proposition 3.2. The equivalence of (2) and (3) is an easy consequence of the definition of a face generated by α . The equivalence of (1) and (4) follows from the definition of \ll_S . ■

PROPOSITION 3.5. *Let D be a subset of C . Then $D^\#$ is the union of all faces of C disjoint from D .*

Proof. For each $\mu \in D^\#$, the face F_μ generated by μ is a subset of $D^\#$, too. According to the definition of a complementary set, $F_\mu \cap D = \emptyset$, so $D^\#$ is a union of faces disjoint from D . Let F be a face of C disjoint from D . Take any $\mu \in F$. Since $F_\mu \subseteq F$, we have $F_\mu \cap D = \emptyset$. By Proposition 3.4(2), this implies $\mu \in D^\#$, so $F \subseteq D^\#$. ■

A proper face may have an empty complementary set:

EXAMPLE 3.6. Let C be the set of all sequences $(x_n)_{n \in \mathbb{N}}$ of nonnegative real numbers such that $\sum_{n \in \mathbb{N}} x_n \leq 1$. Let D be the set of all sequences from C which have only finitely many nonzero elements. Then C is a convex set compact in the product topology and D is its face. Although $D \subsetneq C$, we have $D^\# = \emptyset$. As a consequence, $D^{\#\#} = C \neq D$.

DEFINITION 3.7. Let D be a subset of C . We say that $\mu \in C$ has a *D-decomposition* if one of the following conditions holds:

- $\mu \in D$,
- $\mu \in D^\#$,
- μ is an inner convex combination of an element of D and an element of $D^\#$.

The D -decomposition need not exist in general (e.g., in Example 3.6 the elements of $C \setminus D$ do not have a D -decomposition). In [19] Rüttimann gives a sufficient condition for its existence—the closedness of the face D in a suitable norm, called the base norm. We will only deal with a particular case, defining this norm on the family of the states on an OMP in Section 4 (see Theorem 4.5). Most decompositions studied in this paper are D -decompositions.

Note that if D is a union of faces, then $D \subseteq D^{\#\#}$. In fact $D^{\#\#}$ contains all faces of C disjoint from $D^\#$, and the faces in D have this property. As Example 3.6 shows, the reverse inclusion is not true in general, but it holds in the following important case:

PROPOSITION 3.8. *Let D be a union of faces of C . If each element of C admits a D -decomposition, then $D^{\#\#} = D$.*

Proof. Take any $\alpha \in D^{\#\#} \setminus D$. By the assumption, we know that $\alpha \in \text{conv} \{\beta, \gamma\}$, for some $\beta \in D$ and $\gamma \in D^\#$. Condition $\alpha \notin D^\#$ and statement (4) of Proposition 3.4 imply $\alpha = \beta \in D$, a contradiction. ■

Example 3.6 shows that the existence of the D -decomposition has to be assumed in the latter proposition. Another such example (inspired by Schindler [23]) follows; it is an improvement of an observation from [19].

EXAMPLE 3.9. Let C be the set of all sequences $(x_n)_{n \in \mathbb{N}}$ of real numbers such that $\sum_{n \in \mathbb{N}} |x_n| \leq 1$. We define D as the set of all $(x_n)_{n \in \mathbb{N}} \in C$ such that all coordinates x_n , $n \in \mathbb{N}$, are nonnegative, only finitely many of them are nonzero, and $\sum_{n \in \mathbb{N}} x_n = 1$. Then D is a face of C . Its complementary set, $D^\#$, consists of all $(y_n)_{n \in \mathbb{N}} \in C$ such that all coordinates y_n , $n \in \mathbb{N}$, are nonpositive and $\sum_{n \in \mathbb{N}} |y_n| = 1$. The double complementary set, $D^{\#\#}$, consists of all $(z_n)_{n \in \mathbb{N}} \in C$ such that all coordinates z_n , $n \in \mathbb{N}$, are nonnegative and $\sum_{n \in \mathbb{N}} z_n = 1$. In this case $D \subsetneq D^{\#\#}$.

We shall also ask about the uniqueness of D -decomposition.

Remark 3.10. The D -decomposition of $\alpha \in C$ exists iff

- (1) $\alpha \in D \cup D^\#$ or
- (2) $\alpha = t\beta + (1-t)\gamma$ for some $\beta \in D$, $\gamma \in D^\#$, $t \in (0, 1)$.

The D -decomposition is considered unique if case (1) applies or there are unique t, β, γ satisfying case (2). Also for other decompositions to convex combinations, existence and uniqueness are understood this way.

Uniqueness of D -decomposition is obtained in some trivial cases. Notice that the following proposition requires the compactness of C .

PROPOSITION 3.11. *If D is a singleton consisting of an extreme point of C , then the D -decomposition exists and it is unique.*

Proof. Let $D = \{\beta\}$, $\alpha \in C$. If $\alpha = \beta$, then the D -decomposition of α exists and it is unique. If $\alpha \neq \beta$, then α, β determine a line which intersects with C in a line segment, S (due to the compactness of C). One of its endpoints is β ; we denote by γ the other endpoint of S . Let $\mu \in S \setminus \{\beta\}$. Then the face F_μ of C generated by μ contains β iff $\mu \neq \gamma$. Thus $S \cap D^\# = \{\gamma\}$ and α admits a unique decomposition to a convex combination of β, γ . ■

Some uniqueness properties can be proved for finite-dimensional simplexes and, more generally, for metrizable Choquet simplexes. These can be alternatively defined as compact convex subsets of a Banach space such that each of their elements can be expressed as a unique σ -convex combination of their extreme points.

PROPOSITION 3.12. *Let D be a σ -convex face of a metrizable Choquet simplex C . If an element α of C admits a D -decomposition, then this D -decomposition is unique.*

Proof. If $\alpha \in D \cup D^\#$, the D -decomposition is unique in the sense of Remark 3.10. We assume that $\alpha = t\beta + (1-t)\gamma$ for some $\beta \in D$, $\gamma \in D^\#$, $t \in (0, 1)$. According to the assumptions on C , elements β, γ have unique decompositions to σ -convex combinations of extreme points of C ,

$$\beta = \sum_{i \in I} u_i \beta_i, \quad \gamma = \sum_{j \in J} v_j \gamma_j,$$

where $u_i \in (0, 1]$, $v_j \in (0, 1]$, $\sum_{i \in I} u_i = \sum_{j \in J} v_j = 1$, and β_i, γ_j are extreme points of C . As D is a face, $\beta_i \in D$ for all $i \in I$. As $\gamma \in D^\#$, $\gamma_j \notin D$ for all $j \in J$. We obtain a decomposition

$$\alpha = \sum_{i \in I} t u_i \beta_i + \sum_{j \in J} (1-t) v_j \gamma_j,$$

where β_i, γ_j are extreme points of C . All coefficients in the latter decomposition are nonzero and this decomposition of α is unique. This implies the uniqueness of the D -decomposition, too. ■

In general, the D -decomposition need not be unique.

EXAMPLE 3.13. Let C be a square.

(i) Let D be an edge of C . Then the complementary set $D^\#$ is the edge of C opposite to D , and the D -decomposition is not unique (for the elements of the interior of C).

(ii) Let D be a singleton containing a vertex of C . Then $D^\#$ is the union of the two edges disjoint from D and the D -decomposition is unique (Proposition 3.11).

Example 3.13(i) is a universal example of a non-unique D -decomposition in the following sense ($\text{aff } X$ denotes the affine hull of a set X):

PROPOSITION 3.14. *Let D be a face of C . The D -decomposition is not unique if and only if there is a subspace S of $\text{aff } C$ such that*

- (1) *the affine dimension of S is 2,*
- (2) *the intersections $S \cap D, S \cap D^\#$ have more than one element.*

Proof. Suppose first that the subspace S with the given properties exists. Choose different elements $\alpha, \beta \in S \cap D$, $\gamma, \delta \in S \cap D^\#$. If α is a convex combination of β, γ, δ , then at least one of γ, δ belongs to the face generated by α , a contradiction with the assumption $\gamma, \delta \in D^\#$. An analogous argument can be used to prove that none of the four elements $\alpha, \beta, \gamma, \delta$ is

a convex combination of the remaining three. Thus $\alpha, \beta, \gamma, \delta$ are vertices of a convex quadrangle. Its diagonals intersect, so at least one of the sets

$$\begin{aligned} & \text{conv}\{\alpha, \beta\} \cap \text{conv}\{\gamma, \delta\}, \\ & \text{conv}\{\alpha, \gamma\} \cap \text{conv}\{\beta, \delta\}, \\ & \text{conv}\{\alpha, \delta\} \cap \text{conv}\{\beta, \gamma\} \end{aligned}$$

is nonempty. The first case is impossible because the face generated by the intersection (and hence also its superset D) contains γ or δ . In the latter two cases, the intersection is an element with a non-unique D -decomposition.

To prove the reverse implication, suppose that there is a point $\mu \in C$ with a non-unique D -decomposition. There are $\alpha, \beta \in D$, $\gamma, \delta \in D^\#$, such that $\mu \in \text{conv}\{\alpha, \gamma\} \cap \text{conv}\{\beta, \delta\}$ and $(\alpha, \gamma) \neq (\beta, \delta)$. As $D, D^\#$ are disjoint, $\alpha \neq \gamma$ and $\beta \neq \delta$. Thus the one-dimensional spaces $\text{aff}\{\alpha, \gamma\}$, $\text{aff}\{\beta, \delta\}$, intersecting at μ , determine an at most two-dimensional space, $S = \text{aff}\{\alpha, \beta, \gamma, \delta\}$. Moreover, due to the definition of $D, D^\#$, we have $\beta, \delta \notin \text{conv}\{\alpha, \gamma\}$, $\alpha, \gamma \notin \text{conv}\{\beta, \delta\}$. In particular, $\alpha \neq \beta$ and $\gamma \neq \delta$ and the affine dimension of S is 2. ■

The following example shows that the sufficient conditions for the uniqueness of the D -decomposition given in Propositions 3.11 and 3.12 are not necessary.

EXAMPLE 3.15. (In order to obtain both D and $D^\#$ as faces which are not simplices and which allow a unique D -decomposition, the minimal affine dimension of C is 5.) In the 5-dimensional space \mathbb{R}^5 , we define the sets

$$\begin{aligned} D &= \text{conv}\{(0, 0, 0, 0, 0), (0, 0, 0, 0, 1), (0, 0, 0, 1, 1), (0, 0, 0, 1, 0) \\ &= \{(0, 0, 0, y_4, y_5) : y_4, y_5 \in [0, 1]\}, \\ E &= \text{conv}\{(1, 0, 0, 0, 0), (1, 0, 1, 0, 0), (1, 1, 1, 0, 0), (1, 1, 0, 0, 0) \\ &= \{(1, z_2, z_3, 0, 0) : z_2, z_3 \in [0, 1]\}, \\ C &= \text{conv}(D \cup E). \end{aligned}$$

Then C contains D, E as its complementary faces, $D^\# = E$, $E^\# = D$ (see Fig. 3). Each point $(x_1, x_2, x_3, x_4, x_5) \in C$ is of the form

$$(x_1, x_2, x_3, x_4, x_5) = t \cdot (0, 0, 0, y_4, y_5) + (1 - t) \cdot (1, z_2, z_3, 0, 0)$$

for unique $t, z_2, z_3, y_4, y_5 \in [0, 1]$, so the D -decomposition is unique.

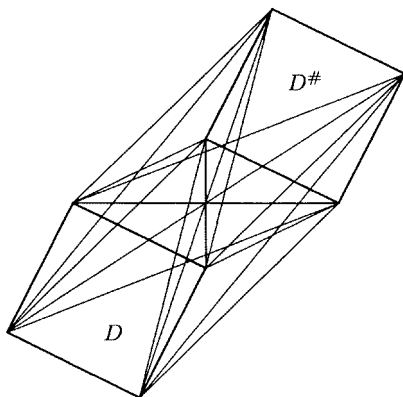


FIG. 3. The 5-dimensional set with unique D -decomposition from Example 3.15.

4. STATES ON ORTHOMODULAR POSETS AND THEIR D -DECOMPOSITION

In this section, we shall apply the D -decomposition to state spaces of OMPs. We also present several observations concerning uniqueness of D -decomposition. As we shall see later, this uniqueness has important consequences for the structure of the state space.

Let L be an OMP. A function $\mu: L \rightarrow \mathbb{R}$ is called *additive* (resp. *completely additive*) if the equality $\sum_{x \in M} \mu(x) = \mu(\bigvee M)$ holds for all orthogonal subsets $M \subset L$ which are finite (resp. which have supremum in L ; in this case the absolute convergence of the series is assumed). An additive function $\mu: L \rightarrow [0, \infty)$ is called a *positive measure*; a difference of two positive measures is called a *Jordan measure*. We denote by $J^+(L)$ the set of all positive measures on L and by $J(L) = J^+(L) - J^+(L)$ the set of all Jordan measures on L . The set of all positive (resp. Jordan) measures on L which are completely additive is denoted by $J_c^+(L)$ (resp. $J_c(L)$). The space $J(L)$ inherits from \mathbb{R}^L the natural ordering of functions,

$$\alpha \leq \beta \iff \forall p \in L : \alpha(p) \leq \beta(p).$$

The elements of $J^+(L)$ which attain 1 on the unit of L are called *states*. The set of all states (resp. completely additive states) on L is denoted by $\Omega(L)$ (resp. $\Omega_c(L)$).

As we already observed, the state space of an OMP is convex. It is a closed subset of the product space $[0, 1]^L$ which is compact due to the Tichonoff theorem. Therefore also the state space is compact (in the

product topology). The characterization of state spaces of OMPs was completed by the following famous theorem by Shultz (originally stated for OMLs in [24]; see also [12, 15] for simplified proofs and Theorem 6.4 for its strengthening):

THEOREM 4.1. *Every compact convex subset of a locally convex Hausdorff topological linear space is affinely homeomorphic to the state space of an OML.*

Assumption 4.2. Unless stated otherwise, we assume in the sequel that L is an OMP and D is a face of $\Omega(L)$.

If we take $\Omega(L)$ for the compact convex set C , we may apply the results of Section 3. For a face $D \subseteq C$, we shall study the D -decomposition of a state $\alpha \in \Omega(L)$ into a convex combination of $\beta \in D$ and $\gamma \in D^\#$. With particular choices of D , we can obtain some important kinds of decompositions, like Yosida–Hewitt or Lebesgue-type decompositions.

In view of Theorem 4.1, all examples from Section 3 can be obtained as examples of the D -decomposition of state spaces of an OML. In particular, Example 3.13 has an easy analogue:

EXAMPLE 4.3. Let L be the OML $MO2$ from Example 2.1. Each state $\mu \in \Omega(L)$ is uniquely determined by $\mu(p), \mu(q)$, which are arbitrary values from $[0, 1]$. The state space is a square. The cases discussed in Example 3.13 correspond to the following choices of D :

- (i) $D = \{\mu \in \Omega(L) : \mu(p) = 1\}$,
- (ii) $D = \{\mu \in \Omega(L) : \mu(p) = 1 \text{ and } \mu(q) = 1\}$.

Results of Section 3 concerning the D -decomposition can be reformulated for state spaces of OMPs; see [9, 21] for some of them. In particular, Propositions 3.11, 3.12, 3.14 are applicable to the uniqueness of the D -decomposition and Proposition 3.11 gives a sufficient condition for its existence. The D -decomposition need not exist even for complete Boolean algebras.

EXAMPLE 4.4. Take the Boolean algebra $L = 2^\mathbb{N}$. For $n \in \mathbb{N}$ we denote by $\chi_n \in \Omega(L)$ the state concentrated in n , i.e., for each $A \subseteq \mathbb{N}$,

$$\chi_n(A) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A. \end{cases}$$

Let D be the set of all (finite) convex combinations of concentrated states. Then D is a face of $\Omega_c(L)$. The set $D^\#$ consists of all states that vanish at all singletons and $D^{\#\#}$ contains all countable convex combinations of concentrated states. Elements of $D^{\#\#} \setminus D$ do not admit a D -decomposition.

The uniqueness of the D -decomposition for complete Boolean algebras was proved in [21, Proposition 5.1]. However, we shall prove it even for Boolean algebras which are not complete.

For each state $\alpha \in \Omega(L)$ we define its *base norm* by putting

$$\|\alpha\| = \inf\{s + t : \alpha = s\beta - t\gamma : s, t \in \mathbb{R}_+, \beta, \gamma \in \Omega\}.$$

As Rüttimann showed, the closedness of a face D with respect to the topology generated by this norm is a sufficient condition for the existence of D -decomposition (then for the equality $D = D^{\#\#}$).

THEOREM 4.5. *Let L be an OMP and let D be a base norm closed face of $\Omega(L)$. Then the D -decomposition exists. If, moreover, D is a subset of $\Omega_c(L)$ and L is a Boolean algebra, then the D -decomposition is also unique.*

Proof. The proof of existence is quite technical, but it can be obtained by a routine translation of the proof of [19, Theorem 2.5] (formulated for cones of measures) into the terms of faces of states. The uniqueness follows from Proposition 9.7 and Corollary 9.10 below. ■

In Theorem 4.5, it is necessary to assume that D is base norm closed as Example 4.4 shows.

EXAMPLE 4.6. There are non-Boolean OMLs having simplices as state spaces, see [13–15, 26]; Proposition 3.12 can be applied to them as in Theorem 4.5, giving OMLs with a unique D -decomposition for each face D .

5. LEBESGUE-TYPE DECOMPOSITION

For particular choices of the face D in the D -decomposition, we obtain the Yosida–Hewitt and Lebesgue-type decompositions as special cases. In this section we present a Lebesgue-type decomposition which has been intensively studied also in the case of classical measure theory. It deals with the possibility of decomposing a state to a convex combination of two states: the first absolutely continuous, the second orthogonal with respect to a fixed state. In the case of completely additive states defined on complete Boolean algebras, it is possible to define absolute continuity and orthogonality with respect to a fixed state in many equivalent ways (see, e.g., [4]). Generalizing to the case of states on OMPs, several definitions are possible (they coincide on Boolean algebras). For our purpose, the most convenient seems to be the following:

DEFINITION 5.1. Let α and λ be states on an OMP L . We say that α is *absolutely continuous* with respect to λ (in symbols, $\alpha \ll \lambda$) if

$$\forall \varepsilon > 0 \exists \delta > 0 : (\lambda(p) < \delta \implies \alpha(p) < \varepsilon).$$

Absolute continuity is weaker than the strong absolute continuity; if $\alpha \ll_S \lambda$ for some $\alpha, \lambda \in \Omega(L)$, then $\alpha \ll \lambda$, too. The reverse implication need not hold:

EXAMPLE 5.2. Let L be the horizontal sum of countably many 4-element Boolean algebras $\{0, 1, p_n, p'_n\}$, $n \in \mathbb{N}$. Let α, λ be the states on L uniquely determined by the following values on p_n , $n \in \mathbb{N}$,

$$\alpha(p_n) = \frac{1}{2^n}, \quad \lambda(p_n) = \frac{1}{n+1}.$$

Then $\alpha \ll \lambda$ and $\lambda \ll \alpha$. We also have $\alpha \ll_S \lambda$, but $\lambda \not\ll_S \alpha$.

From an intuitive point of view, to say that α is absolutely continuous with respect to λ means that “ α is small when λ is small,” or, in the same way, that it is possible to “control” α by using λ ; on the contrary, the orthogonality of a state μ with respect to λ means that it is not possible to find a state ν absolutely continuous with respect to λ for which the relation $\nu \ll_S \mu$ holds.

DEFINITION 5.3. Let μ and λ be states on L . We say that μ is *orthogonal* with respect to λ (in symbols $\mu \perp \lambda$) if there is no $\nu \in \Omega(L)$ satisfying $\nu \ll_S \mu$, $\nu \ll \lambda$.

Let us fix a state λ on L and take for D the set of all states on L which are absolutely continuous with respect to λ ,

$$D = \{\alpha \in \Omega(L) : \alpha \ll \lambda\}.$$

It is easy to check that D is a face of $\Omega(L)$ and its complementary set $D^\#$ is exactly the set of all states on L orthogonal with respect to λ

$$D^\# = \{\mu \in \Omega(L) : \mu \perp \lambda\}.$$

In this case, the D -decomposition is a Lebesgue-type decomposition. (This is not the only possible Lebesgue-type decomposition of states on OMPs. Other types were studied, e.g., in [2, 3, 22].) The existence of the Lebesgue-type decomposition follows from [8, Theorem 3.2]:

PROPOSITION 5.4. Let L be an OMP and let $\lambda \in \Omega(L)$. Each state $\alpha \in \Omega(L)$ can be expressed as a convex combination of states $\beta, \gamma \in \Omega(L)$ such that $\beta \ll \lambda$ and $\gamma \perp \lambda$.

If L is a Boolean algebra, uniqueness of the Lebesgue decomposition follows from [8, Remark 4.3 (2)] (for λ completely additive also from

Theorem 4.5). In the general case, uniqueness need not hold:

EXAMPLE 5.5. Let L be the OML $MO2$ from Example 2.1. Take $\lambda \in \Omega(L)$ such that $\lambda(p) = 0$, $\lambda(q) = 1/2$. It is easy to check that $D = \{\alpha \in \Omega(L) : \alpha \ll \lambda\} = \{\alpha \in \Omega(L) : \alpha(p) = 0\}$ and $D^\# = \{\alpha \in \Omega(L) : \alpha(p) = 1\}$. In this case D and $D^\#$ are opposite edges of the square $\Omega(L)$ and the Lebesgue decomposition is not unique; see Example 3.13.

Using Proposition 5.4, we obtain a result analogous to Propsitions 3.4 and 3.8:

PROPOSITION 5.6. *A state $\alpha \in \Omega(L)$ is absolutely continuous with respect to $\lambda \in \Omega(L)$ iff*

$$\alpha \in \text{conv} \{\beta, \gamma\} \text{ for } \beta, \gamma \in \Omega(L), \beta \perp \lambda \implies \alpha = \gamma.$$

6. YOSIDA–HEWITT DECOMPOSITION

Our original interest was to investigate the *Yosida–Hewitt decomposition*. It is a particular D -decomposition of $\Omega(L)$ where we take for D the set $\Omega_c(L)$ of all completely additive states on L .

PROPOSITION 6.1 [20, Theorem 2.2]. *The space of completely additive states, $\Omega_c(L)$, is a (possibly empty) base norm closed face of $\Omega(L)$.*

The states in the complementary set $\Omega_c(L)^\#$ are called weakly purely finitely additive (wpfa for short, see [21]), and we denote $\Omega_{wpfa}(L) = \Omega_c(L)^\#$. The original definition (equivalent due to Proposition 3.4) was the following:

DEFINITION 6.2. *A state $\mu \in \Omega(L) \setminus \Omega_c(L)$ is weakly purely finitely additive if the condition $\mu \in \text{conv} \{\lambda, \nu\}$ for some $\lambda \in \Omega_c(L)$, $\nu \in \Omega(L)$ implies $\mu = \nu$.*

This definition means that a wpfa state cannot be expressed as an inner convex combination of a completely additive state and an arbitrary state. In [9, 21], numerous results are proved about the Yosida–Hewitt decomposition (and related questions) for OMLs and for orthocomplete OMPs. The same technique can be used for their common generalization, *alternative orthomodular posets* introduced by Ovchinnikov in [18]. Its improved version presented here allows us to prove them also for general OMPs. We formulate several such results for states (probability measures) instead of positive measures. The proofs are omitted if they are analogous to the proofs in [9, 16, 21]. We present those proofs which are completely new or which we succeeded to simplify substantially by a different technique.

The existence of the Yosida–Hewitt decomposition follows from Propositions 4.5, 6.1, and [8, Theorem 4.1]:

COROLLARY 6.3 [8, 19]. *Let L be an OMP. Each state on L can be expressed as a convex combination of a completely additive state and a wpfa state.*

The duality between D and $D^\#$ (see Proposition 3.8) in the particular case of $D = \Omega_c(L)$ says that $\Omega_c(L) = \Omega_{wpfa}(L)^\# = \Omega_c(L)^{\#\#}$ and $\Omega_{wpfa}(L) = \Omega_{wpfa}(L)^{\#\#}$.

To obtain examples of the Yosida–Hewitt decomposition, we need a characterization of spaces of completely additive states. It is given in [16]; we recall the necessary definitions from there. Let C be a compact convex set. Consider the space $A(C)$ of all real-valued continuous affine functions on C and its second dual, $A(C)^{**}$. Let $e \in A(C)$ be the unit function on C . An element $f \in A(C)^{**}$ is said to be an *s-functional* if f is the weak* limit of an isotone sequence of elements of $\{g \in A(C) : 0 \leq g \leq e\}$. A face F of C is said to be *s-exposed* if there exists an s-functional f such that $F = f^{-1}(1) \cap C$, and F is said to be *s-semi-exposed* if F is an intersection of s-exposed faces of C . The following theorem is an improvement of Theorem 4.1.

THEOREM 6.4 [16]. *For every OMP L , the state space $\Omega(L)$ is a compact convex set with respect to the product topology on \mathbb{R}^L and $\Omega_c(L)$ is an s-semi-exposed face of $\Omega(L)$.*

Conversely, let C be a compact convex subset of a locally convex Hausdorff topological linear space and let F be an s-semi-exposed face of C . Then there is an OML L and an affine homeomorphism $h: C \xrightarrow{\text{onto}} \Omega(L)$ such that $h(F) = \Omega_c(L)$.

In the finite-dimensional case, every face is closed [5, Theorem 5.1], so the latter theorem attains a simpler form:

COROLLARY 6.5. *Let C be a polytope (=convex hull of finitely many points) and let F be a face of C . Then there is an OML L and an affine homeomorphism $h: C \xrightarrow{\text{onto}} \Omega(L)$ such that $h(F) = \Omega_c(L)$.*

Due to the latter corollary, many examples from Section 3 (namely Examples 3.13, 3.15) can be obtained as examples of the Yosida–Hewitt decomposition (with $C = \Omega(L)$, $D = \Omega_c(L)$ for an OML L).

EXAMPLE 6.6. Example 3.13 and Corollary 6.5 lead to an example of an OML L with a non-unique Yosida–Hewitt decomposition. This OML is not complete. However, it contains only one infinite maximal Boolean subalgebra (which is isomorphic to the Boolean algebra of all finite and cofinite subsets of a countable set; here we refer to the proof of Theorem 6.4 in

[16]). The MacNeille completion M of L is a complete OML. Each state on L admits an extension to M and this extension is unique for completely additive states. The state space of M becomes larger than the original square, but, according to Proposition 3.14, the Yosida–Hewitt decomposition is not unique on it. We conclude that completeness of an OML is not sufficient for the uniqueness of the Yosida–Hewitt decomposition.

Theorem 4.5 guarantees the uniqueness of the Yosida–Hewitt decomposition for Boolean algebras. Proposition 3.12 and Example 4.6 with Corollary 6.5 give examples of non-Boolean OMPs with a unique Yosida–Hewitt decomposition.

A characterization of the wpfa states follows from Proposition 3.5:

PROPOSITION 6.7. *The set $\Omega_{wpfa}(L)$ is the union of all faces F of $\Omega(L)$ such that $F \cap \Omega_c(L) = \emptyset$.*

The characterization of spaces of completely additive states (Theorem 6.4) induces also a characterization of spaces of wpfa states:

COROLLARY 6.8. *Let C be a compact convex subset of a locally convex Hausdorff topological linear space and let F be an s -semi-exposed face of C . Then there is an OML L and an affine homeomorphism $h: C \xrightarrow{\text{onto}} \Omega(L)$ such that $h(F)^\# = h(F^\#) = \Omega_{wpfa}(L)$.*

We summarize that the Yosida–Hewitt decomposition on an OMP always exists and need not be unique.

7. FILTERING STATES AND RÜTTIMANN DECOMPOSITION

Looking for an alternative to the Yosida–Hewitt decomposition with some advantageous properties, Rüttimann introduced the notion of filtering measure [21]. Here we use this notion for states. The decomposition of a state to a completely additive state and a filtering state (*Rüttimann decomposition*) need not exist, but we shall prove that when it exists, it is unique. In contrast to the Yosida–Hewitt decomposition, the Rüttimann decomposition is not a special case of the D -decomposition in general.

If $\mu \in J(L)$, we define the kernel of μ as

$$\ker \mu = \{p \in L : (\forall q \in L_p : \mu(q) = 0)\}.$$

If μ is positive, this definition reduces to the usual one: $\ker \mu = \{p \in L : \mu(p) = 0\}$.

DEFINITION 7.1. A Jordan measure μ on L is called *filtering* if $\ker \mu$ is a filtering set, i.e., $\forall p \in L \setminus \{0\} \exists q \in \ker \mu \setminus \{0\} : q \leq p$.

In particular, a filtering Jordan measure vanishes at all atoms. If an OMP L contains a maximal Boolean subalgebra which is finite, then the zero measure is the only filtering positive measure and L does not admit any filtering state. We denote by $J_f(L)$ the set of all filtering Jordan measures on L and we extend this notation to positive measures and states: $J_f^+(L) = J_f(L) \cap J^+(L)$, $\Omega_f(L) = J_f(L) \cap \Omega(L)$.

PROPOSITION 7.2. *Let L be an OMP. The set $J_f(L)$ of all filtering Jordan measures on L is a linear subspace of $J(L)$. The set $\Omega_f(L)$ of all filtering states is a (possibly empty) face of $\Omega(L)$.*

Proof. The zero measure is filtering. Let $\lambda, \nu \in J_f(L)$ and let μ be a linear combination of λ, ν . According to Proposition 2.5, $\ker \mu \supseteq \ker \lambda \cap \ker \nu$ is filtering, so $\mu \in J_f(L)$ and $J_f(L)$ is a linear subspace of $J(L)$. It remains to prove that $\Omega_f(L)$ is a face of $\Omega(L)$. Let $\mu \in \Omega_f(L)$ and $\mu \in \text{iconv}\{\lambda, \nu\}$ for some $\lambda \neq \nu$. Then $\ker \mu = \ker \lambda \cap \ker \nu$, so $\lambda, \nu \in \Omega_f(L)$. Apparently, $\Omega_f(L) = \Omega(L) \cap J_f(L)$ is convex. ■

PROPOSITION 7.3. *The only completely additive filtering Jordan measure on an OMP is the zero measure. In particular, there is no completely additive filtering state.*

Proof. Let μ be a completely additive Jordan measure with a filtering kernel. First we prove that $\mu(1) = 0$. Let M be a maximal orthogonal subset of $\ker \mu$. According to Lemma 2.4, $\bigvee M = 1$. The complete additivity of μ ensures that $\mu(1) = \sum_{x \in M} \mu(x) = 0$.

Take now any element $p \in L$. Let M be a maximal orthogonal subset of $\ker \mu \cap L_p$. The union $M \cup \{p'\}$ is a maximal orthogonal subset of L . Therefore it has no upper bound different from the unit 1 of L and 1 is its supremum. Then

$$0 = \mu(1) = \mu\left(\bigvee (M \cup \{p'\})\right) = \sum_{x \in M} \mu(x) + \mu(p') = \mu(p').$$

From $\mu(p') = 0$, since p was an arbitrary element of L , we derive $\mu = 0$. ■

Proposition 7.3 implies that a completely additive Jordan measure on an OMP is fully determined by its values on a filtering set. From the latter proposition, we easily obtain the uniqueness of the Rüttimann decomposition. This completes several results in [21].

COROLLARY 7.4. *Each Jordan measure on an OMP admits at most one Rüttimann decomposition, (i.e., a decomposition to a sum of a completely additive Jordan measure and a filtering Jordan measure). Each state on an OMP admits at most one Rüttimann decomposition, (i.e., a decomposition to a convex combination of a completely additive state and a filtering state).*

Proof. Let μ be a Jordan measure. If $\mu = \alpha + \gamma = \beta + \delta$ for some completely additive Jordan measures α, β and filtering Jordan measures γ, δ , then $\alpha - \beta = \delta - \gamma$ is a Jordan measure which is completely additive and filtering. According to Proposition 7.3, $\alpha = \beta$, $\gamma = \delta$. The conclusion for states follows easily. ■

Propositions 6.7, 7.2, and 7.3 imply that the property of being filtering is a stronger condition for a state than wpfa:

COROLLARY 7.5. *Let L be an OMP. Each filtering state on L is wpfa, i.e., $\Omega_f(L) \subseteq \Omega_{wpfa}(L)$.*

While spaces of wpfa states are characterized in Corollary 6.8, a characterization of spaces of filtering states is not known and seems to be an interesting problem.

8. WPFA-HEREDITY AND THE INTERPLAY OF RÜTTIMANN AND YOSIDA–HEWITT DECOMPOSITION

Filtering states are always wpfa. If filtering states are the only wpfa states, then the Rüttimann decomposition coincides with the Yosida–Hewitt decomposition. This case is of particular importance. In general, the Rüttimann decomposition need not exist for all states, but it is unique. The Yosida–Hewitt decomposition always exists, but it need not be unique. When they coincide, we obtain a unique decomposition for all states:

THEOREM 8.1 (cf. [21, Theorem 6.1]). *For every OMP L , the following conditions are equivalent:*

- (1) $\Omega(L) = \text{conv}(\Omega_c(L) \cup \Omega_f(L))$ (i.e., each state has a Rüttimann decomposition),
- (2) $\Omega_{wpfa}(L) = \Omega_f(L)$ (i.e., the Rüttimann decomposition coincides with the Yosida–Hewitt decomposition).

Proof. The implication (2) \Rightarrow (1) is an easy application of Corollary 6.3. For the reverse implication, let $\alpha \in \Omega_{wpfa}(L) \setminus \Omega_f(L)$. Then the existence of the Rüttimann decomposition implies that $\alpha \in \text{iconv}\{\beta, \gamma\}$ for some $\beta \in \Omega_c(L)$, $\gamma \in \Omega_f(L)$. It follows that $\beta \ll_S \alpha$, hence $\beta \in F_\alpha \subseteq \Omega_{wpfa}(L)$. As β is completely additive, we obtain a contradiction with Proposition 6.7. We proved that $\Omega_{wpfa}(L) \subseteq \Omega_f(L)$; the reverse inclusion holds, too (see Corollary 7.5). ■

In contrast to the latter observation about the Rüttimann decomposition, uniqueness of the Yosida–Hewitt decomposition is a strictly weaker

condition than those from Theorem 8.1:

EXAMPLE 8.2. As in Example 3.13(ii), let C be a square and D be a singleton containing a vertex of C . By Corollary 6.5, there is an OML L and an affine homeomorphism $h: C \xrightarrow{\text{onto}} \Omega(L)$ such that $h(D) = \Omega_c(L)$. This homeomorphism induces a correspondence between the D -decomposition of C and the Yosida–Hewitt decomposition of $\Omega(L)$. By Proposition 3.11, these decompositions are unique. As $\Omega(L)$ is a parallelogram and $\Omega_{wpfa}(L)$ consists of two of its edges, it is not a face. The set $\Omega_f(L)$ of filtering states depends on the particular construction of L , but it must be a face, hence different from $\Omega_{wpfa}(L)$.

Let $\mu \in \Omega(L)$ and let $p \in L \setminus \ker \mu$. The mapping $\mu_p: L_p \rightarrow [0, 1]$ defined by

$$\mu_p(x) = \frac{\mu(x)}{\mu(p)}$$

is a state on L_p called the *normalized restriction* of μ to L_p . Whenever we speak of a state μ_p for some $p \in L$, we assume that it originated this way for some $\mu \in \Omega(L)$ and $p \in L \setminus \ker \mu$.

We ask which properties of states are *hereditary*, i.e., if $\mu \in \Omega(L)$ has a property, then $\mu_p \in \Omega(L_p)$ has this property, too (for all $p \in L \setminus \ker \mu$). Let us show that complete additivity is a hereditary property. (As this was proved in [21, Lemma 4.1] only for OMLs and for orthocomplete OMPs, we present the new proof for the general case in detail.)

PROPOSITION 8.3. *If $\mu \in \Omega_c(L)$ and $p \in L \setminus \ker \mu$, then $\mu_p \in \Omega_c(L_p)$.*

Proof. As each orthogonal subset of L can be extended to a maximal one, it suffices to verify complete additivity on maximal orthogonal subsets. Let M be a maximal orthogonal subset of L_p . As in Proposition 7.3, we have $\vee(M \cup \{p'\}) = 1$. Then

$$\mu(p') + \sum_{x \in M} \mu(x) = 1,$$

$$\mu(p) = 1 - \mu(p') = \sum_{x \in M} \mu(x),$$

and

$$\sum_{x \in M} \mu_p(x) = 1,$$

so μ_p is completely additive. ■

Also the property of being filtering is hereditary. In contrast to this, the property of being wpfa is not hereditary.

EXAMPLE 8.4. Let G be a finite OML without states (see the main result of [10]). Take the product P of countably many factors isomorphic to G . Elements of P are all sequences of the form $(q_n)_{n \in \mathbb{N}}$, where $q_n \in G$. We select a subset Q of P consisting of all sequences $(q_n)_{n \in \mathbb{N}}$ satisfying one of the following conditions:

- (S0) only finitely many elements q_n are nonzero, or
- (S1) only finitely many elements q_n are different from 1_G .

It is easy to verify that Q is a sub-OML of P . We take a four-element Boolean algebra $B = \{0_B, p, p', 1_B\}$, and we define L as the horizontal sum of B and Q .

Let $\mu \in \Omega(L)$. Its restriction to Q is a state on Q . All states on Q vanish at all elements of the form (S0) and attain 1 at all elements of the form (S1), hence there is only one state on Q . The state space of L is one-dimensional; each state is uniquely determined by its value on p (which can be an arbitrary number from $[0, 1]$). In particular, there is only one state, ν , attaining 1 at p . There are no completely additive states on Q , and hence also on L . Therefore all states on L are wpfa.

The interval L_p is the two-element Boolean algebra. It admits only one state—the state attaining 1 at p , i.e., $\nu_p = \nu|_{L_p}$. Obviously, this state on L_p is completely additive, and there are no wpfa states on L_p . Thus L is not wpfa-hereditary: All states on L are wpfa, but the corresponding states on L_p are not wpfa.

Remark 8.5. In the latter example, we may replace one factor in P by an OML admitting exactly one state (e.g., the two-element Boolean algebra). Then the same construction leads to an OML with a two-dimensional state space. We obtain a situation from Examples 2.1 and 3.13—the Yosida–Hewitt decomposition is not unique. This is an explicit example the existence of which was guaranteed by Theorem 6.4.

The normalized restriction of states does not preserve the property of being wpfa. This justifies the following definition:

DEFINITION 8.6 [21]. An OMP L is called *wpfa-hereditary* if

$$\forall \mu \in \Omega_{wpfa}(L) \forall p \in L \setminus \ker \mu : \mu_p \in \Omega_{wpfa}(L_p).$$

Example 8.4 shows an OML which is not wpfa-hereditary. There are important cases of wpfa-hereditary OMPs. (The following proposition was proved in [21, Proposition 5.1] only for complete Boolean algebras, so we present a detailed proof here.)

PROPOSITION 8.7. *Every Boolean algebra is wpfa-hereditary.*

Proof. For $p \in L$, let $\phi^p: L \rightarrow L_p$ be the homomorphism $\phi^p(x) = p \wedge x$.

Let $\mu \in \Omega_{wpfa}(L)$ and $p \in L \setminus \ker \mu$. We assume that $\mu_p \notin \Omega_{wpfa}(L_p)$ and we shall seek for a contradiction.

The function $\mu_p \circ \phi^p$ is additive and $\mu_p \circ \phi^p(1) = \mu_p(p) = 1$, so $\mu_p \circ \phi^p \in \Omega(L)$. We claim that $\mu_p \circ \phi^p \ll_S \mu$. To prove this, we have to distinguish two cases. If $\mu(p) = t < 1$, then $\mu(p') > 0$ and we may express μ as

$$\begin{aligned} \mu(x) &= \mu(p \wedge x) + \mu(p' \wedge x) = t \mu_p(p \wedge x) + (1 - t) \mu_{p'}(p' \wedge x) \\ &= t(\mu_p \circ \phi^p)(x) + (1 - t)(\mu_{p'} \circ \phi^{p'})(x). \end{aligned}$$

If $\mu(p) = 1$, then $\mu = \mu_p \circ \phi^p$. In both cases we obtained $\mu_p \circ \phi^p \ll_S \mu$.

As μ_p is not wpfa, there is a $\nu \in \Omega_c(L_p)$ such that $\nu \ll_S \mu_p$. We have a state $\nu \circ \phi^p$ on L . The mapping $\alpha \mapsto \alpha \circ \phi^p$ on $\Omega(L)$ preserves convex combinations and hence also the relation \ll_S , so $\nu \circ \phi^p \ll_S \mu_p \circ \phi^p$. Due to transitivity of \ll_S , $\nu \circ \phi^p \ll_S \mu$. Moreover, $\nu \circ \phi^p$ is completely additive. To prove this, we use the following distributivity property [25, Sect. 19 (9)]. If $p \wedge \bigvee_{i \in I} x_i$ exists, then $\bigvee_{i \in I} (p \wedge x_i)$ exists and both expressions are equal,

$$p \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (p \wedge x_i).$$

If ν is completely additive and $(x_i)_{i \in I}$ is an orthogonal family in L which has a supremum, then

$$\begin{aligned} (\nu \circ \phi^p)\left(\bigvee_{i \in I} x_i\right) &= \nu\left(p \wedge \bigvee_{i \in I} x_i\right) = \nu\left(\bigvee_{i \in I} (p \wedge x_i)\right) \\ &= \sum_{i \in I} \nu(p \wedge x_i) = \sum_{i \in I} (\nu \circ \phi^p)(x_i). \end{aligned}$$

The complete additivity of $\nu \circ \phi^p$ contradicts the assumption that μ is wpfa, so the proof is finished. ■

PROPOSITION 8.8. *If $\Omega_f(L) = \Omega_{wpfa}(L)$, then L is wpfa-hereditary.*

Proof. Let $\mu \in \Omega_{wpfa}(L) = \Omega_f(L)$ and $p \in L \setminus \ker \mu$. As the property of being filtering is hereditary, $\mu_p \in \Omega_f(L_p)$. All filtering states are wpfa (Corollary 7.5), so $\mu_p \in \Omega_{wpfa}(L_p)$. ■

The reverse implication in Proposition 8.8 does not hold in general:

EXAMPLE 8.9. Let B be the Borel σ -algebra on the real line. We take the σ -ideal Δ of all meager sets in B . The quotient Boolean algebra $A = B/\Delta$ does not admit any completely additive state, so $\Omega_c(A) = \emptyset$, $\Omega_{wpfa}(A) = \Omega(A)$, and the same holds for any nontrivial interval in A , so A is wpfa-hereditary. On the other hand, there is a state μ on A which is strictly positive, i.e., $\mu(x) > 0$ for all $x \in A \setminus \{0\}$ (see [17]). The state μ is not filtering, so $\Omega_{wpfa}(A) \neq \Omega_f(A)$.

9. THE #-HEREDITY AND WEAKLY FILTERING STATES

In this section we introduce two notions—weakly filtering states and weakly D -filtering states—which generalize filtering states. We study the analogues of Rüttimann decomposition and heredity for them.

The Rüttimann decomposition is improved in [9]: Instead of filtering states we use states μ which are *weakly filtering*, i.e., $\ker\mu \cup \bigcap_{\alpha \in \Omega_c(L)} \ker\alpha$ is a filtering set. They form a larger set than filtering states. In comparison to the Rüttimann decomposition, the decomposition to completely additive states and weakly filtering states exists in more general cases and it is still unique. This approach is further generalized by replacing $\Omega_c(L)$ by its arbitrary face D . The analogy to wpfa-heredity cannot be direct: while complete additivity has a unified meaning in all OMPs, in particular in L and its interval L_p , the property of “being an element of D ” has no such analogue.

Let us state the exact definitions and properties.

DEFINITION 9.1. Let $D \subseteq \Omega_c(L)$. A state μ on L is called *weakly D -filtering* if $\ker\mu \cup (\bigcap_{\nu \in D} \ker\nu)$ is a filtering set.

We denote by $\Omega_{f,D}(L)$ the set of all weakly D -filtering states on L . All filtering states are weakly D -filtering, i.e., $\Omega_f(L) \subseteq \Omega_{f,D}(L)$ for all $D \subseteq \Omega_c(L)$ (cf. Definition 7.1). If $D \subseteq E \subseteq \Omega_c(L)$, then $\Omega_{f,D}(L) \supseteq \Omega_{f,E}(L)$. As an extreme case, $\Omega_{f,\emptyset}(L) = \Omega(L)$.

PROPOSITION 9.2. Let L be an OMP and let $D \subseteq \Omega_c(L)$. The set $\Omega_{f,D}(L)$ is a (possibly empty) face of $\Omega(L)$ disjoint from D . As a consequence, $\Omega_{f,D}(L) \subseteq D^\#$.

Proof. First we shall prove that $\Omega_{f,D}(L)$ is a face. Let $\mu, \lambda, \nu \in \Omega(L)$ and $\mu \in \text{conv}\{\lambda, \nu\}$. Then $\ker\mu = \ker\lambda \cap \ker\nu$. If $\mu \in \Omega_{f,D}(L)$, then also λ and ν are weakly D -filtering. Conversely, if $\lambda, \nu \in \Omega_{f,D}(L)$, then Proposition 2.5 implies that μ is weakly D -filtering.

To prove disjointness, suppose that there is a state $\mu \in D \cap \Omega_{f,D}(L)$. Then $\bigcap_{\nu \in D} \ker\nu \subseteq \ker\mu$, hence $\ker\mu \cup \bigcap_{\nu \in D} \ker\nu = \ker\mu$ is a filtering set and μ is a filtering state. As an element of D , μ is completely additive in contradiction with Proposition 7.3. ■

THEOREM 9.3. Let L be an OMP, $D \subseteq \Omega_c(L)$, and $\mu \in \Omega(L)$. If μ can be decomposed to a convex combination of a state from D and a weakly D -filtering state, then this decomposition is unique.

Proof. Obviously the decomposition is unique (in the sense of Remark 3.10) for the elements of $D \cup \Omega_{f,D}(L)$. Assume that a state $\mu \in \Omega(L) \setminus (D \cup \Omega_{f,D}(L))$ admits two decompositions,

$$\mu = t\alpha + (1-t)\beta = u\gamma + (1-u)\delta,$$

where $\alpha, \gamma \in D$, $\beta, \delta \in \Omega_{f,D}(L)$, $t, u \in (0, 1)$. Consider the Jordan measure

$$\varepsilon = t\alpha - u\gamma = (1 - u)\delta - (1 - t)\beta.$$

As a linear combination of $\alpha, \gamma \in D \subseteq \Omega_c(L)$, ε is completely additive. Its kernel satisfies

$$\ker \varepsilon \supseteq \ker \alpha \cap \ker \gamma \supseteq \bigcap_{\nu \in D} \ker \nu.$$

This, together with the fact that ε is a linear combination of $\beta, \delta \in \Omega_{f,D}(L)$, gives

$$\begin{aligned} \ker \varepsilon &= \ker \varepsilon \cup \left(\bigcap_{\nu \in D} \ker \nu \right) \supseteq \left(\ker \beta \cup \left(\bigcap_{\nu \in D} \ker \nu \right) \right) \\ &\quad \cap \left(\ker \delta \cup \left(\bigcap_{\nu \in D} \ker \nu \right) \right). \end{aligned}$$

According to Proposition 2.5, the latter intersection is a filtering set. Thus ε is filtering and completely additive, therefore the zero measure due to Proposition 7.3. From

$$0 = \varepsilon(1) = t\alpha(1) - u\gamma(1) = t - u$$

we obtain $t = u$, $\alpha = \gamma$, and $\beta = \delta$, so the two decompositions must be identical. ■

For a face D of $\Omega_c(L)$ and an element $p \in L$, we define

$$D_p = \{\alpha \in \Omega(L_p) : (\exists \beta \in D : \beta(p) \neq 0 \text{ and } \alpha \ll_S \beta_p)\}.$$

This means that D_p is the union of faces generated by the normalized restrictions to L_p of some of the states from D (precisely, those which attain nonzero value at p). The normalized restrictions of states from D to L_p form a convex set, so D_p is a single face of $\Omega(L_p)$ (see Remark 3.3).

For each element p of L , we have

$$(D_p)^\# = \{\alpha \in \Omega(L_p) : (\forall \beta \in D_p : \beta \not\ll_S \alpha)\}.$$

It is obvious that if μ is an element of D and $\mu(p) \neq 0$, then μ_p belongs to D_p . A similar property is not true, in general, for the elements of $D^\#$.

EXAMPLE 9.4. Let L be the OML $MO2$ from Example 2.1. We take the set $D = \{\mu \in \Omega(L) : \mu(q) = 1\}$. It is easy to check that

$$D^\# = \{\mu \in \Omega(L) : \mu(q) = 0\},$$

$$D_p = \Omega(L_p) \text{ is a singleton because } L_p \text{ is a two-element}$$

Boolean algebra,

$$(D_p)^\# = \emptyset.$$

If we take a state $\mu \in D^\#$ such that $\mu(p) \neq 0$, then $\mu_p \notin (D_p)^\#$.

At this point, the following definition is justified:

DEFINITION 9.5. Let L be an OMP and let D be a face of $\Omega_c(L)$. We say that D is $\#$ -hereditary if $\mu \in D^\#$, $p \in L \setminus \ker \mu$ implies $\mu_p \in (D_p)^\#$.

The notion of $\#$ -heredity is dependent on the choice of the face D . Even for the choice $D = \Omega_c(L)$, it does not coincide with wpfa-heredity. While $\Omega_{wpfa}(L_p) = (\Omega_c(L_p))^\#$, this set need not coincide with $((\Omega_c(L))_p)^\#$. These sets are equal if $\Omega_c(L_p) = (\Omega_c(L))_p$. One inclusion, $\Omega_c(L_p) \supseteq (\Omega_c(L))_p$, always holds. (Indeed, if $\mu \in \Omega_c(L)$ and $p \in L \setminus \ker \mu$, then μ_p is completely additive and hence the face of $\Omega(L_p)$ generated by μ_p is a subset of the face $\Omega_c(L_p)$.) Therefore $(\Omega_c(L_p))^\# \subseteq ((\Omega_c(L))_p)^\#$.

EXAMPLE 9.6. For each OMP L , the least face \emptyset of $\Omega(L)$ is $\#$ -hereditary. On the other hand, if $\Omega_c(L) = \emptyset$, then L need not be wpfa-hereditary; see Example 8.4, where $\Omega_c(L) = \emptyset$, $\Omega_{wpfa}(L) = \Omega(L)$, $\Omega_{wpfa}(L_p) = \emptyset$, although $\Omega(L_p)$ is nonempty.

The latter example demonstrates the difference between wpfa-heredity of L and $\#$ -heredity of $\Omega_c(L)$. A less trivial example will be presented in Example 9.13. The following proposition is a reformulation of [9, Proposition 1].

PROPOSITION 9.7. Let L be a Boolean algebra and D a face of $\Omega_c(L)$ such that each state on L has a D -decomposition. Then D is $\#$ -hereditary.

In questions about heredity, the following lemma plays an important role. It was proved by Rüttimann [21] for OMLs and for orthocomplete OMPs. Here we give its generalization:

LEMMA 9.8. Let L be an OMP. If $p \in L$, $\mu \in J^+(L)$, and $\nu \in J_c^+(L)$ such that $\mu(p) < \nu(p)$, then there exists $q \in L_p \setminus \{0\}$ such that $\mu(x) < \nu(x)$ for all x in $L_q \setminus \{0\}$.

Proof. Let us define $P = \{x \in L_p : \nu(x) \leq \mu(x)\}$. Let M be a maximal orthogonal subset of P . If M is a maximal orthogonal subset of L_p , then $M \cup \{p'\}$ is a maximal orthogonal subset of L . The relation $\nu(1) = \sum_{x \in M} \nu(x) + \nu(p')$ gives $\nu(p) = \sum_{x \in M} \nu(x)$. For any finite subset K of M we have $\nu(\bigvee K) \leq \mu(\bigvee K) \leq \mu(p)$. By passing to the limit, we obtain

$$\nu(p) = \sum_{x \in M} \nu(x) \leq \nu(\bigvee M) \leq \mu(\bigvee M) \leq \mu(p)$$

which is impossible. This ensures that there exists a nonzero element q in L_p which is orthogonal to M . In other words, $\exists q \in L_p \setminus \{0\} : M \leq q'$. If x is an element in $P \cap L_q$, then $x \leq q \Rightarrow x' \geq q' \geq m$ for all $m \in M$; hence $x \perp M$. The maximality of M in P implies that $x = 0$, so $P \cap L_q = \{0\}$ or, in the same way, $\forall x \in L_q \setminus \{0\} : \mu(x) < \nu(x)$. ■

The latter lemma allows us to affirm that in hereditary structures the existence of the D -decomposition ensures its uniqueness. Weakly D -filtering states play a similar role with respect to $\#$ -heredity as filtering states with respect to wpfa-heredity. We have always the inclusion $\Omega_{f,D}(L) \subseteq D^\#$; sometimes we obtain equality. We are ready to state a theorem which says that this equality is equivalent to the uniqueness of D -decomposition and to $\#$ -heredity of D . It is a generalization (to OMPs) of [9, Theorem 2] (which was stated for OMLs and orthocomplete OMPs).

THEOREM 9.9. *Let L be an OMP and let D be a face of $\Omega_c(L)$ such that each state has a D -decomposition. Then the following conditions are equivalent:*

- (1) $\Omega_{f,D}(L) = D^\#$,
- (2) *each state on L is a convex combination of a state from D and a weakly D -filtering state,*
- (3) D is $\#$ -hereditary.

Proof. Conditions (1) and (2) are obviously equivalent.

(3) \Rightarrow (1): Take $\lambda \in D^\#$ and $p \in L \setminus \{0\}$. We want to find an element $q \in \ker \lambda \cup (\bigcap_{\mu \in D} \ker \mu)$ such that $q \in L_p \setminus \{0\}$. If $p \in \bigcap_{\mu \in D} \ker \mu$, we can take $q = p$. If not, we can take $\nu \in D$ such that $\nu(p) > 0$. There is an $r \in \mathbb{R}^+$ such that $\lambda(p) < r\nu(p)$, and, according to Lemma 9.8, there exists a $q \in L_p \setminus \{0\}$ such that $\lambda(x) < r\nu(x)$ for all $x \in L_q \setminus \{0\}$. We want to prove that $q \in \ker \lambda$. Suppose the contrary. We can write $r'\nu_q(x) = \lambda_q(x) + \alpha(x)$ for $r' = r \cdot \nu(q)/\lambda(q)$ and a suitable $\alpha \in J^+(L_q)$, with $\alpha(q) > 0$ and $\alpha \neq r'\nu_q$. This means that $\lambda_q \ll_S \nu_q$, so $\lambda_q \in D_q$. On the other side, the $\#$ -heredity of the face D ensures that $\lambda_q \in (D_q)^\#$ which is a contradiction. We proved that $\lambda(q) = 0$ and $\lambda \in \Omega_{f,D}$.

(1) \Rightarrow (3): Take $\nu \in D^\# = \Omega_{f,D}(L)$ and $p \in L$ such that $\nu(p) > 0$. Observe that $\nu_p \in \Omega_{f,D_p}(L_p)$; in fact, since $\ker \nu \cup \bigcap_{\mu \in D} \ker \mu$ is filtering, for each $q \in L_p \setminus \{0\}$, there exists $x \in L_q \setminus \{0\}$ such that $x \in \ker \nu \cup \bigcap_{\mu \in D} \ker \mu$. Recall that $x \leq p$. If $x \in \ker \nu$, then $x \in \ker \nu_p$. Otherwise, $x \in \bigcap_{\mu \in D} \ker \mu$. Each $\alpha \in D_p$, satisfies $\alpha \ll_S \beta_p$ for some $\beta \in D$. As $x \in \ker \beta$, we obtain $\beta(x) = 0$ and $\alpha(x) = 0$. This means $x \in \bigcap_{\alpha \in D_p} \ker \alpha$. We proved that the set $\ker \nu_p \cup \bigcap_{\alpha \in D_p} \ker \alpha$ is filtering which means $\nu_p \in \Omega_{f,D_p}(L_p)$.

From the inclusion $D_p \subseteq \Omega_c(L_p)$ it follows that $\Omega_{f,D_p}(L_p) \subseteq (D_p)^\#$ which gives the thesis. ■

According to Theorem 9.3, any of these conditions implies the uniqueness of the D -decomposition:

COROLLARY 9.10. *If D is a $\#$ -hereditary face of $\Omega_c(L)$, then the D -decomposition is unique.*

Proof. It follows from Theorems 9.3 and 9.9. ■

In view of Proposition 9.7, the latter proposition gives the uniqueness of D -decomposition for the case of Boolean algebras. The reverse implication in Corollary 9.10 does not hold in general; the D -decomposition may be unique if $D^\#$ is not a face, hence different from $\Omega_{f,D}(L)$ (see Examples 3.13 and 4.3(ii)).

The case of $D = \Omega_c(L)$ is of special importance. We call the states which are weakly $\Omega_c(L)$ -filtering simply *weakly filtering states*. We denote by $\Omega_{wf}(L)$ the set of all weakly filtering states on L . Weakly filtering states need not coincide with filtering states:

EXAMPLE 9.11. The OML L from Example 8.4 has no completely additive states, so all states are weakly filtering (and there exist some). However, L contains a finite maximal Boolean subalgebra B , so there are no filtering states on L . Also in Example 8.9 all states are weakly filtering, but not all are filtering.

For $D = \Omega_c(L)$, Theorem 9.9 and Corollary 9.10 attain the following form:

COROLLARY 9.12. *Let L be an OMP. Then the following conditions are equivalent:*

- (1) $\Omega_{wf}(L) = \Omega_{wpfa}(L)$,
- (2) *each state on L is a convex combination of a completely additive state and a weakly filtering state,*
- (3) $\Omega_c(L)$ *is $\#$ -hereditary.*

Any of these conditions implies the uniqueness of the Yosida–Hewitt decomposition.

The Yosida–Hewitt decomposition may be unique even if the conditions of Corollary 9.12 are not satisfied, e.g., if $\Omega_{wpfa}(L)$ is not a face, hence different from $\Omega_{wf}(L)$. If L is wpfa-hereditary, then $\Omega_c(L)$ is a $\#$ -hereditary face. The reverse implication need not hold.

EXAMPLE 9.13. Take the product, K , of a two-element Boolean algebra and the OML L from Example 8.4, $K = 2 \times L$. Denote by a the atom $(1, 0) \in K$ and by $p \in L$ the same atom as in Example 8.4. The state space $\Omega(K)$ is a triangle, $\Omega_c(K)$ is a singleton containing its vertex, μ (which is the only state satisfying $\mu(a) = 1$), and $\Omega_{wpfa}(K)$ is the opposite edge of the triangle. As K contains a finite maximal Boolean subalgebra, it has no filtering states. Nevertheless, there are weakly filtering states on K . Indeed, $\bigcap_{\lambda \in \Omega_c(L)} \ker \lambda = \ker \mu = \{(0, x) \in K : x \in L\}$, so each state vanishing at a is weakly filtering. In this particular example, weakly filtering states coincide

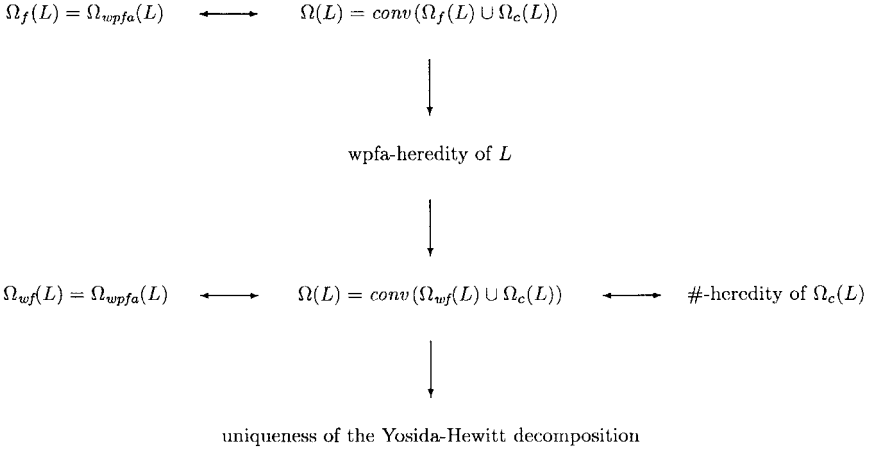


FIG. 4. Implications between conditions related to completely additive states.

with the wpfa states. According to Theorem 9.9, $\Omega_c(K)$ is $\#$ -hereditary. To prove that K is not wpfa-hereditary, take the element $b = (1, p) \in K$. The interval K_b is isomorphic to the four-element Boolean algebra, so there are no wpfa states on K_b . The only state $\nu \in \Omega(K)$ which attains 1 at $(0, p)$ (constructed as in Example 8.4) is wpfa, but $\nu_b \notin \Omega_{wpfa}(K_b)$.

The relations between various conditions studied in this paper are displayed in Fig. 4. Non-oriented arrows connect equivalent conditions. Oriented arrows connect conditions which satisfy one implication and need not satisfy the reverse one.

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