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Periodic boundary value problem for first-order impulsive ordinary differential equations ☆

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Abstract

This paper investigates the existence of minimal and maximal solutions of the periodic boundary value problem for first-order impulsive differential equations by establishing two comparison results and using the method of upper and lower solutions and the monotone iterative technique.

Keywords: Impulsive differential equation; Periodic boundary value problem; Upper and lower solution; Monotone iterative technique

1. Introduction

The theory of impulsive differential equations has become an important area of investigation in recent years (see Refs. [1,2,5,6,9]). In this paper we consider the periodic boundary value problem for first-order impulsive ordinary differential equations (PBVP)

$$\begin{cases} x'(t) = f(t, x(t)), & t \neq t_k, t \in J, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \ldots, p, \\ x(0) = x(T), \end{cases} \tag{1}$$

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where $f \in C(J \times R, R)$, $J = [0, T]$, $I_k \in C(R, R)$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ $(k = 1, 2, \dots, p)$, $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_p < T$.

The method of upper and lower solutions coupled with the monotone iterative technique has been widely used in the treatment of nonlinear differential equations in recent years (see Refs. [3–9]). The basic idea of this method is that using the upper and lower solutions as an initial iteration one can construct monotone sequences from a corresponding linear equation, and these sequences converge monotonically to the maximal and minimal solutions of the nonlinear equation. When the method is applied to impulsive differential equations, it usually need a suitable impulsive differential inequality as a comparison principle.

The results in the paper are inspired by Lakshmikantham and Leela [4], Liu [5], Vatsala and Sun [6]. Here we establish two comparison principles, i.e., Lemmas 2 and 3. Then we discuss the existence and uniqueness of the solutions for linear periodic boundary value problems for impulsive differential equation, i.e., Lemmas 4 and 5. Finally, by use of the monotone iterative technique and the method of upper and lower solutions we obtain the existence theorems of extremal solutions for the PBVP (1).

2. Preliminaries and comparison principles

Let $PC(J, R) = \{x : J \to R;\ x(t)$ is continuous everywhere except some t_k at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k)\}$. Let $J' = J \backslash \{t_1, t_2, \dots, t_p\}$, $\Omega = PC(J, R) \cap C^1(J', R)$. A function $x \in \Omega$ is called a solution of PBVP (1) if it satisfies (1).

Let $t_0 = 0$, $t_{p+1} = T$. We list the following assumptions for convenience.

(A_0) There exist function $\alpha, \beta \in \Omega$, $\beta(t) \leqslant \alpha(t)$ $(\forall t \in J)$ such that

$$\begin{cases} \alpha'(t) \leqslant f(t, \alpha(t)) - M r_\alpha, & t \neq t_k, t \in J, \\ \Delta\alpha(t_k) \leqslant I_k(\alpha(t_k)), & k = 1, 2, \dots, p, \end{cases} \tag{2}$$

and

$$\begin{cases} \beta'(t) \geqslant f(t, \beta(t)) + M r_\beta, & t \neq t_k, t \in J, \\ \Delta\beta(t_k) \geqslant I_k(\beta(t_k)), & k = 1, 2, \dots, p, \end{cases} \tag{3}$$

where $M > 0$, r_α and r_β are given by

$$r_\alpha = \begin{cases} \frac{[\alpha(0) - \alpha(T)]}{\sum_{i=0}^{p} \prod_{t_0 < t_k < t_i} (1 + L_k)^{-1} (e^{-Mt_i} - e^{-Mt_{i+1}})}, & \text{if } \alpha(0) > \alpha(T), \\ 0, & \text{if } \alpha(0) \leqslant \alpha(T), \end{cases}$$

$$r_\beta = \begin{cases} \frac{[\beta(T) - \beta(0)]}{\sum_{i=0}^{p} \prod_{t_0 < t_k < t_i} (1 + L_k)^{-1} (e^{-Mt_i} - e^{-Mt_{i+1}})}, & \text{if } \beta(0) < \beta(T), \\ 0, & \text{if } \beta(0) \geqslant \beta(T); \end{cases}$$

that is, $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of PBVP (1), respectively.

(A_1) The function $f \in C(J \times R, R)$ satisfies

$$f(t,x) - f(t,y) \leqslant M(x-y),$$

whenever $\beta(t) \leqslant y \leqslant x \leqslant \alpha(t)$, $t \in J$, where $M > 0$.

(A_2) The functions $I_k \in C(R, R)$ satisfy

$$I_k(x) - I_k(y) \leqslant L_k(x-y),$$

whenever $\beta(t_k) \leqslant y \leqslant x \leqslant \alpha(t_k)$, and $k = 1, 2, \ldots, p$, where $L_k < 1$, $k = 1, 2, \ldots, p$.

(B_0) There exist functions $\alpha, \beta \in \Omega$, $\alpha(t) \leqslant \beta(t)$ $(\forall t \in J)$ such that

$$\begin{cases} \alpha'(t) \leqslant f(t, \alpha(t)) - M\bar{r}_\alpha, & t \neq t_k, t \in J, \\ \Delta\alpha(t_k) \leqslant I_k(\alpha(t_k)), & k = 1, 2, \ldots, p, \end{cases} \tag{4}$$

and

$$\begin{cases} \beta'(t) \geqslant f(t, \beta(t)) + M\bar{r}_\beta, & t \neq t_k, t \in J, \\ \Delta\beta(t_k) \geqslant I_k(\beta(t_k)), & k = 1, 2, \ldots, p, \end{cases} \tag{5}$$

where $M > 0$, $\bar{r}_\alpha$ and $\bar{r}_\beta$ are given by

$$\bar{r}_\alpha = \begin{cases} \frac{[\alpha(0)-\alpha(T)]e^{MT}}{\sum_{i=0}^{p}\prod_{t_i<t_k<t_{p+1}}(1-L_k)(e^{Mt_{i+1}}-e^{Mt_i})}, & \text{if } \alpha(0) > \alpha(T), \\ 0, & \text{if } \alpha(0) \leqslant \alpha(T), \end{cases}$$
$$\bar{r}_\beta = \begin{cases} \frac{[\beta(T)-\beta(0)]e^{MT}}{\sum_{i=0}^{p}\prod_{t_i<t_k<t_{p+1}}(1-L_k)(e^{Mt_{i+1}}-e^{Mt_i})}, & \text{if } \beta(0) < \beta(T), \\ 0, & \text{if } \beta(0) \geqslant \beta(T); \end{cases}$$

that is, $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of PBVP (1), respectively.

(B_1) The function $f \in C(J \times R, R)$ satisfies

$$f(t,x) - f(t,y) \geqslant -M(x-y),$$

whenever $\alpha(t) \leqslant y \leqslant x \leqslant \beta(t)$, $t \in J$, where $M > 0$.

(B_2) The functions $I_k \in C(R, R)$ satisfy

$$I_k(x) - I_k(y) \geqslant -L_k(x-y),$$

whenever $\alpha(t_k) \leqslant y \leqslant x \leqslant \beta(t_k)$, and $k = 1, 2, \ldots, p$, where $L_k < 1$, $k = 1, 2, \ldots, p$.

Lemma 1 [1]. *Assume that*

(C_0) *the sequence* $\{t_k\}$ *satisfies* $0 \leqslant t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ *with* $\lim_{k\to\infty} t_k = \infty$;

(C_1) *$m \in PC^1(R_+, R)$ is left continuous at t_k for $k = 1, 2, \ldots$;*
(C_2) *for $k = 1, 2, \ldots, t \geqslant t_0$,*

$$m'(t) \leqslant p(t)m(t) + q(t), \quad t \neq t_k, \tag{6}$$

$$m(t_k^+) \leqslant d_k m(t_k) + b_k, \tag{7}$$

where $p, q \in C(R_+, R)$, $d_k \geqslant 0$ and b_k are real constants.

Then

$$\begin{aligned} m(t) \leqslant{}& m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^{t} p(s)\,ds\right) \\ &+ \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp\left(\int_{s}^{t} p(\sigma)\,d\sigma\right) q(s)\,ds \\ &+ \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^{t} p(s)\,ds\right) b_k. \end{aligned} \tag{8}$$

Remark 1. If the inequalities (6) and (7) are reversed then in the conclusion the inequality (8) is also reversed.

Lemma 2. *Assume that $m \in \Omega$ satisfies*

$$\begin{cases} m'(t) \geqslant Mm(t) + Mr_m, & t \neq t_k, t \in J, \\ \Delta m(t_k) \geqslant L_k m(t_k), & k = 1, 2, \ldots, p, \end{cases} \tag{9}$$

where $M > 0$, $L_k > -1$ for $k = 1, 2, \ldots, p$, $\prod_{k=1}^{p} (1 + L_k)^{-1} e^{-MT} < 1$, and

$$r_m = \begin{cases} \dfrac{[m(T) - m(0)]}{\sum_{i=0}^{p} \prod_{t_0 < t_k < t_i} (1 + L_k)^{-1} (e^{-Mt_i} - e^{-Mt_{i+1}})}, & \text{if } m(0) < m(T), \\ 0, & \text{if } m(0) \geqslant m(T). \end{cases}$$

Then $m(t) \leqslant 0$ for $t \in J$.

Proof. Consider inequalities (9). In view of Lemma 1, we get

$$\begin{aligned} m(T) \geqslant{}& m(t) \prod_{t < t_k < T} (1 + L_k) e^{M(T-t)} \\ &+ Mr_m \int_{t}^{T} \prod_{s < t_k < T} (1 + L_k) e^{M(T-s)}\,ds. \end{aligned} \tag{10}$$

From (10), we have

$$m(t) \leqslant m(T) \prod_{t<t_k<T} (1+L_k)^{-1} e^{-M(T-t)} - \frac{M r_m \int_t^T \prod_{s<t_k<T} (1+L_k) e^{M(T-s)}\, ds}{\prod_{t<t_k<T} (1+L_k) e^{M(T-t)}}. \tag{11}$$

Since $r_m \geqslant 0,$ it is enough to show $m(T) \leqslant 0,$ from which the lemma follows.

Let $t = 0$ in (11). We get

$$\begin{aligned}
m(0) &\leqslant m(T) \prod_{k=1}^{p} (1+L_k)^{-1} e^{-MT} \\
&\quad - \frac{M r_m \int_0^T \prod_{s<t_k<T} (1+L_k) e^{M(T-s)}\, ds}{\prod_{k=1}^{p} (1+L_k) e^{MT}} \\
&= m(T) \prod_{k=1}^{p} (1+L_k)^{-1} e^{-MT} \\
&\quad - M r_m \prod_{k=1}^{p} (1+L_k)^{-1} \Biggl[\int_0^{t_1} \prod_{s<t_k<T} (1+L_k) e^{-Ms}\, ds \\
&\quad + \int_{t_1^+}^{t_2} \prod_{s<t_k<T} (1+L_k) e^{-Ms}\, ds + \cdots \\
&\quad + \int_{t_i^+}^{t_{i+1}} \prod_{s<t_k<T} (1+L_k) e^{-Ms}\, ds + \cdots \\
&\quad + \int_{t_p^+}^{T} \prod_{s<t_k<T} (1+L_k) e^{-Ms}\, ds \Biggr] \\
&= m(T) \prod_{k=1}^{p} (1+L_k)^{-1} e^{-MT} \\
&\quad + r_m \prod_{k=1}^{p} (1+L_k)^{-1} \Biggl(\sum_{i=0}^{p} \prod_{t_i<t_k<t_{p+1}} (1+L_k) \bigl(e^{-Mt_{i+1}} - e^{-Mt_i} \bigr) \Biggr) \\
&= m(T) \prod_{k=1}^{p} (1+L_k)^{-1} e^{-MT} \\
&\quad - r_m \sum_{i=0}^{p} \prod_{t_0<t_k<t_i} (1+L_k)^{-1} \bigl(e^{-Mt_i} - e^{-Mt_{i+1}} \bigr).
\end{aligned}$$

Consider the case $m(0) \geqslant m(T)$; then $r_m = 0$. Suppose $m(T) > 0$. We have

$$m(T) \leqslant m(0) \leqslant m(T) \prod_{k=1}^{p} (1+L_k)^{-1} e^{-MT} < m(T),$$

which is a contradiction. So $m(T) \leqslant 0$.

If $m(0) < m(T)$, then $r_m > 0$. Suppose $m(T) > 0$. Then

$$\begin{aligned} m(0) &\leqslant m(T) \prod_{k=1}^{p} (1+L_k)^{-1} e^{-MT} \\ &\quad - \frac{[m(T)-m(0)]}{\sum_{i=0}^{p} \prod_{t_0<t_k<t_i} (1+L_k)^{-1} (e^{-Mt_i} - e^{Mt_{i+1}})} \\ &\quad \times \sum_{i=0}^{p} \prod_{t_0<t_k<t_i} (1+L_k)^{-1} \left(e^{-Mt_i} - e^{-Mt_{i+1}}\right) \\ &= m(T) \left[\prod_{k=1}^{p} (1+L_k)^{-1} e^{-MT} - 1 \right] + m(0) \\ &< m(0), \end{aligned}$$

which is also a contradiction. Therefore $m(T) \leqslant 0$. The proof of Lemma 2 is complete. □

Lemma 3. *Assume that $m \in \Omega$ satisfies*

$$\begin{cases} m'(t) \leqslant -Mm(t) - M\bar{r}_m, & t \neq t_k,\ t \in J, \\ \Delta m(t_k) \leqslant -L_k m(t_k), & k = 1, 2, \ldots, p, \end{cases} \tag{12}$$

where $M > 0$, $L_k < 1$ for $k = 1, 2, \ldots, p$, $\prod_{k=1}^{p} (1-L_k) e^{-MT} < 1$, and

$$\bar{r}_m = \begin{cases} \frac{[m(0)-m(T)]e^{MT}}{\sum_{i=0}^{p} \prod_{t_i<t_k<t_{p+1}} (1-L_k)(e^{Mt_{i+1}} - e^{Mt_i})}, & \text{if } m(0) > m(T), \\ 0, & \text{if } m(0) \leqslant m(T). \end{cases}$$

Then $m(t) \leqslant 0$ for $t \in J$.

Proof. Consider inequalities (12). By Lemma 1 we get

$$\begin{aligned} m(t) &\leqslant m(0) \prod_{0<t_k<t} (1-L_k) e^{-Mt} \\ &\quad - M\bar{r}_m \int_0^t \prod_{s<t_k<t} (1-L_k) e^{M(s-t)}\, ds. \end{aligned} \tag{13}$$

Since $\bar{r}_m \geqslant 0$, it is enough to show $m(0) \leqslant 0$, from which the lemma follows.

Let $t = T$ in (13). We have

$$\begin{aligned}
m(T) &\leqslant m(0)\prod_{k=1}^{p}(1-L_k)e^{-MT} - M\bar{r}_m\int_0^T \prod_{s<t_k<T}(1-L_k)e^{M(s-T)}\,ds \\
&= m(0)\prod_{k=1}^{p}(1-L_k)e^{-MT} - M\bar{r}_m\Bigg[\int_0^{t_1}\prod_{s<t_k<T}(1-L_k)e^{M(s-T)}\,ds \\
&\quad + \int_{t_1^+}^{t_2}\prod_{s<t_k<T}(1-L_k)e^{M(s-T)}\,ds + \cdots \\
&\quad + \int_{t_i^+}^{t_{i+1}}\prod_{s<t_k<T}(1-L_k)e^{M(s-T)}\,ds + \cdots \\
&\quad + \int_{t_p^+}^{T}\prod_{s<t_k<T}(1-L_k)e^{M(s-T)}\,ds\Bigg] \\
&= m(0)\prod_{k=1}^{p}(1-L_k)e^{-MT} \\
&\quad - \bar{r}_m\sum_{i=0}^{p}\prod_{t_i<t_k<t_{p+1}}(1-L_k)\big(e^{Mt_{i+1}}-e^{Mt_i}\big)e^{-MT}.
\end{aligned}$$

Consider the case $m(0) \leqslant m(T)$; then $\bar{r}_m = 0$. Suppose $m(0) > 0$. We have

$$m(0) \leqslant m(T) \leqslant m(0)\prod_{k=1}^{p}(1-L_k)e^{-MT} < m(0),$$

which is a contradiction. So $m(0) \leqslant 0$.

If $m(0) > m(T)$, then $\bar{r}_m > 0$. Suppose $m(0) > 0$. Then

$$\begin{aligned}
m(T) &\leqslant m(0)\prod_{k=1}^{p}(1-L_k)e^{-MT} \\
&\quad - \frac{[m(0)-m(T)]e^{MT}}{\sum_{i=0}^{p}\prod_{t_i<t_k<t_{p+1}}(1-L_k)(e^{Mt_{i+1}}-e^{Mt_i})} \\
&\quad \times \sum_{i=0}^{p}\prod_{t_i<t_k<t_{p+1}}(1-L_k)\big(e^{Mt_{i+1}}-e^{Mt_i}\big)e^{-MT}
\end{aligned}$$

$$
\begin{aligned}
&= m(0)\left[\prod_{k=1}^{p}(1-L_k)e^{-MT} - 1\right] + m(T) \\
&< m(T),
\end{aligned}
$$

which is also a contradiction. Therefore $m(0) \leqslant 0$. The proof of Lemma 3 is complete. □

Let us consider the following periodic boundary value problems of linear impulsive differential equations (PBVP):

$$
\begin{cases}
u'(t) - Mu(t) = \sigma(t), & t \neq t_k,\ t \in J, \\
\Delta u(t_k) = L_k u(t_k) + I_k(\eta(t_k)) - L_k\eta(t_k), & k = 1, 2, \ldots, p, \\
u(0) = u(T),
\end{cases}
\tag{14}
$$

and

$$
\begin{cases}
u'(t) + Mu(t) = \sigma(t), & t \neq t_k,\ t \in J, \\
\Delta u(t_k) = -L_k u(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k), & k = 1, 2, \ldots, p, \\
u(0) = u(T),
\end{cases}
\tag{15}
$$

where M, L_k $(k = 1, 2, \ldots, p)$ are constants, $I_k \in C(J, R)$ $(k = 1, 2, \ldots, p)$, $\sigma \in PC(J, R)$ and $\eta \in \Omega$.

In view of Lemma 1, we can show the following two lemmas easily.

Lemma 4. *Let $M > 0$, $L_k > -1$ for $k = 1, 2, \ldots, p$. If*

$$
\prod_{k=1}^{p}(1+L_k)^{-1}e^{-MT} < 1,
$$

then the PBVP (14) *has a unique solution*

$$
\begin{aligned}
u(t) = u(0)\prod_{0<t_k<t}(1+L_k)e^{Mt} &+ \int_0^t \prod_{s<t_k<t}(1+L_k)e^{M(t-s)}\sigma(s)\,ds \\
&+ \sum_{0<t_k<t}\prod_{t_k<t_j<t}(1+L_j)e^{M(t-t_k)}\bigl[I_k(\eta(t_k)) - L_k\eta(t_k)\bigr],
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
u(0) &= u(T) \\
&= \left(1 - \prod_{k=1}^{p}(1+L_k)e^{MT}\right)^{-1}\Biggl\{\int_0^T \prod_{s<t_k<T}(1+L_k)e^{M(T-s)}\sigma(s)\,ds \\
&\quad + \sum_{0<t_k<T}\prod_{t_k<t_j<T}(1+L_j)e^{M(T-t_k)}\bigl[I_k(\eta(t_k)) - L_k\eta(t_k)\bigr]\Biggr\}.
\end{aligned}
\tag{17}
$$

Lemma 5. *Let $M > 0$, $L_k < 1$ for $k = 1, 2, \dots, p$. If*

$$\prod_{k=1}^{p}(1 - L_k)e^{-MT} < 1,$$

then the PBVP (15) *has a unique solution*

$$u(t) = u(0) \prod_{0<t_k<t} (1 - L_k)e^{-Mt} + \int_0^t \prod_{s<t_k<t} (1 - L_k)e^{-M(t-s)}\sigma(s)\,ds + \sum_{0<t_k<t} \prod_{t_k<t_j<t} (1 - L_j)e^{-M(t-t_k)}\bigl[I_k(\eta(t_k)) + L_k\eta(t_k)\bigr], \tag{18}$$

where

$$u(0) = u(T) = \left(1 - \prod_{k=1}^{p}(1 - L_k)e^{-MT}\right)^{-1} \left\{\int_0^T \prod_{s<t_k<T} (1 - L_k)e^{-M(T-s)}\sigma(s)\,ds + \sum_{0<t_k<T} \prod_{t_k<t_j<T} (1 - L_j)e^{-M(T-t_k)}\bigl[I_k(\eta(t_k)) + L_k\eta(t_k)\bigr]\right\}. \tag{19}$$

3. The main results

Theorem 1. *Assume that conditions* (A_0)–(A_2) *hold and*

$$\prod_{k=1}^{p}(1 + L_k)^{-1}e^{-MT} < 1.$$

Then there exist monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, such that $\lim_{n\to\infty} \alpha_n(t) = r(t)$, $\lim_{n\to\infty} \beta_n(t) = \rho(t)$ uniformly on J, and $\rho(t)$, $r(t)$ are the minimal and the maximal solutions of the PBVP (1)*, respectively, such that*

$$\beta_0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_n \leqslant \rho \leqslant x \leqslant r \leqslant \alpha_n \leqslant \cdots \leqslant \alpha_2 \leqslant \alpha_1 \leqslant \alpha_0 \quad \text{on } J,$$

where x is any solution of the PBVP (1) *such that $\beta(t) \leqslant x(t) \leqslant \alpha(t)$ on J.*

Proof. Let $[\beta, \alpha] = \{x \in \Omega\colon\ \beta(t) \leqslant x(t) \leqslant \alpha(t),\ t \in J\}$. For any $\eta \in [\beta, \alpha]$, consider the PBVP (14), where

$$\sigma(t) = f\bigl(t, \eta(t)\bigr) - M\eta(t).$$

By Lemma 4, PBVP (14) possesses a unique solution $u \in \Omega$. We define an operator A by $u = A\eta$, then the operator A has the following properties:

(i) $\beta \leqslant A\beta$, $A\alpha \leqslant \alpha$;
(ii) A is monotone nondecreasing in $[\beta, \alpha]$; i.e., for any $\eta_1, \eta_2 \in [\beta, \alpha]$,

$$\eta_1 \leqslant \eta_2 \quad \text{implies} \quad A\eta_1 \leqslant A\eta_2.$$

We consider here only the case where $\alpha(0) > \alpha(T)$ and $\beta(0) < \beta(T)$. To prove (i), set $m = \beta_0 - \beta_1$, where $\beta_1 = A\beta_0$. Then, from (A_0) and (14), we have

$$\begin{aligned} m'(t) &= \beta_0'(t) - \beta_1'(t) \\ &\geqslant f\big(t, \beta_0(t)\big) + M\frac{[\beta_0(T) - \beta_0(0)]}{\sum_{i=0}^{p}\prod_{t_0<t_k<t_i}(1+L_k)^{-1}(e^{-Mt_i} - e^{-Mt_{i+1}})} \\ &\quad - \big[M\beta_1(t) + f\big(t, \beta_0(t)\big) - M\beta_0(t)\big] \\ &= Mm(t) + M\frac{[(\beta_0(T) - \beta_1(T)) - (\beta_0(0) - \beta_1(0))]}{\sum_{i=0}^{p}\prod_{t_0<t_k<t_i}(1+L_k)^{-1}(e^{-Mt_i} - e^{-Mt_{i+1}})} \\ &= Mm(t) + Mr_m, \quad t \neq t_k,\ t \in J, \\ \Delta m(t_k) &= \Delta\beta_0(t_k) - \Delta\beta_1(t_k) \\ &\geqslant I_k\big(\beta_0(t_k)\big) - \big[L_k\beta_1(t_k) + I_k\big(\beta_0(t_k)\big) - L_k\beta_0(t_k)\big] \\ &= L_k m(t_k), \quad k = 1, 2, \ldots, p, \\ m(0) &< m(T). \end{aligned}$$

By Lemma 2, we get $m(t) \leqslant 0$ on J, i.e., $\beta \leqslant A\beta$. Similar arguments show that $A\alpha \leqslant \alpha$.

To prove (ii), let $u_1 = A\eta_1$, $u_2 = A\eta_2$, where $\eta_1 \leqslant \eta_2$ on J and $\eta_1, \eta_2 \in [\beta, \alpha]$. Set $m = u_1 - u_2$. Using (A_1), (A_2) and (14), we get

$$\begin{aligned} m'(t) &= u_1'(t) - u_2'(t) \\ &= \big[Mu_1(t) + f\big(t, \eta_1(t)\big) - M\eta_1(t)\big] \\ &\quad - \big[Mu_2(t) + f\big(t, \eta_2(t)\big) - M\eta_2(t)\big] \\ &\geqslant M\big(u_1(t) - u_2(t)\big) = Mm(t), \quad t \neq t_k,\ t \in J, \\ \Delta m(t_k) &= \Delta u_1(t_k) - \Delta u_2(t_k) \\ &= \big[L_k u_1(t_k) + I_k\big(\eta_1(t_k)\big) - L_k\eta_1(t_k)\big] \\ &\quad - \big[L_k u_2(t_k) + I_k\big(\eta_2(t_k)\big) - L_k\eta_2(t_k)\big] \\ &\geqslant L_k m(t_k), \quad k = 1, 2, \ldots, p, \\ m(0) &= m(T). \end{aligned}$$

In view of Lemma 2, we have $m(t) \leqslant 0$ on J, i.e., $u_1 \leqslant u_2$.

It is now easy to define the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\alpha_{n+1} = A\alpha_n$, $\beta_{n+1} = A\beta_n$. From (i) and (ii), the functions $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ satisfy the inequalities

$$\beta_0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_n \leqslant \cdots \leqslant \alpha_n \leqslant \cdots \leqslant \alpha_2 \leqslant \alpha_1 \leqslant \alpha_0 \quad \text{on } J,$$

and each $\alpha_n, \beta_n \in \Omega$ $(n = 1, 2, \ldots)$ satisfies

$$\begin{cases} \alpha_n'(t) - M\alpha_n(t) = \sigma_{n-1}(t), & t \neq t_k,\, t \in J, \\ \Delta\alpha_n(t_k) = L_k\alpha_n(t_k) + I_k(\alpha_{n-1}(t_k)) - L_k\alpha_{n-1}(t_k), & k = 1, 2, \ldots, p, \\ \alpha_n(0) = \alpha_n(T), \end{cases}$$

and

$$\begin{cases} \beta_n'(t) - M\beta_n(t) = \bar{\sigma}_{n-1}(t), & t \neq t_k,\, t \in J, \\ \Delta\beta_n(t_k) = L_k\beta_n(t_k) + I_k(\beta_{n-1}(t_k)) - L_k\beta_{n-1}(t_k), & k = 1, 2, \ldots, p, \\ \beta_n(0) = \beta_n(T), \end{cases}$$

where

$$\sigma_{n-1}(t) = f\big(t, \alpha_{n-1}(t)\big) - M\alpha_{n-1}(t),$$
$$\bar{\sigma}_{n-1}(t) = f\big(t, \beta_{n-1}(t)\big) - M\beta_{n-1}(t).$$

Therefore there exist ρ, r such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly on J. Clearly ρ, r satisfy the PBVP (1). To prove that ρ, r are extreme solutions of PBVP (1), let $x(t)$ be any solution of the PBVP (1) such that $x \in [\beta, \alpha]$. Suppose that there exists a positive integer n such that $\beta_n(t) \leqslant x(t) \leqslant \alpha_n(t)$ on J. Then, setting $m = \beta_{n+1} - x$, we have

$$\begin{aligned} m'(t) &= \beta_{n+1}'(t) - x'(t) \\ &= \big[M\beta_{n+1}(t) + f\big(t, \beta_n(t)\big) - M\beta_n(t)\big] - f\big(t, x(t)\big) \\ &\geqslant Mm(t), \quad t \neq t_k,\ t \in J, \end{aligned}$$

$$\begin{aligned} \Delta m(t_k) &= \Delta\beta_{n+1}(t_k) - \Delta x(t_k) \\ &= \big[L_k\beta_{n+1}(t_k) + I_k\big(\beta_n(t_k)\big) - L_k\beta_n(t_k)\big] - I_k\big(x(t_k)\big) \\ &\geqslant L_k m(t_k), \quad k = 1, 2, \ldots, p, \end{aligned}$$

$$m(0) = m(T).$$

By Lemma 2, $m(t) \leqslant 0$ on J, i.e., $\beta_{n+1}(t) \leqslant x(t)$ on J. Similarly, we obtain $x(t) \leqslant \alpha_{n+1}(t)$ on $[0, T]$. Since $\beta_0(t) \leqslant x(t) \leqslant \alpha_0(t)$ on J, by induction we get $\beta_n(t) \leqslant x(t) \leqslant \alpha_n(t)$ on J for every n. Therefore, $\rho(t) \leqslant x(t) \leqslant r(t)$ on J by taking limit as $n \to \infty$. The proof of the theorem is complete. □

Theorem 2. *Assume that conditions* (B_0)–(B_2) *hold and*

$$\prod_{k=1}^{p}(1 - L_k)e^{-MT} < 1.$$

Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ *with* $\alpha_0 = \alpha,\ \beta_0 = \beta$, *such that* $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim_{n\to\infty} \beta_n(t) = r(t)$ *uniformly on* J, *and* $\rho(t), r(t)$ *are the minimal and the maximal solutions of the PBVP* (1), *respectively, such that*

$$\alpha_0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n \leqslant \rho \leqslant x \leqslant r \leqslant \beta_n \leqslant \cdots \leqslant \beta_2 \leqslant \beta_1 \leqslant \beta_0 \quad \text{on } J,$$

where x *is any solution of the PBVP* (1) *such that* $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ *on* J.

We leave out the proof of this theorem because it can be complete in the same way as the proof of Theorem 1.

References

[1] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[2] D.D. Bainov, P.S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, Harlow, Longman, 1993.

[3] G.S. Ladde, V. Lakshmikantham, A.S. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman, London, 1985.

[4] V. Lakshmikantham, S. Leela, Remarks on first and second order periodic boundary value problems, Nonlinear Anal. 8 (1984) 281–287.

[5] X.Z. Liu, Monotone iterative techniques for impulsive differential equations in a Banach space, J. Math. Phys. Sci. 24 (1990) 183–191.

[6] A.S. Vatsala, Y. Sun, Periodic boundary value problems for impulsive differential equations, Appl. Anal. 44 (1992) 145–158.

[7] G.R. Shendge, Monotone method for boundary value problems describing periodic transport processes, J. Math. Anal. Appl. 106 (1985) 286–292.

[8] J.J. Nieto, N. Alvarez-Noriega, Periodic boundary problems for nonlinear first order ordinary differential equations, Acta Math. Hungar. 71 (1996) 49–58.

[9] J.J. Nieto, Basic theory for nonresonance impulsive periodic problems of first order equations, J. Math. Anal. Appl. 205 (1997) 423–433.