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J. Math. Anal. Appl. 301 (2005) 187–207

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

# Permanence extinction and global asymptotic stability in a stage structured system with distributed delays

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Received 22 October 2003

Available online 22 September 2004

Submitted by K. Gopalsamy

**Abstract**

In this paper we consider a nonautonomous stage-structured competitive system of $n$-species population growth with distributed delays which takes into account the delayed feedback in both interspecific and intraspecific interactions. We obtain, by using the method of repeated replace, sufficient conditions for permanence and extinction of the species. The global attractivity of the unique positive equilibrium is proved in the autonomous case. Our results extend previous ones obtained by Liu et al. in [Nonlinear Anal. 51 (2002) 1347–1361; J. Math. Anal Appl. 274 (2002) 667–684].


*Keywords:* Distributed delays; Extinction; Global asymptotic stability; Interspecific competition; Permanence; Stage structure

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0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2004.07.017

## 1. Introduction

Stage-structured models had already received much attention before 1990. But the great progress on the stage structured models had not been obtained until 1990, when Aiello and Freedman [1] proposed and studied their, by now well-known, single species model with time delayed stage structure. In the model of Aiello and Freedman the population has a life history and is divided into two stages: immature and mature. In particular we have in mind mammalian populations which exhibit these two distinct stages: the mature species are the adult animals and the immature represent their babies. The model is formulated mathematically by the following system of two delay differential equations:

$$\begin{aligned} \frac{dx}{dt}(t) &= \alpha e^{-\gamma\tau} x(t-\tau) - \beta x^2(t), \\ \frac{dy}{dt}(t) &= \alpha x(t) - \gamma y(t) - \alpha e^{-\gamma\tau} x(t-\tau), \end{aligned} \tag{1.1}$$

for $t > 0$, where $x(t)$ and $y(t)$ represent the density of mature and immature species at time $t$, respectively. The model is derived under the following hypotheses:

(i) Only mature species can reproduce immatures, and the immature born at time $t-\tau$ that survives to time $t$ exit from the immature population and enters the mature population.
(ii) The birth of the immature population is proportional to the existing mature population with proportionality constant $\alpha$.
(iii) The death rate of the immature population is proportional to the existing immature population with proportionality constant $\gamma$.
(iv) The death rate of the mature population is of logistic nature, i.e., it is proportional to the square of the population with proportionality constant $\beta$.

$\tau$ is said to be the constant time to maturity and $\xi = \tau\gamma$ is the degree of the stage-structure. For the model to make sense it is assumed that both mature and immature populations are known in the interval $(-\tau, 0)$. Aiello and Freedman associate to the system (1.1) the following initial data:

$$x(t) = \varphi(t) \geqslant 0, \quad y(t) = \xi(t) \geqslant 0, \quad -\tau \leqslant t \leqslant 0. \tag{1.2}$$

The model (1.1)–(1.2) predicts a positive steady state as the global attractor (see [1]). It then suggests that stage-structure does not generate the sustained oscillations frequently observed in nature.

In [5], Freedman and Wu constructed a single stage-structured model in a somewhat complicated environment. They proved the permanence and the global asymptotic stability of a positive equilibrium in a multi-patch environment when the species can disperse between the patches. In particular, they proved that the heterogeneity of the environment may change the size of the positive equilibrium but cannot change its global asymptotic stability. We also quote the works of Aiello et al. [2] in which the time delay to maturity $\tau$ depends on the population density and that of Freedman et al. [4] which drafts the cases of cooperation and cannibalism interactions.

Inspired by the work on competitive Lotka–Volterra systems [22], Liu et al. [14] derived a competitive stage-structured model of two species where only mature species can compete with the other species. This model was extended in [15] to $n$-competing species where only mature species can compete with themselves for the common resource, so the competition with the immature species can be ignored. They proved that the system with stage-structure has similar behavior to that without stage-structure. These works unify and extend those on Lotka–Volterra systems to the case with stage-structure.

All these models are formulated in the autonomous case, i.e., the coefficient parameters of the model are constant (not depending on time). This situation is unrealistic since the parameters of the environment do vary with time in general. It is then very important to consider models of population interactions which take into account both the seasonality of the changing environment and the effects of time delays. These models can provide some explanation about the often fluctuated behavior of the population densities observed in nature. Mathematically speaking, to model the fluctuation of the environment it is assumed that the coefficients of the system are functions of time (see [11]). In this regard, Liu et al. considered in [16] the nonautonomous version of their autonomous competitive system in [15], i.e., for $i = 1, \dots, n$,

$$\begin{cases} \frac{dx_i}{dt} = b_i(t-\tau_i)e^{-\int_{t-\tau_i}^t d_i(s)\,ds} x_i(t-\tau_i) - x_i(t)\sum_{j=1}^n a_{ij}(t)x_j(t), \\ \frac{dy_i}{dt} = b_i(t)x_i(t) - d_i(t)y_i(t) - b_i(t-\tau_i)e^{-\int_{t-\tau_i}^t d_i(s)\,ds} x_i(t-\tau_i). \end{cases} \tag{1.3}$$

They established sufficient conditions under which the species are permanent. They also proved that the increase of the stage-structure degree of the species $x_2, \dots, x_n$ can lead to their extinction. This result shows that stage-structure of the species has negative effect on its permanence. We refer the reader to the survey paper of Liu et al. [13] on stage-structured population dynamics.

In view of the fact that in real-life competition interactions, instantaneous responses are rare or weak relatively to delayed responses, more realistic models should consist of delay differential systems instead of systems with instantaneous feedbacks. Some authors (Volterra [24], Kostitzin [25]) consider continuously distributed delays as ecologically and biologically more realistic than discrete delays to model the species interactions, which is proved true in 1969 by Caperon [26] who tried discrete and continuous delay models to fit data obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuously distributed delay. Later, Gopalsamy and He [23] point out that in several cases of animal populations, the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animals and in such cases distributed time delays will be appropriate. We note that in the previous studied stage-structured systems (see, for instance, [14–16]), delayed feedback responses in competition among the mature species are ignored. Motivated by the works on competitive Lotka–Volterra systems with distributed delays (see [7,9,10,12,19]), it is our goal here in this paper to introduce distributed delay terms into the $n$-species nonautonomous competitive stage-structured system (1.3). These nonlocal terms will represent feedback responses from the past life history of the species. To be precise, we shall consider the following nonautonomous competitive stage-structured system with distributed delays:

$$\begin{cases} \frac{dx_i}{dt} = b_i(t-\tau_i)e^{-\int_{t-\tau_i}^t d_i(s)\,ds} x_i(t-\tau_i) - x_i(t)\sum_{j=1}^n a_{ij}(t)x_j(t) \\ \qquad - x_i(t)\sum_{j=1}^n \int_{-\tau_j}^0 x_j(t+s)\,dh_{ij}(s), \quad t>0, \\ \frac{dy_i}{dt} = b_i(t)x_i(t) - d_i(t)y_i(t) - b_i(t-\tau_i)e^{-\int_{t-\tau_i}^t d_i(s)\,ds} x_i(t-\tau_i), \quad t>0, \end{cases} \tag{1.4}$$

with the following initial conditions:

$$x_i(t) = \varphi_i(t), \quad y_i(t) = \xi_i(t), \quad -\tau_i \leqslant t \leqslant 0. \tag{1.5}$$

The system (1.4) can be regarded as a generalization of (1.3) since it takes into account both interspecific and intraspecific interactions. We will make the following assumptions on the coefficient parameters of the system:

(H1) The functions $b_i(t), d_i(t), a_{ij}(t)$ are assumed to be nonnegative continuous and bounded such that $\inf_{t>0} a_{ii}(t) > 0$, $i = 1, \dots, n$.
(H2) The functions $h_{ij}(t)$ are continuous and of bounded variations over $[-\tau, 0]$, with $\tau = \max_{1\leqslant i\leqslant n}\{\tau_i\}$.
(H3) The initial functions $\varphi_i(t), \xi_i(t)$ are positive continuous and bounded over $[-\tau_i, 0]$, $i = 1, \dots, n$.

For the continuity of the initial conditions we assume that

(H4) $y_i(0) = \int_{-\tau_i}^0 b_i(s)\varphi_i(s)e^{-\int_s^0 d_i(u)\,du}\,ds$ for any $i = 1, \dots, n$.

This paper is organized as follows. In Section 2 we prove that system (1.4)–(1.5) has a positive solution which is ultimately bounded. In Section 3 we obtain sufficient conditions for permanence and extinction of the species. These conditions extend to the case of distributed delays those in [16]. In Section 4 we consider system (1.4) into the autonomous case, i.e., when the coefficients are constants. We establish a similar conditions to that in [15] assuring the existence of a positive global asymptotic equilibrium. We end the paper by some concluding remarks.

## 2. Preliminaries

In this section we make some notations and state some results which will be used later. Put for $1 \leqslant i, j \leqslant n$,

$$H_{ij}(t) = \mathrm{Var}(h_{ij}|_{[-\tau,t]}),$$
$$h_{ij}^-(t) = \frac{1}{2}(H_{ij} - h_{ij})(t),$$
$$h_{ij}^+(t) = \frac{1}{2}(H_{ij} + h_{ij})(t),$$

for $t \in [-\tau, 0]$, where $\tau = \max_{1\leqslant i\leqslant n}\{\tau_i\}$ and $\mathrm{Var}(g|_{[-\tau,t]}.)$ denotes the total variation of $g$ on $[-\tau, t]$. It is easy to see that $H_{ij}, h_{ij}^+, h_{ij}^-$ are nondecreasing functions. Moreover

$$h_{ij}(t) = h_{ij}^+(t) - h_{ij}^-(t), \quad -\tau \leqslant t \leqslant 0.$$

Let

$$C_{ij}^+ = \int_{-\tau_j}^{0} dh_{ij}^+(s), \qquad C_{ij}^- = \int_{-\tau_j}^{0} dh_{ij}^-(s).$$

If we denote by $B_i(t)$ the expression $b_i(t-\tau_i)e^{-\int_{t-\tau_i}^{t} d_i(s)\,ds}$, then system (1.4) can be rewritten as

$$\begin{cases} \frac{dx_i}{dt} = B_i(t)x_i(t-\tau_i) - x_i(t)\sum_{j=1}^n a_{ij}(t)x_j(t) \\ \qquad - x_i(t)\sum_{j=1}^n \int_{-\tau_j}^0 x_j(t+s)\,dh_{ij}^+(s) + x_i(t)\sum_{j=1}^n \int_{-\tau_j}^0 x_j(t+s)\,dh_{ij}^-(s), \\ \frac{dy_i}{dt} = b_i(t)x_i(t) - d_i(t)y_i(t) - B_i(t)x_i(t-\tau_i), \end{cases} \tag{2.1}$$

for $t > 0$ and $i = 1, \dots, n$.

Denote by $C_\tau = C([-\tau, 0], R^n)$ the Banach space of continuous functions mapping $[-\tau, 0]$ into $R^n$ with the supremum norm. Define $C = \prod_{i=1}^n C_{\tau_i}$ and the positive cone $C^+ = \{\phi = (\phi_1, \dots, \phi_n) \in C\colon \phi_i \geqslant 0,\ i = 1, \dots, n\}$. If $A > 0$ is a positive real number and $x \in C([-\tau, A], R^n)$, then for each $t \in [0, A]$ we denote by $x_t$ the function defined by $x_t(s) = x(t+s)$ for $-\tau \leqslant s \leqslant 0$. If $g$ is a real valued function of $t$, we put

$$g^l = \inf_{t\geqslant 0} g(t), \qquad g^m = \sup_{t\geqslant 0} g(t)$$

and make the assumption

(H5) $a_{ii}^l > \sum_{j=1}^n C_{ij}^-$ for $i = 1, \dots, n$.

**Proposition 1.** *Under the assumptions* (H1)–(H5)*, system* (2.1)–(1.5) *has a unique positive solution* $(x_i(t), y_i(t))$*,* $i = 1, \dots, n$*, for* $t > 0$*, which is ultimately bounded.*

**Proof.** Let $f = (f_1, \dots, f_n)$, $f : R_+ \times C \to R^n$ be defined by

$$f_i(t,\phi) = B_i(t)\phi_i(-\tau_i) - \phi_i(0)\sum_{j=1}^n a_{ij}(t)\phi_j(0) - \phi_i(0)\sum_{j=1}^n \int_{-\tau_j}^0 \phi_j(s)\,dh_{ij}^+(s)$$
$$+ \phi_i(0)\sum_{j=1}^n \int_{-\tau_j}^0 \phi_j(s)\,dh_{ij}^-(s).$$

It is easy to check that $f$ is continuous from $R_+ \times C$ into $R^n$. Further denoting by $D_\phi f_i(t,\phi)$ the differential of $f_i$ in $\phi$ we find

$$\begin{aligned}
D_\phi f_i(t,\phi)(h)n &= B_i(t)h_i(-\tau_i) - h_i(0)\sum_{j=1}^n a_{ij}(t)\phi_j(0) \\
&\quad - \phi_i(0)\sum_{j=1}^n a_{ij}(t)h_j(0) - \phi_i(0)\sum_{j=1}^n \int_{-\tau_j}^0 h_j(s)\,dh_{ij}^+(s) \\
&\quad + h_i(0)\sum_{j=1}^n \int_{-\tau_j}^0 \phi_j(s)\,dh_{ij}^-(s) - h_i(0)\sum_{j=1}^n \int_{-\tau_j}^0 \phi_j(s)\,dh_{ij}^+(s) \\
&\quad + \phi_i(0)\sum_{j=1}^n \int_{-\tau_j}^0 h_j(s)\,dh_{ij}^+(s),
\end{aligned}$$

where $h = (h_1,\dots,h_n) \in C$. Therefore $f$ is continuously differentiable in $\phi$. Thus by [8, Theorem 2.3, p. 42] the first equation of (2.1) has a unique solution $(x_1(t),\dots,x_n(t))$ for $t > 0$. It follows by the classical results on ordinary differential equations that the second equation of (2.1) has also a unique solution $y(t) = (y_1(t),\dots,y_n(t))$ for $t > 0$. Now if $\phi \in C^+$ with $\phi_i(0) = 0$ for some $i$, then $f_i(t,\phi) = B_i(t)\phi_i(-\tau_i) \geqslant 0$, so by Proposition 1.2 of [17] we obtain that the solution $x(t) = (x_1(t),\dots,x_n(t))$ of system (2.1)–(1.5) satisfies $x_i(t) \geqslant 0$ for $t > 0$.

We now prove that $x(t) > 0$ for $t > 0$. By (H3) $\phi_i, \xi_i > 0$ on $[-\tau_i, 0]$. Suppose on the contrary that there is $t_1 > 0$ such that $x_i(t_1) = 0$ and define $t_0 = \inf\{t > 0\colon\ x_i(t) = 0\}$, then by (H3) $t_0 > 0$ and $x_i(t_0) = 0$. From the first equation of (2.1),

$$\begin{aligned}
&\frac{dx_i}{dt}(t_0) = B_i(t_0)x_i(t_0 - \tau_i) > 0, \quad \text{if } t_0 > \tau_i, \\
&\frac{dx_i}{dt}(t_0) = B_i(t_0)\varphi_i(t_0 - \tau_i) > 0, \quad \text{if } t_0 < \tau_i.
\end{aligned}$$

Clearly in both cases $\frac{dx_i}{dt}(t_0) > 0$. This a contradiction since by the definition of $t_0$ we know that $\frac{dx_i}{dt}(t_0) \leqslant 0$.

We now prove the boundedness of solutions of system (2.1)–(1.5). Choose $M$ such that

$$M = \max_{1\leqslant i\leqslant n}\left\{\frac{B_i^m}{a_{ii}^l - \sum_{j=1}^n C_{ij}^-},\ \sup_{s\in[-\tau_i,0]}\varphi_i(s)\right\}. \tag{2.2}$$

We claim that $x_i(t) \leqslant M$ for $t > 0$ and $i = 1,\dots,n$. Otherwise there would exist a $\tilde t > 0$ and $i_0 \in \{1,\dots,n\}$ such that $x_{i_0}(\tilde t) = M$, $\frac{dx_{i_0}}{dt}(\tilde t) \geqslant 0$ and $x_i(t) \leqslant M$ for $t \leqslant \tilde t$ and $i = 1,\dots,n$. We have from (2.1) and (2.2),

$$\begin{aligned}
\frac{dx_{i_0}}{dt}(\tilde t) &\leqslant B_{i_0}(\tilde t)x_{i_0}(\tilde t - \tau_{i_0}) - a_{i_0i_0}(\tilde t)x_{i_0}^2(\tilde t) + x_{i_0}(\tilde t)\sum_{j=1}^n \int_{-\tau_j}^0 x_j(\tilde t + s)\,dh_{i_0j}^-(s) \\
&\leqslant M\left\{B_{i_0}^m - M\left(a_{i_0i_0}^l - \sum_{j=1}^n C_{i_0j}^-\right)\right\} < 0,
\end{aligned}$$

which is a contradiction. □

Arguing as in the proof of the Corollaries 3.1 and 3.2 in [16] (see also [14, Corollary 4.1, p. 133], we can prove the following comparison result.

**Lemma 1.** *$a, b, c$ and $d$ be positive constants and let $x(t)$ be continuously differentiable function such that*

$$\begin{cases} \frac{dx}{dt}(t) \leqslant bx(t-\tau) - cx(t) + dx(t) - ax^2(t), & t > 0, \\ x(t) = \varphi(t), & -\tau \leqslant t \leqslant 0, \end{cases} \tag{2.3}$$

*where the initial function $\varphi$ is assumed to be in $C_\tau^+ = \{\phi \in C_\tau \colon \phi \geqslant 0\}$. Then*

(1) (i) *If $b > c - d$, then for any $\varepsilon > 0$ sufficiently small there exist $T_\varepsilon > 0$ such that*

$$x(t) < \frac{b - c + d}{a} + \varepsilon \quad \text{for } t > T_\varepsilon.$$

(ii) *Further if $\frac{dx}{dt}(t) \geqslant bx(t-\tau) - cx(t) + dx(t) - ax^2(t)$ for $t > 0$, then for any $\varepsilon > 0$ (sufficiently small) there exist $T'_\varepsilon > 0$ such that*

$$x(t) > \frac{b - c + d}{a} - \varepsilon \quad \text{for } t > T'_\varepsilon.$$

(2) *If $b < c - d$ then $\lim_{t\to\infty} x(t) = 0$.*

## 3. Permanence and extinction

In this section we study the permanence and the extinction of the species $x_i$ of system (2.1). We begin first by the following theorem which gives sufficient conditions for permanence of the species.

**Theorem 1.** *Assume that the assumptions* (H1)–(H5) *hold and*

$$B_i^l > \sum_{j=1,\, j\neq i}^{n} a_{ij}^m \gamma_j + \sum_{j=1}^{n} C_{ij}^+ \gamma_j,$$

*where $\gamma_i$ is the unique positive solution of the equation*

$$\gamma_i = \frac{1}{a_{ii}^l}\left(B_i^m + \sum_{j=1}^{n} C_{ij}^- \gamma_j\right), \quad i = 1, \ldots, n. \tag{3.1}$$

*Then $(x_i, y_i)$ are uniformly permanent, $i = 1, \ldots, n$.*

**Proof.** Let $M$ be as in (2.2) and choose $\gamma^{(0)}$ such that $\gamma^{(0)} = M + 1$. Then there exists $T > 0$ such that

$$x_i(t) < \gamma^{(0)}, \quad t > T,\ i = 1, \ldots, n. \tag{3.2}$$

From (2.1) and the properties of $h_{ij}^-$ and $h_{ij}^+$ we see that

$$\frac{dx_i}{dt} \leqslant B_i^m x_i(t-\tau_i) - a_{ii}^l x_i^2(t) + x_i(t) \sum_{j=1}^n \int_{-\tau_j}^0 x_j(t+s)\, dh_{ij}^-(s)$$

$$\leqslant B_i^m x_i(t-\tau_i) - a_{ii}^l x_i^2(t) + x_i(t) \sum_{j=1}^n C_{ij}^- \gamma^{(0)}, \tag{3.3}$$

for $t > T + \tau$, where $\tau = \max_{1 \leqslant i \leqslant n}\{\tau_i\}$. Pick $\varepsilon^{(1)} > 0$, then by (3.3) and Lemma 1 there is a large time $T_i^{(1)} > T + \tau$ such that

$$x_i \leqslant \frac{1}{a_{ii}^l}\left(B_i^m + \gamma^{(0)} \sum_{j=1}^n C_{ij}^-\right) + \varepsilon^{(1)} = \gamma_i^{(1)} > 0,$$

for any $t > T_i^{(1)}$. Repeating the above process $k$-times for $\varepsilon^{(1)} > \varepsilon^{(2)} > \cdots > \varepsilon^{(k)}$, we obtain two sequences $\gamma_i^{(k)}$ and $T_i^{(k)}$ such that

$$x_i \leqslant \frac{1}{a_{ii}^l}\left(B_i^m + \sum_{j=1}^n C_{ij}^- \gamma_i^{(k-1)}\right) + \varepsilon^{(k)} = \gamma_i^{(k)},$$

for $t > T_i^{(k)}$. Moreover, we have

$$\gamma_i^{(k)} - \gamma_i^{(k-1)} = \frac{1}{a_{ii}^l} \sum_{j=1}^n C_{ij}^- \bigl(\gamma_i^{(k-1)} - \gamma_i^{(k-2)}\bigr) + \varepsilon^{(k)} - \varepsilon^{(k-1)}, \tag{3.4}$$

since $\varepsilon^{(k)} - \varepsilon^{(k-1)} < 0$, it suffices to prove that $\gamma_i^{(1)} - \gamma^{(0)} < 0$ to conclude by induction that $\gamma_i^{(k)} - \gamma_i^{(k-1)} < 0$, for any $k \geqslant 1$. We have by definition of $\gamma_i^{(1)}$ that

$$\gamma_i^{(1)} - \gamma^{(0)} = \frac{1}{a_{ii}^l}\left[B_i^m + \left(\sum_{j=1}^n C_{ij}^- - a_{ii}^l\right)\gamma^{(0)} - a_{ii}^l \varepsilon^{(1)}\right] < 0.$$

Therefore, the sequence $\gamma_i^{(k)}$ is decreasing in $k$. So there is $\gamma_i > 0$ such that

$$\lim_{k\to\infty} \gamma_i^{(k)} = \gamma_i, \quad i = 1, \ldots, n.$$

It follows that

$$\limsup_{t\to\infty} x_i(t) \leqslant \gamma_i, \quad i = 1, \ldots, n, \tag{3.5}$$

with

$$\gamma_i = \frac{1}{a_{ii}^l}\left(B_i^m + \sum_{j=1}^n C_{ij}^- \gamma_j\right). \tag{3.6}$$

By the assumption of Theorem 1 we can select $\varepsilon > 0$ so that

$$B_i^l - \sum_{j=1,\, j\neq i}^n a_{ij}^m (\gamma_j + \varepsilon) - \sum_{j=1}^n C_{ij}^+ (\gamma_j + \varepsilon) > 0. \tag{3.7}$$

For this $\varepsilon$ we entail from (3.5) that there are $T_i > 0$ such that

$$x_i(t) \leqslant \gamma_i + \varepsilon, \quad t > T_i.$$

Put $T' = \max_{1 \leqslant i \leqslant n} T_i$. We have then by (2.1),

$$\frac{dx_i}{dt} \geqslant B_i^l x_i(t-\tau_i) - x_i \sum_{j=1,\, j\neq i}^{n} a_{ij}^m(\gamma_j + \varepsilon) - x_i \sum_{j=1}^{n} C_{ij}^+(\gamma_j + \varepsilon) - a_{ii}^m x_i^2, \tag{3.8}$$

for $t > T' + \tau$. Let $\varepsilon' > 0$ be small. By Lemma 1, (3.7) and (3.8) there exists $T_i'' > T'$ such that

$$x_i \geqslant \frac{1}{a_{ii}^m}\left(B_i^l - \sum_{j\neq i}^{n} a_{ij}^m(\gamma_j + \varepsilon) - \sum_{j=1}^{n} C_{ij}^+(\gamma_j + \varepsilon)\right) - \varepsilon',$$

for any $t > T_i''$. Since $\varepsilon$ and $\varepsilon'$ are arbitrarily small then taking the limit as $t \to \infty$, we obtain

$$\liminf_{t\to\infty} x_i \geqslant \frac{1}{a_{ii}^m}\left(B_i^l - \sum_{j=1,\, j\neq i}^{n} a_{ij}^m \gamma_j - \sum_{j=1}^{n} C_{ij}^+ \gamma_j\right) = \delta_i > 0.$$

We now turn to prove the permanence of $y_i(t)$. From the second equation in (2.1) we have

$$\frac{dy_i}{dt} = -d_i(t)y_i(t) + f_i(t), \tag{3.9}$$

where $f_i(t) = b_i(t)x_i(t) - b_i(t-\tau_i)e^{-\int_{t-\tau_i}^t d_i(s)\,ds} x_i(t-\tau_i)$. Integrating both sides of (3.9) on $(0,t)$, we obtain (as in [15])

$$y_i(t) = y_i(0)e^{-\int_0^t d_i(s)\,ds} + \left(\int_{t-\tau_i}^{t} b_i(s)x_i(s)e^{\int_0^s d_i(u)\,du}\,ds\right) \times e^{-\int_0^t d_i(s)\,ds}, \tag{3.10}$$

we deduce that

$$\liminf_{t\to\infty} y_i(t) \geqslant b_i^l \tau_i \delta_i e^{-\tau_i d_i^m} > 0.$$

Consequently, $y_i(t)$, $i = 1, \ldots, n$, are permanent. □

If we put $N = (C_{ij}^- / a_{ii}^l)$ $(1 \leqslant i, j \leqslant n)$, then by (H5),

$$\|N\| = \max_{1\leqslant i\leqslant n}\left(\frac{1}{a_{ii}^l}\sum_{j=1}^{n} C_{ij}^-\right) < 1, \tag{3.11}$$

the matrix $(I - N)$ is then invertible, if we denote by $(\alpha_{ij})$ $(1 \leqslant i, j \leqslant n)$ its elements, then $\alpha_{ij} \geqslant 0$ for $i, j = 1, \ldots, n$ and we can explicit $\gamma_i$ as

$$\gamma_i = \sum_{j=1}^{n} \alpha_{ij} \frac{B_j^m}{a_{jj}^l} \quad \text{for } i = 1, \ldots, n.$$

The following theorem gives sufficient conditions for the extinction of the species.

**Theorem 2.** *Assume that* (H1)–(H5) *hold and*

(i) $$B_1^l > \sum_{j=2}^n \bigl(C_{1j}^+ + a_{1j}^l\bigr)\gamma_j - C_{11}^+\gamma_1,$$

(ii) $$\frac{B_i^m + C_{i1}^-\gamma_1}{a_{i1}^l} < \frac{B_1^l - C_{11}^+\gamma_1}{a_{11}^m} \quad \textit{for } i = 2, \ldots, n.$$

*Then* $(x_i(t), y_i(t)) \to (0,0)$ *as* $t \to \infty$ *for* $i = 2, \ldots, n$ *while* $(x_1(t), y_1(t))$ *gets permanent.*

**Proof.** The proof is divided into several steps. Define two sequences $(v_i^{(m)})_{m\geqslant 0}$ and $(u^{(m)})_{m\geqslant 1}$ as follows:

$$v_i^{(m+1)} = \frac{1}{a_{ii}^l}\left(B_i^m - a_{i1}^l u^{(m+1)} + \sum_{j=2}^n C_{ij}^- v_j^{(m)} + C_{i1}^-\gamma_1\right),$$
$$u^{(m+1)} = \frac{1}{a_{11}^m}\left(B_1^l - \sum_{j=2}^n (C_{1j}^+ + a_{1j}^l) v_j^{(m)} - C_{11}^+\gamma_1\right), \tag{3.12}$$

for $m \geqslant 0$ with $v_i^{(0)} = \gamma_i$, where $\gamma_i$ is defined in Theorem 1. We have for $m \geqslant 1$,

$$v_i^{(m+1)} - v_i^{(m)} = \frac{1}{a_{ii}^l}\left(-a_{i1}^l\bigl(u^{(m+1)} - u^{(m)}\bigr) + \sum_{j=2}^n C_{ij}^-\bigl(v_j^{(m)} - v_j^{(m-1)}\bigr)\right),$$
$$u^{(m+1)} - u^{(m)} = -\frac{1}{a_{11}^m}\left(\sum_{j=2}^n \bigl(C_{1j}^+ + a_{1j}^l\bigr)\bigl(v_j^{(m)} - v_j^{(m-1)}\bigr)\right), \tag{3.13}$$

for $i = 2, \ldots, n$.

**Claim 1.** There is $i_0$, $2 \leqslant i_0 \leqslant n$ and $m_{i_0} \geqslant 1$ such that

$$v_{i_0}^{(m_{i_0})} < 0 \quad \text{and} \quad v_i^{(m_{i_0}-1)} > 0 \quad \text{for } i = 2, \ldots, n. \tag{3.14}$$

By assumption (i) of the theorem,

$$u^{(1)} = \frac{1}{a_{11}^m}\left(B_1^l - \sum_{j=2}^n \bigl(C_{1j}^+ + a_{1j}^l\bigr)\gamma_j - C_{11}^+\gamma_1\right) > 0,$$

hence

$$v_i^{(1)} - v_i^{(0)} = -\frac{a_{i1}^l}{a_{ii}^l} u^{(1)} < 0 \quad \text{for } i = 2, \ldots, n,$$
$$u^{(2)} - u^{(1)} = -\frac{1}{a_{11}^m}\left(\sum_{j=2}^n \bigl(C_{1j}^+ + a_{1j}^l\bigr)\bigl(v_j^{(1)} - v_j^{(0)}\bigr)\right) > 0.$$

By induction we conclude that the sequences $(v_i^{(m)})_{m\geqslant 0}$ and $(u^{(m)})_{m\geqslant 1}$ are respectively decreasing and increasing. There are then two cases to distinguish.

**Case 1.** $\lim_{m\to\infty} u^{(m)} = +\infty$, we have by the first equation of (3.12) $\lim_{m\to\infty} v_i^{(m)} = -\infty$ for any $i = 2,\dots,n$. Since $v_i^{(0)} > 0$ for $i = 2,\dots,n$ we can define $i_0$ and $m_{i_0} \geqslant 1$ as follows:

$$m_{i_0} = \min_{2\leqslant i\leqslant n} \bigl(m_i = \min\bigl\{m \geqslant 1\colon\ v_i^{(m)} < 0\bigr\}\bigr). \tag{3.15}$$

**Case 2.** $\lim_{n\to\infty} u^{(m)} = u < \infty$. Since $\|N\| < 1$ (see (3.11)), the linear map

$$v = (v_2,\dots,v_n) \to \left(\frac{1}{a_{ii}^l}\sum_{j=2}^n C_{ij}^- v_j\right)_{2\leqslant i\leqslant n}^T$$

is a contraction and by the first equation of (3.12) the sequence $v_i^{(m)}$ converges to some finite number $v_i$ as $m\to\infty$. Substituting the second equation of (3.12) into the first one we find

$$\begin{aligned}
v_i^{(m+1)} &= \frac{B_i^m + C_{i1}^-\gamma_1}{a_{ii}^l} - \frac{a_{i1}^l}{a_{ii}^l}\left[\frac{B_1^l - C_{11}^+\gamma_1}{a_{11}^m} - \frac{1}{a_{11}^m}\sum_{j=2}^n \bigl(C_{1j}^+ + a_{1j}^l\bigr)v_j^{(m)}\right] \\
&\quad + \frac{1}{a_{ii}^l}\sum_{j=2}^n C_{ij}^- v_j^{(m)} \\
&= \frac{a_{i1}^l}{a_{ii}^l}\left(\frac{B_i^m + C_{i1}^-\gamma_1}{a_{ii}^l} - \frac{B_1^l - C_{11}^+\gamma_1}{a_{11}^m}\right) \\
&\quad + \sum_{j=2}^n \left(\frac{a_{i1}^l}{a_{ii}^l}\frac{C_{1j}^+}{a_{11}^m} + \frac{a_{i1}^l}{a_{ii}^l}\frac{a_{1j}^l}{a_{11}^m} + \frac{C_{ij}^-}{a_{ii}^l}\right)v_j^{(m)}.
\end{aligned} \tag{3.16}$$

Denote by

$$\begin{aligned}
\beta &= \left(\frac{a_{i1}^l}{a_{ii}^l}\left(\frac{B_i^m + C_{i1}^-\gamma_1}{a_{ii}^l} - \frac{B_1^l - C_{11}^+\gamma_1}{a_{11}^m}\right)\right)_{2\leqslant i\leqslant n}^T, \\
C &= \left(\frac{a_{i1}^l}{a_{ii}^l}\frac{C_{1j}^+}{a_{11}^m} + \frac{a_{i1}^l}{a_{ii}^l}\frac{a_{1j}^l}{a_{11}^m} + \frac{C_{ij}^-}{a_{ii}^l}\right)_{2\leqslant i,j\leqslant n}, \\
v^{(m)} &= \bigl(v_2^{(m)},\dots,v_n^{(m)}\bigr)^T \quad \text{for } m \geqslant 0.
\end{aligned}$$

Then inequality (3.16) can be rewritten as

$$v^{(m+1)} = \beta + Cv^{(m)} \quad \text{for } m \geqslant 0. \tag{3.17}$$

So $v^{(m)} - v^{(m+1)} = C(v^{(m-1)} - v^{(m)}) = \cdots = C^m(v^{(0)} - v^{(1)})$ for $m \geqslant 0$. Since $v^{(m)} - v^{(m+1)} \to 0$ as $m\to\infty$, then $C^m(v^{(0)} - v^{(1)}) \to 0$ as $m\to\infty$. Put $\xi = (v^{(0)} - v^{(1)})^T > 0$ elementwise. Since $C^m\xi \to 0$ as $m\to\infty$, then by the matrix theory (see [3]) $\sigma(C) < 1$, where $\sigma(C)$ denote the spectral radius of $C$, thus $C^m\zeta \to 0$ as $m \to \infty$ for any positive vector $\zeta \in R^{n-1}$. From (3.17) we have by induction

$$\begin{aligned} v^{(m)} &= \beta + Cv^{(m-1)} = \beta + C\beta + C^2 v^{(m-2)} = \cdots \\ &= \beta + C\beta + \cdots + C^{m-1}\beta + C^m v^{(0)}, \end{aligned}$$

and since $C^k\beta < 0$ (recall that $\beta < 0$) elementwise for any $k \geqslant 1$, we obtain

$$v^{(m)} < \beta + C^m v^{(0)}, \tag{3.18}$$

for any $m \geqslant 1$. Now since $v^{(m)} \to v$ and $C^m v^{(0)} \to 0$ as $m \to \infty$, we obtain from (3.18) that $v < \beta < 0$. We can then define $i_0$, $2 \leqslant i_0 \leqslant n$ and $m_{i_0} \geqslant 1$ as in (3.15) such that (3.14) holds.

**Claim 2.** $\lim_{t\to\infty} x_{i_0}(t) = 0$. By assumption (i) of the theorem we can select $\varepsilon > 0$ such that

$$B_1^l > \sum_{j=2}^{n} \bigl(C_{1j}^+ + a_{1j}^l\bigr)(\gamma_j + \varepsilon) - C_{11}^+(\gamma_1 + \varepsilon), \tag{3.19}$$

therefore there is $T_1$ such that for $t > T_1$,

$$\begin{aligned} \frac{dx_1}{dt} \geqslant{}& B_1^l x_1(t-\tau_1) - x_1 \sum_{j=2}^{n} a_{1j}^m(\gamma_j + \varepsilon) - x_1 \sum_{j=2}^{n} C_{1j}^+(\gamma_j + \varepsilon) \\ & - a_{11}^m x_1^2 - x_1 C_{1j}^+(\gamma_1 + \varepsilon), \end{aligned}$$

from Lemma 1 and (3.19) there are $\bar{T}_1 > T_1$ and $\varepsilon_1 > 0$ small enough such that

$$x_1(t) \geqslant \frac{1}{a_{11}^m}\left(B_1^l - \sum_{j=2}^{n}\bigl(C_{1j}^+ + a_{1j}^l\bigr)(\gamma_j + \varepsilon) - C_{11}^+(\gamma_1 + \varepsilon)\right) - \varepsilon_1 \quad \text{for } t > \bar{T}_1,$$

thus

$$\liminf_{t\to\infty} x_1(t) \geqslant u^{(1)} > 0.$$

Now if $v_i^{(1)} > 0$ for any $i = 2, \ldots, n$, pick $\varepsilon' > 0$ such that

$$\begin{aligned} & u^{(1)} - \varepsilon' > 0, \\ & B_i^m - a_{i1}^l(u - \varepsilon') + \sum_{j=2}^{n} C_{ij}^-(\gamma_j + \varepsilon') + C_{i1}^-(\gamma_1 + \varepsilon') > 0, \end{aligned}$$

for any $i = 2, \ldots, n$. There is $T_i^{(1)}$ such that

$$\begin{aligned} \frac{dx_i}{dt} \leqslant{}& B_i^m x_i(t-\tau_i) - a_{i1}^l x_i(u^{(1)} - \varepsilon') + x_i \sum_{j=2}^{n} C_{ij}^-(\gamma_j + \varepsilon') \\ & + x_i C_{i1}^-(\gamma_1 + \varepsilon') - a_{ii}^l x_i^2, \end{aligned}$$

for $t > T_i^{(1)}$. By Lemma 1 once again there is $\varepsilon'' > 0$ small enough and $\bar{T}_i^{(1)} > T_i^{(1)}$ such that

$$x_i(t) \leqslant \frac{1}{a_{ii}^l}\left(B_i^m - a_{i1}^l(u-\varepsilon') + \sum_{j=2}^n C_{ij}^-(\gamma_j+\varepsilon') + C_{i1}^-(\gamma_1+\varepsilon')\right) + \varepsilon'',$$

for $t > \bar{T}_i^{(1)}$, we infer that

$$\limsup_{t\to\infty} x_i(t) \leqslant v_i^{(1)} \quad \text{for } i = 2,\dots,n.$$

Continuing in this way until the step $(m_{i_0}-1)$, where $m_{i_0}$ is given in Claim 1, we obtain

$$\limsup_{t\to\infty} x_i(t) \leqslant v_i^{(m_{i_0}-1)} \quad \text{and} \quad \liminf_{t\to\infty} x_1(t) \geqslant u^{(m_{i_0})} \quad \text{for } i = 2,\dots,n.$$

Now since $v_{i_0}^{(m_{i_0})} < 0$, we can select $\varepsilon_{m_0} > 0$ such that

$$B_{i_0}^m - a_{i_01}^l(u^{(m_0)} - \varepsilon_{m_0}) + \sum_{j=2}^n C_{i_0j}^-\left(v_j^{(m_0-1)} + \varepsilon_{m_0}\right) + C_{i_01}^-(\gamma_1+\varepsilon_{m_0}) < 0. \tag{3.20}$$

Then there is $T_{i_0}^{(m_0)}$ such that

$$\begin{aligned}\frac{dx_{i_0}}{dt} \leqslant{}& B_{i_0}^m x_{i_0}(t-\tau_{i_0}) - a_{i_01}^l x_{i_0}(u^{(m_0)} - \varepsilon_{m_0}) + x_{i_0}\sum_{j=2}^n C_{i_0j}^-\left(v_j^{(m_0-1)} + \varepsilon_{m_0}\right)\\ &+ x_{i_0}C_{i_01}^-(\gamma_1+\varepsilon_{m_0}) - a_{i_0i_0}^l x_{i_0}^2,\end{aligned}$$

for $t > T_{i_0}^{(m_0)}$. Lemma 1(2) and (3.20) leads to

$$\lim_{t\to\infty} x_{i_0}(t) = 0.$$

**Claim 3.** $\lim_{t\to\infty} x_i(t) = 0$ for any $i = 2,\dots,n$.

By a permutation of the indices $\{2, \dots, n\}$ we can suppose that $i_0 = n$. So $\lim_{t\to\infty} x_n(t) = 0$. Define the new sequences

$$\begin{aligned}v_i'^{(m+1)} &= \frac{1}{a_{ii}^l}\left(B_i^m - a_{i1}^l u'^{(m+1)} + \sum_{j=2}^{n-1} C_{ij}^- v_j'^{(m)} + C_{i1}^-\gamma_1\right), \quad i = 2,\dots,n,\\ u'^{(m+1)} &= \frac{1}{a_{11}^m}\left(B_1^l - \sum_{j=2}^{n-1}\left(C_{1j}^+ + a_{1j}^l\right)v_j'^{(m)} - C_{11}^+\gamma_1\right).\end{aligned}$$

We can prove as in Claim 1 that the sequences $v_i'^{(m)}$ and $u'^{(m)}$ are respectively nonincreasing and nondecreasing and that there are $i_1$ $(2 \leqslant i_1 \leqslant n-1)$ and $m_{i_1} \geqslant 1$, such that $v_i'^{(m_{i_1}-1)} > 0$ for $i = 2,\dots,n-1$ and $v_{i_1}'^{(m_{i_1})} < 0$. We also prove as in Claim 2 that

$$\limsup_{t\to\infty} x_i(t) \leqslant v_i'^{(m_{i_1}-1)} \quad \text{for } i = 2,\dots,n-1$$

and

$$\liminf_{t\to\infty} x_1(t) \geqslant u'^{(m_{i_1})} > 0.$$

Now arguing as in Claim 2 we deduce that $\lim_{t\to\infty} x_{i_1}(t) = 0$. Continuing in this way, we obtain after a finite number of steps that $\lim_{t\to\infty} x_i(t) = 0$ for any $i = 2, \ldots, n$.

Now by (3.10) we obtain $\lim_{t\to\infty} y_i(t) = 0$ for $i = 2, \ldots, n$ and $\liminf_{t\to\infty} y_1(t) \geqslant b_1^l \tau_1 u^{(1)} e^{-\tau_1 d_1^m} > 0$. The proof of the theorem is complete. □

Consider the limit system of system (2.1),

$$\begin{cases} \frac{dx}{dt}(t) = B_1(t)x(t-\tau_1) - a_{11}(t)x^2(t) + x(t)\int_{-\tau_1}^0 x(t+s)\,dh_{11}^-(s) \\ \qquad\qquad - x(t)\int_{-\tau_1}^0 x(t+s)\,dh_{11}^+(s), \quad t > 0, \\ \frac{dy}{dt}(t) = b_1(t)x(t) - d_1(t)y(t) - B_1(t)x(t-\tau_1), \quad t > 0. \end{cases} \tag{3.21}$$

**Corollary 1.** *Assume that the coefficients $b_1(t), d_1(t), a_{11}(t)$ are periodic functions with the same period $\omega > 0$. Then under the hypotheses of Theorem* 2 *the solution $(x_i(t), y_i(t))$ of system* (2.1) *is such that $x_1(t) - x^*(t) \to 0$ and $y_1(t) - y^*(t) \to 0$ as $t \to \infty$ while $(x_i(t), y_i(t))$ go to extinction as $t \to \infty$ for $i = 2, \ldots, n$, where $(x^*(t), y^*(t))$, $t > 0$, is some positive solution of system* (3.21).

**Proof.** Define a sequence of functions as follows:

$$x_m(t) = x_1(t + m\omega), \quad t > 0,\ m \geqslant 1.$$

Since $\frac{dx_m(t)}{dt}$ are uniformly in $(m)$ bounded for $t > 0$, the sequence $(x_m(t))_{m\geqslant 1}$ is equicontinuous on compact sub-intervals of $(0, \infty)$. Thus by the Ascoli–Arzéla theorem, there is a sub-sequence $(x_{m_k}(t))_{k\geqslant 1}$ such that $x_{m_k}(t) \to x^*(t)$, $\frac{dx_{m_k}(t)}{dt} \to \frac{dx^*(t)}{dt}$ uniformly for $t > 0$ as $k \to \infty$. Taking the limit in system (2.1) as $k \to \infty$, by the fact that $x_i(t + m_k\omega) \to 0$ as $k \to \infty$ for $i = 2, \ldots, n$ and the periodicity of $B_1(t), a_{11}(t)$ we obtain that $x^*(t)$ is some positive solution of (3.21).

Let $\varepsilon > 0$ be given; there is $m_{k_\varepsilon} \geqslant 1$ so that $|x_1(t + m_k\omega) - x^*(t)| < \varepsilon$ for $m_k \geqslant m_{k_\varepsilon}$ and $t > 0$. Then if $t > T_\varepsilon = m_{k_\varepsilon}\omega$, we have

$$\left|x_1(t) - x^*(t)\right| < \varepsilon, \quad t > T_\varepsilon,$$

thus $x_1(t) - x^*(t) \to 0$ as $t \to \infty$.

Arguing as in the proof of Theorem 4.1 in [15] we can prove that $\lim_{t\to\infty}(y_1(t) - y^*(t)) = 0$. This completes the proof of the corollary. □

**Remark 1.** If $C_{ij}^+ = C_{ij}^- = 0$, then by (3.1), $\gamma_i = B_i^m / a_{ii}^l$. Assumption of Theorem 1 is reduced to

$$B_i^l > \sum_{j=1,\, j\neq i}^{n} a_{ij}^m \frac{B_j^m}{a_{jj}^l}, \quad i = 1, \ldots, n, \tag{3.22}$$

and those of Theorem 2 are reduced to

$$B_1^l > \sum_{j=2}^{n} a_{1j}^l \frac{B_j^m}{a_{jj}^l},$$
$$\frac{B_i^m}{a_{i1}^l} < \frac{B_1^l}{a_{11}^m}, \quad i = 2, \dots, n. \tag{3.23}$$

Under (3.22) we have that the species $(x_i(t), y_i(t))$ are permanent for $i = 1, \dots, n$ and under (3.23) we obtain the extinction of the species $(x_i(t), y_i(t))$ as $t \to \infty$ for $i = 2, \dots, n$ while $(x_1(t), y_1(t))$ gets permanent. In this sense our Theorems 1 and 2 extend Theorems 2.1 and 2.2 of [16].

**Remark 2.** If $C_{11}^- = C_{11}^+ = 0$, it is easy to check that the conditions of Theorem 8 in [20] are satisfied and by this Theorem we obtain that the limit system (3.21) has a unique positive periodic solution which is globally attractive. Thus in this case and by Corollary 1, the solution $(x_1, y_1)$ is attracted by the periodic solution of the limit system (3.21). Corollary 1 extends in this case Theorem 2.2 in [16]. Note here that conditions of Theorem 8 of [20] are not satisfied by our limit system (3.21) in general.

## 4. Global attractivity of a positive equilibrium

In this section we intend to study existence and global attractivity of a positive equilibrium for the autonomous case of system (1.7), namely

$$\begin{cases} \frac{dx_i}{dt} = b_i e^{-d_i\tau_i} x_i(t-\tau_i) - x_i(t)\sum_{j=1}^n a_{ij}x_j(t) \\ \qquad - x_i(t)\sum_{j=1}^n \int_{-\tau_j}^0 x_j(t+s)\,dh_{ij}^+(s) \\ \qquad + x_i(t)\sum_{j=1}^n \int_{-\tau_j}^0 x_j(t+s)\,dh_{ij}^-(s), \\ \frac{dy_i}{dt} = -d_i y_i(t) + b_i x_i(t) - b_i e^{-d_i\tau_i} x_i(t-\tau_i), \end{cases} \tag{4.1}$$

for $t > 0$, with the initial data (1.5). The coefficients $b_i, d_i, a_{ij}$ are assumed to be nonnegative such that $a_{ii} > 0$, $i = 1, \dots, n$. If $a_{ii} > \sum_{j=1}^n C_{ij}^-$ we can define $\gamma_i$ as in Theorem 1 by

$$\gamma_i = \frac{1}{a_{ii}}\left(B_i + \sum_{j=1}^n C_{ij}^- \gamma_j\right), \quad i = 1, \dots, n,$$

and let the matrices $A, C^+, C^-$ be defined by $A = (a_{ij})$, $C^+ = (C_{ij}^+)$, $C^- = (C_{ij}^-)$ for $1 \leqslant i, j \leqslant n$.

**Theorem 3.** *Assume that the assumptions* (H1)–(H5) *hold. Assume further that*

$$B_i > \sum_{j=1,\, j\neq i}^n a_{ij}\gamma_j + \sum_{j=1}^n C_{ij}^+\gamma_j, \quad 1 \leqslant i \leqslant n. \tag{4.2}$$

*Then system* (4.1) *has a unique positive equilibrium* $(x_i^*, y_i^*)$ *which is globally attractive, i.e.,*

$$\lim_{t\to\infty}\big(x_i(t), y_i(t)\big) = \big(x_i^*, y_i^*\big), \quad i = 1, \dots, n,$$

*where $x_i^*$ is the unique positive solution of the equation*

$$a_{ii}x_i^* = B_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}x_j^* - \sum_{j=1}^{n} C_{ij}^+ x_j^* + \sum_{j=1}^{n} C_{ij}^- x_j^*, \quad i = 1, \ldots, n. \tag{4.3}$$

**Proof.** Let $\bar{w}_i^{(0)} = \gamma_i$ and $\underline{w}_i^{(0)} = \delta_i$, where $\gamma_i$ and $\delta_i$ are as in Theorem 1. Choose $\varepsilon^{(0)} > 0$ such that

$$B_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}\big(\bar{w}_j^{(0)} + \varepsilon^{(0)}\big) - \sum_{j=1}^{n} C_{ij}^+\big(\bar{w}_j^{(0)} + \varepsilon^{(0)}\big) > 0, \quad i = 1, \ldots, n. \tag{4.4}$$

This is possible by assumption (4.2). By Theorem 1 there exists $T_i^{(0)} > 0$ such that

$$\underline{w}_i^{(0)} - \varepsilon^{(0)} \leqslant x_i(t) \leqslant \bar{w}_i^{(0)} + \varepsilon^{(0)},$$

for $t > T_i^{(0)}$. These relations allow us to derive that

$$\begin{aligned} \frac{dx_i}{dt} \geqslant{}& B_i x_i(t-\tau_i) - a_{ii}x_i^2 - x_i \sum_{j=1,\, j\neq i}^{n} a_{ij}\big(\bar{w}_j^{(0)} + \varepsilon^{(0)}\big) \\ & - x_i \sum_{j=1}^{n} C_{ij}^+\big(\bar{w}_j^{(0)} + \varepsilon^{(0)}\big) + x_i \sum_{j=1}^{n} C_{ij}^-\big(\underline{w}_j^{(0)} - \varepsilon^{(0)}\big) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \frac{dx_i}{dt} \leqslant{}& B_i x_i(t-\tau_i) - a_{ii}x_i^2 - x_i \sum_{j=1,\, j\neq i}^{n} a_{ij}\big(\underline{w}_j^{(0)} - \varepsilon^{(0)}\big) \\ & - x_i \sum_{j=1}^{n} C_{ij}^+\big(\underline{w}_j^{(0)} - \varepsilon^{(0)}\big) + x_i \sum_{j=1}^{n} C_{ij}^-\big(\bar{w}_j^{(0)} + \varepsilon^{(0)}\big), \end{aligned} \tag{4.6}$$

for $t > T^{(0)} = \max_{1\leqslant i\leqslant n}(T_i^{(0)}) + \tau$. Let $\varepsilon^{(1)} > 0$ be small. Using Lemma 1 we deduce from (4.4)–(4.6) that there is $T_i^{(1)} > T^{(0)}$ such that

$$\begin{aligned} x_i \geqslant{}& \Bigg(B_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}\big(\bar{w}_j^{(0)} + \varepsilon^{(0)}\big) - \sum_{j=1}^{n} C_{ij}^+\big(\bar{w}_j^{(0)} + \varepsilon^{(0)}\big) \\ & + \sum_{j=1}^{n} C_{ij}^-\big(\underline{w}_j^{(0)} - \varepsilon^{(0)}\big)\Bigg)\Big/ a_{ii} - \varepsilon^{(1)} \end{aligned}$$

and

$$\begin{aligned} x_i \leqslant{}& \Bigg(B_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}\big(\underline{w}_j^{(0)} - \varepsilon^{(0)}\big) - \sum_{j=1}^{n} C_{ij}^+\big(\underline{w}_j^{(0)} - \varepsilon^{(0)}\big) \\ & + \sum_{j=1}^{n} C_{ij}^-\big(\bar{w}_j^{(0)} + \varepsilon^{(0)}\big)\Bigg)\Big/ a_{ii} + \varepsilon^{(1)}, \end{aligned}$$

for $t > T_i^{(1)}$. Taking the limit as $t \to \infty$ and since $\varepsilon^{(0)}, \varepsilon^{(1)}$ are arbitrarily small we obtain

$$\underline{w}_i^{(1)} \leqslant \liminf_{t\to\infty} x_i(t) \leqslant \limsup_{t\to\infty} x_i(t) \leqslant \bar{w}_i^{(1)},$$

where

$$\underline{w}_i^{(1)} = \frac{1}{a_{ii}}\left(B_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}\bar{w}_j^{(0)} - \sum_{j=1}^{n} C_{ij}^{+}\bar{w}_j^{(0)} + \sum_{j=1}^{n} C_{ij}^{-}\underline{w}_j^{(0)}\right) > 0,$$

$$\bar{w}_i^{(1)} = \frac{1}{a_{ii}}\left(B_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}\underline{w}_j^{(0)} - \sum_{j=1}^{n} C_{ij}^{+}\underline{w}_j^{(0)} + \sum_{j=1}^{n} C_{ij}^{-}\bar{w}_j^{(0)}\right) > 0,$$

for $i = 1, \ldots, n$. Furthermore, from the definition of $\bar{w}_i^{(1)}, \underline{w}_i^{(1)}, \gamma_i$ and $\delta_i$ we may write

$$\bar{w}_i^{(1)} - \bar{w}_i^{(0)} = \frac{1}{a_{ii}}\left(-\sum_{j=1,\, j\neq i}^{n} a_{ij}\delta_j - \sum_{j=1}^{n} C_{ij}^{+}\delta_j\right) \leqslant 0, \tag{4.7}$$

$$\underline{w}_i^{(1)} - \underline{w}_i^{(0)} = \frac{1}{a_{ii}}\sum_{j=1}^{n} C_{ij}^{-}\delta_j \geqslant 0. \tag{4.8}$$

Repeating the above process $k$-times we obtain two sequences $\underline{w}_i^{(k)}, \bar{w}_i^{(k)}$ such that

$$\underline{w}_i^{(k)} \leqslant \liminf_{t\to\infty} x_i(t) \leqslant \limsup_{t\to\infty} x_i(t) \leqslant \bar{w}_i^{(k)}, \tag{4.9}$$

where

$$\bar{w}_i^{(k)} = \frac{1}{a_{ii}}\left(B_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}\underline{w}_j^{(k-1)} - \sum_{j=1}^{n} C_{ij}^{+}\underline{w}_j^{(k-1)} + \sum_{j=1}^{n} C_{ij}^{-}\bar{w}_j^{(k-1)}\right),$$

$$\underline{w}_i^{(k)} = \frac{1}{a_{ii}}\left(B_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}\bar{w}_j^{(k-1)} - \sum_{j=1}^{n} C_{ij}^{+}\bar{w}_j^{(k-1)} + \sum_{j=1}^{n} C_{ij}^{-}\underline{w}_j^{(k-1)}\right), \tag{4.10}$$

for $i = 1, \ldots, n$. Further we have

$$\begin{aligned}\bar{w}_i^{(k)} - \bar{w}_i^{(k-1)} = &-\frac{1}{a_{ii}}\left(\sum_{j=1,\, j\neq i}^{n} a_{ij}\big(\underline{w}_j^{(k-1)} - \underline{w}_j^{(k-2)}\big) + \sum_{j=1}^{n} C_{ij}^{+}\big(\underline{w}_j^{(k-1)} - \underline{w}_j^{(k-2)}\big)\right)\\ &+ \frac{1}{a_{ii}}\left(\sum_{j=1}^{n} C_{ij}^{-}\big(\bar{w}_j^{(k-1)} - \bar{w}_j^{(k-2)}\big)\right)\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\underline{w}_i^{(k)} - \underline{w}_i^{(k-1)} = &-\frac{1}{a_{ii}}\left(\sum_{j=1,\, j\neq i}^{n} a_{ij}\big(\bar{w}_j^{(k-1)} - \bar{w}_j^{(k-2)}\big) + \sum_{j=1}^{n} C_{ij}^{+}\big(\bar{w}_j^{(k-1)} - \bar{w}_j^{(k-2)}\big)\right)\\ &+ \frac{1}{a_{ii}}\left(\sum_{j=1}^{n} C_{ij}^{-}\big(\underline{w}_j^{(k-1)} - \underline{w}_j^{(k-2)}\big)\right).\end{aligned} \tag{4.12}$$

The inequalities (4.7), (4.8), (4.11), (4.12) and the induction process allow us to conclude that the sequences $\bar{w}_i^{(k)}$ and $\underline{w}_i^{(k)}$ are respectively decreasing and increasing, so from (4.9) we deduce that these sequences are convergent. Put $\alpha_i = \lim_{k\to\infty} \underline{w}_i^{(k)}$ and $\beta_i = \lim_{k\to\infty} \bar{w}_i^{(k)}$ then from (4.9),

$$\alpha_i \leqslant \liminf_{t\to\infty} x_i(t) \leqslant \limsup_{t\to\infty} x_i(t) \leqslant \beta_i,$$

where $\alpha_i$ and $\beta_i$ are given by

$$\beta_i = \frac{1}{a_{ii}}\left(B_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}\alpha_j - \sum_{j=1}^{n} C_{ij}^{+}\alpha_j + \sum_{j=1}^{n} C_{ij}^{-}\beta_j\right),$$
$$\alpha_i = \frac{1}{a_{ii}}\left(B_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}\beta_j - \sum_{j=1}^{n} C_{ij}^{+}\beta_j + \sum_{j=1}^{n} C_{ij}^{-}\alpha_j\right).$$

If we put $w_i = \beta_i - \alpha_i$ we obtain that

$$w_i = \frac{1}{a_{ii}}\left(\sum_{j=1,\, j\neq i}^{n} a_{ij}w_j + \sum_{j=1}^{n} C_{ij}^{+}w_j + \sum_{j=1}^{n} C_{ij}^{-}w_j\right).$$

Define the matrix $M = (M_{ij})$ by

$$M_{ij} = \begin{cases} a_{ii} - C_{ii}^{+} - C_{ii}^{-}, & i = j, \\ -a_{ij} - C_{ij}^{+} - C_{ij}^{-}, & i \neq j, \end{cases}$$

then

$$Mw = 0,$$

where $w = (w_1, \ldots, w_n)^T$. Using (3.6),

$$a_{ii}\gamma_i - \sum_{i=1}^{n} C_{ij}^{-}\gamma_j = B_i,$$

which with the relation (4.2) yields

$$\sum_{j=1}^{n} M_{ij}\gamma_j = a_{ii}\gamma_i - \sum_{j=1}^{n} C_{ij}^{-}\gamma_j - \sum_{j=1}^{n} C_{ij}^{+}\gamma_j - \sum_{j=1,\, j\neq i}^{n} a_{ij}\gamma_j$$
$$= B_i - \sum_{j=1}^{n} C_{ij}^{+}\gamma_j - \sum_{j=1,\, j\neq i}^{n} a_{ij}\gamma_j > 0. \tag{4.13}$$

Now since $M_{ij} \leqslant 0$ for $i \neq j$ and $\gamma_i > 0$ $(i = 1, \ldots, n)$ we conclude by (4.13) that $M$ is an $M$-matrix (see [6, Proposition 3.6.13, p. 228]). Consequently, $\det M > 0$ and then $w = 0$. This means that $\beta_i = \alpha_i$ for $i = 1, \ldots, n$.

To prove the uniqueness of $x_i^*$ it suffices to prove that the matrix $A + C^{+} - C^{-}$ is nonsingular. To this end note that (4.13) imply

$$\left(a_{ii} - C_{ii}^{+} - C_{ii}^{-}\right)\gamma_i > \sum_{j=1,\, j\neq i}^{n} \left(a_{ij} + C_{ij}^{+} + C_{ij}^{-}\right)\gamma_j$$

and hence

$$\big(a_{ii}+C_{ii}^{+}-C_{ii}^{-}\big)\gamma_i > \big(a_{ii}-C_{ii}^{+}-C_{ii}^{-}\big)\gamma_i > \sum_{j=1,\, j\neq i}^{n}\big(a_{ij}+C_{ij}^{+}+C_{ij}^{-}\big)\gamma_j$$
$$> \sum_{j=1,\, j\neq i}^{n}\big|a_{ij}+C_{ij}^{+}-C_{ij}^{-}\big|\gamma_j,$$

$(A+C^{+}-C^{-})$ is then a diagonally dominant matrix, consequently $A+C^{+}-C^{-}$ is nonsingular.

We now prove that $y_i(t)$ also has a positive equilibrium. Noticing from the second equation of (4.1) that

$$\lim_{t\to\infty} f_i(t) = \lim_{t\to\infty}\big(b_i x_i(t) - b_i e^{-d_i\tau_i} x_i(t-\tau_i)\big) = b_i x_i^* - b_i e^{-d_i\tau_i} x_i^*$$
$$= b_i x_i^*(1-e^{-d_i\tau_i}) > 0,$$

and by the well-known theory of ODE there is $y_i^* > 0$ such that

$$\lim_{t\to\infty} y_i(t) = y_i^*,$$

for $i = 1,\dots,n$. □

**Remark 3.** If $C_{ij}^{-} = C_{ij}^{+} = 0$ then system (4.1) is reduced to the well-known autonomous stage-structured system in [15]. The assumption of Theorem 3 is reduced to

$$b_i e^{-d_i\tau_i} > \sum_{j=1,\, j\neq i}^{n} a_{ij}\frac{b_j e^{-d_j\tau_j}}{a_{jj}},$$

which is the well-known hypothesis stated in [15]. We extend Theorem 2.2 of [15].

## 5. Discussion

In this paper, we extend the model in Liu et al. [16] to the case of a nonautonomous multispecies competitive stage-structured system with distributed delays. Biologically, this case would embody delayed feedback (rather than instantaneous feedback) of the competition among the mature species from their past life history. Using the method of repeated replace we obtained sufficient conditions for their permanence and extinction. These results extend those in [16] to the case of distributed delays. We also proved that under some assumptions similar to those in [16, Theorem 2.2 ] the last $(n-1)$ species go to extinction while the first species converge to some positive solution of the limit system. This result is weaker than the one in [16] but it is obtained under weaker conditions. Then we considered the autonomous case and we build sufficient conditions for the global attractivity of the positive equilibrium, which directly extends the analogous one in [15].

The sufficient conditions of permanence in nonautonomous case, i.e., Theorem 1 require that the maximum effect of the distributed delays $(C_{ij}^{+}+C_{ij}^{-})$ $(i,j=1,\dots,n)$ be small and that the below boundary of intraspecific competition coefficients $a_{ii}^l$ $(i = 1, \ldots, n)$ be large compared with the upper boundary of the interspecific competition coefficients $a_{ij}^m$ $(j = 1, \ldots, n$, and $j \neq i)$. For the global attractivity of positive equilibrium in autonomous case, sufficient conditions of Theorem 3 require that the maximum effect of the distributed delays $(C_{ij}^+ + C_{ij}^-)$, $i, j = 1, \ldots, n$, be small and that the intraspecific competition coefficients $a_{ii}$ $(i = 1, \ldots, n)$ be large compared with the interspecific competition coefficients $a_{ij}$ $(j = 1, \ldots, n$, and $j \neq i)$. Our conclusions are very similar to those for two species Lotka–Volterra delay differential equation by Gopalsamy and He [23].

Although much has been done in this domain some questions remain unsolved. In particular we mention the following two questions: Is the solution of our $n$-species competitive nonautonomous stage-structured system with distributed delays globally attractive? Does stage-structure affect the dynamics of the system?

Recently, Xu and Zhao [21] considered an asymptotically periodic competitive model with stage-structure, which extended those stage-structured systems in Liu et al. [15,16] but again they ignored the delayed feedback in competition among the mature species. By appealing to the theory of autonomous and nonautonomous semiflows (see [16,18]), they established sufficient conditions for the existence of periodic solutions, coexistence, global persistence and extinction in terms of spectral radii of Poincaré maps associated with linear periodic delay equations. The methods in [21] gives us an insight on the kind of problems we are treating. This will be the subject of one of our future work.

## Acknowledgments

The first author is supported by Academy of Finland and the Chinese Postdoctoral Science Foundation. The third author thanks King Fahd University of Petroleum and Minerals for its financial support.

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