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Ultimate boundedness and periodicity for some partial functional differential equations with infinite delay ☆

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Abstract

In this work we study the existence of periodic solutions for some partial functional differential equation with infinite delay. We assume that the linear part is not necessarily densely defined and satisfies the known Hille–Yosida condition. Firstly, we give some estimates of the solutions. Secondly, we prove that the Poincaré map is condensing which allows us to prove the existence of periodic solutions when the solutions are ultimately bounded.

Keywords: Hille–Yosida condition; Integral solutions; Semigroup; Uniform fading memory space; Ultimate boundedness; Condensing map; Hale and Lunel's fixed point theorem; Periodic solution

1. Introduction

The aim here is to investigate the existence of a periodic solution for the following partial functional differential equation with infinite delay:

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) + F(t, u_t) \quad \text{for } t \geqslant 0, \\ u_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.1}$$

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where $A : D(A) \subset X \to X$ is not necessarily densely defined linear operator on a Banach space X, here we assume that A satisfies the Hille–Yosida condition, which means that there exist $\bar{M} \geqslant 1$, $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\left|R(\lambda, A)^n\right| \leqslant \frac{\bar{M}}{(\lambda - \omega)^n} \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$. The phase space $\mathcal{B}$ is the space of functions mapping $(-\infty, 0]$ to X and satisfying some axioms which will be described below. For every $t \geqslant 0$, the history function $u_t \in \mathcal{B}$ is defined by

$$u_t(\theta) = u(t + \theta) \quad \text{for } \theta \in (-\infty, 0].$$

F is a continuous function from $\mathbb{R} \times \mathcal{B}$ into X and ω-periodic in t.

For functional differential equations with finite delay, the phase spaces are L^p-spaces, for $1 \leqslant p < \infty$, or the space of continuous functions from $[-r, 0]$ to X, for more details about this topics we refer to [3,15,23]. When the delay is infinite, the selection of the phase space $\mathcal{B}$ plays a crucial role to study the quantitative behavior of solutions. A large variety of phase spaces $\mathcal{B}$ has been used in the theory of functional differential equation with infinite delay. Usual choice is normed spaces $\mathcal{B}$ satisfying suitable axioms which have been introduced by Hale and Kato [14]. Partial functional differential equations with infinite delay has been the subject of many works, we refer to [1,2,18] and the references therein. In [1,2], the authors established the existence, regularity and stability of solutions of Eq. (1.1) where A is not necessarily densely defined and satisfies the Hille–Yosida condition. In [18], the author proved the existence and regularity of solutions of Eq. (1.1) where A is the infinitesimal generator of analytic semigroup on X.

Periodic solutions of ordinary and partial functional differential equations are of great interest in the qualitative analysis of that kind of equations. There is an extensive literature related to this topics, for instance, we refer to [6–9,11–13,16,17,20–22]. Fixed point theorems is a powerful tool to investigate this problem. The standard approach to prove the existence of periodic solutions is to consider the Poincaré map $\mathcal{P}$ which is defined by

$$\mathcal{P}\varphi = u_\omega(., \varphi),$$

where $u(., \varphi)$ is the solution of Eq. (1.1). Since the existence of a periodic solution is equivalent to the existence of a fixed point of $\mathcal{P}$. In [11], the authors proved the existence of periodic solutions for partial functional differential equations with finite delay, when the solutions are bounded and ultimate bounded, they proved the existence of a periodic solution by using Horn's fixed point theorem, which requires the compactness of the Poincaré map. In [6], the authors investigated the existence of periodic solutions for nonhomogeneous partial functional differential equations with infinite delay, they proved that the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of periodic solutions, when the phase space is uniform fading memory space. In [20], the author discussed the existence of periodic solutions for nonautonomous partial functional differential equations with infinite delay, it has been proved that the Poincaré map is condensing on C_g, which allows to prove the existence of fixed point of the Poincaré map by using Sadovskii's fixed point theorem. The present work is a continuation of papers [12,13,20,21], we use the ultimate boundedness to prove the existence of a periodic solution of Eq. (1.1) when $\mathcal{B}$ is a uniform fading memory space.

The work is organized as follows, in Section 2 we recall the axioms and properties about the phase space $\mathcal{B}$ and some results on the spectral analysis of linear operators which will be used in the whole of this work. In Section 3 we give some definitions and results about the solutions of

Eq. (1.1). In Section 4 we prove the existence of a periodic solution by using Hale and Lunel's fixed point theorem when the solutions are ultimate bounded. Finally, we propose to study the existence of periodic solution for some nonlinear partial differential equations arising in physical systems.

2. Preliminary results

Throughout this work, we assume that $\mathcal{B}$ is a normed linear space consisting of functions mapping $(-\infty, 0]$ into X endowed with a norm $|\cdot|_{\mathcal{B}}$, and satisfies the following axioms which have been introduced at first by Hale and Kato [14]:

(A) There exist a positive constant H and functions $K(\cdot), M(\cdot) : [0, +\infty) \to [0, +\infty)$, with K is continuous and M is locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x : (-\infty, \sigma + a] \to X$, $x_\sigma \in \mathcal{B}$, and $x(\cdot)$ is continuous on $[\sigma, \sigma + a]$, then for all t in $[\sigma, \sigma + a]$, the following conditions hold:
   (i) $x_t \in \mathcal{B}$,
   (ii) $|x(t)| \leqslant H|x_t|_{\mathcal{B}}$,
   (iii) $|x_t|_{\mathcal{B}} \leqslant K(t-\sigma)\sup_{\sigma \leqslant s \leqslant t} |x(s)| + M(t-\sigma)|x_\sigma|_{\mathcal{B}}$,
   (iv) $t \to x_t$ is a $\mathcal{B}$-valued continuous function for t in $[\sigma, \sigma + a]$.

(B) The space $\mathcal{B}$ is complete.

Let $\mathcal{C}_{00}$ be the space of continuous functions mapping $(-\infty, 0]$ into X with compact supports. We assume that $\mathcal{B}$ satisfies:

(C) If a uniformly bounded sequence $(\phi_n)_{n \geqslant 0}$ in $\mathcal{C}_{00}$ converges compactly to ϕ on $(-\infty, 0]$, then ϕ is in $\mathcal{B}$ and $|\phi_n - \phi|_{\mathcal{B}} \to 0$.

For $\phi \in \mathcal{B}$, $t \geqslant 0$ and $\theta \leqslant 0$, we define the linear operator $W(t)$ by

$$[W(t)\phi](\theta) = \begin{cases} \phi(0), & \text{if } t + \theta \geqslant 0, \\ \phi(t + \theta), & \text{if } t + \theta < 0. \end{cases}$$

$(W(t))_{t \geqslant 0}$ is exactly the solution semigroup associated to the following equation

$$\begin{cases} \frac{d}{dt}u(t) = 0, \\ u_0 = \varphi. \end{cases}$$

Let

$$W_0(t) = W(t)/\mathcal{B}_0, \quad \text{where } \mathcal{B}_0 := \{\phi \in \mathcal{B}\colon \phi(0) = 0\}.$$

Let $\mathcal{BC}$ be the space of bounded continuous functions mapping $(-\infty, 0]$ into X, provided with the uniform norm topology.

Proposition 2.1. [19, Proposition 1.5, p. 190] *If $\mathcal{B}$ satisfies axiom* (C)*, then $BC \subset \mathcal{B}$ and there exists a positive constant J such that $|\phi|_{\mathcal{B}} \leqslant J|\phi|_{BC}$. Moreover,*

$$|x_t|_{\mathcal{B}} \leqslant J \sup_{\sigma \leqslant s \leqslant t} \bigl|x(s)\bigr| + (1 + JH)\bigl|W_0(t-\sigma)\bigr||x_\sigma|_{\mathcal{B}}, \quad \sigma > 0,$$

for any function x satisfying axiom (A).

Definition 2.2. $\mathcal{B}$ is called a uniform fading memory space if it satisfies axioms (A)–(C) and $|W_0(t)| \to 0$ as $t \to +\infty$.

As a consequence of Proposition 2.1, the functions $K(.)$ and $M(.)$ can be chosen as $K(t) = J$ and $M(t) = (1 + JH)|W_0(t)|$. Moreover, if $\mathcal{B}$ is a uniform fading memory space, then $M(t) \to 0$ as $t \to +\infty$.

In the next, we introduce the Kuratowski's measure of noncompactness, $\alpha(.)$ of bounded sets K on a Banach space Y which is defined by

$$\alpha(K) = \inf\{\epsilon > 0\colon\ K \text{ has a finite cover of ball of diameter} < \epsilon\}.$$

Some basic properties of $\alpha(.)$ are given in the following lemma.

Lemma 2.3. *Let A_1 and A_2 be bounded sets of a bounded Banach space Y. Then*

(i) $\alpha(A_1) \leqslant dia(A_1)$, *where* $dia(A_1) = \sup_{x,y \in A_1} |x - y|$,
(ii) $\alpha(A_1) = 0$ *if and only if A_1 is relatively compact in Y,*
(iii) $\alpha(A_1 \cup A_2) = \max\{\alpha(A_1), \alpha(A_2)\}$,
(iv) *if* $A_1 \subseteq A_2$, *then* $\alpha(A_1) \leqslant \alpha(A_2)$,
(v) $\alpha(A_1 + A_2) \leqslant \alpha(A_1) + \alpha(A_2)$.

Let $\mathcal{K}: Y \to Y$ be a closed linear operator with a dense domain $D(\mathcal{K})$ in a Banach space Y. We denote by $\sigma(\mathcal{K})$ the spectrum of $\mathcal{K}$.

Definition 2.4. [23] The essential spectrum $\sigma_{\text{ess}}(\mathcal{K})$ of $\mathcal{K}$ is the set of all $\lambda \in \mathbb{C}$ such that at least one of the following holds:

(i) the range $\operatorname{Im}(\lambda I - \mathcal{K})$ is not closed,
(ii) the generalized eigenspace $M_\lambda(\mathcal{K}) = \bigcup_{n \geqslant 1} \ker(\lambda I - \mathcal{K})^n$ of λ is infinite dimensional,
(iii) λ is a limit point of $\sigma(\mathcal{K})$, that is $\lambda \in \overline{\sigma(\mathcal{K}) \setminus \{\lambda\}}$.

Theorem 2.5. [10] *Let $\lambda \in \sigma(\mathcal{K}) \setminus \sigma_{\text{ess}}(\mathcal{K})$. Then, λ is a pole of $R(\lambda, \mathcal{K})$ and the residue is an operator of finite rank. In particular, λ is an eigenvalue of finite algebraic multiplicity.*

For a bounded linear operator $\mathcal{K}$ on Y, we define the Kuratowski measure of noncompactness of $\mathcal{K}$ by

$$|\mathcal{K}|_\alpha = \inf\bigl\{\epsilon > 0\colon\ \alpha\bigl(\mathcal{K}(\Omega)\bigr) \leqslant \epsilon\alpha(\Omega) \text{ for every bounded subset } \Omega \text{ of } Y\bigr\}.$$

Let $(\mathcal{T}(t))_{t \geqslant 0}$ be a strongly continuous semigroup on Y and $\mathcal{A}_{\mathcal{T}}$ its infinitesimal generator.

Definition 2.6. The growth bound of $(\mathcal{T}(t))_{t \geqslant 0}$ is the real number $\omega_0(\mathcal{T})$ defined by

$$\omega_0(\mathcal{T}) := \inf\bigl\{\omega \in \mathbb{R}\colon\ \text{there exists a constant } M \geqslant 1 \text{ such that } \bigl|\mathcal{T}(t)\bigr| \leqslant Me^{\omega t}\bigr\}.$$

Definition 2.7. The essential growth bound $\omega_{\text{ess}}(\mathcal{T})$ of $(\mathcal{T}(t))_{t \geqslant 0}$ is defined by

$$\omega_{\text{ess}}(\mathcal{T}) := \lim_{t \to +\infty} \frac{1}{t} \log\bigl|\mathcal{T}(t)\bigr|_\alpha = \inf_{t > 0} \frac{1}{t} \log\bigl|\mathcal{T}(t)\bigr|_\alpha.$$

Set

$$s'(\mathcal{A}_{\mathcal{T}}) := \sup\{\mathrm{Re}(\lambda):\ \lambda \in \sigma(\mathcal{A}_{\mathcal{T}}) - \sigma_{\mathrm{ess}}(\mathcal{A}_{\mathcal{T}})\}.$$

The following lemma gives the relationship between the growth bound and the essential growth bound.

Lemma 2.8. [10, Corollary 5.2.11, p. 258]

$$\omega_0(\mathcal{T}) = \max\big(\omega_{\mathrm{ess}}(\mathcal{T}), s'(\mathcal{A}_{\mathcal{T}})\big).$$

Definition 2.9. [15] A continuous mapping $P : Y \to Y$ is said to be an α-contraction if P maps bounded sets into bounded sets and if there exists a constant $k \in (0, 1)$ such that

$$\alpha\big(P(\Omega)\big) \leqslant k\alpha(\Omega)$$

for every bounded subset Ω of Y.

Definition 2.10. [15] A continuous mapping $P : Y \to Y$ is condensing on Y if P maps bounded sets into bounded sets and

$$\alpha\big(P(\Omega)\big) < \alpha(\Omega)$$

for every bounded subset Ω of Y such that $\alpha(\Omega) > 0$.

3. Existence and estimation of solutions

Throughout this work, we suppose that:

(H_0) A satisfies the Hille–Yosida condition.

The following results are taken from [1,2].

Definition 3.1. A function $u : (-\infty, T] \to X$ with $T > 0$, is said to be an integral solution of Eq. (1.1) if the following conditions hold:

(i) $u : [0, T] \to X$ is continuous,
(ii) $\int_0^t u(s)\,ds \in D(A)$ for $t \in [0, T]$,
(iii) $u(t) = \varphi(0) + A\int_0^t u(s)\,ds + \int_0^t F(s, u_s)\,ds$ for $t \in [0, T]$,
(iv) $u_0 = \varphi$.

Remark 3.2. From the closedness property of A, one can see that if u is an integral solution of Eq. (1.1), then $u(t) \in \overline{D(A)}$ for all $t \in [0, T]$. In particular, $\varphi(0) \in \overline{D(A)}$. It has been proved in [1] and [2], that the condition $\varphi(0) \in \overline{D(A)}$ is enough for the existence of the integral solutions of Eq. (1.1).

Let A_0 be the part of the operator A in $\overline{D(A)}$ which is defined by

$$\begin{cases} D(A_0) = \{x \in D(A):\ Ax \in \overline{D(A)}\}, \\ \mathcal{A}_0 x = Ax \quad \text{for } x \in D(A_0). \end{cases}$$

Lemma 3.3. [5, Lemma 3.3.12, p. 140] *A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geqslant 0}$ on $\overline{D(A)}$.*

For the existence of the integral solutions, we suppose that:

(H_1) F is continuous and Lipschitzian with respect to the second argument, there exists a positive constant μ such that

$$\left|F(t,\phi)-F(t,\psi)\right|\leqslant \mu|\phi-\psi|_{\mathcal{B}} \quad \text{for } \phi,\psi\in\mathcal{B} \text{ and } t\geqslant 0.$$

Theorem 3.4. [1, Theorem 19] *Assume that* (H_0) *and* (H_1) *hold. Then, for any $\varphi\in\mathcal{B}$ such that $\varphi(0)\in\overline{D(A)}$, Eq.* (1.1) *has a unique integral solution u on* $(-\infty,+\infty)$. *Moreover, u is given by*

$$u(t)=T_0(t)\varphi(0)+\lim_{\lambda\to+\infty}\int_0^t T_0(t-s)\lambda R(\lambda,A)F(s,u_s)\,ds \quad \textit{for } t\geqslant 0.$$

In the whole of this work, the integral solutions of Eq. (1.1) will be called solutions.
The phase space $\mathcal{B}_0$ of Eq. (1.1) is defined by

$$\mathcal{B}_0=\left\{\varphi\in\mathcal{B}\colon\ \varphi(0)\in\overline{D(A)}\right\}.$$

For each $t\geqslant 0$, we define the linear operator $\mathcal{U}(t)$ on $\mathcal{B}_0$ by

$$\mathcal{U}(t)\varphi=x_t(.,\varphi),$$

where $x(.,\varphi)$ is the solution of the following equation:

$$\begin{cases}\frac{d}{dt}u(t)=Au(t) & \text{for } t\geqslant 0,\\ u_0=\varphi.\end{cases} \tag{3.1}$$

Proposition 3.5. [6] *$(\mathcal{U}(t))_{t\geqslant 0}$ is a linear strongly continuous semigroup on $\mathcal{B}_0$, that is*:

(i) *for all $t\geqslant 0$, $\mathcal{U}(t)$ is a bounded linear operator on $\mathcal{B}_0$,*
(ii) *$\mathcal{U}(0)=I$,*
(iii) *$\mathcal{U}(t+s)=\mathcal{U}(t)\mathcal{U}(s)$ for all $t,s\geqslant 0$,*
(iv) *for all $\varphi\in\mathcal{B}_0$, $\mathcal{U}(t)\varphi$ is a continuous function of $t\geqslant 0$ with values in $\mathcal{B}_0$,*
(v) *$(\mathcal{U}(t))_{t\geqslant 0}$ satisfies for $t\geqslant 0$ and $\theta\in(-\infty,0]$ the translation property*:

$$\left[\mathcal{U}(t)\phi\right](\theta)=\begin{cases}[\mathcal{U}(t+\theta)\phi](0), & \text{if } t+\theta\geqslant 0,\\ \phi(t+\theta), & \text{if } t+\theta<0.\end{cases}$$

Without loss of generality, we assume that:

(H_2) $(T_0(t))_{t\geqslant 0}$ is exponentially stable, which means that there exist $\alpha_0>0$ and $M_0\geqslant 1$ such that

$$\left|T_0(t)\right|\leqslant M_0e^{-\alpha_0 t} \quad \text{for } t\geqslant 0.$$

Otherwise, we can replace A by $A-\delta I$, where $\delta>0$ is chosen such that the semigroup generated by the part of $A-\delta I$ in $\overline{D(A)}$ is exponentially stable.
In the following, we suppose that:

(H_3) $T_0(t)$ is compact on $\overline{D(A)}$, whenever $t > 0$.

The following fundamental lemma plays an important role for the existence of periodic solutions.

Proposition 3.6. *Assume that* (H_0), (H_2) *and* (H_3) *hold. If $\mathcal{B}$ is a uniform fading memory space, then $(\mathcal{U}(t))_{t\geqslant 0}$ is an exponentially stable semigroup on $\mathcal{B}_0$, that is there exist $\eta > 0$ and $\widetilde{M} \geqslant 1$ such that*:

$$\left|\mathcal{U}(t)\right| \leqslant \widetilde{M} e^{-\eta t} \quad \textit{for } t \geqslant 0.$$

For the proof, we need the following fundamental lemma.

Lemma 3.7. [6] *Assume that* (H_0) *and* (H_3) *hold. If $\mathcal{B}$ is a uniform fading memory space. Then, for any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$\left|\mathcal{U}(t)\right|_\alpha \leqslant C_\varepsilon M(t-\varepsilon) \quad \textit{for } t > \varepsilon.$$

Proof of Proposition 3.6. Since $\mathcal{B}$ is a uniform fading memory space, then the function $M(.)$ can be chosen such that $M(t) \to 0$ as $t \to +\infty$. Let $\varepsilon > 0$ and $t_0 > 0$ such that $C_\varepsilon M(t_0 - \varepsilon) < 1$. Then, by Lemma 3.7, we have

$$\omega_{\mathrm{ess}}(\mathcal{U}) \leqslant \frac{1}{t_0} \log\left|\mathcal{U}(t_0)\right|_\alpha < 0.$$

Let $\lambda \in \sigma(\mathcal{A}_\mathcal{U}) - \sigma_{\mathrm{ess}}(\mathcal{A}_\mathcal{U})$. By Theorem 2.5, $\lambda \in \sigma_p(\mathcal{A}_\mathcal{U})$ and there exists $\phi \in D(\mathcal{A}_\mathcal{U})$, $\phi \neq 0$ such that $\mathcal{A}_\mathcal{U}\phi = \lambda\phi$, which implies that

$$\mathcal{U}(t)\phi = e^{\lambda t}\phi \quad \text{for } t \geqslant 0.$$

Let $t \geqslant 0$ and $\theta \leqslant 0$ such that $t + \theta \geqslant 0$. By the translation property of $(\mathcal{U}(t))_{t\geqslant 0}$, we get that

$$e^{\lambda t}\phi(\theta) = \big(\mathcal{U}(t)\phi\big)(\theta) = \big(\mathcal{U}(t+\theta)\phi\big)(0) = e^{\lambda(t+\theta)}\phi(0).$$

Then it follows

$$\phi(\theta) = e^{\lambda\theta}\phi(0) \quad \text{with } \phi(0) \neq 0,$$

consequently

$$T_0(t)\phi(0) = e^{\lambda t}\phi(0) \quad \text{for } t \geqslant 0 \text{ and } \phi(0) \neq 0.$$

Assumption (H_2) gives that

$$e^{\mathrm{Re}(\lambda)t} \leqslant M_0 e^{-\alpha_0 t} \quad \text{for } t \geqslant 0,$$

we deduce that

$$\mathrm{Re}(\lambda) \leqslant -\alpha_0,$$

and

$$s'(\mathcal{A}_\mathcal{U}) < 0.$$

By Lemma 2.8 we get $\omega_0 < 0$, and we conclude that the semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ is exponentially stable. □

For any $\varphi \in \mathcal{B}_0$, we introduce the new norm on $\mathcal{B}_0$ by

$$|\varphi|_\eta = \sup_{t \geqslant 0} e^{\eta t} \big|\mathcal{U}(t)\varphi\big|_{\mathcal{B}},$$

where η is the positive constant obtained in Proposition 3.6. Clearly,

$$|\varphi|_{\mathcal{B}} \leqslant |\varphi|_\eta \leqslant \widetilde{M}|\varphi|_{\mathcal{B}},$$

which implies that $|.|_\eta$ and $|.|_{\mathcal{B}}$ are equivalent norms in $\mathcal{B}_0$.

As an consequence, we obtain the following corollary.

Corollary 3.8. *Assume that* (H_0)*,* (H_2) *and* (H_3) *hold. Then,*

$$\big|\mathcal{U}(t)\big|_\eta \leqslant e^{-\eta t} \quad \textit{for } t \geqslant 0.$$

Proof. Since for every $t \geqslant 0$, we have

$$\begin{aligned}\big|\mathcal{U}(t)\varphi\big|_\eta &= \sup_{s \geqslant 0} e^{\eta s} \big|\mathcal{U}(s)\mathcal{U}(t)\varphi\big|_{\mathcal{B}} = e^{-\eta t} \sup_{s \geqslant 0} e^{\eta(t+s)} \big|\mathcal{U}(s+t)\varphi\big|_{\mathcal{B}} \\ &\leqslant e^{-\eta t} \sup_{s \geqslant 0} e^{\eta s} \big|\mathcal{U}(s)\varphi\big|_{\mathcal{B}} = e^{-\eta t}|\varphi|_\eta,\end{aligned}$$

which implies that

$$\big|\mathcal{U}(t)\big|_\eta \leqslant e^{-\eta t} \quad \text{for all } t \geqslant 0. \qquad \square$$

Theorem 3.9. [6, Proposition 5] *Assume that* (H_0)–(H_3) *hold. Then, the solution* $u(.,\varphi)$ *of Eq.* (1.1) *is decomposed as follows*:

$$u_t(.,\varphi) = \mathcal{U}(t)\varphi + \mathcal{W}(t)\varphi \quad \textit{for } t \geqslant 0,$$

where $\mathcal{W}(t)$ *is a compact operator in* $\mathcal{B}_0$ *for each* $t \geqslant 0$*.*

4. Boundedness, ultimate boundedness and periodicity

To discuss the existence of periodic solutions of Eq. (1.1), we use the concept of boundedness and ultimate boundedness of solutions.

Definition 4.1. The solutions of Eq. (1.1) are locally bounded if for each $N_0 > 0$ and $T > 0$, there exists a constant $\overline{N}_0 > 0$ such that $|\varphi|_{\mathcal{B}} \leqslant N_0$ implies $|u(t,\varphi)| \leqslant \overline{N}_0$ for $t \in [0,T]$.

Definition 4.2. The solutions of Eq. (1.1) are bounded if for each $N_1 > 0$, there exists a constant $\overline{N}_1 > 0$ such that $|\varphi|_{\mathcal{B}} \leqslant N_1$ implies $|u(t,\varphi)| \leqslant \overline{N}_1$ for $t \geqslant 0$.

Definition 4.3. The solutions of Eq. (1.1) are ultimate bounded if there is a bound $N > 0$ such that for each $N_2 > 0$, there exists a constant $k > 0$ such that $|\varphi|_{\mathcal{B}} \leqslant N_2$ and $t \geqslant k$ imply that $|u(t,\varphi)| \leqslant N$.

The following proposition gives the relationship between the local boundedness, bounded and ultimate boundedness.

Proposition 4.4. *The local boundedness and the ultimate boundedness of solutions of Eq.* (1.1) *imply the boundedness of solutions.*

Proof. Let N be given by the ultimate boundedness, then for any $N_1 > 0$, there exists a constant $k > 0$ such that $|\varphi|_{\mathcal{B}} \leqslant N_1$ and $t \geqslant k$ imply that $|u(t,\varphi)| \leqslant N$. From local boundedness of solutions we get that there exists a constant $N_2 > N$ such that $|\varphi|_{\mathcal{B}} \leqslant N_1$ implies that $|u(t,\varphi)| < N_2$, for $t \in [0,k]$. It follows that for any positive constant N_1, there exists a constant $N_2 > N$ such that $|\varphi|_{\mathcal{B}} \leqslant N_1$ implies that $|u(t,\varphi)| < N_2$ for $t \geqslant 0$. □

The next result gives the local boundedness of solutions of Eq. (1.1).

Proposition 4.5. *Assume that* (H_0) *and* (H_1) *hold. Then, the solutions of Eq.* (1.1) *are locally bounded.*

The proof is an immediate consequence of the following proposition.

Proposition 4.6. [2, Proposition 2] *Assume that* (H_0) *and* (H_1) *hold. Let u and v be solutions of Eq.* (1.1) *on* $(-\infty, T_0]$ *for some $T_0 > 0$. Then, there exist positive constants ρ and $\tilde{\rho}$ such that*:

$$|u_t - v_t|_{\mathcal{B}} \leqslant \tilde{\rho} e^{\rho t} |u_0 - v_0|_{\mathcal{B}} \quad \text{for } t \in [0, T_0].$$

In the following we study the existence of periodic solutions of Eq. (1.1). To achieve this goal, we use the Poincaré map $\mathcal{P}$ which is defined by

$$\begin{aligned} \mathcal{P} : \mathcal{B}_0 &\to \mathcal{B}_0 \\ \varphi &\to u_\omega(.,\varphi), \end{aligned}$$

where $u(.,\varphi)$ is the solution of Eq. (1.1).

Proposition 4.7. *Assume that* (H_0)–(H_3) *hold. Then the Poincaré map $\mathcal{P}$ is an α-contraction map on $\mathcal{B}_0$.*

Proof. By Theorem 3.9, $\mathcal{P}$ is decomposed as follows

$$\mathcal{P}\varphi = \mathcal{U}(\omega)\varphi + \mathcal{W}(\omega)\varphi,$$

where $\mathcal{W}(\omega)$ is a compact operator on $\mathcal{B}_0$. Let Ω a bounded set in $\mathcal{B}_0$. It follows that

$$\alpha\big(\mathcal{P}(\Omega)\big) \leqslant \alpha\big(\mathcal{U}(\omega)(\Omega)\big).$$

Corollary 3.8 implies that

$$\alpha\big(\mathcal{P}(\Omega)\big) < \exp(-\eta\omega)\alpha(\Omega) \quad \text{for any bounded set } \Omega \text{ in } \mathcal{B}_0,$$

which gives that $\mathcal{P}$ is an α-contraction map on $\mathcal{B}_0$. □

In the following, we assume that:

(H_4) F is ω-periodic in t.

Theorem 4.8. *Assume that* (H_0)–(H_4) *hold. If $\mathcal{B}$ is a uniform fading memory space and the solutions of Eq.* (1.1) *are ultimately bounded. Then Eq.* (1.1) *has an ω-periodic solution.*

We use Hale and Lunel's fixed point theorem which is an extension of Horn's fixed point theorem for condensing maps.

Theorem 4.9. [15, Hale and Lunel's fixed point theorem] *Suppose $S_0 \subseteq S_1 \subseteq S_2$ are convex bounded subsets of a Banach space Y, such that S_0, S_2 are closed and S_1 is open in S_2. Let P be a condensing map on Y such that $P^j(S_1) \subseteq S_2$ for $j \geqslant 0$, and there is a number $N(S_1)$ such that $P^k(S_1) \subseteq S_0$, for $k \geqslant N(S_1)$, then P has a fixed point.*

Proof of Theorem 4.8. Let N be the bound from the ultimate boundedness. By the boundedness of solutions, there exists a constant $N_1 > \overline{N} = \max(1, J)N + 1$ such that for $|\varphi|_{\mathcal{B}} \leqslant \overline{N}$ and $t \geqslant 0$, one has $|u(t,\varphi)| < N_1$, where J is the constant given in Proposition 2.1. Moreover, there exists a constant $N_2 > N_1$ such that for $|\varphi|_{\mathcal{B}} \leqslant N_1$ and $t \geqslant 0$, one has $|u(t,\varphi)| < N_2$. By the ultimate boundedness, we can see that there exists a positive integer $m = m(N_1)$ such that for $|\varphi|_{\mathcal{B}} \leqslant N_1$ and $t \geqslant m\omega$, one has $|u(t,\varphi)| < N$. On the other hand,

$$\mathcal{P}^k\varphi = u_{k\omega}(.,\varphi) \quad \text{for } k \in \mathbb{N}.$$

Since $\mathcal{B}$ is a uniform fading memory space, then for $t \geqslant \sigma$

$$\left|u_t(.,\varphi)\right|_{\mathcal{B}} \leqslant J \sup_{s\in[\sigma,t]} \left|u(s)\right| + M(t-\sigma)\left|u_\sigma(.,\varphi)\right|_{\mathcal{B}}, \quad \text{where } M(t) \to 0 \text{ as } t \to +\infty.$$

Let $M_1 = \sup_{t\in\mathbb{R}^+} M(t)$. Then for $\varphi \in \mathcal{B}_0$ such that $|\varphi|_{\mathcal{B}} \leqslant N_1$ and $k \geqslant 0$ we have

$$\begin{aligned} \left|\mathcal{P}^k(\varphi)\right|_{\mathcal{B}} = \left|u_{k\omega}(.,\varphi)\right|_{\mathcal{B}} &\leqslant J \sup_{s\in[0,k\omega]} \left|u(s)\right| + M_1|\varphi|_{\mathcal{B}} \\ &\leqslant \max(1,J)N_2 + \max(1,M_1)N_1. \end{aligned} \tag{4.1}$$

Since $M(t) \to 0$ as $t \to +\infty$, there exists an integer $m_1 \geqslant m$ such that

$$M(t) \leqslant \frac{1}{M_1N_1 + JN_2} \quad \text{for } t \geqslant m_1\omega. \tag{4.2}$$

For $\varphi \in \mathcal{B}_0$ such that $|\varphi|_{\mathcal{B}} \leqslant N_1$ and $k \geqslant 2m_1$, one has

$$\left|\mathcal{P}^k(\varphi)\right|_{\mathcal{B}} = \left|u_{k\omega}(.,\varphi)\right|_{\mathcal{B}} \leqslant J \sup_{s\in[m_1\omega,k\omega]} \left|u(s)\right| + M\big((k-m_1)\omega\big)\left|u_{m_1\omega}(.,\varphi)\right|_{\mathcal{B}}.$$

Since $|u_{m_1\omega}(.,\varphi)|_{\mathcal{B}} \leqslant M_1N_1 + JN_2$, it follows from (4.2) that

$$\left|\mathcal{P}^k(\varphi)\right|_{\mathcal{B}} \leqslant \max(1,J)N + 1 = \overline{N}. \tag{4.3}$$

Let $\overline{N}_2 = \max(1,J)N_2 + \max(1,M_1)N_1$. Define the following sets:

$$\begin{aligned} S_0 &= \left\{\varphi \in \mathcal{B}_0\colon |\varphi|_{\mathcal{B}} \leqslant \overline{N}\right\}, \\ S_1 &= \left\{\varphi \in \mathcal{B}_0\colon |\varphi|_{\mathcal{B}} < N_1\right\}, \\ S_2 &= \left\{\varphi \in \mathcal{B}_0\colon |\varphi|_{\mathcal{B}} \leqslant \overline{N}_2\right\}. \end{aligned}$$

Then S_0, S_1 and S_2 are convex bounded subsets of $\mathcal{B}_0$. Moreover, $S_0 \subseteq S_1 \subseteq S_2$, S_0 and S_2 are closed and S_1 is open in S_2. Moreover, inequality (4.1) gives that

$$P^k(S_1) \subseteq S_2 \quad \text{for } k \geqslant 0,$$

and by (4.3), we deduce that there exists a positive integer $m_2 = m_2(S_1)$ such that

$$P^k(S_1) \subseteq S_0 \quad \text{for } k \geqslant m_2.$$

By Proposition 4.7, $\mathcal{P}$ is an α-contraction map on $\mathcal{B}_0$. Consequently, fixed point Theorem 4.9 gives that the Poincaré map $\mathcal{P}$ has at least one fixed point which gives an ω-periodic solution of Eq. (1.1). $\square$

5. Applications

To illustrate the previous results, we consider the following Lotka–Volterra model with diffusion:

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) = \frac{\partial^2}{\partial x^2} v(t,x) + \int_{-\infty}^{t} g(t,s,v(s,x))\,ds + h(t,x) & \text{for } t \geqslant 0 \text{ and } x \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0 & \text{for } t \geqslant 0, \\ v(\theta,x) = \varphi_0(\theta,x) & \text{for } \theta \in \mathbb{R}^- \text{ and } x \in [0,\pi], \end{cases} \tag{5.1}$$

where $g : \Delta \times \mathbb{R} \to \mathbb{R}$, $\varphi_0 : \mathbb{R}^- \times [0,\pi] \to \mathbb{R}$ and $h : \mathbb{R} \times [0,\pi] \to \mathbb{R}$ are given functions, here $\Delta = \{(t,s) \in \mathbb{R}^2\colon t \geqslant s\}$.

In order to rewrite Eq. (5.1) in the abstract form we introduce the space $X = C([0,\pi];\mathbb{R})$ of continuous functions from $[0,\pi]$ to $\mathbb{R}$ endowed with the uniform norm topology and the linear operator $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = \{y \in C^2([0,\pi];\mathbb{R})\colon y(0) = y(\pi) = 0\}, \\ Ay = y''. \end{cases}$$

Lemma 5.1. [6]

$$(0,+\infty) \subset \rho(A) \quad \textit{and} \quad \left|(\lambda I - A)^{-1}\right| \leqslant \frac{1}{\lambda} \quad \textit{for } \lambda > 0.$$

Lemma 5.1 implies that assumption (H_0) is satisfied. Moreover, one can see that

$$\overline{D(A)} = \bigl\{y \in X\colon y(0) = y(\pi) = 0\bigr\}.$$

The part A_0 of A in $\overline{D(A)}$ is given by

$$\begin{cases} D(A_0) = \{y \in C^2([0,\pi];\mathbb{R})\colon y(0) = y(\pi) = y''(0) = y''(\pi) = 0\}, \\ \mathcal{A}_0 y = y''. \end{cases}$$

Lemma 5.2. [6] *A_0 generates a compact strongly continuous semigroup $(T_0(t))_{t\geqslant 0}$ on $\overline{D(A)}$. Moreover,*

$$\left|T_0(t)\right| \leqslant e^{-t} \quad \textit{for } t \geqslant 0.$$

Consequently, (H_2) and (H_3) hold.

Let $\gamma > 0$. We introduce the following phase space:

$$\mathcal{B} = C_\gamma(X) := \Bigl\{\phi \in C\bigl((-\infty,0];X\bigr)\colon \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } X\Bigr\},$$

provided with the norm

$$|\phi|_\gamma = \sup_{-\infty < \theta \leqslant 0} e^{\gamma\theta}\left|\phi(\theta)\right|.$$

Lemma 5.3. [19, Proposition 1.4.2, p. 22] *The space $C_\gamma(X)$ satisfies axioms* (A)–(C) *with $H = K(t) = 1$ and $M(t) = e^{-\gamma t}$ for $t \geqslant 0$. Moreover, $C_\gamma(X)$ is a uniform fading memory space.*

We introduce the following notations:

$$u(t)(x) = v(t,x) \quad \text{for } t \geqslant 0,\ x \in [0,\pi],$$
$$\varphi(\theta)(x) = \varphi_0(\theta,x) \quad \text{for } \theta \leqslant 0,\ x \in [0,\pi],$$

and define the function $F : \mathbb{R} \times \mathcal{B} \to X$ by

$$F(t,\phi)(x) = \int_{-\infty}^{0} g\big(t, t+s, \phi(s)(x)\big)\,ds + h(t,x) \quad \text{for } t \in \mathbb{R},\ x \in [0,\pi] \text{ and } \phi \in \mathcal{B}.$$

Then, Eq. (5.1) takes the abstract form:

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) + F(t,u_t) & \text{for } t \geqslant 0, \\ u_0 = \varphi. \end{cases} \tag{5.2}$$

We assume that $g : \Delta \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \times [0,\pi] \to \mathbb{R}$ are continuous functions such that:

(H_5) $|g(t,s,0)| \leqslant \beta(s-t)$ for $(t,s) \in \Delta$,
(H_6) $|g(t,s,\xi) - g(t,s,\zeta)| \leqslant \chi(s-t)|\xi - \zeta|$ for $(t,s) \in \Delta$ and $\xi, \zeta \in \mathbb{R}$,

where $\beta, \chi : (-\infty, 0] \to [0,+\infty)$ are two measurable functions such that $\beta(.)$ and $e^{-\gamma \cdot}\chi(.)$ are integrable on $(-\infty, 0]$.

Under the above conditions, $F : \mathbb{R} \times \mathcal{B} \to X$ satisfies condition (H_1). In fact, for given $t \in \mathbb{R}$, $\phi \in \mathcal{B}$ and sequences $(t_n)_{n \geqslant 0}$ of $\mathbb{R}$ and $(\phi_n)_{n \geqslant 0}$ of $\mathcal{B}$ such that $t_n \to t$ and $\phi_n \to \phi$ as $n \to +\infty$, we have

$$\begin{aligned} \big|F(t_n,\phi_n) - F(t,\phi)\big| &= \sup_{x \in [0,\pi]} \big|F(t_n,\phi_n)(x) - F(t,\phi)(x)\big| \\ &\leqslant \sup_{x \in [0,\pi]} \int_{-\infty}^{0} \big|g\big(t_n, t_n+s, \phi_n(s)(x)\big) - g\big(t, t+s, \phi(s)(x)\big)\big|\,ds \\ &\quad + \sup_{x \in [0,\pi]} \big|h(t_n,x) - h(t,x)\big|. \end{aligned}$$

It follows that

$$\begin{aligned} &\big|g\big(t_n, t_n+s, \phi_n(s)(x)\big) - g\big(t, t+s, \phi(s)(x)\big)\big| \\ &\quad \leqslant \big|g\big(t_n, t_n+s, \phi_n(s)(x)\big) - g\big(t_n, t_n+s, \phi(s)(x)\big)\big| \\ &\qquad + \big|g\big(t_n, t_n+s, \phi(s)(x)\big) - g\big(t, t+s, \phi(s)(x)\big)\big|, \end{aligned}$$

by assumption (H_6), we get that

$$\begin{aligned} &\big|g\big(t_n, t_n+s, \phi_n(s)(x)\big) - g\big(t, t+s, \phi(s)(x)\big)\big| \leqslant \chi(s)e^{-\gamma s}|\phi_n - \phi|_\gamma \\ &\quad + \big|g\big(t_n, t_n+s, \phi(s)(x)\big) - g\big(t, t+s, \phi(s)(x)\big)\big|. \end{aligned}$$

Then

$$\sup_{x \in [0,\pi]} \int_{-\infty}^{0} \big|g\big(t_n, t_n+s, \phi_n(s)(x)\big) - g\big(t, t+s, \phi(s)(x)\big)\big|\,ds$$

$$\leqslant \left(\int_{-\infty}^{0} \chi(s) e^{-\gamma s}\, ds \right) |\phi_n - \phi|_{\mathcal{B}}$$
$$+ \sup_{x \in [0,\pi]} \int_{-\infty}^{0} \left| g\big(t_n, t_n + s, \phi(s)(x)\big) - g\big(t, t+s, \phi(s)(x)\big) \right| ds.$$

Since for $s \in \mathbb{R}^-$, $\{\phi(s)(x)\colon\ x \in [0,\pi]\}$ is a compact set in $\mathbb{R}$, it follows from the continuity of g that

$$\lim_{n \to +\infty} \sup_{x \in [0,\pi]} \left| g\big(t_n, t_n + s, \phi(s)(x)\big) - g\big(t, t+s, \phi(s)(x)\big) \right| = 0 \quad \text{for } s \in \mathbb{R}^-.$$

Assumptions (H_5) and (H_6) imply that

$$\left| g(\tau, \tau + s, \psi(s)(x) \right| \leqslant \beta(s) + \chi(s) e^{-\gamma s} |\psi|_\gamma \quad \text{for } \tau \in \mathbb{R},\ s \in \mathbb{R}^- \text{ and } \psi \in \mathcal{B},$$

and

$$\sup_{x \in [0,\pi]} \left| g\big(t_n, t_n + s, \phi(s)(x)\big) - g\big(t, t+s, \phi(s)(x)\big) \right| \leqslant 2\big(\beta(s) + \chi(s) e^{-\gamma s} |\phi|_\gamma\big).$$

Lebesgue's dominated convergence theorem gives that

$$\lim_{n \to +\infty} \int_{-\infty}^{0} \sup_{x \in [0,\pi]} \left| g\big(t_n, t_n + s, \phi(s)(x)\big) - g\big(t, t+s, \phi(s)(x)\big) \right| ds = 0,$$

we deduce that

$$\lim_{n \to +\infty} \left| F(t_n, \phi_n) - F(t, \phi) \right| = 0.$$

Then F is continuous on $\mathbb{R} \times \mathcal{B}$ with values in X. We claim that F is Lipschitz continuous with respect to the second argument. In fact, let $\phi, \psi \in \mathcal{B}$ and $t \geqslant 0$. Then,

$$\begin{aligned}
\left| F(t,\phi) - F(t,\psi) \right| &= \sup_{x \in [0,\pi]} \left| F(t,\phi)(x) - F(t,\psi)(x) \right| \\
&\leqslant \sup_{x \in [0,\pi]} \left(\int_{-\infty}^{0} \left| g\big(t, t+s, \phi(s)(x)\big) - g\big(t, t+s, \psi(s)(x)\big) \right| ds \right).
\end{aligned}$$

By assumption (H_6), we obtain

$$\begin{aligned}
\left| F(t,\phi) - F(t,\psi) \right| &\leqslant \int_{-\infty}^{0} \chi(s) \sup_{x \in [0,\pi]} \left| \phi(s)(x) - \psi(s)(x) \right| ds \\
&\leqslant \int_{-\infty}^{0} \chi(s) e^{-\gamma s} e^{\gamma s} \sup_{x \in [0,\pi]} \left| \phi(s)(x) - \psi(s)(x) \right| ds \\
&\leqslant a |\phi - \psi|_\gamma, \qquad (5.3)
\end{aligned}$$

where $a = \int_{-\infty}^{0} e^{-\gamma s} \chi(s)\, ds$.

Consequently F is Lipschitz continuous with respect to the second argument.

If we assume that $\varphi_0 \in C((-\infty, 0] \times [0, \pi]; \mathbb{R})$ such that $\lim_{\theta \to -\infty} \sup_{e^{\gamma\theta}} \sup_{x \in [0,\pi]} |\varphi_0(\theta, x)|$ exists and $\varphi_0(0, 0) = \varphi_0(0, \pi) = 0$, then $\varphi \in \mathcal{B}_0$ and by Theorem 3.4, we conclude that Eq. (5.2) has a unique solution.

To discuss the existence of periodic solutions of Eq. (5.2), we suppose that:

(H_7) g is ω-periodic in the first and second variables and h is ω-periodic in the first variable.

Proposition 5.4. *Assume that* (H_5)–(H_7) *hold and* $\int_{-\infty}^0 e^{-\gamma s} \chi(s)\, ds < 1$. *Then, the solutions of Eq.* (5.2) *are ultimately bounded.*

The following technical lemma is needed for the proof.

Lemma 5.5. [4, p. 89] *If f, h and y are positive continuous functions on* $[t_0, t_1]$ *such that*

$$y(t) \leqslant f(t) + \int_{t_0}^{t} h(s)y(s)\, ds \quad \text{for } t_0 \leqslant t \leqslant t_1.$$

Then for $t_0 \leqslant t \leqslant t_1$

$$y(t) \leqslant f(t) + \int_{t_0}^{t} f(s)h(s) \exp\left(\int_{s}^{t} h(\mu)\, d\mu \right) ds.$$

Proof. Let $u(., \varphi)$ be the solution of Eq. (5.2), then

$$u(t, \varphi) = T_0(t)\varphi(0) + \lim_{\lambda \to +\infty} \int_0^t T_0(t-s) B_\lambda F\big(s, u_s(., \varphi)\big)\, ds \quad \text{for } t \geqslant 0.$$

It follows from (5.3) that for $t \in \mathbb{R}^+$ and $\phi \in \mathcal{B}$

$$\big|F(t, \phi)\big| < a|\phi|_\gamma + b,$$

where $b = \sup_{t \in [0,\omega]} |F(t, 0)|$.

Since $|T_0(t)| \leqslant e^{-t}$ for $t \geqslant 0$, then

$$\big|u(t, \varphi)\big| \leqslant e^{-t}|\varphi|_\gamma + \int_0^t e^{-(t-s)}\big[a\big|u_s(., \varphi)\big|_\gamma + b\big]\, ds \quad \text{for } t \geqslant 0.$$

Moreover, since

$$\big|u_t(., \varphi)\big|_\gamma \leqslant \sup_{s \in [0,t]} \big|u(s, \varphi)\big| + e^{-\gamma t}|\varphi|_\gamma \quad \text{for } t \geqslant 0,$$

one has

$$\big|u(t, \varphi)\big| \leqslant e^{-t}|\varphi|_\gamma + b\big(1 - e^{-t}\big) + ae^{-t} \int_0^t \sup_{\xi \in [0,s]} e^{\xi} \big|u(\xi, \varphi)\big|\, ds + ae^{-t} \int_0^t e^{(1-\gamma)s}\, ds |\varphi|_\gamma$$

and

$$\begin{aligned} e^t\big|u(t,\varphi)\big| &\leqslant \left(1+a\int_0^t e^{(1-\gamma)s}\,ds\right)|\varphi|_\gamma \\ &\quad + b\big(e^t-1\big)+a\int_0^t \sup_{\xi\in[0,s]} e^{\xi}\big|u(\xi,\varphi)\big|\,ds \quad \text{for } t\geqslant 0. \end{aligned}$$

Let $g(t)=\sup_{s\in[0,t]} e^s|u(s,\varphi)|$. Then,

$$g(t)\leqslant \left(1+a\int_0^t e^{(1-\gamma)s}\,ds\right)|\varphi|_\gamma + b\big(e^t-1\big)+a\int_0^t g(s)\,ds \quad \text{for } t\geqslant 0. \tag{5.4}$$

If $\gamma=1$, we get

$$g(t)\leqslant (1+at)|\varphi|_\gamma + b\big(e^t-1\big)+a\int_0^t g(s)\,ds \quad \text{for } t\geqslant 0.$$

By Lemma 5.5 we obtain that

$$g(t)\leqslant (1+at)|\varphi|_\gamma + b\big(e^t-1\big)+a\int_0^t \big[(1+as)|\varphi|_\gamma + b\big(e^s-1\big)\big]e^{a(t-s)}\,ds \quad \text{for } t\geqslant 0,$$

which gives that

$$g(t)\leqslant \big(2e^{at}-1\big)|\varphi|_\gamma + b\big(e^t-e^{at}\big)+\frac{ab}{1-a}\big(e^t-e^{at}\big) \quad \text{for } t\geqslant 0.$$

We arrive at

$$\begin{aligned} \big|u(t,\varphi)\big| &\leqslant \big(2e^{(a-1)t}-e^{-t}\big)|\varphi|_\gamma + b + \frac{ab}{1-a} \\ &\quad - be^{(a-1)t} - \frac{ab}{1-a}e^{(a-1)t} \quad \text{for } t\geqslant 0. \end{aligned} \tag{5.5}$$

If $\gamma\neq 1$, then by (5.4), we get that

$$g(t)\leqslant \left(1-\frac{a}{1-\gamma}+\frac{a}{1-\gamma}e^{(1-\gamma)t}\right)|\varphi|_\gamma + b\big(e^t-1\big)+a\int_0^t g(s)\,ds \quad \text{for } t\geqslant 0.$$

By Lemma 5.5 we obtain that

$$\begin{aligned} g(t) &\leqslant \left(1-\frac{a}{1-\gamma}+\frac{a}{1-\gamma}e^{(1-\gamma)t}\right)|\varphi|_\gamma + b\big(e^t-1\big) \\ &\quad + a\int_0^t \left[\left(1-\frac{a}{1-\gamma}+\frac{a}{1-\gamma}e^{(1-\gamma)s}\right)|\varphi|_\gamma + b\big(e^s-1\big)\right]e^{a(t-s)}\,ds \quad \text{for } t\geqslant 0, \end{aligned}$$

which gives that

$$g(t) \leqslant \left(e^{at} - \frac{a}{1-\gamma}e^{at} + \frac{a}{1-\gamma}e^{(1-\gamma)t} + \frac{a^2}{1-\gamma}e^{at}\int_0^t e^{(1-\gamma-a)s}\,ds\right)|\varphi|_\gamma$$
$$+ b\big(e^t - e^{at}\big) + \frac{ab}{1-a}\big(e^t - e^{at}\big) \quad \text{for } t \geqslant 0. \tag{5.6}$$

If $\gamma + a = 1$, then

$$g(t) \leqslant \left(e^{at} - \frac{a}{1-\gamma}e^{at} + \frac{a}{1-\gamma}e^{(1-\gamma)t} + \frac{a^2}{1-\gamma}te^{at}\right)|\varphi|_\gamma$$
$$+ b\big(e^t - e^{at}\big) + \frac{ab}{1-a}\big(e^t - e^{at}\big) \quad \text{for } t \geqslant 0.$$

Consequently, for $t \geqslant 0$

$$\big|u(t,\varphi)\big| \leqslant \left(e^{(a-1)t} - \frac{a}{1-\gamma}e^{(a-1)t} + \frac{a}{1-\gamma}e^{-\gamma t} + \frac{a^2}{1-\gamma}te^{(a-1)t}\right)|\varphi|_\gamma + b + \frac{ab}{1-a}$$
$$- be^{(a-1)t} - \frac{ab}{1-a}e^{(a-1)t}. \tag{5.7}$$

If $\gamma + a \neq 1$, by (5.6), we obtain that

$$g(t) \leqslant \left(e^{at} - \frac{a}{1-\gamma}e^{at} + \frac{a}{1-\gamma}e^{(1-\gamma)t} + \frac{a^2}{(1-\gamma)(1-\gamma-a)}\big(e^{(1-\gamma)t} - e^{at}\big)\right)|\varphi|_\gamma$$
$$+ b\big(e^t - e^{at}\big) + \frac{ab}{1-a}\big(e^t - e^{at}\big) \quad \text{for } t \geqslant 0,$$

and

$$\big|u(t,\varphi)\big| \leqslant \left(e^{(a-1)t} - \frac{a}{1-\gamma}e^{(a-1)t} + \frac{a}{1-\gamma}e^{-\gamma t}\right.$$
$$\left. + \frac{a^2}{(1-\gamma)(1-\gamma-a)}\big(e^{-\gamma t} - e^{(a-1)t}\big)\right)|\varphi|_\gamma$$
$$+ b + \frac{ab}{1-a} - be^{(a-1)t} - \frac{ab}{1-a}e^{(a-1)t} \quad \text{for } t \geqslant 0. \tag{5.8}$$

From (5.5), (5.7), (5.8) and the fact that $a < 1$, we conclude that for all $\varphi \in \mathcal{B}_0$

$$\overline{\lim_{t\to+\infty}}\big|u(t,\varphi)\big| < b + \frac{ab}{1-a} + \varepsilon \quad \text{for all } \varepsilon > 0,$$

which implies that the solutions of Eq. (5.2) are ultimately bounded. □

As a direct consequence of Theorem 4.8 and Proposition 5.4, we obtain the following result.

Proposition 5.6. *Assume that* (H$_5$)–(H$_7$) *hold and* $\int_{-\infty}^0 e^{-\gamma s}\chi(s)\,ds < 1$. *Then, Eq.* (5.2) *has an* ω*-periodic solution.*

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