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Note

Some starlikeness criterions for analytic functions ☆

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Abstract

We determine a condition on M, α, λ and μ for which

$$\left|(1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu}+\alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-1\right|<M$$

implies $f \in S_n^*(\lambda)$, where $S_n^*(\lambda)$ is the set of starlike functions of order λ. We also give some application of them. Some results of Mocanu and Ponnusamy are improved. We also obtain some new results on starlikeness criterions.

Keywords: Differential subordination; Starlike function; Convex function

1. Introduction

Let n be a natural number, and let A_n denote the class of function $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ that are analytic in the unit disk $U = \{z\colon |z| < 1\}$. Let $0 \leqslant \lambda < 1$, and let

$$S_n^*(\lambda) = \left\{ f \in A_n\colon \operatorname{Re} \frac{zf'(z)}{f(z)} > \lambda \text{ for } z \text{ in } U \right\}.$$

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Function f is called starlike function of order λ.

Let f, g be analytic in U. Function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function w such that $|w(z)| \leqslant |z|$ and $f(z) = g(w(z))$ for z in U. Let

$$S_M = \left\{ f \in A_1 \colon \left| f'(z) - 1 \right| < M \text{ for } z \in U \right\}.$$

Singh [10] has proved that $S_M \subset S_1^*(0)$ if $0 < M \leqslant 2/\sqrt{5}$. Fournier [1,2] has proved that

$$S_M \subset S_1^*(0) \quad \Leftrightarrow \quad 0 \leqslant M \leqslant \frac{2}{\sqrt{5}},$$

and

$$\rho_M = \begin{cases} \dfrac{(1-M)(1-M/2)}{1-M^2/4}, & \text{if } 0 \leqslant M \leqslant \frac{2}{3}, \\ \dfrac{\frac{1}{2}(1-\frac{5}{4}M^2)}{1-M^2/4}, & \text{if } \frac{2}{3} \leqslant M \leqslant 1, \end{cases}$$

is the order of starlikeness of S_M. These results were extended by Mocanu, Ponnusamy and others.

In this article we are mainly interested in determining a condition on M, α, λ and μ for which

$$\left| (1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1 \right| < M$$

implies $f \in S_n^*(\lambda)$. We improved results of Mocanu and Ponnusamy. We also obtain some new results on starlikeness criterions.

2. Main results

Lemma 1. *(See [7].) Let B, C and D be complex function on U and let n be a positive integer. Suppose that $D(0) = 0$, $B(z) \neq 0$ and $\operatorname{Re} \frac{C(z)}{B(z)} \geqslant -n$ for $z \in U$. If $p(z) = p_n z^n + \cdots$ is analytic in U and satisfies*

$$\left| B(z) z p'(z) + C(z) p(z) + D(z) \right| < M$$

for $z \in U$, then $|p(z)| < N$ for $z \in U$, where

$$N = \sup\left\{ \frac{M + |D(z)|}{|nB(z) + C(z)|} \colon |z| < 1 \right\}.$$

Theorem 1. *Let $\alpha > 0$, $\mu > 0$, and let*

$$M_n(\alpha,\lambda,\mu) = \begin{cases} \dfrac{(\mu+n\alpha)(1-\lambda)}{n+\mu(1-\lambda)}, & \text{if } \alpha \geqslant \alpha_2, \\ \dfrac{(\mu+n\alpha)\sqrt{2\alpha(1-\lambda)-1}}{\sqrt{n^2\alpha^2+2[n\mu+(1-\lambda)\mu^2]\alpha}}, & \text{if } \alpha_1 \leqslant \alpha \leqslant \alpha_2, \\ \dfrac{\alpha(\mu+n\alpha)(1-\lambda)}{2\mu+(n-\mu+\mu\lambda)\alpha}, & \text{if } 0 < \alpha < \alpha_1, \end{cases} \tag{1}$$

where $\alpha_2 = [n+\mu(1-\lambda)]/[n(1-\lambda)]$, and

$$\alpha_1 = \frac{\sqrt{9\mu^2+2n\mu+n^2-(18\mu^2+2n\mu)\lambda+9\mu^2\lambda^2}-3\mu+n+3\mu\lambda}{2n(1-\lambda)}.$$

If $p(z)$ and $q(z)$ are analytic in U with $p(z) = 1 + p_n z^n + \cdots$ and $q(z) = 1 + q_n z^n + \cdots$, and satisfy

$$q(z) \prec 1 + \frac{\mu M z}{n\alpha+\mu},$$

then

$$q(z)\big[1-\alpha+\alpha p(z)\big] \prec 1 + Mz, \tag{2}$$

where $0 < M \leqslant M_n(\alpha,\lambda,\mu)$, then $\operatorname{Re} p(z) > \lambda$ for $z \in U$.

Proof. Let

$$N = \frac{\mu M}{\mu+n\alpha} \quad \text{and} \quad N_n(\alpha,\lambda,\mu) = \frac{\mu M_n(\alpha,\lambda,\mu)}{\mu+n\alpha}.$$

If there exists $z_0 \in U$ such that $\operatorname{Re} p(z_0) = \lambda$, then we will show

$$\big|q(z_0)\big[1-\alpha+\alpha p(z_0)\big]-1\big| \geqslant M. \tag{3}$$

Note that $|q(z_0)-1| \leqslant N$ for $z \in U$, it is sufficient to check the inequality

$$\alpha\big|p(z_0)-1\big| - N\big|1-\alpha+\alpha p(z_0)\big| \geqslant M. \tag{4}$$

Since $\mu M = N(\mu+n\alpha)$, then inequality (4) is equivalent to

$$N \leqslant \frac{\alpha\mu\sqrt{x^2+1-2\lambda}}{\mu+n\alpha+\mu\sqrt{\alpha^2x^2+2(1-\alpha)\alpha\lambda+(1-\alpha)^2}}, \tag{5}$$

where $x = |p(z_0)| \geqslant \lambda$. If we let

$$\varphi(x) = \frac{\sqrt{x^2+1-2\lambda}}{\mu+n\alpha+\mu\sqrt{\alpha^2x^2+2(1-\alpha)\alpha\lambda+(1-\alpha)^2}},$$

then we have

$$\varphi'(x) = \frac{x\{(\mu+n\alpha)\sqrt{\alpha^2x^2+2(1-\alpha)\alpha\lambda+(1-\alpha)^2}+\mu[1-2\alpha(1-\lambda)]\}}{\sqrt{x^2+1-2\lambda}\sqrt{\alpha^2x^2+2(1-\alpha)\alpha\lambda+(1-\alpha)^2}(\mu+n\alpha+\mu\sqrt{\alpha^2x^2+2(1-\alpha)\alpha\lambda+(1-\alpha)^2})^2}.$$

Noted that

$$\sqrt{\alpha^2x^2+2(1-\alpha)\alpha\lambda+(1-\alpha)^2} \geqslant \left|1-\alpha(1-\lambda)\right| \quad \text{for } x \geqslant \lambda,$$

we let

$$T=(\mu+n\alpha)\left|\alpha(1-\lambda)-1\right|+\mu\left[1-2\alpha(1-\lambda)\right].$$

When

$$\frac{1}{1-\lambda} \leqslant \alpha,$$

we have

$$T=\alpha\left[n\alpha(1-\lambda)-n-\mu(1-\lambda)\right].$$

Hence we obtain

$$T \geqslant 0 \quad \text{for } \alpha \geqslant \frac{n+\mu(1-\lambda)}{n(1-\lambda)} \quad \text{and}$$
$$T<0 \quad \text{for } \frac{1}{1-\lambda} \leqslant \alpha < \frac{n+\mu(1-\lambda)}{n(1-\lambda)}.$$

When

$$\frac{1}{2(1-\lambda)} \leqslant \alpha < \frac{1}{1-\lambda},$$

we have

$$\begin{aligned} T &= (\mu+n\alpha)\left[1-\alpha(1-\lambda)\right]+\mu\left[1-2\alpha(1-\lambda)\right] \\ &= 2\mu-[3\mu-n-3\mu\lambda]\alpha-n(1-\lambda)\alpha^2. \end{aligned}$$

Hence we obtain

$$T<0 \quad \text{for } \alpha_1 \leqslant \alpha < \frac{1}{1-\lambda} \quad \text{and}$$
$$T \geqslant 0 \quad \text{for } \frac{1}{2(1-\lambda)} \leqslant \alpha < \alpha_1.$$

When

$$0<\alpha<\frac{1}{2(1-\lambda)},$$

we have $1-2(1-\lambda)\alpha>0$. It follows that $T>0$. Therefore we are obtain $\varphi'(x) \geqslant 0$ for $x \geqslant \lambda$ when $0<\alpha<\alpha_1$ or

$$\alpha \geqslant \frac{n+\mu(1-\lambda)}{n(1-\lambda)}=\alpha_2.$$

It follows that

$$\min_{x \geqslant \lambda} \varphi(x)=\varphi(\lambda)=\begin{cases} \dfrac{(1-\lambda)}{\alpha[n+\mu(1-\lambda)]}, & \text{if } \alpha \geqslant \alpha_2, \\ \dfrac{1-\lambda}{2\mu+(n-\mu+\mu\lambda)\alpha}, & \text{if } 0<\alpha<\alpha_1. \end{cases}$$

If $\alpha_1 \leqslant \alpha < \alpha_2$, then we have $\varphi'(\lambda) < 0$. Hence there exists a unique $x_0 \in (\lambda, +\infty)$ such that

$$(\mu + n\alpha)\sqrt{\alpha^2 x_0^2 + 2\alpha(1-\alpha)\lambda + (1-\alpha)^2} = \mu\bigl[2\alpha(1-\lambda) - 1\bigr].$$

By a simple calculation, we may obtain

$$\min_{x \geqslant \lambda} \varphi(x) = \varphi(x_0) = \frac{\sqrt{2\alpha(1-\lambda) - 1}}{\alpha\sqrt{n^2\alpha^2 + 2[n\mu + (1-\lambda)\mu^2]\alpha}}.$$

This shows that inequality (5) holds. It follows that inequality (3) holds, which contradicts (2). Hence we must have $\operatorname{Re} p(z) > \lambda$ for $z \in U$. $\square$

Theorem 2. *Let α, μ, λ, M and $M_n(\alpha, \lambda, \mu)$ be defined as in Theorem* 1. *If $f \in A_n$ satisfies*

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec 1 + Mz, \tag{6}$$

then $f \in S_n^*(\lambda)$.

Proof. If we let

$$q(z) = \left(\frac{f(z)}{z}\right)^{\mu},$$

then we have

$$q(z) + \frac{\alpha}{\mu} z q'(z) \prec 1 + Mz.$$

By Lemma 1, we obtain

$$q(z) \prec 1 + \frac{\mu M}{\mu + n\alpha} z.$$

Let

$$p(z) = \frac{z f'(z)}{f(z)}.$$

Then we have

$$q(z)\bigl[1 - \alpha + \alpha p(z)\bigr] \prec 1 + Mz.$$

By Theorem 1, we have $\operatorname{Re} p(z) > \lambda$ for $z \in U$. It follows $f \in S_n^*(\lambda)$. $\square$

Corollary 1. *Let $\alpha > 0$ and let*

$$M_n(\alpha) = \begin{cases} \dfrac{1 + n\alpha}{n+1}, & \text{if } \alpha \geqslant \dfrac{n+1}{n}, \\[2ex] \dfrac{(1+n\alpha)\sqrt{2\alpha - 1}}{\sqrt{n^2\alpha^2 + 2(n+1)\alpha}}, & \text{if } \dfrac{\sqrt{9 + 2n + n^2} - 3 + n}{2n} \leqslant \alpha < \dfrac{n+1}{n}, \\[2ex] \dfrac{\alpha(1 + n\alpha)}{2 + (n-1)\alpha}, & \text{if } 0 < \alpha < \dfrac{\sqrt{9 + 2n + n^2} - 3 + n}{2n}. \end{cases}$$

If $f \in A_n$ satisfies

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec 1 + Mz,$$

where $0 < M \leqslant M_n(\alpha)$*, then* $f \in S_n^*(0)$.

Remark 1. It is easy to show that

$$\frac{1-\lambda}{1+\mu(1-\lambda)} > \frac{\sqrt{2\alpha(1-\lambda)-1}}{\sqrt{\alpha^2 + 2[\mu + (1-\lambda)\mu^2]\alpha}}$$

for $\alpha > \frac{1+\mu(1-\lambda)}{1-\lambda}$. By straightforward calculation, the inequality

$$\frac{\alpha(1-\lambda)}{2\mu + (1-\mu+\mu\lambda)\alpha} \geqslant \frac{\sqrt{2\alpha(1-\lambda)-1}}{\sqrt{\alpha^2 + 2[\mu + (1-\lambda)\mu^2]\alpha}}$$

is equivalent to

$$\big[2\mu - \big(3\mu(1-\lambda)-1\big)\alpha - (1-\lambda)\alpha\big]^2 \geqslant 0.$$

Taking $n = 1$ in Theorems 1 and 2, we improve [8, Theorems 2 and 3]. Taking $n = 1$ in Corollary 1, we improve the results of Mocanu in [4–6] (or see [3, Theorems 5.5a, 5.5c]).

Theorem 3. *Let* $\mu > 0$ *and*

$$0 < \beta \leqslant \frac{\mu + n}{\sqrt{\mu^2 + (\mu+n)^2}}.$$

If $f \in A_n$ satisfies

$$\left| f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1 \right| < \beta$$

for $z \in U$, then $f \in S_n^*(\lambda)$, where

$$\lambda = \begin{cases} \dfrac{(\mu+n)(1-\beta)}{\mu+n+\mu\beta}, & \textit{if } 0 < \beta < \dfrac{\mu+n}{2\mu+n}, \\[2ex] \dfrac{(\mu+n)^2 - [\mu^2 + (\mu+n)^2]\beta^2}{2(\mu+n)^2 - 2\mu^2\beta^2}, & \textit{if } \dfrac{\mu+n}{2\mu+n} \leqslant \beta \leqslant \dfrac{\mu+n}{\sqrt{\mu^2+(\mu+n)^2}}. \end{cases} \tag{7}$$

Proof. From (7), we have

$$\beta = \begin{cases} \dfrac{(\mu+n)\sqrt{1-2\lambda}}{\sqrt{n^2 + 2[n\mu + (1-\lambda)\mu^2]}}, & \text{if } 0 \leqslant \lambda \leqslant \dfrac{\mu}{3\mu+n}, \\[2ex] \dfrac{(\mu+n)(1-\lambda)}{n+\mu+\mu\lambda}, & \text{if } \dfrac{\mu}{3\mu+n} < \lambda < 1. \end{cases}$$

It is easy to show that the inequality

$$\sqrt{9\mu^2 + 2n\mu + n^2 - (18\mu^2 + 2\mu n)\lambda + 9\mu^2\lambda^2} \leqslant 2n(1-\lambda) + 3\mu - n - 3\mu\lambda$$

is equivalent to

$$\alpha \leqslant \frac{\mu}{3\mu+n}.$$

Hence we obtain $\beta = M_n(1, \lambda, \mu)$. By Theorem 2, we have $f \in S_n^*(\lambda)$, where λ is given by (7). □

Remark 2. By a simple calculation, we have

$$\frac{\mu+n+\sqrt{\mu^2+(\mu+n)^2\beta}}{2(\mu+n-\mu\beta)} > 1 \quad \text{for } \beta \geqslant \frac{\mu+n}{2\mu+n}$$

and

$$\frac{(\mu+n)(1-\beta)}{\mu+n+\mu\beta} > \frac{\mu+n-\sqrt{\mu^2+(\mu+n)^2}\beta^2}{\mu+n+\mu\beta} \quad \text{for } 0 < \beta < \frac{\mu+n}{2\mu+n}.$$

Hence we obtain

$$\lambda > \frac{\mu+n-\sqrt{\mu^2+(n+\mu)^2\beta}}{\mu+n+\mu\beta},$$

where λ is given by (7). Taking $n = 1$ in Theorem 3, we improve [9, Theorems 2 and 3].

Theorem 4. *Let $0 \leqslant \lambda < 1$, $\alpha > 0$, and let*

$$M_n(\alpha,\lambda) = \begin{cases} \dfrac{(1+n\alpha)(1-\lambda)}{n+1-\lambda}, & \text{if } \alpha \geqslant \alpha_2, \\ \dfrac{(1+n\alpha)\sqrt{2\alpha(1-\lambda)-1}}{\sqrt{n^2\alpha^2+2(n+1-\lambda)\alpha}}, & \text{if } \alpha_1 \leqslant \alpha < \alpha_2, \\ \dfrac{\alpha(1+n\alpha)(1-\lambda)}{2+(n-1+\lambda)\alpha}, & \text{if } 0 < \alpha < \alpha_1, \end{cases}$$

where $\alpha_2 = (n+1-\lambda)/(n-n\lambda)$,

$$\alpha_1 = \frac{\sqrt{9+2n+n^2-(18+2n)\lambda+9\lambda^2}-3+n+3\lambda}{2n(1-\lambda)}.$$

If f satisfies $|f'(z)+\alpha z f''(z)-1| \leqslant M$ for $z \in U$, where $0 < M \leqslant M_n(\alpha,\lambda)$, then $zf'(z) \in S_n^*(\lambda)$, i.e., f is a convex function of order λ.

Proof. If we let $g(z) = zf'(z)$, then we have

$$(1-\alpha)\frac{g(z)}{z} + \alpha g'(z) \prec 1 + Mz.$$

By Theorem 2, we must have $g \in S_n^*(\lambda)$. Hence f is a convex function of order λ. □

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