

Inequalities for a polynomial and its derivative

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Abstract

Let $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, be a polynomial of degree n such that $p(z) \neq 0$ in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$, Dewan, Yadav and Pukhta [K.K. Dewan, R.S. Yadav, M.S. Pukhta, Inequalities for a polynomial and its derivative, Math. Inequal. Appl. 2 (2) (1999) 203–205] proved

$$\max_{|z|=r} |p(z)| \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=R} |p(z)| + \left[1 - \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \right] \min_{|z|=k} |p(z)|.$$

Equality holds for the polynomial $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ .

In this paper, we obtain an improvement of the above inequality by involving some of the coefficients. As an application of our result, we further improve upon a result recently proved by Aziz and Shah [A. Aziz, W.M. Shah, Inequalities for a polynomial and its derivative, Math. Inequal. Appl. 7 (3) (2004) 379–391].

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1. Introduction and statement of results

Let $p(z)$ be a polynomial of degree n , then

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)| \tag{1.1}$$

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and

$$\max_{|z|=r} |p(z)| \geq r^n \max_{|z|=1} |p(z)| \quad \text{for } r \leq 1. \quad (1.2)$$

Inequality (1.1) is a famous result known as Bernstein's inequality [12] whereas inequality (1.2) is due to Zarantonello and Varga [13]. It is noted that in both (1.1) and (1.2) equality holds if and only if $p(z)$ has all its zeros at the origin and so it is natural to seek improvements under appropriate assumptions on the zeros of $p(z)$.

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then the inequalities (1.1) and (1.2) can be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)| \quad (1.3)$$

and

$$\max_{|z|=r} |p(z)| \geq \left(\frac{r+1}{2}\right)^n \max_{|z|=1} |p(z)| \quad \text{for } r \leq 1. \quad (1.4)$$

Inequality (1.3) was conjectured by Erdős and later proved by Lax [9], whereas inequality (1.4) is due to Rivlin [11].

As an extension of (1.3), it was shown by Malik [10] that if $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \quad (1.5)$$

Equality in (1.5) holds for $p(z) = (z+k)^n$.

Whereas, as a generalization of (1.4), Govil [8, Theorem 1] proved that if $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zero in $|z| < 1$, then for $0 < r \leq \rho \leq 1$,

$$\max_{|z|=r} |p(z)| \geq \left(\frac{1+r}{1+\rho}\right)^n \max_{|z|=\rho} |p(z)|. \quad (1.6)$$

Inequality (1.6) is best possible and equality holds for the polynomial $p(z) = (z+1)^n$.

Bidkham and Dewan [3] obtained a generalization of inequality (1.5) for the class of polynomials $p(z) = \sum_{v=0}^n a_v z^v \neq 0$ in $|z| < k$, $k \geq 1$, by proving

$$\max_{|z|=r} |p'(z)| \leq n \frac{(r+k)^{n-1}}{(1+k)^n} \max_{|z|=1} |p(z)|, \quad (1.7)$$

where $1 \leq r \leq k$.

Equality in (1.7) occurs for $p(z) = (z+k)^n$.

By considering a more general class of polynomials $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, not vanishing in $|z| < k$, $k \geq 1$, inequalities (1.6) and (1.7) were generalized respectively by Dewan [4] (see also [3]) and Aziz and Zargar [2, Theorem 2] by proving

$$\max_{|z|=r} |p(z)| \geq \left(\frac{k^\mu + r^\mu}{k^\mu + \rho^\mu}\right)^{\frac{n}{\mu}} \max_{|z|=\rho} |p(z)| \quad \text{for } 0 < r \leq \rho \leq 1, \quad (1.8)$$

and

$$\max_{|z|=R} |p'(z)| \leq n \frac{R^{\mu-1} (R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} \max_{|z|=r} |p(z)| \quad \text{for } 0 < r \leq R \leq k. \quad (1.9)$$

Inequalities (1.8) and (1.9) are best possible and equalities hold for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ .

Dewan, Yadav and Pukhta [5] further improved upon as well as generalized inequality (1.8) by involving $\min_{|z|=k} |p(z)|$.

Theorem A. *If $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n such that $p(z) \neq 0$ in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,*

$$\max_{|z|=r} |p(z)| \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=R} |p(z)| + \left[1 - \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \right] \min_{|z|=k} |p(z)|. \quad (1.10)$$

Whereas, the corresponding generalization and improvement of (1.9) was made recently by Aziz and Shah [1]. In fact, they proved

Theorem B. *If $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n such that $p(z) \neq 0$ in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,*

$$\max_{|z|=R} |p'(z)| \leq n \frac{R^{\mu-1} (R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} \left\{ \max_{|z|=r} |p(z)| - \min_{|z|=k} |p(z)| \right\}. \quad (1.11)$$

Both (1.10) and (1.11) are best possible and equalities hold for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ .

In this paper, we first obtain an improvement of Theorem A by involving some of the coefficients, and so also of inequalities (1.4), (1.6) and (1.8), respectively. Our result in this direction is

Theorem 1. *If $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n such that $p(z) \neq 0$ in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,*

$$\begin{aligned} \max_{|z|=r} |p(z)| &\geq \exp \left\{ -n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \max_{|z|=R} |p(z)| \\ &+ \left[1 - \exp \left\{ -n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \right] \\ &\times \min_{|z|=k} |p(z)|. \end{aligned} \quad (1.12)$$

Equality holds in (1.12) for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ .

Remark 1. As mentioned earlier, Theorem 1 is, in general, an improvement of Theorem A, except in the case when $\frac{|a_\mu|}{|a_0|-m} k^\mu = \frac{n}{\mu}$. To see this, we note that for every $R \leq k$, by Lemma 2.1, $\max_{|z|=R} |p(z)| \geq \min_{|z|=k} |p(z)|$ and hence the function $x \max_{|z|=R} |p(z)| + (1-x) \min_{|z|=k} |p(z)|$ is a non-decreasing function of x . If we combine this fact with Lemma 2.4, according to which

$$\exp \left\{ -n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}},$$

it is easy to conclude that Theorem 1 is an improvement of Theorem A.

Next, as an application of Theorem 1, we prove the following theorem which is an improvement of Theorem B, which in turn, is an improvement and a generalization of (1.7). In fact, we prove

Theorem 2. If $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in the disk $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,

$$\begin{aligned} \max_{|z|=R} |p'(z)| &\leq n \left\{ \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} R^\mu + k^{2\mu} R)} \right\} \\ &\times \exp \left\{ n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \\ &\times \left\{ \max_{|z|=r} |p(z)| - m \right\}, \end{aligned} \quad (1.13)$$

where $m = \min_{|z|=k} |p(z)|$.

Remark 2. To show that Theorem 2 is, in general, an improvement of Theorem B, it is sufficient to prove that

$$\begin{aligned} &\left\{ \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} R^\mu + k^{2\mu} R)} \right\} \\ &\times \exp \left\{ n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \\ &\leq \frac{R^{\mu-1} (R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}. \end{aligned} \quad (1.14)$$

Since $R \leq k$, if we put $t = R$ in (2.5), we have

$$\frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} R^\mu + k^{2\mu} R)} \leq \frac{R^{\mu-1}}{R^\mu + k^\mu} \quad (1.15)$$

and Lemma 2.4 equivalently yields

$$\exp \left\{ n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \leq \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right)^{\frac{n}{\mu}}. \quad (1.16)$$

Inequality (1.14) is true because it is obtained on multiplying the inequalities (1.15) and (1.16).

2. Lemmas

The following lemmas are needed for the proofs of the theorems.

Lemma 2.1. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n , $p(z) \neq 0$ in $|z| < k$, $k > 0$, then

$$|p(z)| \geq m \quad \text{for } |z| \leq k, \quad (2.1)$$

where $m = \min_{|z|=k} |p(z)|$.

This lemma is due to Gardner, Govil and Musukula [6, proof of Lemma 2.6].

Lemma 2.2. If $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n , $p(z) \neq 0$ for $|z| < k$, $k \geq 1$, and if $m = \min_{|z|=k} |p(z)|$, then

$$\frac{|a_\mu| k^\mu}{|a_0| - m} \leq \frac{n}{\mu}. \quad (2.2)$$

The above result is due to Gardner, Govil and Weems [7, proof of Lemma 3].

Lemma 2.3. If $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$ where $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+s_0} \left(\max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right), \quad (2.3)$$

where

$$s_0 = k^{\mu+1} \left(\frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu-1} + 1}{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} + 1} \right).$$

The above lemma is due to Gardner, Govil and Weems [7].

Lemma 2.4. If $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n such that $p(z) \neq 0$ in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,

$$\exp \left\{ -n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}}.$$

Proof. Since $p(z) \neq 0$ in $|z| < k$, $k > 0$, the polynomial $P(z) = p(tz) \neq 0$ in $|z| < \frac{k}{t}$, $\frac{k}{t} \geq 1$, where $0 < t \leq k$. Hence applying Lemma 2.2 to $P(z)$, we get

$$\frac{|a_\mu| t^\mu}{|a_0| - m} \left(\frac{k}{t} \right)^\mu \leq \frac{n}{\mu}, \quad (2.4)$$

where $m = \min_{|z|=\frac{k}{t}} |P(z)| = \min_{|z|=\frac{k}{t}} |p(tz)| = \min_{|z|=k} |p(z)|$.

Now, (2.4) becomes

$$\frac{\mu}{n} \frac{|a_\mu|}{|a_0| - m} k^\mu \leq 1,$$

which is equivalent to

$$\frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t) + k^{\mu+1}} \leq \frac{t^{\mu-1}}{t^\mu + k^\mu}. \quad (2.5)$$

Integrating both sides of (2.5) with respect to t from r to R where $0 < r \leq R \leq k$, we have

$$\int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t) + k^{\mu+1}} dt \leq \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} dt,$$

which is equivalent to

$$-n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t) + k^{\mu+1}} dt \geq -n \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} dt,$$

which implies

$$\begin{aligned} & \exp \left\{ -n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t) + k^{\mu+1}} dt \right\} \\ & \geq \exp \left\{ -\frac{n}{\mu} \int_r^R \frac{\mu t^{\mu-1}}{t^\mu + k^\mu} dt \right\} \\ & = \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}}, \end{aligned}$$

which proves Lemma 2.4 completely. \square

3. Proofs of the theorems

Proof of Theorem 1. Since $p(z)$ has no zero in $|z| < k$, $k > 0$, then for $0 < t \leq k$, $P(z) = p(tz)$ has no zero in $|z| < \frac{k}{t}$, $\frac{k}{t} \geq 1$. Hence using Lemma 2.3, we have

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + \left(\frac{k}{t}\right)^{\mu+1} \left\{ \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} \left(\frac{k}{t}\right)^{\mu-1} + 1}{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} \left(\frac{k}{t}\right)^{\mu+1} + 1} \right\}} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=\frac{k}{t}} |P(z)| \right\},$$

where $m = \min_{|z|=\frac{k}{t}} |P(z)| = \min_{|z|=\frac{k}{t}} |p(tz)| = \min_{|z|=k} |p(z)|$.

This gives

$$t \max_{|z|=t} |p'(z)| \leq n \left\{ \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} \frac{k^{\mu+1}}{t} + 1}{1 + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} \frac{k^{\mu+1}}{t} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} \frac{k^{2\mu}}{t^\mu} + \frac{k^{\mu+1}}{t^{\mu+1}}} \right\} \left\{ \max_{|z|=t} |p(z)| - m \right\},$$

which is clearly equivalent to

$$\begin{aligned} \max_{|z|=t} |p'(z)| & \leq n \left\{ \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^\mu + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{2\mu} t + k^{\mu+1}} \right\} \\ & \times \left\{ \max_{|z|=t} |p(z)| - m \right\}. \end{aligned} \quad (3.1)$$

Now, for $0 < r \leq R \leq k$, and $0 \leq \theta < 2\pi$, we have

$$|p(Re^{i\theta}) - p(re^{i\theta})| \leq \int_r^R |p'(te^{i\theta})| dt,$$

which implies

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R |p'(te^{i\theta})| dt,$$

from which it follows that

$$\max_{|z|=R} |p(z)| \leq \max_{|z|=r} |p(z)| + \int_r^R \max_{|z|=t} |p'(z)| dt. \quad (3.2)$$

Denote $\max_{|z|=r} |p(z)|$ by $M(p, r)$.

Using (3.1) to (3.2), we get

$$\begin{aligned} M(p, R) &\leq M(p, r) + n \left[\int_r^R \left\{ \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^\mu + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{2\mu} t + k^{\mu+1}} \right\} \right. \\ &\quad \left. \times M(p, t) dt - \int_r^R \left\{ \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^\mu + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{2\mu} t + k^{\mu+1}} \right\} m dt \right]. \end{aligned} \quad (3.3)$$

If we denote the R.H.S. of (3.3) by $\phi(R)$, then

$$\begin{aligned} \phi'(R) &= n \left[\left\{ \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu}{R^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^\mu + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{2\mu} R + k^{\mu+1}} \right\} M(p, R) \right. \\ &\quad \left. - \left\{ \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu}{R^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^\mu + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{2\mu} R + k^{\mu+1}} \right\} m \right]. \end{aligned} \quad (3.4)$$

Also, by (3.3), we have

$$M(p, R) \leq \phi(R). \quad (3.5)$$

Using (3.5) to (3.4), we conclude that

$$\phi'(R) - n \left\{ \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu}{R^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^\mu + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{2\mu} R + k^{\mu+1}} \right\} \{\phi(R) - m\} \leq 0. \quad (3.6)$$

Multiplying both sides of (3.6) by

$$\exp \left\{ -n \int \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu}{R^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^\mu + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{2\mu} R + k^{\mu+1}} dR \right\}$$

we get

$$\frac{d}{dR} \left[\{ \phi(R) - m \} \times \exp \left\{ -n \int \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu}{R^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^\mu + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{2\mu} R + k^{\mu+1}} dR \right\} \right] \leq 0. \quad (3.7)$$

It is concluded from (3.7) that the function

$$\psi(R) = \{ \phi(R) - m \} \exp \left\{ -n \int \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu}{R^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^\mu + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{2\mu} R + k^{\mu+1}} dR \right\}$$

is a non-increasing function of R in $(0, k)$. Hence for $0 < r \leq R \leq k$,

$$\psi(r) \geq \psi(R). \quad (3.8)$$

Since $\phi(R) \geq M(p, R)$ and $\phi(r) = M(p, r)$, it follows from (3.8) that

$$M(p, r) \geq M(p, R) \exp \left\{ -n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^\mu + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{2\mu} t + k^{\mu+1}} dt \right\} \\ + \left[1 - \exp \left\{ -n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^\mu + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{2\mu} t + k^{\mu+1}} dt \right\} \right] m.$$

This completes the proof of Theorem 1. \square

Proof of Theorem 2. Since $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, does not vanish in $|z| < k$, $k > 0$, the polynomial $P(z) = p(Rz)$ has no zero in $|z| < \frac{k}{R}$, $\frac{k}{R} \geq 1$. Now, applying Lemma 2.3 to the polynomial $P(z)$, we have

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + (\frac{k}{R})^{\mu+1}} \left\{ \max_{|z|=1} |P(z)| - m \right\},$$

$$1 + (\frac{k}{R})^{\mu+1} \left\{ \frac{\frac{\mu}{n} \frac{|a_\mu| R^\mu}{|a_0|-m} (\frac{k}{R})^{\mu-1} + 1}{\frac{\mu}{n} \frac{|a_\mu| R^\mu}{|a_0|-m} (\frac{k}{R})^{\mu+1} + 1} \right\}$$

where $m = \min_{|z|=\frac{k}{R}} |P(z)| = \min_{|z|=\frac{k}{R}} |p(Rz)| = \min_{|z|=k} |p(z)|$. This gives

$$R \max_{|z|=1} |p'(Rz)| \leq n \frac{\left\{ \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} \frac{k^{\mu+1}}{R} + 1 \right\}}{\left\{ \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} \frac{k^{\mu+1}}{R} + 1 + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} \frac{k^{2\mu}}{R^\mu} + \frac{k^{\mu+1}}{R^{\mu+1}} \right\}} \left\{ \max_{|z|=1} |p(Rz)| - m \right\},$$

which is equivalent to

$$\max_{|z|=R} |p'(z)| \leq n \frac{\left\{ \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu \right\}}{\left\{ R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} R^\mu + k^{2\mu} R) \right\}} \left\{ \max_{|z|=R} |p(z)| - m \right\}. \quad (3.9)$$

Now, if $0 < r \leq R \leq k$, then by Theorem 1, we obtain

$$\begin{aligned} \max_{|z|=R} |p(z)| \leq & \exp \left\{ n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \max_{|z|=r} |p(z)| \\ & + \left[1 - \exp \left\{ n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \right] m. \end{aligned} \quad (3.10)$$

Combining (3.9) and (3.10), we get

$$\begin{aligned} \max_{|z|=R} |p'(z)| \leq & n \frac{\left\{ \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu \right\}}{\left\{ R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} R^\mu + k^{2\mu} R) \right\}} \\ & \times \exp \left\{ n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \\ & \times \left\{ \max_{|z|=r} |p(z)| - m \right\}, \end{aligned}$$

which proves Theorem 2. \square

References

- [1] A. Aziz, W.M. Shah, Inequalities for a polynomial and its derivative, *Math. Inequal. Appl.* 7 (3) (2004) 379–391.
- [2] A. Aziz, B.A. Zargar, Inequalities for a polynomial and its derivative, *Math. Inequal. Appl.* 1 (4) (1998) 543–550.
- [3] M. Bidkham, K.K. Dewan, Inequalities for a polynomial and its derivative, *J. Math. Anal. Appl.* 166 (1992) 319–324.
- [4] K.K. Dewan, Inequalities for a polynomial and its derivative II, *J. Math. Anal. Appl.* 190 (1995) 625–629.
- [5] K.K. Dewan, R.S. Yadav, M.S. Pukhta, Inequalities for a polynomial and its derivative, *Math. Inequal. Appl.* 2 (2) (1999) 203–205.
- [6] Robert B. Gardner, N.K. Govil, Srinath R. Musukula, Rate of growth of polynomials not vanishing inside a circle, *JIPAM. J. Inequal. Pure Appl. Math.* 6 (2) (2005) 1–9.
- [7] Robert B. Gardner, N.K. Govil, Amy Weems, Some results concerning rate of growth of polynomials, *East J. Approx.* 10 (2004) 301–312.
- [8] N.K. Govil, On the maximum modulus of polynomials, *J. Math. Anal. Appl.* 112 (1985) 253–258.
- [9] P.D. Lax, Proof of a conjecture of P. Erdős on the derivative of a polynomial, *Bull. Amer. Math. Soc.* 50 (1944) 509–513.
- [10] M.A. Malik, On the derivative of a polynomial, *J. London Math. Soc.* 1 (1969) 57–60.
- [11] T.J. Rivlin, On the maximum modulus of polynomials, *Amer. Math. Monthly* 67 (1960) 251–253.
- [12] A.C. Schaeffer, Inequalities of A. Markoff and S. Bernstein for polynomials and related functions, *Bull. Amer. Math. Soc.* 47 (1941) 565–579.
- [13] R.S. Varga, A comparison of the successive overrelaxation method and semi-iterative methods using Chebyshev polynomials, *J. Soc. Indust. Appl. Math.* 5 (1957) 44.