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The equations of thermoelasticity with time-dependent coefficients

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ABSTRACT

We consider an inhomogeneous thermoelastic system with second sound in one space dimension where the coefficients are space- and time-dependent. For Dirichlet–Neumann type boundary conditions the global existence of smooth solutions is proved by using the theory of Kato. Then the asymptotic behavior of the solutions is discussed.

1. Introduction

The equations of thermoelasticity describe the elastic and the thermal behavior of elastic, heat conductive media, in particular the reciprocal actions between elastic stresses and temperature differences. This paper is concerned with global existence, uniqueness, and asymptotic behavior of solutions to the linear inhomogeneous equations of one-dimensional thermoelasticity that model the second sound effect. Let $u \equiv u(t,x)$, $\theta \equiv \theta(t,x)$ and $q \equiv q(t,x)$ for $t \geqslant 0$, $x \in \Omega := (0,L) \subset \mathbb{R}$ for some fixed $L > 0$, denote the unknown functions representing the displacement, the temperature difference to a fixed reference temperature, and the heat flux. Then the differential equations for u, θ, q are represented as

$$u_{tt} - au_{xx} + b\theta_x = f_1, \tag{1}$$

$$\theta_t + gq_x + du_{tx} = f_2, \tag{2}$$

$$\tau q_t + q + k\theta_x = f_3. \tag{3}$$

We emphasize that the coefficients are space- and time-dependent, i.e., $a \equiv a(t,x)$, $b \equiv b(t,x)$, $g \equiv g(t,x)$, $d \equiv d(t,x)$, $\tau \equiv \tau(t,x)$, and $k \equiv k(t,x)$. Initial data and boundary conditions are given by

$$u(0,\cdot) = u_0, \qquad u_t(0,\cdot) = u_1, \qquad \theta(0,\cdot) = \theta_0, \qquad q(0,\cdot) = q_0, \tag{4}$$

and

$$au_x(t,0) + b\theta(t,0) = 0, \qquad \theta_x(t,0) = 0, \qquad u(t,L) = \theta(t,L) = 0. \tag{5}$$

The boundary conditions (5) arise in the pulsed laser heating of solids, for instance in laser assisted particle removal from silicon wafers, cf. Refs. [8,14,16].

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Note that the time-dependent coefficients a and b also appear in the first boundary condition. It is difficult to deal with such time-dependent boundary conditions because in general they lead to a time-dependent domain of an associated evolution operator. We succeed in finding a transformation of our problem into an evolution system

$$V_t + A(t)V = F(t), \quad 0 \leqslant t \leqslant T,$$
$$V(0) = V_0$$

where the domain $D(A(t))$ of $A(t)$ is independent of t. In view of the fact that we have to introduce certain Sobolev spaces to present $D(A(t))$ in detail, we refer to Section 3.2. Utilizing our transformation, we prove the existence of a unique, global solution to our problem using the classical theory of Kato.[1] After that we discuss the asymptotic behavior of our solution. In particular we prove that the solution to (1)–(5) decays exponentially if

$$\Lambda(t) := \left(\|f_1\|_{L^2}^2 + \left\|(f_1)_t\right\|_{L^2}^2 + \|f_2\|_{L^2}^2 + \left\|(f_2)_t\right\|_{L^2}^2 + \left\|(f_2)_x\right\|_{L^2}^2 + \|f_3\|_{L^2}^2 + \left\|(f_3)_t\right\|_{L^2}^2 + \left\|(f_3)_x\right\|_{L^2}^2\right)$$

decays exponentially. It will be necessary to construct a certain Lyapunov function and to combine techniques from energy methods and boundary control, cf. [5,10–12].

For the classical homogeneous equations of thermoelasticity with constant coefficients

$$u_{tt} - \alpha u_{xx} + \beta\theta_x = 0,$$
$$\theta_t - \kappa\theta_{xx} + \delta u_{tx} = 0,$$

it is well known that their solutions are exponentially stable for various types of boundary conditions. The latter equations result from replacing Cattaneo's law (3) by Fourier's law

$$q + k\theta_x = 0, \qquad \kappa = gk. \tag{6}$$

This classical model for example is treated in [2] and [5]. For a discussion of the second sound model see Refs. [3,4,13,15]. In [10] Racke gives a detailed discussion of the problem (1)–(4) with one of the following boundary conditions in the homogeneous case with constant coefficients.

(i) $u(t,0) = u(t,L) = q(t,0) = q(t,L) = 0$ for $t \geqslant 0$;
(ii) $u(t,0) = u(t,L) = \theta(t,0) = \theta(t,L) = 0$ for $t \geqslant 0$;
(iii) $\alpha u_x(t,0) + \beta\theta(t,0) = 0$, $\theta_x(t,0) = 0$, $u(t,L) = \theta(t,L) = 0$ for $t \geqslant 0$.

A few remarks on notation: we denote by $L(X,Y)$ (X, Y are Banach spaces) the set of all bounded linear operators from X to Y. The spaces $L^p(\Omega)$, $H^m(\Omega) = H^{m,2}(\Omega)$ and $W^m(\Omega) = W^{m,2}(\Omega)$ denote the standard Lebesgue and Sobolev spaces, cf. [1]. The standard inner product in L^2 is denoted by $\langle\cdot,\cdot\rangle$ and the standard L^2 norm is denoted by $\|\cdot\| = \|\cdot\|_{L^2}$.

The organization of this work is as follows: In Section 2 we will give some technical results. In particular we summarize some of the main results of the theory of Kato. Section 3 is dedicated to the well-posedness of our problem (1)–(5). In Section 4 we discuss the asymptotic behavior of the solutions.

This work extends a diploma thesis at the University of Konstanz [17] where the homogeneous case of (1)–(5) is discussed.

2. Some technical results

In this section, we summarize some technical results that we need to prove the well-posedness of our problem (1)–(5). In particular, we present a theorem of Kato concerning the existence and regularity of solutions to the following abstract linear evolution system:

$$V_t + A(t)V = F(t), \quad 0 \leqslant t \leqslant T,$$

with given initial data

$$V(0) = V_0,$$

where $T > 0$ is an arbitrary but fixed constant.

Definition 2.1. A triple $(A; X_0, Y_1)$, consisting of a family $A = (A(t);\ t \in [0,T])$, and a pair of real separable Banach spaces $Y_1 \subset X_0$, is called a CD-system (introduced by Kato [7] see also [5]) if the following conditions are satisfied:

(i) $A = (A(t);\ t \in [0,T])$ is a stable family of (negative) generators of a C_0 semigroup on X_0, with stability constants M and β.

[1] Note that these ideas can be generalized to three space-dimensions. This will be the content of a forthcoming paper.

(ii) The domain $D(A(t)) = Y_1$ of $A(t)$ is independent of t.
(iii) $\partial_t A \in L^\infty([0,T]; L(Y_1, X_0))$.

Lemma 2.2. *Let $(X, \|\cdot\|)$ be a Banach space. Furthermore let $\|\cdot\|_t$ $(t \in [0,T])$ be equivalent norms to the given norm on X such that*

$$\exists c > 0,\ \forall s, t \in [0,T],\ \forall x \neq 0: \quad \frac{\|x\|_t}{\|x\|_s} \leqslant e^{c|t-s|}.$$

For each fixed $t \in [0,T]$ let $A(t) : D(A(t)) \subset X_t \to X_t$ be the generator of a C_0 semigroup $S_t(s)$, $s \geqslant 0$, satisfying $\|S_t(s)\|_t \leqslant e^{\beta s}$. Then the family $(A(t))_t$ is stable on $(X, \|\cdot\|)$ and also stable in $(X, \|\cdot\|_t)$ for arbitrary $t \in [0,T]$.

A proof for this result can be found in [6].

Lemma 2.3. *Let $s_i(t,\cdot)$ $(i = 1,\dots,4;\ t \in [0,T])$ be real valued functions defined on Ω such that the following properties hold:*

(1) $\forall t \in [0,T]$: $s_i(t,\cdot) \in L^1(\Omega)$,
(2) $\exists C_1, C_2 > 0$: $\forall t \in [0,T]$, $\forall x \in \Omega$: $C_1^2 \leqslant s_i(t,x) \leqslant C_2^2$.

Define $S(t,x) := \operatorname{diag}(s_1, s_2, s_3, s_4)$ and for $V, W \in (L^2(\Omega))^4$ the inner product $\langle U, V\rangle_t := \langle U, SV\rangle$. Then we have

(i) *$C_1\|V\| \leqslant \|V\|_t \leqslant C_2\|V\|$ for $t \in [0,T]$ and $V \in (L^2(\Omega))^4$. In particular $(L^2(\Omega))^4$, $\langle\cdot,\cdot\rangle_t)$ is a Hilbert space.*
(ii) *$(L^2(\Omega))^4 = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_t}$ for $t \in [0,T]$.*

Proof. Let $t \in [0,T]$ be fixed and $V \in (L^2(\Omega))^4$ then we have

$$\begin{aligned} C_1^2\|V\|^2 &= C_1^2 \sum_{i=1}^4 \langle V^i, V^i\rangle = \sum_{i=1}^4 \int_\Omega C_1^2 |V^i(x)|^2\,dx \leqslant \sum_{i=1}^4 \int_\Omega c_i(t,x)|V^i(x)|^2\,dx \\ &= \|V\|_t^2 \leqslant \sum_{i=1}^4 \int_\Omega C_2^2 |V^i(x)|^2\,dx = C_2^2 \sum_{i=1}^4 \langle V^i, V^i\rangle \leqslant C_2^2\|V\|^2; \end{aligned}$$

in detail $C_1^2\|V\|^2 \leqslant \|V\|_t^2 \leqslant C_2^2\|V\|^2$. The second claim is obvious. □

In [9] Pazy gives a proof for the following:

Theorem 2.4. *Let $(A(t))_{t\in[0,T]}$ be a stable family of infinitesimal generators with stability constants M and ω. Let $(B(t))_{t\in[0,T]}$ be bounded linear operators on X. If $\|B(t)\| \leqslant K$ for all $0 \leqslant t \leqslant T$, then $(A(t) + B(t))_{t\in[0,T]}$ is a stable family of infinitesimal generators with stability constants M and $\omega + KM$.*

Theorem 2.5. *Suppose that X_0, Y_1 are real, separable Hilbert spaces. Let $(A; X_0, Y_1)$ be a CD-system. Let $V^0 \in Y_1$, $F \in \mathcal{C}^0([0,T], X_0)$ and $F_t \in L^1([0,T], X_0)$. Then there exists a unique solution*

$$V \in \mathcal{C}^0([0,T], Y_1) \cap \mathcal{C}^1([0,T], X_0), \qquad V(0) = V^0,$$

for the initial value problem

$$V_t + A(t)V = F(t), \quad 0 \leqslant t \leqslant T, \qquad V(0) = V^0. \tag{7}$$

A proof for this result is given in [6]. Next we want to gain more regularity of the solution given in Theorem 2.5. Therefore we introduce a double scale of real Banach spaces X_j, Y_j $(0 \leqslant i, j \leqslant s-1)$ of the following structure

$$\begin{array}{ccccccccc} X_0 & \supset & X_1 & \supset & X_2 & \supset & \cdots & \supset & X_{s-1} \\ X_0 = Y_0 & \supset & Y_1 & \supset & Y_2 & \supset & \cdots & \supset & Y_{s-1}. \end{array}$$

Here it is assumed that all the inclusions are continuous and dense and that, if $s \geqslant 2$, Y_1 is a closed subspace of X_1 and $Y_j = Y_1 \cap X_j$ for $1 \leqslant j \leqslant s-1$. We introduce the following assumptions:

(L1) (Stability) The triple $(A; X_0, Y_1)$ is a CD-system with stability constants M and β.
(L2) (Smoothness) We have

$$\partial_t^r A \in Lip([0,T], L(Y_{j+r+1}; X_j)), \quad 0 \leqslant j \leqslant s-r-1,$$

for $0 \leqslant r \leqslant s-1$. This implies that $\partial_t^{r+1} A \in L^\infty([0,T], L(Y_{j+r+1}; X_j))$ for the same range of r and j.

(L3) (Ellipticity) For a.e. $t \in [0, T]$ and $0 \leqslant j \leqslant s-1$,

$$\phi \in Y_1, \quad A(t)\phi \in X_j \quad \Longrightarrow \quad \phi \in Y_{j+1}, \qquad \|\phi\|_{Y_{j+1}} \leqslant K\big(\|A(t)\phi\|_{X_j} + \|\phi\|_{X_0}\big),$$

where $K > 0$ is a constant.

(L4) Let $\partial_t^k F \in \mathcal{C}^0([0,T], X_{s-1-k})$, $k = 0, \dots, s-1$; $\partial_t^s F \in L^1([0,T], X_0)$.

(A1) (Compatibility condition)

$$V^r := \partial_t^{r-1} F(0) - \sum_{k=0}^{r-1} \binom{r-1}{k} \big(\partial_t^k A\big)(0) V^{r-1-k} \in Y_{s-r}, \quad 0 \leqslant r \leqslant s.$$

A proof for the following result is given in [5].

Theorem 2.6. *Let X_0 and Y_1 be real separable Hilbert spaces. Let the triple (A, X_0, Y_1) be a CD-System such that the conditions* (L1)–(L4) *hold. If $V^0 \in Y_s$, then the solution given by Theorem* 2.5 *belongs to $\mathcal{C}^0([0,T], Y_s)$ (hence $\partial_t^k V \in \mathcal{C}^0([0,T], Y_{s-k})$, $k = 0, \dots, s-1$) if and only if V^0 and F satisfy the compatibility condition* (A1) *with respect to the family A and F.*

3. Well-posedness

We consider the system of hyperbolic thermoelasticity

$$u_{tt} - a(t,x)u_{xx} + b(t,x)\theta_x = f_1, \tag{8}$$

$$\theta_t + g(t,x)q_x + d(t,x)u_{tx} = f_2, \tag{9}$$

$$\tau(t,x)q_t + q + k(t,x)\theta_x = f_3, \tag{10}$$

together with initial conditions

$$u(0,\cdot) = u_0, \qquad u_t(0,\cdot) = u_1, \qquad \theta(0,\cdot) = \theta_0, \qquad q(0,\cdot) = q_0, \tag{11}$$

and boundary conditions

$$(au_x)(t,0) - (b\theta)(t,0) = 0, \qquad u(t,L) = \theta(t,L) = 0, \qquad q(t,0) = 0. \tag{12}$$

Here $a \equiv a(t,x)$, $b \equiv b(t,x)$, $g = g(t,x)$, $d \equiv d(t,x)$, $\tau \equiv \tau(t,x)$, and $k \equiv k(t,x)$ are real-valued functions defined on $[0,T] \times \Omega$. The given functions $f_1 \equiv f_1(t,x)$, $f_2 \equiv f_2(t,x)$, and $f_3 \equiv f_3(t,x)$ are also defined on $[0,T] \times \Omega$.

3.1. Assumption

Let $s \geqslant 1$ and

$$\partial_t^r a(t,\cdot), \partial_t^r b(t,\cdot), \partial_t^r g(t,\cdot), \partial_t^r d(t,\cdot), \partial_t^r k(t,\cdot), \partial_t^r \tau(t,\cdot) \in L^\infty\big([0,T], H^{s-r+1}(\Omega)\big)$$

for $0 \leqslant r \leqslant s+1$ as well as

$$\partial_t^r \partial_x a(t,\cdot), \partial_t^r \partial_x b(t,\cdot), \partial_t^r \partial_x g(t,\cdot), \partial_t^r \partial_x d(t,\cdot) \in L^\infty\big([0,T], H^{s-r}(\Omega)\big)$$

for $0 \leqslant r \leqslant s$.

Furthermore, let C_a,C^a, C_b, C^b, C_g, C^g, C_d, C^d, C_τ, C^τ, C_k, C^k be positive constants such that for all $(t,x) \in [0,T] \times \Omega$ the following inequalities hold:

$$\begin{aligned} &C_a \leqslant a(t,x) \leqslant C^a, \qquad C_b \leqslant b(t,x) \leqslant C^b, \\ &C_g \leqslant g(t,x) \leqslant C^g, \qquad C_d \leqslant d(t,x) \leqslant C^d, \\ &C_k \leqslant k(t,x) \leqslant C^k, \qquad C_\tau \leqslant \tau(t,x) \leqslant C^\tau. \end{aligned} \tag{13}$$

3.2. Existence

Let (u, θ, q) be a solution to (8)–(12) and let

$$V \equiv V(t,x) := \begin{pmatrix} \frac{a}{b} u_x \\ u_t \\ \theta \\ \frac{g}{d} q \end{pmatrix}(t,x), \qquad V_0 \equiv V_0(x) := \begin{pmatrix} (\frac{a}{b})(0,x) u_{0,x}(x) \\ u_1(x) \\ \theta_0(x) \\ (\frac{g}{d})(0,x) q_0(x) \end{pmatrix}.$$

Then V satisfies

$$V_t + AV = F, \qquad V(0) = V_0, \tag{14}$$

where $A \equiv A(t,x)$ is defined as $A = Q^{-1}(N_0 + N_1)$. Here F, Q, N_0, and N_1 are defined as follows:

$$F \equiv F(t,x) = \begin{pmatrix} 0 \\ f_1 \\ f_2 \\ \frac{g}{d\tau} f_3 \end{pmatrix}, \qquad N_0 \equiv N_0(t,x) = \begin{pmatrix} -(\frac{a}{b})_t \frac{b^2}{a^2} & 0 & 0 & 0 \\ (\frac{a}{b})_x \frac{b}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\frac{g}{d})_x \frac{d}{g} \\ 0 & 0 & 0 & -(\frac{g}{d})_t \frac{d^2\tau}{g^2 k} \end{pmatrix}(t,x),$$

$$Q^{-1} \equiv Q^{-1}(t,x) = \begin{pmatrix} \frac{a}{b} & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & \frac{gk}{d\tau} \end{pmatrix}(t,x), \qquad N_1 \equiv N_1(t,x) = \begin{pmatrix} 0 & -\partial_x & 0 & 0 \\ -\partial_x & 0 & \partial_x & 0 \\ 0 & \partial_x & 0 & \partial_x \\ 0 & 0 & \partial_x & \frac{d}{gk} \end{pmatrix}(t,x).$$

In the following we will prove an existence theorem for (14) under the assumption in Section 3.1.

In view of (13) we can choose $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \leqslant \frac{b}{a}, \frac{1}{b}, \frac{1}{d}, \frac{d\tau}{gk} \leqslant C_2$$

holds for arbitrary $(t,x) \in [0,T] \times \Omega$. Now we define for $U, V \in (L^2(\Omega))^4$ the inner product

$$\langle U, V\rangle_t := \big\langle U, Q(t,\cdot)V\big\rangle.$$

Let $t \in [0,T]$ be fixed. In view of Lemma 2.3 we conclude that $((L^2(\Omega))^4, \langle\cdot,\cdot\rangle_t)$ is a Hilbert space, which we denote by $\mathcal{H}_t$. With the help of the spaces

$$\mathcal{C}^\infty_{\mathfrak{l}}(\Omega) := \big\{u \in \mathcal{C}^\infty(\Omega) \bigm| \exists\lambda \in \Omega\colon \forall x \in (0,\lambda)\colon u(x) = 0\big\},$$
$$\mathcal{C}^\infty_{\mathfrak{r}}(\Omega) := \big\{u \in \mathcal{C}^\infty(\Omega) \bigm| \exists\lambda \in \Omega\colon \forall x \in (\lambda,L)\colon u(x) = 0\big\}$$

we define the Sobolev spaces

$$W^1_{\mathfrak{l}}(\Omega) := \left\{u \in L^2(\Omega) \,\middle|\, u \in H^1(\Omega),\ \forall\varphi \in \mathcal{C}^\infty_{\mathfrak{r}}(\Omega) \cap H^1(\Omega)\colon \int_\Omega u\varphi'\,\mathrm{d}x = -\int_\Omega u'\varphi\,\mathrm{d}x\right\},$$

$$H^1_{\mathfrak{l}}(\Omega) := \overline{\mathcal{C}^\infty_{\mathfrak{l}}(\Omega) \cap W^1(\Omega)}^{\|\cdot\|_{W^1}}$$

and analogously $W^1_{\mathfrak{r}}(\Omega)$ and $H^1_{\mathfrak{r}}(\Omega)$. These spaces are Hilbert spaces generalizing one-sided boundary conditions. By using standard arguments we obtain that $W^1_{\mathfrak{l}}(\Omega) = H^1_{\mathfrak{l}}(\Omega)$ and that

$$W^1_{\mathfrak{l}}(\Omega) = \left\{u \in L^2(\Omega) \,\middle|\, u \in H^1(\Omega),\ \forall\varphi \in W^1_{\mathfrak{r}}(\Omega)\colon \int_\Omega u\varphi'\,\mathrm{d}x = -\int_\Omega u'\varphi\,\mathrm{d}x\right\}.$$

By utilizing the matrices Q^{-1} and N_1 we define the operator

$$A_1(t) : D\big(A_1(t)\big) \subset \mathcal{H}_t \to \mathcal{H}_t$$

with domain

$$D\big(A_1(t)\big) := \big\{\big(V^1, V^2, V^3, V^4\big) \in \mathcal{H}_t\colon\ V^1 - V^3 \in H^1_{\mathfrak{l}}(\Omega),\ V^2, V^3 \in H^1_{\mathfrak{r}}(\Omega),\ V^4 \in H^1_{\mathfrak{l}}(\Omega)\big\}$$

by

$$A_1(t)f := Q^{-1}(t,\cdot)N_1(t,\cdot)f. \tag{15}$$

Our next aim is to show that the operator $-A_1(t)$ generates a C_0 semigroup of contractions on $\mathcal{H}_t$ for every fixed $t \in [0,T]$. Then we conclude that $-A(t)$ generates a C_0 semigroup of contractions on $(L^2(\Omega))^4$ for every fixed $t \in [0,T]$. Finally we show that the family $(-A_1(t))_t$ is a stable family of generators of a C_0 semigroup on $(L^2(\Omega))^4$.

In order to show that the operator $-A_1(t)$ generates a C_0 semigroup of contractions we show that $-A_1(t)$ is densely defined and closed. Furthermore, we show that both $-A_1(t)$ and its adjoint operator $(-A_1(t))^*$ are dissipative.

Lemma 3.1. *Let $A_1(t)$ be defined as in* (15)*. Then $-A_1(t)$ is densely defined.*

Proof. It is easy to see that $\mathcal{C}_0^\infty(\Omega) \subset D(A_1(t))$. Thus the density of $-A_1(t)$ is a consequence of Lemma 2.3. $\square$

Lemma 3.2. *Let $A_1(t)$ be defined as in* (15). *Then $-A_1(t)$ is closed.*

Proof. Let $(V_n)_{n\in\mathbb{N}} \subset D(A_1(t))$ be a sequence with $V_n \to V \in \mathcal{H}_t$ and $A_1(t)V_n \to W \in \mathcal{H}_t$ as $n \to \infty$. Then we have that

$$\forall \Phi \in \mathcal{H}_t\colon \quad \langle -A_1(t)V_n, \Phi\rangle_t \to \langle W, \Phi\rangle_t \tag{16}$$

as $n \to \infty$. In other words,

$$\begin{aligned} &\langle \partial_x V_n^2, \Phi^1\rangle + \langle \partial_x V_n^1 - \partial_x V_n^3, \Phi^2\rangle - \langle \partial_x V_n^2 + \partial_x V_n^4, \Phi^3\rangle - \left\langle \partial_x V_n^3 + \frac{d}{gk} V_n^4, \Phi^4\right\rangle \\ &\quad \to \left\langle W^1, \frac{b}{a}\Phi^1\right\rangle + \left\langle W^2, \frac{1}{b}\Phi^2\right\rangle + \left\langle W^3, \frac{1}{d}\Phi^3\right\rangle + \left\langle W^4, \frac{d\tau}{gk}\Phi^4\right\rangle \end{aligned} \tag{17}$$

as $n \to \infty$.

(a) Choosing $\Phi = (\Phi^1, 0, 0, 0)$ with $\Phi^1 \in L^2(\Omega)$ we obtain with the help of (17):

$$\langle \partial_x V_n^2, \Phi^1\rangle \to \left\langle W^1, \frac{b}{a}\Phi^1\right\rangle$$

as $n \to \infty$. Now, choosing $\Phi^1 \in \mathcal{C}_0^\infty(\Omega)$, we conclude that

$$-\langle V^2, \partial_x \Phi^1\rangle = -\lim_{n\to\infty}\langle V_n^2, \partial_x\Phi^1\rangle = \lim_{n\to\infty}\langle \partial_x V_n^2, \Phi^1\rangle = \left\langle \frac{b}{a} W^1, \Phi^1\right\rangle.$$

So, we have $V^2 \in H^1(\Omega)$ with $\partial_x V^2 = \frac{b}{a} W^1$. Choosing $\Phi^1 \in H^1_{\mathfrak{l}}(\Omega)$ we obtain $V^2 \in H^1_{\mathfrak{r}}(\Omega)$. Summarizing results in

$$V^2 \in H^1_{\mathfrak{r}}(\Omega) \quad \text{and} \quad \partial_x V^2 = \frac{b}{a} W^1.$$

(b) Choosing $\Phi := (0, 0, \Phi^3, 0)$ with $\Phi^3 \in L^2(\Omega)$ we obtain with the help of (17) that

$$-\langle \partial_x V_n^2 + \partial_x V_n^4, \Phi^3\rangle \to \left\langle W^3, \frac{1}{d}\Phi^3\right\rangle.$$

Assuming that $\Phi^3 \in \mathcal{C}_0^\infty(\Omega)$ yields

$$\langle V^4, \partial_x\Phi^3\rangle = \left\langle \frac{b}{a} W^1 + \frac{1}{d} W^3, \Phi^3\right\rangle.$$

Now, choosing $\Phi^3 \in H^1_{\mathfrak{r}}(\Omega)$ we get that

$$V^4 \in H^1_{\mathfrak{l}}(\Omega) \quad \text{and} \quad -\partial_x V^4 = \frac{1}{d} W^3 + \partial_x V^2.$$

(c) Now, choosing $\Phi := (0, 0, 0, \Phi^4)$ with $\Phi^4 \in L^2(\Omega)$ we obtain with the help of (17) that

$$-\left\langle \partial_x V_n^3 + \frac{d}{gk} V_n^4, \Phi^4\right\rangle \to \left\langle W^4, \frac{d\tau}{gk}, \Phi^4\right\rangle.$$

Similarly to (a) and (b) we deduce

$$V^3 \in H^1_{\mathfrak{r}}(\Omega) \quad \text{and} \quad -\partial_x V^3 = \frac{d\tau}{gk} W^4 + \frac{d}{gk} V^4.$$

(d) Finally, choosing $\Phi := (0, \Phi^2, 0, 0)$ with $\Phi^2 \in L^2(\Omega)$ we get with the help of (17) that

$$\langle \partial_x V_n^1 - \partial_x V_n^3, \Phi^2\rangle \to \left\langle W^2, \frac{1}{b}\Phi^2\right\rangle.$$

Summarizing then yields

$$V^1 - V^3 \in H^1_{\mathfrak{l}}(\Omega) \quad \text{and} \quad \partial_x\big(V^1 - V^3\big) = \frac{1}{b} W^2.$$

Overall we have $V \in D(A_1(t))$ and $-A_1(t)V = W$. This completes the proof. $\square$

Lemma 3.3. *Let $A_1(t)$ be defined as in* (15). *Then we have that $D(-A_1(t)) = D((-A_1(t))^*)$ and that*

$$\big(-A_1(t)\big)^* = Q^{-1}\begin{pmatrix} 0 & -\partial_x & 0 & 0 \\ -\partial_x & 0 & \partial_x & 0 \\ 0 & \partial_x & 0 & \partial_x \\ 0 & 0 & \partial_x & -\frac{d}{gk} \end{pmatrix}.$$

Proof. Similarly to the proof of Lemma 3.2 we obtain this claim. □

Lemma 3.4. *Let $A_1(t)$ be defined as in* (15). *Then both $-A_1(t)$ and $(-A_1(t))^*$ are dissipative.*

Proof. Let $V \in D(A_1(t))$.

$$\begin{aligned}\big\langle -A_1(t)V, V\big\rangle_t &= \big\langle -Q^{-1}(t)N_1(t)V, V\big\rangle_t = \big\langle -N_1(t)V, V\big\rangle \\ &= -\bigg(-\big\langle \partial_x V^2, V^1\big\rangle + \big\langle V^1, \partial_x V^2\big\rangle + \big\langle V^3, \partial_x V^2\big\rangle - \big\langle \partial_x V^2, V^3\big\rangle - \big\langle V^4, \partial_x V^3\big\rangle + \big\langle \partial_x V^3, V^4\big\rangle + \bigg\langle \frac{d}{gk}V^4, V^4\bigg\rangle\bigg).\end{aligned}$$

We conclude that $\mathrm{Re}\langle -A_1(t)V, V\rangle_t = -\int_\Omega \frac{d}{gk}|V^4|^2\,dx$ and the proof is completed. □

This implies

Theorem 3.5. *Let $t \in [0, T]$ be fixed and*

$$A_1(t) : D\big(A_1(t)\big) \subset \mathcal{H}_t \to \mathcal{H}_t$$

with

$$D\big(A_1(t)\big) := \big\{\big(V^1, V^2, V^3, V^4\big) \in \mathcal{H}_t\colon\ V^1 - V^3 \in H^1_{\mathfrak{l}}(\Omega),\ V^2, V^3 \in H^1_{\mathfrak{r}}(\Omega),\ V^4 \in H^1_{\mathfrak{l}}(\Omega)\big\}$$

be defined as $A_1(t)f := Q^{-1}(t,\cdot)N_1(t,\cdot)f$. Then the following statements hold:

(i) *$-A_1(t)$ is a generator of a C_0 semigroup of contractions on $(\mathcal{H}_t, \langle\cdot,\cdot\rangle_t)$.*
(ii) *The family $(-A_1(t))_{t\in[0,T]}$ is a stable family of generators of a C_0 semigroup on the Hilbert space $((L^2(\Omega))^4, \langle\cdot,\cdot\rangle)$.*

Proof.

(i) This statement is a direct consequence of Lemmas 3.1–3.4.
(ii) Let $V \in (L^2(\Omega))^4$ with $V \neq 0$. We consider the function

$$f_V : [0, T] \to \mathbb{R}, \qquad t \mapsto \ln\big(\|V\|_t^2\big).$$

Obviously f_V is continuously differentiable. Furthermore, we have

$$\big|f_V'(t)\big| \leqslant \frac{\int_\Omega (|(\frac{b}{a})_t||V^1|^2 + |(\frac{1}{b})_t||V^2|^2 + |(\frac{1}{d})_t||V^3|^2 + |(\frac{d\tau}{gk})_t||V^4|^2)\,dx}{\int_\Omega (\frac{b}{a}|V^1|^2 + \frac{1}{b}|V^2|^2 + \frac{1}{d}|V^3|^2 + \frac{d\tau}{gk}|V^4|^2)\,dx} \leqslant C$$

for a certain constant $C > 0$. The existence of such a constant follows direct from the assumption in Section 3.1. Now we conclude for $s, t \in [0, T]$ with $s < t$ by using the mean value theorem that

$$\exists \xi \in [s,t]\colon \quad \frac{|\ln(\|V\|_t^2) - \ln(\|V\|_s^2)|}{|t-s|} = \left|\frac{f_V(t) - f_V(s)}{t-s}\right| = \big|f_V'(\xi)\big|.$$

This implies

$$\ln\big(\|V\|_t^2\big) - \ln\big(\|V\|_s^2\big) \leqslant \big|\ln\big(\|V\|_t^2\big) - \ln\big(\|V\|_s^2\big)\big| \leqslant C|t-s|$$

for $s, t \in [0, T]$. Since the exponential function is monotone we deduce

$$\frac{\|V\|_t^2}{\|V\|_s^2} \leqslant e^{c|t-s|}$$

for $s, t \in [0, T]$. Applying Lemma 2.2 the claim follows. □

Our next aim is to show that $(-A(t))_{t\in[0,T]}$ is also a stable family of generators of a C_0 semigroup. For this purpose we intend to apply Theorem 2.4. Thus, we have to show that

$$A_0(t,x) := Q^{-1}(t,x)N_0(t,x) \tag{18}$$

is a bounded linear operator for every fixed $t \in [0,T]$ and that the family $(A_0(t))_{t\in[0,T]}$ is uniformly bounded in t.

Lemma 3.6. *For every fixed $t \in [0,T]$ the operator $A_0(t,x)$ as defined above is bounded. Furthermore, there exists a constant $K > 0$ such that $\|A_0(t)\| \leqslant K$ for every $t \in [0,T]$.*

Proof. Let $t \in [0,T]$ be fixed. Obviously we have that

$$A_0(t,x) = \begin{pmatrix} -(\frac{a}{b})_t \frac{b}{a} & 0 & 0 & 0 \\ (\frac{a}{b})_x \frac{b^2}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\frac{g}{d})_x \frac{d^2}{g} \\ 0 & 0 & 0 & -(\frac{g}{d})_t \frac{d}{g} \end{pmatrix}.$$

Choosing $\xi \in (L^2(\Omega))^4$ we obtain

$$\|A_0(t,\cdot)\xi(\cdot)\|^2 = \left\|\left(\frac{a}{b}\right)_t \frac{b}{a}\xi_1\right\|^2 + \left\|\left(\frac{a}{b}\right)_x \frac{b^2}{a}\xi_1\right\|^2 + \left\|\left(\frac{g}{d}\right)_x \frac{d^2}{g}\xi_4\right\|^2 + \left\|\left(\frac{g}{d}\right)_t \frac{d}{g}\xi_4\right\|^2 \leqslant 4C\|\xi\|^2.$$

Observe, that here the constant C can be chosen independently of t in view of the assumption in Section 3.1. Setting $K := 2C$, we obtain $\|A_0(t,\cdot)\xi\| \leqslant K\|\xi\|$. □

We define for

$$V, W \in \mathcal{D} := \{(V^1, V^2, V^3, V^4) \in \mathcal{H}_t\colon\ V^1 - V^3 \in H^1_{\mathfrak{l}}(\Omega),\ V^2, V^3 \in H^1_{\mathfrak{r}}(\Omega),\ V^4 \in H^1_{\mathfrak{l}}(\Omega)\}$$

the inner product $\langle V, W\rangle_{\mathcal{D}}$ as

$$\langle U, V\rangle_{\mathcal{D}} := \sum_{i=1}^{4} \langle U^i, V^i\rangle_{H^1(\Omega)}.$$

Lemma 3.7. *$(\mathcal{D}, \langle\cdot,\cdot\rangle_{\mathcal{D}})$ is a Hilbert space.*

Proof. Note that for $V \in \mathcal{D}$ the norm $\|V\|_{\mathcal{D}}$ is given by

$$\|V\|^2_{\mathcal{D}} = \|V^1\|^2_{H^1(\Omega)} + \|V^2\|^2_{H^1(\Omega)} + \|V^3\|^2_{H^1(\Omega)} + \|V^4\|^2_{H^1(\Omega)}.$$

Let $(V_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{D}$. Then the $(V^i_n)_{n\in\mathbb{N}}$ for $i = 2,3$ are Cauchy sequences in $H^1_{\mathfrak{r}}(\Omega)$, $(V^4_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1_{\mathfrak{l}}(\Omega)$, and $(V^1_n - V^3_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1_{\mathfrak{l}}(\Omega)$. This implies the convergence of $(V_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$. □

Employing Lemma 2.5 we obtain the existence of a solution of our problem, if we assume that $A \in \mathit{Lip}([0,T]; L(Y_1, X_0))$. We will discuss this point later. First we want to gain more regularity for our solution. To this end we define for $s \geqslant 2$,

$$X_0 := (L^2(\Omega))^4, \qquad X_1 := (H^1(\Omega))^4, \qquad X_2 := (H^2(\Omega))^4, \quad \ldots, \quad X_{s-1} := (H^{s-1}(\Omega))^4,$$

and

$$Y_0 := X_0, \qquad Y_1 := \mathcal{D} \quad \text{as well as} \quad Y_j := Y_1 \cap X_j \quad \text{for } 1 \leqslant j \leqslant s-1.$$

It is clear that

$$X_{s-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0$$

and

$$Y_{s-1} \subset \cdots \subset Y_2 \subset Y_1 \subset Y_0 = X_0$$

and that all the inclusions are continuous.

Lemma 3.8. *For arbitrary $i = 1, \dots, s-1$ we have*

$$X_{i-1} = \overline{X_i}^{\|\cdot\|_{X_{i-1}}} \quad \text{and} \quad Y_{i-1} = \overline{Y_i}^{\|\cdot\|_{Y_{i-1}}}.$$

Proof. The first claim is obvious. In order to see the second claim we define

$$\mathcal{D}^\infty := \{(V^1, V^2, V^3, V^4):\ V^1 - V^3 \in \mathcal{C}_{\mathfrak{l}}^\infty(\Omega),\ V^2, V^3 \in \mathcal{C}_{\mathfrak{r}}^\infty(\Omega),\ V^4 \in \mathcal{C}_{\mathfrak{l}}^\infty(\Omega)\}.$$

We easily deduce that $\mathcal{D}^\infty \subset Y_i$ for $i = 1, \dots, s-1$. Let $1 \leqslant i \leqslant s-1$ be fixed and $(V^1, V^2, V^3, V^4) \in Y_i$. Obviously there exists a sequence $(V_n^1)_n \subset \mathcal{C}_{\mathfrak{l}}^\infty(\Omega)$ with $V_n^1 \to_{H^i} V^1 - V^3$, and sequences $(V_n^2)_n \subset \mathcal{C}_{\mathfrak{r}}^\infty(\Omega)$, $(V_n^3)_n \subset \mathcal{C}_{\mathfrak{r}}^\infty(\Omega)$ and $(V_n^4)_n \subset \mathcal{C}_{\mathfrak{l}}^\infty(\Omega)$ with $V_n^2 \to_{H^i} V^2$, $V_n^3 \to_{H^i} V^3$ and $V_n^4 \to_{H^i} V^4$. For $((V_n^1 + V_n^3, V_n^2, V_n^3, V_n^4))_n$ we have that $((V_n^1 + V_n^3, V_n^2, V_n^3, V_n^4))_n \subset \mathcal{D}^\infty$, that $(V_n^1 + V_n^3, V_n^2, V_n^3, V_n^4) \to_{H^i} (V^1, V^2, V^3, V^4)$. □

Lemma 3.9. *Let $s \geqslant 1$ and $0 \leqslant r \leqslant s-1$. Then we have*

$$\partial_t^{r+1} A \in L^\infty([0,T], L(Y_{j+r+1}; X_j)), \quad \text{for } 0 \leqslant j \leqslant s-r-1.$$

Proof. Observe that

$$\partial_t^{r+1} A(t) = \partial_t^{r+1} \begin{pmatrix} -(\frac{a}{b})_t \frac{b}{a} & -\frac{a}{b}\partial_x & 0 & 0 \\ (\frac{a}{b})_x \frac{b^2}{a} - b\partial_x & 0 & b\partial_x & 0 \\ 0 & d\partial_x & 0 & -(\frac{g}{d})_x \frac{d^2}{g} + d\partial_x \\ 0 & 0 & \frac{gk}{d\tau} & -(\frac{g}{d})_t \frac{d}{g} + \frac{1}{\tau} \end{pmatrix}.$$

In the following we prove that there is a constant $C > 0$ such that

$$\|\partial_t^{r+1} A(t) V\|_{H^j}^2 \leqslant C \|V\|_{Y_{j+r+1}} \tag{19}$$

for all $t \in [0,T]$ and arbitrary $V \in Y_{j+r+1}$. In the next estimate we order the terms with respect to the order of the derivatives, i.e., first we write the ones with lowest order, then the ones with second lowest order, a.s.o. Note that the terms that are not written explicitly can be treated in a similar way.

$$\begin{aligned} \|\partial_t^{r+1} A(t) V\|_{H^j}^2 \leqslant{}& \sum_{l=0}^{j} \sum_{k=0}^{l} \binom{l}{k} \left\| \partial_x^k \left[\partial_t^{r+1} \left[\left(\frac{a}{b}\right)_t \frac{b}{a} \right] \right] \partial_x^{l-k} V^1 \right\|^2 + \dots + \sum_{l=0}^{j} \sum_{k=0}^{l} \binom{l}{k} \left\| \partial_x^k \left[\partial_t^{r+1} \left[\frac{1}{\tau} \right] \right] \partial_x^{l-k} V^4 \right\|^2 \\ &+ \sum_{l=0}^{j} \sum_{k=0}^{l} \binom{l}{k} \|\partial_x^k [\partial_t^{r+1} b] \partial_x^{l-k+1} V^1\|^2 + \dots + \sum_{l=0}^{j} \sum_{k=0}^{l} \binom{l}{k} \|\partial_x^k [\partial_t^{r+1} d] \partial_x^{l-k+1} V^4\|^2. \end{aligned}$$

No we distinguish the following two cases:

(i) The case $r > 0$. Here we have $V_i \in H^{j+2}(\Omega)$ for $i = 1, \dots, 4$. Therefore in the terms above there appear only derivatives of order less or equal to $j+1$. So, relation (19) follows by Sobolev's imbedding theorem.

(ii) The case $r = 0$. In the case that $j \neq s-1$ we can estimate the derivatives of the coefficients again by Sobolev's imbedding theorem. In the case that $j = s+1$ the derivatives of V^i in the terms with highest order derivatives of the coefficients only have order one. Hence they also can be estimated by Sobolev's imbedding theorem. □

Lemma 3.10. *For $t \in [0,T]$ and $0 \leqslant j \leqslant s-1$ we have the following statement: Let $V \in Y_1$ and $A(t)V \in X_j$. Then we have $V \in Y_{j+1}$ and that*

$$\|V\|_{Y_{j+1}} \leqslant K(\|A(t)V\|_{X_j} + \|V\|_{X_0}).$$

Proof. The first claim is obtained successively. We prove the estimate by induction over j.

1. $j = 0$. Note that by

$$A(t)V = \begin{pmatrix} (\frac{a}{b})_t \frac{b}{a} V^1 - \frac{a}{b}\partial_x V^2 \\ b\partial_x V^1 + (\frac{a}{b})_x \frac{b^2}{a} V^1 + b\partial_x V^3 \\ d\partial_x V^2 + d\partial_x V^4 - (\frac{g}{d})_x \frac{d^2}{g} V^4 \\ \frac{gk}{d\tau}\partial_x V^3 - (\frac{g}{d})_t \frac{d}{g} V^4 + \frac{1}{\tau} V^4 \end{pmatrix}$$

we can estimate

$$\left\|-\frac{a}{b}\partial_x V^2\right\|^2 = \left\|\left(\frac{a}{b}\right)_t\frac{b}{a}V^1 - \frac{a}{b}\partial_x V^2 - \left(\frac{a}{b}\right)_t\frac{b}{a}V^1\right\|^2 \leqslant 2\left\|A(t)V\right\|^2 + 2\left\|\left(\frac{a}{b}\right)_t\frac{b}{a}V^1\right\|^2.$$

This implies

$$\exists K_1 > 0:\quad \left\|\partial_x V^2\right\|^2 \leqslant K_1\left(\left\|A(t)V\right\|^2 + \|V\|^2\right). \tag{20}$$

Furthermore,

$$\begin{aligned}\left\|d\partial_x V^4\right\|^2 &= \left\|d\partial_x V^2 + d\partial_x V^4 - \left(\frac{g}{d}\right)_x\frac{d^2}{g}V^4 - d\partial_x V^2 + \left(\frac{g}{d}\right)_x\frac{d^2}{g}V^4\right\|^2\\ &\leqslant 2\left\|A(t)V\right\|^2 + 4\left\|d\partial_x V^2\right\|^2 + 4\left\|\left(\frac{g}{d}\right)_x\frac{d^2}{g}V^4\right\|^2\end{aligned}$$

which yields

$$\exists K_2 > 0:\quad \left\|\partial_x V^4\right\|^2 \leqslant K_2\left(\left\|A(t)V\right\|^2 + \|V\|^2\right). \tag{21}$$

Next we estimate

$$\begin{aligned}\left\|\frac{gk}{d\tau}\partial_x V^3\right\|^2 &= \left\|\frac{gk}{d\tau}\partial_x V^3 - \left(\frac{g}{d}\right)_t\frac{d}{g}V^4 + \frac{1}{\tau}V^4 + \left(\frac{g}{d}\right)_t\frac{d}{g}V^4 - \frac{1}{\tau}V^4\right\|^2\\ &\leqslant 2\left\|\frac{gk}{d\tau}\partial_x V^3 - \left(\frac{g}{d}\right)_t\frac{d}{g}V^4 + \frac{1}{\tau}V^4\right\|^2 + \left\|\left(\frac{g}{d}\right)_t\frac{d}{g}V^4 - \frac{1}{\tau}V^4\right\|^2\end{aligned}$$

and obtain

$$\exists K_3 > 0:\quad \left\|\partial_x V^3\right\|^2 \leqslant K_3\left(\left\|A(t)V\right\|^2 + \|V\|^2\right). \tag{22}$$

Finally, by

$$\begin{aligned}\left\|b\partial_x V^1\right\|^2 &= \left\|b\partial_x V^1 + \left(\frac{a}{b}\right)_x\frac{b^2}{a}V^1 + b\partial_x V^3 - \left(\frac{a}{b}\right)_x\frac{b^2}{a}V^1 - b\partial_x V^3\right\|^2\\ &\leqslant 2\left\|b\partial_x V^1 + \left(\frac{a}{b}\right)_x\frac{b^2}{a}V^1 + b\partial_x V^3\right\|^2 + 2\left\|-\left(\frac{a}{b}\right)_x\frac{b^2}{a}V^1 - b\partial_x V^3\right\|^2\end{aligned}$$

we may conclude that

$$\exists K_4 > 0:\quad \left\|\partial_x V^1\right\|^2 \leqslant K_4\left(\left\|A(t)V\right\|^2 + \|V\|^2\right). \tag{23}$$

By the estimates (20)–(23) we obtain the existence of a constant $K > 0$ such that

$$\|V\|_{H^1}^2 \leqslant K\left(\left\|A(t)V\right\|^2 + \|V\|^2\right).$$

2. For the step $j \rightsquigarrow j+1$ we assume that $\|V\|_{H^m} \leqslant K(\|A(t)V\|_{H^{m-1}} + \|V\|)$ for all $0 \leqslant m \leqslant j$. First we conclude the following identity:

$$\partial_x^j\left(A(t)V\right) = \begin{pmatrix} \sum_{k=0}^j \binom{j}{k}\partial_x^k((\frac{a}{b})_t\frac{b}{a})\partial_x^{j-k}V^1 - \sum_{k=1}^j \binom{j}{k}\partial_x^k(\frac{a}{b})\partial_x^{j-k+1}V^2 - \frac{a}{b}\partial_x^{j+1}V^2 \\ \sum_{k=1}^j \binom{j}{k}\partial_x^k b\partial_x^{j-k+1}V^1 + b\partial_x^{j+1}V^1 + \sum_{k=0}^j \binom{j}{k}\partial_x^k((\frac{a}{b})_x\frac{b^2}{a})\partial_x^{j-k}V^1 \\ \quad + \sum_{k=1}^j \binom{j}{k}\partial_x^k b\partial_x^{j-k+1}V^3 + b\partial_x^{j+1}V^3 \\ \sum_{k=1}^j \binom{j}{k}\partial_x^k d\partial_x^{j-k+1}V^2 + d\partial_x^{j+1}V^2 + \sum_{k=1}^j \binom{j}{k}\partial_x^k d\partial_x^{j-k+1}V^4 \\ \quad + d\partial_x^{j+1}V^4 - \sum_{k=0}^j \partial_x^k((\frac{g}{d})_x\frac{d^2}{g})\partial_x^{j-k}V^4 \\ \sum_{k=1}^j \binom{j}{k}\partial_x^k(\frac{gk}{d\tau})\partial_x^{j-k+1}V^3 - \sum_{k=0}^j \binom{j}{k}\partial_x^k((\frac{g}{d})_t\frac{d}{g})\partial_x^{j-k}V^4 \\ \quad + \frac{gk}{d\tau}\partial_x^{j+1}V^3 + \sum_{k=0}^j \binom{j}{k}\partial_x^k(\frac{1}{\tau})\partial_x^{j-k}V^4 \end{pmatrix}.$$

We can estimate with the help of the induction hypothesis

$$\left\|\frac{a}{b}\partial_x^{j+1}V^2\right\|^2 \leqslant 2\left\|\sum_{k=0}^{j}\binom{j}{k}\partial_x^k\left(\left(\frac{a}{b}\right)_t\frac{b}{a}\right)\partial_x^{j-k}V^1-\sum_{k=1}^{j}\binom{j}{k}\partial_x^k\left(\frac{a}{b}\right)\partial_x^{j-k+1}V^2-\frac{a}{b}\partial_x^{j+1}V^2\right\|$$
$$+2\left\|\sum_{k=0}^{j}\binom{j}{k}\partial_x^k\left(\left(\frac{a}{b}\right)_t\frac{b}{a}\right)\partial_x^{j-k}V^1-\sum_{k=1}^{j}\binom{j}{k}\partial_x^k\left(\frac{a}{b}\right)\partial_x^{j-k+1}V^2\right\|.$$

This implies the existence of a constant $K_1>0$ with

$$\left\|\partial_x^{j+1}V^2\right\|^2 \leqslant K_1\left(\left\|\partial_x^jA(t)\right\|^2+\left\|A(t)V\right\|_{H^{j-1}}^2+\|V\|^2\right). \tag{24}$$

Furthermore, we have

$$\left\|d\partial_x^{j+1}V^4\right\|^2 \leqslant \left\|\sum_{k=1}^{j}\binom{j}{k}\partial_x^k d\partial_x^{j-k+1}V^2+d\partial_x^{j+1}V^2+\sum_{k=1}^{j}\binom{j}{k}\partial_x^k d\partial_x^{j-k+1}V^4+d\partial_x^{j+1}V^4-\sum_{k=0}^{j}\partial_x^k\left(\left(\frac{g}{d}\right)_x\frac{d^2}{g}\right)\partial_x^{j-k}V^4\right\|$$
$$+\left\|\sum_{k=1}^{j}\binom{j}{k}\partial_x^k d\partial_x^{j-k+1}V^2+d\partial_x^{j+1}V^2+\sum_{k=1}^{j}\binom{j}{k}\partial_x^k d\partial_x^{j-k+1}V^4-\sum_{k=0}^{j}\partial_x^k\left(\left(\frac{g}{d}\right)_x\frac{d^2}{g}\right)\partial_x^{j-k}V^4\right\|.$$

By (24) and the induction hypothesis we therefore obtain the existence of a constant $K_2>0$ such that

$$\left\|\partial_x^{j+1}V^4\right\|^2 \leqslant K_2\left(\left\|\partial_x^jA(t)\right\|^2+\left\|A(t)V\right\|_{H^{j-1}}^2+\|V\|^2\right). \tag{25}$$

Next observe that

$$\left\|\frac{gk}{d\tau}\partial_x^{j+1}V^3\right\|^2$$
$$\leqslant \left\|\sum_{k=1}^{j}\binom{j}{k}\partial_x^k\left(\frac{gk}{d\tau}\right)\partial_x^{j-k+1}V^3-\sum_{k=0}^{j}\binom{j}{k}\partial_x^k\left(\left(\frac{g}{d}\right)_t\frac{d}{g}\right)\partial_x^{j-k}V^4+\frac{gk}{d\tau}\partial_x^{j+1}V^3+\sum_{k=0}^{j}\binom{j}{k}\partial_x^k\left(\frac{1}{\tau}\right)\partial_x^{j-k}V^4\right\|$$
$$+\left\|\sum_{k=1}^{j}\binom{j}{k}\partial_x^k\left(\frac{gk}{d\tau}\right)\partial_x^{j-k+1}V^3-\sum_{k=0}^{j}\binom{j}{k}\partial_x^k\left(\left(\frac{g}{d}\right)_t\frac{d}{g}\right)\partial_x^{j-k}V^4+\sum_{k=0}^{j}\binom{j}{k}\partial_x^k\left(\frac{1}{\tau}\right)\partial_x^{j-k}V^4\right\|.$$

Again the induction hypothesis implies that

$$\left\|\partial_x^{j+1}V^3\right\|^2 \leqslant K_3\left(\left\|\partial_x^jA(t)\right\|^2+\left\|A(t)V\right\|_{H^{j-1}}^2+\|V\|^2\right) \tag{26}$$

for a fixed $K_3>0$. Finally we consider the estimate

$$\left\|b\partial_x^{j+1}V^1\right\|^2$$
$$\leqslant \left\|\sum_{k=1}^{j}\binom{j}{k}\partial_x^k b\partial_x^{j-k+1}V^1+b\partial_x^{j+1}V^1+\sum_{k=0}^{j}\binom{j}{k}\partial_x^k\left(\left(\frac{a}{b}\right)_x\frac{b^2}{a}\right)\partial_x^{j-k}V^1+\sum_{k=1}^{j}\binom{j}{k}\partial_x^k b\partial_x^{j-k+1}V^3+b\partial_x^{j+1}V^3\right\|$$
$$+\left\|\sum_{k=1}^{j}\binom{j}{k}\partial_x^k b\partial_x^{j-k+1}V^1+\sum_{k=0}^{j}\binom{j}{k}\partial_x^k\left(\left(\frac{a}{b}\right)_x\frac{b^2}{a}\right)\partial_x^{j-k}V^1+\sum_{k=1}^{j}\binom{j}{k}\partial_x^k b\partial_x^{j-k+1}V^3+b\partial_x^{j+1}V^3\right\|$$

and obtain by the induction hypothesis and (26) the inequality

$$\left\|\partial_x^{j+1}V^1\right\|^2 \leqslant K_4\left(\left\|\partial_x^jA(t)\right\|^2+\left\|A(t)V\right\|_{H^{j-1}}^2+\|V\|^2\right) \tag{27}$$

for a fixed $K_4>0$. Altogether we obtain using (24), (25), (26) and (27) the claim. □

This implies the following result.

Theorem 3.11. *Let $s\geqslant 2$. Furthermore let A and V_0 be defined as in* (14) *and $V_0\in Y_s$. Suppose that condition* (L4) *holds. Then there is a unique solution V to*

$$V_t+AV=F,\qquad V(0)=V_0$$

with

$$\partial_t^kV\in C^0\left([0,T],Y_{s-k}\right),\quad k=0,\dots,s-1,$$

if and only if V_0 and F satisfy the compatibility condition (A1).

4. Asymptotic behavior

In this part we want to discuss the asymptotic behavior of solutions to the system (8)–(12). Our aim is to find a Lyapunov function G for which there exist positive constants C_1, C_2 and d_0 such that $C_1E(t) \leqslant G(t) \leqslant C_2E(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}G(t) \leqslant -d_0E(t) + \|F(t)\|_*^2$ for some norm $\|\cdot\|_*$.

Definition 4.1. Let (u,θ,q) be a solution to (8)–(12) and $V := (\frac{a}{b}u_x, u_t, \theta, \frac{g}{d}q)^t$. We define

$$E_1(t) := \frac{1}{2}\langle V, V\rangle_t = \frac{1}{2}\int_\Omega \frac{a}{b}|u_x|^2 + \frac{1}{b}|u_t|^2 + \frac{1}{d}|\theta|^2 + \frac{g\tau}{dk}|q|^2\,\mathrm{d}x$$

and

$$E_2(t) := \frac{1}{2}\langle V_t, V_t\rangle_t = \frac{1}{2}\int_\Omega \frac{b}{a}\left|\left(\frac{a}{b}u_x\right)_t\right|^2 + \frac{1}{b}|u_{tt}|^2 + \frac{1}{d}|\theta_t|^2 + \frac{d\tau}{gk}\left|\left(\frac{g}{d}q\right)_t\right|^2\,\mathrm{d}x.$$

The following assumption will be made in this section.

Assumption 4.2. *There exists a positive constant $\mu > 0$, such that all the functions*

$$\begin{aligned}&a_t(t,x),\quad a_{tt}(t,x),\quad a_{xt}(t,x),\quad b_t(t,x),\quad b_{tt}(t,x),\quad b_{xt}(t,x),\quad d_t(t,x),\quad d_{tt}(t,x),\quad d_{xt}(t,x),\\ &g_t(t,x),\quad g_{tt}(t,x),\quad g_{xt}(t,x),\quad k_t(t,x),\quad \tau_t(t,x)\end{aligned} \tag{28}$$

are uniformly bounded by $\mu > 0$ on $[0,T]\times\Omega$.

Lemma 4.3. *There exist constants $K_1 = K_1(a,b,d,g,k,\tau)$, $K_2 = K_2(a,b,d,g,k,\tau)$ such that*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) &= -\int_\Omega \frac{g}{dk}|q|^2\,\mathrm{d}x + \mathrm{Re}\langle -N_0V, V\rangle + \frac{1}{2}\left\langle V, \left(\frac{\mathrm{d}}{\mathrm{d}t}Q\right)V\right\rangle + 2\,\mathrm{Re}\langle F, QV\rangle\\ &\leqslant -\int_\Omega \frac{g}{dk}|q|^2\,\mathrm{d}x + \mu K_1\big(\|u_x\|^2 + \|u_t\|^2 + \|\theta\|^2 + \|q\|^2\big) + \frac{1}{\mu}\|F\|^2 \quad (t \geqslant 0)\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) \leqslant{}& -\int_\Omega \frac{d}{gk}\left|\left(\frac{g}{d}q\right)_t\right|^2\,\mathrm{d}x + \mu K_2\big(\|u_t\|^2 + \|\theta\|^2 + \|q\|^2\\ &+ \|u_{tx}\|^2 + \|u_{tt}\|^2 + \|\theta_t\|^2 + \|q_t\|^2 + \|u_x\|^2\big) + \frac{1}{\mu}\|F_t\|^2 \quad (t \geqslant 0).\end{aligned}$$

Proof. First we estimate the time-derivative of the first energy term with the help of the dissipative character of the operator $(-A_1)(t)$. This gives us

$$\begin{aligned}2\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) &= \langle -N_0V, V\rangle + \langle -N_1V, V\rangle + \langle V, -N_0V\rangle + \langle V, -N_1V\rangle + \left\langle V, \left(\frac{\mathrm{d}}{\mathrm{d}t}Q\right)V\right\rangle + \langle F, QV\rangle + \langle V, QF\rangle\\ &= 2\,\mathrm{Re}\langle -N_1V, V\rangle + 2\,\mathrm{Re}\langle -N_0V, V\rangle + \left\langle V, \left(\frac{\mathrm{d}}{\mathrm{d}t}Q\right)V\right\rangle + 2\,\mathrm{Re}\langle F, QV\rangle\\ &= -2\int_\Omega \frac{g}{dk}|q|^2\,\mathrm{d}x + 2\,\mathrm{Re}\langle -N_0V, V\rangle + \left\langle V, \left(\frac{\mathrm{d}}{\mathrm{d}t}Q\right)V\right\rangle + 2\,\mathrm{Re}\langle F, QV\rangle.\end{aligned}$$

The calculation of the time-derivative of the second energy term will be more complicated since we have to apply the product rule more than one time. So we will get more perturbation terms than above

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle V_t, V_t\rangle_t = 2\,\mathrm{Re}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(-AV), QV_t\right\rangle + \left\langle V_t, \left(\frac{\mathrm{d}}{\mathrm{d}t}Q\right)V_t\right\rangle + 2\,\mathrm{Re}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}F, QV_t\right\rangle.$$

In view of the fact that we need a term that is equivalent to $-\langle q_t, q_t\rangle$, it makes sense to apply the dissipativity of $(-A_1)(t)$ again. We obtain

$$\mathrm{Re}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(-AV), QV_t \right\rangle = -\mathrm{Re}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(Q^{-1}N_0V\right), QV_t \right\rangle + \mathrm{Re}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(-Q^{-1}N_1V\right), QV_t \right\rangle \tag{29}$$

and

$$\mathrm{Re}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(-Q^{-1}N_1V\right), QV_t \right\rangle = -\mathrm{Re}\left\langle \left(\frac{\mathrm{d}}{\mathrm{d}t}Q^{-1}\right)N_1V, QV_t \right\rangle + \mathrm{Re}\left\langle -\frac{\mathrm{d}}{\mathrm{d}t}(N_1V), V_t \right\rangle. \tag{30}$$

Using the dissipativity of the operator $-A_1(t)$, we get

$$\mathrm{Re}\left\langle -\frac{\mathrm{d}}{\mathrm{d}t}(N_1V), V_t \right\rangle \leqslant -\int_\Omega \frac{d}{gk}\left|\left(\frac{g}{d}q\right)_t\right|^2 \mathrm{d}x + \mu C(g,d,k)\left(\|q\|^2 + \|q_t\|^2\right). \tag{31}$$

By virtue of Assumption 4.2 and the boundedness of the coefficients there exists a constant $C_1(a,b,d,g,k,\tau)$ such that

$$-\mathrm{Re}\left\langle \left(\frac{\mathrm{d}}{\mathrm{d}t}Q^{-1}\right)N_1V, QV_t \right\rangle \leqslant \mu C_1(a,b,d,g,k,\tau)\big(\|u_{xx}\|^2 + \|u_x\|^2 + \|\theta_x\|^2 + \|q_x\|^2 + \|u_{tx}\|^2 + \|u_{tt}\|^2 + \|\theta_t\|^2 + \|q_t\|^2 + \|q\|^2\big). \tag{32}$$

Furthermore, we have that

$$-\mathrm{Re}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(Q^{-1}N_0V\right), QV_t \right\rangle = -\mathrm{Re}\left\langle \left(\frac{\mathrm{d}}{\mathrm{d}t}Q^{-1}\right)N_0V, QV_t \right\rangle - \mathrm{Re}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(N_0V), V_t \right\rangle \tag{33}$$

and that

$$-\mathrm{Re}\left\langle \left(\frac{\mathrm{d}}{\mathrm{d}t}Q^{-1}\right)N_0V, QV_t \right\rangle \leqslant \mu C_2(a,b,d,g,k,\tau)\left(\|u_x\|^2 + \|q\|^2 + \|u_{tx}\|^2 + \|u_{tt}\|^2 + \|\theta_t\|^2 + \|q_t\|^2\right) \tag{34}$$

where $C_2(a,b,d,g,k,\tau) > 0$ is a constant. We obtain

$$-\mathrm{Re}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(N_0V), V_t \right\rangle \leqslant \mu C_3(a,b,d,g,k,\tau)\left(\|u_{tx}\|^2 + \|u_x\|^2 + \|u_{tt}\|^2 + \|\theta_t\|^2 + \|q\|^2 + \|q_t\|^2\right) \tag{35}$$

where $C_3(a,b,d,g,k,\tau) > 0$ is a constant. There also exists a constant $C_4(a,b,d,g,k,\tau) > 0$ such that

$$\left\langle V_t, \left(\frac{\mathrm{d}}{\mathrm{d}t}Q\right), V_t \right\rangle \leqslant \mu C_4(a,b,d,g,k,\tau)\left(\|u_{tx}\|^2 + \|u_x\|^2 + \|u_{tt}\|^2 + \|\theta_t\|^2 + \|q\|^2 + \|q_t\|^2\right). \tag{36}$$

Combining (29)–(36) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) \leqslant -\int_\Omega \frac{d}{gk}\left|\left(\frac{g}{d}q\right)_t\right|^2 \mathrm{d}x + \mu K_2\big(\|u_x\|^2 + \|u_t\|^2 + \|\theta\|^2 + \|\theta_x\|^2 + \|q\|^2 + \|q_x\|^2 + \|u_x\|^2 + \|u_{tx}\|^2 + \|u_{tt}\|^2 + \|\theta_t\|^2 + \|q_t\|^2\big) + \frac{1}{\mu}\|F_t\|^2,$$

where $K_2 > 0$ is a constant. □

Multiplying (8) by $\bar{u}_{xx}$ we obtain

$$u_{tt}\bar{u}_{xx} - a|u_{xx}|^2 + b\theta_x\bar{u}_{xx} = f_1\bar{u}_{xx}$$

and we may conclude that

$$\begin{aligned}\|u_{xx}\|^2 &\leqslant \frac{1}{C_a}\mathrm{Re}\left([u_{tt}\bar{u}_x]_0^L - \frac{\mathrm{d}}{\mathrm{d}t}\langle u_{tx}, u_x\rangle + \|u_{tx}\|^2 + \langle b\theta_x, u_{xx}\rangle - \langle f_1, u_{xx}\rangle\right)\\ &\leqslant \frac{1}{C_a}\mathrm{Re}\left([u_{tt}\bar{u}_x]_0^L - \frac{\mathrm{d}}{\mathrm{d}t}\langle u_{tx}, u_x\rangle + \|u_{tx}\|^2\right) + \frac{3}{2C_a^2}\|b\theta_x\|^2 + \frac{1}{6}\|u_{xx}\|^2 + \frac{3}{2C_a^2}\|f_1\|^2 + \frac{1}{6}\|u_{xx}\|^2.\end{aligned}$$

Consequently

$$\frac{2}{3}\|u_{xx}\|^2 + \frac{1}{C_a}\mathrm{Re}\frac{\mathrm{d}}{\mathrm{d}t}\langle u_{tx}, u_x\rangle \leqslant -\frac{1}{C_a}\mathrm{Re}(u_{tt}\bar{u}_x)(0) + \frac{1}{C_a}\|u_{tx}\|^2 + \frac{3}{2C_a^2}\left(\|b\theta_x\|^2 + \|f_1\|^2\right). \tag{37}$$

Multiplication of (9) by $\bar{u}_{tx}$ yields

$$\langle\theta_t, u_{tx}\rangle + \langle gq_x, u_{tx}\rangle + \langle du_{tx}, u_{tx}\rangle = \langle f_2, u_{tx}\rangle.$$

Next we consider $\langle gq_x, u_{tx}\rangle$. Because of the boundary condition on q in (12) we obtain

$$\langle gq_x, u_{tx}\rangle = \langle q_x, gu_{tx}\rangle = (qg\bar{u}_{tx})(L) - \big\langle q, (gu_{tx})_x\big\rangle = (qg\bar{u}_{tx})(L) - \langle q, g_x u_{tx}\rangle - \frac{\mathrm{d}}{\mathrm{d}t}\langle q, gu_{xx}\rangle + \big\langle (qg)_t, u_{xx}\big\rangle.$$

Furthermore, we have

$$\langle \theta_t, u_{tx}\rangle = -(\theta_t\bar{u}_t)(0) - \frac{\mathrm{d}}{\mathrm{d}t}\langle \theta_x, u_t\rangle + \langle \theta_x, u_{tt}\rangle$$

and we obtain

$$\langle du_{tx}, u_{tx}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle \theta_x, u_t\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\langle q, gu_{xx}\rangle - \langle \theta_x, u_{tt}\rangle + \langle q, g_x u_{tx}\rangle - \big\langle (qg)_t, u_{xx}\big\rangle - (qg\bar{u}_{tx})(L) + (\theta_t\bar{u}_t)(0) + \langle f_2, u_{tx}\rangle. \tag{38}$$

From (38), we get

$$\begin{aligned} C_d\|u_{tx}\|^2 \leqslant{}& \operatorname{Re}\frac{\mathrm{d}}{\mathrm{d}t}\big(\langle \theta_x, u_t\rangle + \langle q, gu_{xx}\rangle\big) - \operatorname{Re}(qg\bar{u}_{tx})(L) + \operatorname{Re}(\theta_t\bar{u}_t)(0) + \frac{1}{C_d}\|g_x q\|^2 + \frac{C_d}{4}\|u_{tx}\|^2 + \frac{1}{C_d}\|f_2\|^2 + \frac{C_d}{4}\|u_{tx}\|^2 \\ &+ \big|\big\langle (qg)_t, u_{xx}\big\rangle\big| + \big|\langle \theta_x, au_{xx}\rangle\big| + \big|\langle \theta_x, b\theta_x\rangle\big| + \big|\langle \theta_x, f_1\rangle\big|. \end{aligned} \tag{39}$$

The four latter terms we treat as follows:

$$\begin{aligned} &\big|\big\langle (qg)_t, u_{xx}\big\rangle\big| + \big|\langle \theta_x, au_{xx}\rangle\big| + \big|\langle \theta_x, b\theta_x\rangle\big| + \big|\langle \theta_x, f_1\rangle\big| \\ &\quad\leqslant \frac{C_a C_d}{24}\|u_{xx}\|^2 + \frac{24}{C_a C_d}\big(C^{g_t}\big)^2\|q\|^2 + \frac{24}{C_a C_d}\big(C^{g}\big)^2\|q_t\|^2 + \left(\frac{12}{C_a C_d}\big(C^a\big)^2 + C^b + \frac{1}{2}\right)\|\theta_x\|^2 + \frac{1}{2}\|f_1\|^2. \end{aligned} \tag{40}$$

Combining (37), (39) and (40), we conclude that

$$\begin{aligned} &\operatorname{Re}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{C_a}\langle u_{tx}, u_x\rangle - \frac{4}{C_a C_d}\langle \theta_x, u_t\rangle - \frac{4}{C_a C_d}\langle q, gu_{xx}\rangle\right) \\ &\quad\leqslant -\frac{1}{2}\|u_{xx}\|^2 - \frac{1}{C_a}\|u_{tx}\|^2 + \frac{3}{2C_a^2}\|b\theta_x\|^2 + \frac{4}{C_a C_d^2}\|g_x q\|^2 + \frac{96}{C_a^2 C_d^2}\big(C^{g_t}\big)^2\|q\|^2 + \frac{96}{C_a^2 C_d^2}\big(C^{g}\big)^2\|q_t\|^2 \\ &\qquad + \left(\frac{48}{C_a^2 C_d^2}\big(C^a\big)^2 + \frac{4C^b}{C_a C_d} + \frac{2}{C_a C_d}\right)\|\theta_x\|^2 - \frac{1}{C_a}\operatorname{Re}(u_{tt}\bar{u}_x)(0) - \frac{4}{C_d C_a}\operatorname{Re}(qg\bar{u}_{tx})(L) + \frac{4}{C_d C_a}\operatorname{Re}(\theta_t\bar{u}_t)(0) \\ &\qquad + \frac{2}{C_a C_d}\|f_1\|^2 + \frac{3}{2C_a^2}\|f_1\|^2 + \frac{4}{C_a C_d^2}\|f_2\|^2. \end{aligned} \tag{41}$$

The differential equations (8) and (10) together with the Poincaré inequality yield

$$\begin{aligned} \|u_{tt}\|^2 + \|u_t\|^2 + \|\theta\|^2 &= \|au_{xx} - b\theta_x + f_1\|^2 + \|u_t\|^2 + \|\theta\|^2 \\ &\leqslant 2\|au_{xx}\|^2 + K_3\big(\|u_{tx}\|^2 + \|q\|^2 + \|q_t\|^2 + \|f_1\|^2 + \|f_3\|^2\big) \end{aligned} \tag{42}$$

where $K_3 > 0$ is a constant. Multiplying (8) by $\bar{u}$, we get

$$\langle f_1, u\rangle = \langle u_{tt}, u\rangle + \big\langle (au_x - b\theta), u_x\big\rangle + \langle a_x u_x - b_x\theta, u\rangle.$$

We conclude

$$\langle au_x, u_x\rangle \leqslant \|u_{tt}\|\|u\| + \|b\theta\|\|u_x\| + \|a_x u_x\|\|u\| + \|b_x\theta\|\|u\| + \|f_1\|\|u\|.$$

The differential equation (10) together with the Poincaré inequality yields

$$\begin{aligned} \|u_x\|^2 &\leqslant \frac{1}{C_a}\|u_{tt}\|\|u\| + \frac{1}{C_a}\|b\theta\|\|u_x\| + \frac{1}{C_a}\|a_x u_x\|\|u\| + \frac{1}{C_a}\|b_x\theta\|\|u\| + \frac{1}{C_a}\|f_1\|\|u\| \\ &\leqslant C(a)\|u_{tt}\|^2 + \frac{1}{8}\|u_x\|^2 + C(a,b)\|\theta_x\|^2 + \frac{1}{16}\|u_x\|^2 + \frac{CC^{a_x}}{C_a}\|u_x\|^2 \\ &\quad + C(a,b_x)\|\theta\|^2 + \frac{1}{8}\|u_x\|^2 + C(a)\|f_1\|^2 + \frac{1}{16}\|u_x\|^2 \\ &\leqslant \frac{1}{2}\|u_x\|^2 + K_4\big(\|q\|^2 + \|q_t\|^2 + \|u_{tt}\|^2 + \|f_1\|^2 + \|f_3\|^2\big). \end{aligned}$$

Observe that here we had to assume

$$\frac{CC^{a_x}}{C_a} \leqslant \frac{1}{8} \tag{43}$$

where the constant C arises from the Poincaré inequality. Thus we obtain

$$\frac{1}{2}\|u_x\|^2 \leqslant K_4\big(\|q\|^2+\|q_t\|^2+\|u_{tt}\|^2+\|f_1\|^2+\|f_3\|^2\big). \tag{44}$$

The estimates (42) and (44) yield

$$\frac{1}{2}\|u_{tt}\|^2+\|u_t\|^2+\frac{1}{4K_4}\|u_x\|^2+\|\theta\|^2 \leqslant K_5\big(\|u_{xx}\|^2+\|u_{tx}\|^2+\|q\|^2+\|q_t\|^2+\|f_1\|^2+\|f_3\|^2\big) \tag{45}$$

where K_5 is a constant. Multiplying (9) by $\frac{1}{g}\bar{\theta}_t$, we obtain

$$\left\langle \theta_t, \frac{1}{g}\theta_t\right\rangle + \langle q_x, \theta_t\rangle + \left\langle du_{tx}, \frac{1}{g}\theta_t\right\rangle = \left\langle f_2, \frac{1}{g}\theta_t\right\rangle.$$

In other words,

$$\left\langle \theta_t, \frac{1}{g}\theta_t\right\rangle - \frac{\mathrm{d}}{\mathrm{d}t}\langle q, \theta_x\rangle + \langle q_t, \theta_x\rangle + \left\langle du_{tx}, \frac{1}{g}\theta_t\right\rangle = \left\langle f_2, \frac{1}{g}\theta_t\right\rangle$$

and we can estimate

$$\begin{aligned}\|\theta_t\|^2 - C^g \operatorname{Re}\frac{\mathrm{d}}{\mathrm{d}t}\langle q, \theta_x\rangle &\leqslant -C^g \operatorname{Re}\langle q_t, \theta_x\rangle - C^g \operatorname{Re}\left\langle \frac{d}{g}u_{tx}, \theta_t\right\rangle + C^g\left\langle f_2, \frac{1}{g}\theta_t\right\rangle \\ &\leqslant \frac{C^g}{2}\|\theta_x\|^2 + \frac{C^g}{2}\|q_t\|^2 + C^g\left(\frac{C^d}{C_g}\right)^2\|u_{tx}\|^2 + \frac{1}{2}\|\theta_t\|^2 + C(g)\|f_2\|^2.\end{aligned}$$

This implies

$$\|\theta_t\|^2 - 2C^g\frac{\mathrm{d}}{\mathrm{d}t}\langle q, \theta_x\rangle \leqslant C^g\|\theta_x\|^2 + C^g\|q_t\|^2 + 2C^g\left(\frac{C^d}{C_g}\right)^2\|u_{tx}\|^2 + 2C(g)\|f_2\|^2. \tag{46}$$

The boundary terms arising in (41) are dealt with as follows. Using (9) we obtain

$$\begin{aligned}\frac{4}{C_aC_d}\operatorname{Re}(q\bar{u}_{tx}g)(L) &\leqslant \frac{8(C^g)^2}{C_a^3C_d^2\hat{\varepsilon}}\left(1+\frac{L}{\hat{\varepsilon}^2}\right)\|q\|^2 + \frac{16(C^g)^2}{C_a^3C_d^2L}\hat{\varepsilon}\left\|\frac{\theta_t}{g}\right\|^2 \\ &\quad + \frac{32(C^g)^2\hat{\varepsilon}}{C_a^3C_d^2L}\left\|\frac{du_{tx}}{g}\right\|^2 + \frac{32(C^g)^2\hat{\varepsilon}}{C_a^3C_d^2L}\|f_2\|^2 + C_a\hat{\varepsilon}|u_{tx}|^2(L).\end{aligned} \tag{47}$$

We have that

$$\begin{aligned}\frac{1}{C_a}\operatorname{Re}(u_{tt}\bar{u}_x)(0) &\leqslant \frac{1}{C_a}\left|\frac{b}{a}u_{tt}\theta\right|(0) \leqslant \frac{1}{2}\hat{\varepsilon}|u_{tt}|^2(0) + \frac{(C^b)^2}{C_a^4\hat{\varepsilon}}|\theta|^2(0) \\ &\leqslant \frac{1}{2}\hat{\varepsilon}|u_{tt}|^2(0) + \frac{(C^b)^2C(k,\tau)}{C_a^4\hat{\varepsilon}}\big(\|q\|^2+\|q_t\|^2+\|f_3\|^2\big).\end{aligned} \tag{48}$$

The differential equation (10) and the Poincaré inequality yield

$$\frac{4}{C_aC_d}\operatorname{Re}(\theta_t\bar{u}_t)(0) \leqslant \frac{4}{C_aC_d}\frac{\mathrm{d}}{\mathrm{d}t}(\operatorname{Re}\theta\bar{u}_t)(0) + \frac{1}{2}\hat{\varepsilon}|u_{tt}|^2(0) + \frac{8C(k,\tau)}{C_a^2C_d^2\hat{\varepsilon}}\big(\|q\|^2+\|q_t\|^2+\|f_3\|^2\big). \tag{49}$$

By differentiating (8) with respect to t and multiplying by $\varphi\bar{u}_{tx}$, where $\varphi:\mathbb{R}\to\mathbb{R}$, $\varphi(x):=L-2x$, we obtain

$$\langle u_{ttt}, \varphi u_{tx}\rangle - \big\langle (au_{xx})_t, \varphi u_{tx}\big\rangle + \big\langle (b\theta_x)_t, \varphi u_{tx}\big\rangle = \langle f_1, \varphi u_{tx}\rangle.$$

This is equivalent to

$$\big\langle (f_1)_t, \varphi u_{tx}\big\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle u_{tt}, \varphi u_{tx}\rangle - \langle u_{tt}, \varphi u_{ttx}\rangle - \langle au_{txx}, \varphi u_{tx}\rangle + \langle b\theta_{tx}, \varphi u_{tx}\rangle - \langle a_tu_{xx}, \varphi u_{tx}\rangle + \langle b_t\theta_x, \varphi u_{tx}\rangle. \tag{50}$$

Furthermore, we have

$$-\langle u_{tt}\varphi, u_{ttx}\rangle = -\big[|u_{tt}|^2\varphi\big]_0^L + \langle u_{ttx}\varphi, u_{tt}\rangle - 2\|u_{tt}\|^2$$

and we conclude

$$-\operatorname{Re}\langle u_{ttx}\varphi, u_{tt}\rangle = \frac{L}{2}\big(|u_{tt}|^2(L)+|u_{tt}|^2(0)\big) - \|u_{tt}\|^2. \tag{51}$$

We have that

$$-\langle au_{txx}, \varphi u_{tx}\rangle = -\big[\varphi a|u_{tx}|^2\big]_0^L - 2\langle au_{tx}, u_{tx}\rangle + \langle u_{tx}, \varphi au_{txx}\rangle + \langle u_{tx}, \varphi a_x u_{tx}\rangle$$

and we conclude that

$$-\operatorname{Re}\langle au_{txx}, \varphi u_{tx}\rangle = \frac{L}{2}\big(\big(a|u_{tx}|^2\big)(L) + \big(a|u_{tx}|^2\big)(0)\big) - \langle au_{tx}, u_{tx}\rangle + \frac{1}{2}\langle u_{tx}, \varphi a_x u_{tx}\rangle. \tag{52}$$

Combining (50)–(52) yields

$$\begin{aligned}\operatorname{Re}\langle f_1, \varphi u_{tx}\rangle = {}&\operatorname{Re}\frac{\mathrm{d}}{\mathrm{d}t}\langle u_{tt}\varphi, u_{tx}\rangle + \frac{L}{2}\big(|u_{tt}|^2(L) + |u_{tt}|^2(0)\big) - \|u_{tt}\|^2\\ &+ \frac{L}{2}\big(\big(a|u_{tx}|^2\big)(L) + \big(a|u_{tx}|^2\big)(0)\big) - \langle au_{tx}, u_{tx}\rangle + \frac{1}{2}\langle u_{tx}, \varphi a_x u_{tx}\rangle\\ &+ \operatorname{Re}\langle b\theta_{tx}, \varphi u_{tx}\rangle - \operatorname{Re}\langle a_t u_{xx}, \varphi u_{tx}\rangle + \operatorname{Re}\langle b_t\theta_x, \varphi u_{tx}\rangle.\end{aligned} \tag{53}$$

Multiplying (9) by $-\varphi\frac{b}{d}\theta_{tx}$ yields

$$-\left\langle\theta_t, \varphi\frac{b}{d}\theta_{tx}\right\rangle - \left\langle gq_x, \varphi\frac{b}{d}\theta_{tx}\right\rangle - \left\langle du_{tx}, \varphi\frac{b}{d}\theta_{tx}\right\rangle = -\left\langle f_2, \varphi\frac{b}{d}\theta_{tx}\right\rangle.$$

We have that

$$-\left\langle\theta_t, \varphi\frac{b}{d}\theta_{tx}\right\rangle = -\left[\frac{b}{d}\varphi|\theta_t|^2\right]_0^L - 2\left\langle\frac{b}{d}\theta_t, \theta_t\right\rangle + \left\langle\theta_{tx}\frac{b}{d}\varphi, \theta_t\right\rangle + \left\langle\theta_t\left(\frac{b}{d}\right)_x\varphi, \theta_t\right\rangle.$$

Here we may conclude that

$$-\operatorname{Re}\left\langle\theta_t, \varphi\frac{b}{d}\theta_{tx}\right\rangle = \frac{L}{2}\left(\left(\frac{b}{d}|\theta_t|^2\right)(L) + \left(\frac{b}{d}|\theta_t|^2\right)(0)\right) - \left\langle\frac{b}{d}\theta_t, \theta_t\right\rangle + \frac{1}{2}\left\langle\theta_t\left(\frac{b}{d}\right)_x\varphi, \theta_t\right\rangle.$$

The differential equation (9) yields

$$\begin{aligned}0 = {}&\frac{L}{2}\left(\left(\frac{b}{d}|\theta_t|^2\right)(L) + \left(\frac{b}{d}|\theta_t|^2\right)(0)\right) - \operatorname{Re}\left\langle\frac{b}{d}\theta_t, \theta_t\right\rangle + \frac{1}{2}\operatorname{Re}\left\langle\theta_t\left(\frac{b}{d}\right)_x\varphi, \theta_t\right\rangle - \operatorname{Re}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\frac{gb}{d}\varphi q_x, \theta_x\right\rangle\\ &+ \operatorname{Re}\left\langle\left(\frac{gb}{d}\varphi q_x\right)_t, \theta_x\right\rangle - \operatorname{Re}\langle u_{tx}, \varphi b\theta_{tx}\rangle + \left\langle f_2, \varphi\frac{b}{d}\theta_{tx}\right\rangle.\end{aligned} \tag{54}$$

We conclude from (10) that

$$\left\langle\left(\frac{gb}{d}\varphi q_x\right)_t, \theta_x\right\rangle = \left\langle\left(\frac{gb}{d}\right)_t\varphi q_x, \theta_x\right\rangle - \left\langle\frac{gb}{d}\left(\frac{1}{\tau}q\right)_x, \varphi\theta_x\right\rangle - \left\langle\frac{gb}{d}\left(\frac{k}{\tau}\theta_x\right)_x, \varphi\theta_x\right\rangle + \left\langle\frac{gb}{d}\left(\frac{1}{\tau}f_3\right)_x, \varphi\theta_x\right\rangle. \tag{55}$$

Finally we get

$$-\left\langle\left(\frac{k}{\tau}\theta_x\right)_x, \varphi\frac{gb}{d}\theta_x\right\rangle = -\left\langle\left(\frac{k}{\tau}\right)_x\theta_x, \varphi\frac{gb}{d}\theta_x\right\rangle - \left\langle\frac{k}{\tau}\theta_{xx}, \varphi\frac{gb}{d}\theta_x\right\rangle.$$

On the other hand observe that

$$-\left\langle\left(\frac{k}{\tau}\theta_x\right)_x, \varphi\frac{gb}{d}\theta_x\right\rangle = -\left[\frac{kgb}{\tau d}\varphi|\theta_x|^2\right]_0^L - 2\left\langle\frac{kgb}{d\tau}\theta_x, \theta_x\right\rangle + \left\langle\frac{k}{\tau}\theta_x, \varphi\left(\frac{gb}{d}\right)_x\theta_x\right\rangle + \left\langle\frac{k}{\tau}\theta_x, \varphi\frac{gb}{d}\theta_{xx}\right\rangle.$$

This yields

$$\begin{aligned}-\operatorname{Re}\left\langle\left(\frac{k}{\tau}\theta_x\right)_x, \varphi\frac{gb}{d}\theta_x\right\rangle = {}&\frac{L}{2}\left(\left(\frac{kgb}{\tau d}|\theta_x|^2\right)(L) + \left(\frac{kgb}{\tau d}|\theta_x|^2\right)(0)\right) - \left\langle\frac{kgb}{\tau d}\theta_x, \theta_x\right\rangle\\ &+ \frac{1}{2}\left\langle\frac{k}{\tau}\theta_x, \varphi\left(\frac{gb}{d}\right)_x\theta_x\right\rangle - \frac{1}{2}\left\langle\left(\frac{k}{\tau}\right)_x\theta_x, \varphi\frac{gb}{d}\theta_x\right\rangle.\end{aligned} \tag{56}$$

Combining (54)–(56) we obtain

$$0=\frac{L}{2}\left(\left(\frac{b}{d}|\theta_t|^2\right)(L)+\left(\frac{b}{d}|\theta_t|^2\right)(0)\right)-\operatorname{Re}\left\langle\frac{b}{d}\theta_t,\theta_t\right\rangle+\frac{1}{2}\operatorname{Re}\left\langle\theta_t\left(\frac{b}{d}\right)_x\varphi,\theta_t\right\rangle$$
$$-\operatorname{Re}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\frac{gb}{d}\varphi q_x,\theta_x\right\rangle-\operatorname{Re}\langle u_{tx},\varphi b\theta_{tx}\rangle+\operatorname{Re}\left\langle\left(\frac{gb}{d}\right)_t\varphi q_x,\theta_x\right\rangle+\left\langle f_2,\varphi\frac{b}{d}\theta_{tx}\right\rangle$$
$$-\operatorname{Re}\left\langle\frac{gb}{d}\left(\frac{1}{\tau}q\right)_x,\varphi\theta_x\right\rangle+\frac{L}{2}\left(\left(\frac{kgb}{\tau d}|\theta_x|^2\right)(L)+\left(\frac{kgb}{\tau d}|\theta_x|^2\right)(0)\right)+\left\langle\frac{gb}{d}\left(\frac{1}{\tau}f_3\right)_x,\varphi\theta_x\right\rangle$$
$$-\operatorname{Re}\left\langle\frac{kgb}{\tau d}\theta_x,\theta_x\right\rangle+\frac{1}{2}\operatorname{Re}\left\langle\frac{k}{\tau}\theta_x,\varphi\left(\frac{gb}{d}\right)_x\theta_x\right\rangle-\frac{1}{2}\operatorname{Re}\left\langle\left(\frac{k}{\tau}\right)_x\theta_x,\varphi\frac{gb}{d}\theta_x\right\rangle. \tag{57}$$

We also have

$$\left|\left\langle f_2,\varphi\frac{b}{d}\theta_{tx}\right\rangle\right|\leqslant C(b,d,b_x,d_x)\left(\|f_2\|^2+\left\|(f_2)_x\right\|^2+\|\theta_t\|^2\right)+\frac{1}{2}\left(\frac{b}{d}L|\theta_t|^2\right)(0). \tag{58}$$

Combining (53), (57) and (58), we obtain

$$\operatorname{Re}\frac{\mathrm{d}}{\mathrm{d}t}\langle u_{tt}\varphi,u_{tx}\rangle-\operatorname{Re}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\frac{gb}{d}\varphi q_x,\theta_x\right\rangle+\frac{L}{2}\left(\left(\frac{kgb}{\tau d}|\theta_x|^2\right)(L)+\left(\frac{kgb}{\tau d}|\theta_x|^2\right)(0)\right)$$
$$+\frac{L}{2}\left(\left(a|u_{tx}|^2\right)(L)+\left(a|u_{tx}|^2\right)(0)\right)+\frac{L}{2}\left(|u_{tt}|^2(L)+|u_{tt}|^2(0)\right)$$
$$=\operatorname{Re}\left\langle\frac{b}{d}\theta_t,\theta_t\right\rangle-\frac{1}{2}\operatorname{Re}\left\langle\theta_t\left(\frac{b}{d}\right)_x\varphi,\theta_t\right\rangle+\operatorname{Re}\langle u_{tx},\varphi b\theta_{tx}\rangle-\operatorname{Re}\left\langle\left(\frac{gb}{d}\right)_t\varphi q_x,\theta_x\right\rangle$$
$$+\operatorname{Re}\left\langle\frac{gb}{d}\left(\frac{1}{\tau}q\right)_x,\varphi\theta_x\right\rangle+\operatorname{Re}\left\langle\frac{kgb}{\tau d}\theta_x,\theta_x\right\rangle-\frac{1}{2}\operatorname{Re}\left\langle\frac{k}{\tau}\theta_x,\varphi\left(\frac{gb}{d}\right)_x\theta_x\right\rangle+\frac{1}{2}\operatorname{Re}\left\langle\left(\frac{k}{\tau}\right)_x\theta_x,\varphi\frac{gb}{d}\theta_x\right\rangle$$
$$+\|u_{tt}\|^2+\operatorname{Re}\langle au_{tx},u_{tx}\rangle-\operatorname{Re}\langle u_{tx},\varphi a_x u_{tx}\rangle-\operatorname{Re}\langle b\theta_{tx},\varphi u_{tx}\rangle+\operatorname{Re}\langle a_t u_{xx},\varphi u_{tx}\rangle-\operatorname{Re}\langle b_t\theta_x,\varphi u_{tx}\rangle$$
$$+\operatorname{Re}\langle f_1,\varphi u_{tx}\rangle-\operatorname{Re}\left\langle f_2,\varphi\frac{b}{d}\theta_{tx}\right\rangle-\operatorname{Re}\left\langle\frac{gb}{d}\left(\frac{1}{\tau}f_3\right)_x,\varphi\theta_x\right\rangle-\frac{L}{2}\left(\frac{b}{d}|\theta_t|^2\right)(0)$$
$$\leqslant K_6\left(\|u_{xx}\|^2+\|\theta_x\|^2+\|u_{tx}\|^2+\|\theta_t\|^2+\|q\|^2+\|f_1\|^2+\|f_2\|^2+\left\|(f_2)_x\right\|^2+\|f_3\|^2+\left\|(f_3)_x\right\|^2\right) \tag{59}$$

for a certain constant $K_6=K_6(a,b,g,d,\tau,k)$.

Now we define:

$$W(t):=\operatorname{Re}\left(\frac{1}{C_a}\langle u_{tx},u_x\rangle-\frac{4}{C_dC_a}\langle\theta_x,u_t\rangle-\frac{4}{C_aC_d}\langle q,gu_{xx}\rangle-2C^g\hat{\mu}\langle\theta_x,q\rangle\right.$$
$$\left.-\frac{4}{C_dC_a}(\theta\bar{u}_t)(0)+\frac{2\hat{\varepsilon}}{L}\langle u_{tt}\varphi,u_{tx}\rangle-\frac{2\hat{\varepsilon}}{L}\left\langle\frac{gb}{d}\varphi q_x,\theta_x\right\rangle\right).$$

Then we conclude from (41), (46)–(49), and (59) that

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t)\leqslant\left[-\frac{1}{2}+\frac{2K_6\hat{\varepsilon}}{L}\right]\|u_{xx}\|^2+\left[-\frac{1}{C_a}+\frac{32(C^gC^d)^2\hat{\varepsilon}}{C_a^3C_g^2C_d^2}+\hat{\mu}C^g\left(\frac{C^d}{C_g}\right)^2+\frac{2K_6\hat{\varepsilon}}{L}\right]\|u_{tx}\|^2$$
$$+\left[\frac{16(C^g)^2}{C_a^3C_d^2C_g^2}\hat{\varepsilon}-\hat{\mu}+\frac{2K_6\hat{\varepsilon}}{L}\right]\|\theta_t\|^2+\left[\frac{3(C^b)^2}{2C_a^2}+\frac{48}{C_a^2C_d^2}\left(C^a\right)^2+\frac{4C^b}{C_aC_d}+\frac{2}{C_aC_d}+\hat{\mu}C^g+\frac{2K_6\hat{\varepsilon}}{L}\right]\|\theta_x\|^2$$
$$+\left[\frac{4(C^{g_x})^2}{C_aC_d^2}+\frac{96}{C_a^2C_d^2}\left(C^{g_t}\right)^2+\frac{(C^b)^2C(k,\tau)}{C_a^4\hat{\varepsilon}}+\frac{8(C^g)^2}{C_a^3C_d^2\hat{\varepsilon}}\left(1+\frac{L}{\hat{\varepsilon}^2}\right)+\frac{8C(k,\tau)}{C_a^2C_d^2\hat{\varepsilon}}+\frac{2K_6\hat{\varepsilon}}{L}\right]\|q\|^2$$
$$+\left[\frac{96}{C_a^2C_d^2}\left(C^g\right)^2+\frac{(C^b)^2C(k,\tau)}{C_a^4\hat{\varepsilon}}+\hat{\mu}C^g+\frac{8C(k,\tau)}{C_a^2C_d^2\hat{\varepsilon}}\right]\|q_t\|^2+\left[\frac{2}{C_aC_d}+\frac{3}{2C_a^2}\right]\|f_1\|^2+\left[\frac{2K_6\hat{\varepsilon}}{L}\right]\left\|(f_1)_t\right\|^2$$
$$+\left[\frac{4}{C_aC_d^2}+\frac{32(C^g)^2\hat{\varepsilon}}{C_a^3C_d^2L}+C(g)+\frac{2K_6\hat{\varepsilon}}{L}\right]\|f_2\|^2+\left[\frac{2K_6\hat{\varepsilon}}{L}\right]\left\|(f_2)_x\right\|^2$$
$$+\left[\frac{(C^b)^2C(k,\tau)}{C_a^4\hat{\varepsilon}}+\frac{2K_6\hat{\varepsilon}}{L}\right]\|f_3\|^2+\left[\frac{2K_6\hat{\varepsilon}}{L}\right]\left\|(f_3)_x\right\|^2$$
$$\leqslant-\frac{1}{4}\|u_{xx}\|^2-\frac{1}{2C_a}\|u_{tx}\|^2-\frac{1}{2}\hat{\mu}\|\theta_t\|^2+K_7\left(\|q\|^2+\|q_t\|^2\right)$$
$$+K_8\left(\|f_1\|^2+\left\|(f_1)_t\right\|^2+\|f_2\|^2+\left\|(f_2)_x\right\|^2+\|f_3\|^2+\left\|(f_3)_x\right\|^2\right), \tag{60}$$

where

$$\hat{\mu} < \mu_0 := \frac{1}{4C_a}\left(\frac{(C_g)^2}{C^g(C^d)^2}\right)$$

and

$$\hat{\varepsilon} < \hat{\varepsilon}_0 := \min\left\{\frac{1}{4}\frac{L}{2K_6}, \frac{1}{4C_a}\left(\frac{32(C^gC^d)^2}{C_a^3C_d^2C_g^2} + \frac{2K_6}{L}\right)^{-1}, \frac{1}{2}\hat{\mu}\left(\frac{16(C^g)^2}{C_a^3C_d^2C_g^2} + \frac{2K_6}{L}\right)^{-1}\right\}.$$

Without loss of generality we assume $C_a < 1$. Now we are able to define our desired Lyapunov function as

$$G_\varepsilon(t) := \frac{1}{\varepsilon}\big(E_1(t) + E_2(t)\big) + W(t).$$

Using (45) and (60) as well as Lemma 4.3 we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}G_\varepsilon(t) &= \frac{1}{\varepsilon}\left(\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}E_2(t)\right) + \frac{\mathrm{d}}{\mathrm{d}t}W(t) \\
&\leqslant -\frac{1}{\varepsilon}\int_\Omega \frac{g}{dk}|q|^2\,\mathrm{d}x - \frac{1}{\varepsilon}\int_\Omega \frac{d}{gk}\left|\left(\frac{g}{d}\right)_t q + \frac{g}{d}q_t\right|^2\mathrm{d}x + \frac{\mu}{\varepsilon}(K_1 + K_2)\big(\|u_t\|^2 + \|\theta\|^2 + \|q\|^2 + \|u_{tx}\|^2 \\
&\quad + \|u_{tt}\|^2 + \|\theta_t\|^2 + \|q_t\|^2 + \|u_x\|^2\big) + \frac{1}{\mu\varepsilon}\big(\|F\|^2 + \|F_t\|^2\big) - \frac{1}{8}\|u_{xx}\|^2 - \frac{3}{8}\|u_{tx}\|^2 - \frac{1}{2}\hat{\mu}\|\theta_t\|^2 \\
&\quad - \frac{1}{16K_5}\|u_{tt}\|^2 - \frac{1}{8K_5}\|u_t\|^2 - \frac{1}{32K_4K_5}\|u_x\|^2 - \frac{1}{8K_5}\|\theta\|^2 + \left(K_7 + \frac{1}{8}\right)\big(\|q\|^2 + \|q_t\|^2\big) \\
&\quad + K_9\big(\|f_1\|^2 + \big\|(f_1)_t\big\|^2 + \|f_2\|^2 + \big\|(f_2)_x\big\|^2 + \|f_3\|^2 + \big\|(f_3)_x\big\|^2\big).
\end{aligned}$$

Choosing ε and μ sufficiently small we obtain the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}G_\varepsilon(t) \leqslant -\tilde{d}_0E(t) + K\Lambda(t),$$

where

$$\Lambda(t) := \big(\|f_1\|^2 + \big\|(f_1)_t\big\|^2 + \|f_2\|^2 + \big\|(f_2)_t\big\|^2 + \big\|(f_2)_x\big\|^2 + \|f_3\|^2 + \big\|(f_3)_t\big\|^2 + \big\|(f_3)_x\big\|^2\big).$$

It is easy to see that $E(t)$ and $G_\varepsilon(t)$ are equivalent. Using Gronwall's lemma we obtain

Theorem 4.4. *Let (u, θ, q) be the solution of* (8)–(12) *given in Theorem* 3.11. *Suppose condition* (43) *is satisfied. Then there exist constants $C, d_0, K > 0$ such that*

$$E(t) \leqslant Ce^{-d_0t}\left(E(0) + \int_0^t K\Lambda(t)e^{d_0\tau}\,\mathrm{d}\tau\right),$$

if condition (28) *holds for sufficiently small $\mu > 0$.*

In particular, if $f = 0$ then the energy $E(t)$ decays exponentially. The energy $E(t)$ also decays exponentially, if $\Lambda(t)$ decays exponentially.

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