



Singular perturbations for variational inequalities with time-dependent constraints

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ABSTRACT

We study an abstract hyperbolic variational inequality with a (small) parameter and with time-dependent subdifferentials in a real Hilbert space. We prove that its solution converges to a solution of a parabolic variational inequality as the parameter tends to zero. We also explain how to apply the abstract theory to concrete unilateral problems.

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1. Introduction

Consider the following hyperbolic variational inequality with a parameter $\varepsilon > 0$ in a real Hilbert space H

$$\varepsilon u''(t) + \partial\varphi^t(u'(t)) + Au(t) \ni f(t), \quad 0 < t < T, \quad (1.1)$$

$$u(0) = u_0, \quad u'(0) = u_1. \quad (1.2)$$

We study the convergence of its solution, as $\varepsilon \rightarrow 0$, to the solution of the following parabolic variational inequality

$$\partial\varphi^t(u'(t)) + Au(t) \ni f(t), \quad 0 < t < T, \quad (1.3)$$

$$u(0) = u_0. \quad (1.4)$$

Here $\partial\varphi^t$ is the subdifferential of a proper lower-semicontinuous (l.s.c.) convex function φ^t dependent on $t \in [0, T]$; A is a nonnegative self-adjoint operator on H ; and f , u_0 , and u_1 are given data.

J.L. Lions [15, Chap. 6] initiated the study of singular perturbations ($\varepsilon \rightarrow 0$) for variational inequalities (1.1)–(1.4) in the case of a time-independent constraint, i.e., $\partial\varphi^t \equiv \partial\varphi = I + \partial I_K$, where I_K is the indicator function of a convex set K . The results obtained by Lions are also mentioned by Barbu in [3, Chap. V].

In spite of the importance of the problem, no further results have been reported since [15]. In particular, the generalization to problems with time-dependent constraints is important both from a theoretical point of view and with respect to the application thereof (cf. Lions [16, Open Problem 9.7] and Duvaut and Lions [9]).

The aim of this paper is to establish a general framework for studying the singular perturbation problem (1.1)–(1.4) and thus to extend Lions' result to include the case of time-dependent subdifferential constraints. Our results also provide a

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viewpoint that unifies the study of hyperbolic variational inequalities of the type (1.1) and parabolic variational inequalities of the type (1.3) (see Lions [14,15], Brézis [4], Barbu [3], Senba [19] and Kubo [13]).

Since the original research (with respect to time-independent constraints) by Lions, Brézis and Barbu, hyperbolic variational inequalities of the type (1.1) have attracted renewed interest in the light of their connection to the hyperbolic Stefan problem, which is an ice–water phase change problem with the energy balance law given by the Cattaneo equation

$$\varepsilon \theta'' + \theta' - \Delta \theta = f \quad (1.5)$$

instead of the classical heat equation ($\varepsilon = 0$). We refer, for instance, to Showalter and Walkington [21], Colli et al. [6] and Durand [7], who considered various weak formulations similar to ours (1.1). However, in these studies only a homogeneous boundary condition, $\theta = 0$, is considered, while our abstract result can deal with non-homogeneous and time-dependent boundary conditions by using an appropriate transformation (see Section 5.2).

Furthermore, the singular limit of the hyperbolic Stefan problem to the usual parabolic Stefan problem was studied by Friedman and Hu [10] and Shemetov [20] using a classical solution framework (without appealing to a variational inequality) in a domain with one spatial dimension. Our result, on the other hand, is applicable to problems in a multi-dimensional domain by using the variational formulation (1.1) and we show (Theorem 5.4) that the solution of the hyperbolic variational inequality converges to that of the parabolic Stefan problem studied in classical papers by Duvaut [8] and Friedman and Kinderlehrer [11].

In addition, our result can be viewed as a method of constructing a solution for (1.3) by employing (1.1) as an approximate or regularized problem. From this viewpoint, the singular limit construction of our solution leads to the regularity property $u' \in L^\infty(0, T; H)$, obtained by testing (1.1) by u'' and deriving an estimate uniform in ε and which is stronger than that obtained by Senba [19] with a different method. Our method can also be applied to obstacles with weaker assumptions than in [19] (see Remark 5.5). These merits stem from our systematic usage of the Yosida approximation in the time-dependence condition on $\partial \varphi^t$ and in the compatibility condition of A and $\partial \varphi^t$ (conditions (A)(iii) and (B)(i)), which play an important role in deriving uniform bounds.

Regarding Eq. (1.3), there is a vast body of literature (cf. [17,18], for instance) that deals with abstract equations of the form

$$Mu' + Au \ni f,$$

where the operators M and A are either linear or nonlinear, time-independent or time-dependent. However, to the best of our knowledge, only our result and that of Senba [19] (with a linear A) are applicable to variational inequalities with time-dependent constraints imposed on u' as in (1.3), which arise in a weak formulation of the parabolic Stefan problem (see Section 5.2).

The main theorem, Theorem 2.2, and its Corollary 2.3 are stated in Section 2.

In Section 3, we provide the proof of Theorem 2.2, assuming a uniform estimate of an approximate solution (Proposition 3.2), which is proved in Section 4. Our idea is based on the method introduced in [13], but we have to modify and refine the argument as well as the assumptions to derive uniform (in ε) estimates of the solution of (1.1). A fundamental role is played by Lemma 4.3 derived by the time-dependence condition of convex functions ((A)(iii)), which is a variant of that developed in the theory of the time-dependent subdifferential evolution equation (see [12] and its references).

In Section 5, we explain how the abstract theory can be applied to concrete problems. We provide useful criterions (Lemmas 5.1 and 5.2) for conditions assumed in Theorem 2.2 in the case where $\partial \varphi^t = I + \partial I_{K(t)}$ with time-dependent convex sets $K(t)$. With the help of these criterions, we apply Theorem 2.2 and Corollary 2.3 to an obstacle problem that arises in the Stefan problem (Theorem 5.4).

The norm and inner product of H are denoted by $|\cdot|_H$ and (\cdot, \cdot) , respectively. The domains $D(A)$ and $D(A^{1/2})$ of A and its fractional power $A^{1/2}$ are Hilbert spaces by graph norms. The Yosida approximation of A is denoted by A_λ for $\lambda > 0$. We write $A_\lambda^{1/2}$ for the fractional power $(A_\lambda)^{1/2}$ of A_λ . The effective domain of φ^t is denoted by $D(\varphi^t)$. For $\lambda > 0$, the Yosida approximations of $\partial \varphi^t$ and φ^t are denoted by $\partial \varphi_\lambda^t$ and φ_λ^t , respectively. We refer to Brézis [5] for additional definitions and fundamental properties.

2. Main theorem

First, we list the assumptions for the main theorem.

- (A) (i) The mapping $t \mapsto \varphi^t$ is continuous in the sense of Mosco (see Attouch [1, Chap. I] and [2, Chap. 3]).
(ii) There exists a constant $C_1 > 0$ such that for all $t \in [0, T]$ and $z \in H$, it holds that

$$\varphi^t(z) \geq C_1 |z|_H^2.$$

- (iii) For all $\lambda > 0$ and $z \in H$, the function $t \mapsto \varphi_\lambda^t(z)$ is of bounded variation with an absolutely continuous positive variation on $[0, T]$. Moreover, there exist nonnegative functions $a \in L^2(0, T)$, $b, c \in L^1(0, T)$ and a constant $d \geq 0$, such that for all $\lambda > 0$, $z \in H$, and a.e. $t \in (0, T)$, it holds that

$$\frac{d}{dt} \varphi_\lambda^t(z) \leq a(t) |\partial \varphi_\lambda^t(z)|_H + b(t) + c(t) \varphi_\lambda^t(z) + d(\varphi_\lambda^t(z))^2.$$

(B) (i) There exist a constant $C_2 > 0$ and a nonnegative function $e \in L^2(0, T)$, such that for all $t \in [0, T]$, $\lambda \in (0, 1]$, and $z \in H$ it holds that

$$(\partial\varphi_\lambda^t(z), A_\lambda z) \geq C_2 |A_\lambda^{1/2} z|_H^2 - e(t) |\partial\varphi_\lambda^t(z)|_H.$$

(ii) There exists $h \in W^{1,1}(0, T; D(A^{1/2}))$ such that the function $t \mapsto \varphi^t(h(t))$ belongs to $L^1(0, T)$.
 (C) $f \in W^{1,2}(0, T; H)$, $u_0 \in D(A)$, $u_1 \in D(A^{1/2}) \cap D(\varphi^0)$.

We denote by $(E)_\varepsilon$ where $\varepsilon > 0$, and by (E) the problems $\{(1.1), (1.2)\}$ and $\{(1.3), (1.4)\}$, respectively. The notion of a solution is defined below.

Definition 2.1. A function $u : [0, T] \rightarrow H$ is called a solution of $(E)_\varepsilon$ (resp. (E)), if items (a)–(d) below hold.

- (a) $u \in W^{2,2}(0, T; H) \cap W^{1,\infty}(0, T; D(A^{1/2})) \cap L^\infty(0, T; D(A))$ (resp. $u \in W^{1,\infty}(0, T; H) \cap W^{1,2}(0, T; D(A^{1/2})) \cap L^\infty(0, T; D(A))$).
- (b) $\sup_{0 \leq t \leq T} \varphi^t(u'(t)) < \infty$ (resp. $\text{ess. sup}_{0 \leq t \leq T} \varphi^t(u'(t)) < \infty$).
- (c) Eq. (1.1) (resp. (1.3)) is satisfied for a.e. $t \in (0, T)$.
- (d) The initial condition (1.2) (resp. (1.4)) is satisfied.

The main result of this paper is stated below.

Theorem 2.2. Assume (A)–(C). Then, for each $\varepsilon > 0$, there exists a unique solution u_ε of $(E)_\varepsilon$, and we have for some sequence $\varepsilon_n \rightarrow 0$ ($n \rightarrow \infty$)

$$u_{\varepsilon_n} \rightarrow u \quad \text{weakly-}^* \text{ in } W^{1,\infty}(0, T; H) \cap L^\infty(0, T; D(A)) \text{ and weakly in } W^{1,2}(0, T; D(A^{1/2}))$$

for a function u , that is a solution of (E).

This theorem is proved in Sections 3 and 4. The uniqueness of a solution to $(E)_\varepsilon$ can be proved by a standard argument of monotonicity, and hence is left to the reader.

We note that the uniform bound given in Proposition 3.3 (see Section 3) holds. In particular, we have that $\{\sqrt{\varepsilon} u_\varepsilon''\}$ is bounded in $L^2(0, T; H)$. Therefore, we can easily show the following corollary, which generalizes [15, Chap. 6, Théorème 2.3] (or [3, Chap. V, Theorem 1.3]) to the case of time-dependent constraints.

Corollary 2.3. In addition to (A)–(C), assume that (D) below is satisfied.

(D) There exists a constant $\nu > 0$ such that for all $t \in [0, T]$ and $z_i, z_i^* \in H$ with $z_i^* \in \partial\varphi^t(z_i)$, $i = 1, 2$, it holds that

$$(z_1^* - z_2^*, z_1 - z_2) \geq \nu |z_1 - z_2|_H^2.$$

Then, the solution u of (E) is unique, the whole family u_ε converges as $\varepsilon \rightarrow 0$ to u in the sense of Theorem 2.2, and we have

$$\nu \int_0^T |u'_\varepsilon(t) - u'(t)|_H^2 dt + \sup_{0 \leq t \leq T} |A^{1/2}(u_\varepsilon(t) - u(t))|_H^2 \leq M'_0 \sqrt{\varepsilon},$$

where the constant $M'_0 > 0$ depends only on the constant M_0 in Proposition 3.3.

3. Proof of Theorem 2.2

First, note that the Yosida approximations $\partial\varphi_\lambda^t(z)$ and $A_\lambda z$ are Lipschitz continuous in $z \in H$ (see [5]), and that for each $\lambda > 0$ and $z \in H$, the H -valued function $t \mapsto \partial\varphi_\lambda^t(z)$ is continuous by the Mosco continuity (A)(i) (see [1, Chap. I] and [2, Chap. 3]). Therefore, we can prove the following proposition by a standard argument.

Proposition 3.1. For each $\varepsilon > 0$ and $\lambda \in (0, 1]$, there exists a unique solution $u_{\varepsilon,\lambda} \in C^2([0, T]; H)$ of the problem

$$\varepsilon u_{\varepsilon,\lambda}''(t) + \partial\varphi_\lambda^t(u'_{\varepsilon,\lambda}(t)) + A_\lambda u_{\varepsilon,\lambda}(t) = f(t) \quad \text{a.e. } t \in (0, T),$$

$$u_{\varepsilon,\lambda}(0) = u_0, \quad u'_{\varepsilon,\lambda}(0) = u_1.$$

The crucial step in the proof of Theorem 2.2 is to derive the following uniform estimate.

Proposition 3.2. *There exists a constant $M_0 > 0$ independent of $\varepsilon \in [0, \varepsilon_0]$ (with a fixed $\varepsilon_0 > 0$) and of $\lambda \in (0, 1]$, such that the solution $u_{\varepsilon, \lambda}$ in Proposition 3.1 has the bound*

$$\begin{aligned} & \sqrt{\varepsilon} |u''_{\varepsilon, \lambda}|_{L^2(0, T; H)} + \sqrt{\varepsilon} |A^{1/2} u'|_{L^\infty(0, T; H)} + |u_{\varepsilon, \lambda}|_{W^{1, \infty}(0, T; H)} \\ & + |A^{1/2} u_{\varepsilon, \lambda}|_{W^{1, 2}(0, T; H)} + |A u_{\varepsilon, \lambda}|_{L^\infty(0, T; H)} + \sup_{0 \leq t \leq T} \varphi^t_\lambda(u'_{\varepsilon, \lambda}(t)) \leq M_0. \end{aligned}$$

The proof of this proposition is given in Section 4.

Here, in exactly the same way as in [13, Section 3], we use the limit $\lambda \rightarrow 0$ to obtain the following.

Proposition 3.3. *For each $\varepsilon > 0$, there exists a unique solution u_ε of $(E)_\varepsilon$ with the bound*

$$\begin{aligned} & \sqrt{\varepsilon} |u''_\varepsilon|_{L^2(0, T; H)} + \sqrt{\varepsilon} |A^{1/2} u'|_{L^\infty(0, T; H)} + |u_\varepsilon|_{W^{1, \infty}(0, T; H)} \\ & + |A^{1/2} u_\varepsilon|_{W^{1, 2}(0, T; H)} + |A u_\varepsilon|_{L^\infty(0, T; H)} + \sup_{0 \leq t \leq T} \varphi^t(u'_\varepsilon(t)) \leq M_0, \end{aligned}$$

where M_0 is the same constant as in Proposition 3.2.

We are now in a position to prove Theorem 2.2. From Proposition 3.3, for some sequence $\varepsilon_n \rightarrow 0$ ($n \rightarrow \infty$), we have (for simplicity, we write u_ε and $\varepsilon \rightarrow 0$ for u_{ε_n} and $\varepsilon_n \rightarrow 0$, respectively)

$$\begin{aligned} \varepsilon u''_\varepsilon &\rightarrow 0 \quad \text{in } L^2(0, T; H), \\ u_\varepsilon &\rightarrow u \quad \text{weakly-* in } W^{1, \infty}(0, T; H), \\ A^{1/2} u_\varepsilon &\rightarrow A^{1/2} u \quad \text{weakly in } W^{1, 2}(0, T; H), \\ A u_\varepsilon &\rightarrow A u \quad \text{weakly-* in } L^\infty(0, T; H), \\ u^*_\varepsilon &\rightarrow u^* \quad \text{weakly in } L^2(0, T; H) \end{aligned}$$

for some u and u^* , where u^*_ε is defined by

$$\varepsilon u''_\varepsilon + u^*_\varepsilon + A u_\varepsilon = f \quad \text{and} \quad u^*_\varepsilon \in \partial \varphi^t(u'_\varepsilon) \quad \text{a.e. in } (0, T).$$

It is straightforward that

$$u^* + A u = f \quad \text{a.e. in } (0, T)$$

and

$$u(0) = u_0.$$

Now let us derive the relation $u^*(t) \in \partial \varphi^t(u'(t))$ for a.e. $t \in (0, T)$ and the bound $\text{ess. sup}_{0 \leq t \leq T} \varphi^t(u'(t)) \leq M_0 < \infty$.

To prove $u^*(t) \in \partial \varphi^t(u'(t))$ for a.e. $t \in (0, T)$, take an arbitrary $v \in L^2(0, T; H)$ with

$$\int_0^T \varphi^t(v(t)) dt < \infty.$$

Then, we have by (1.1)

$$\int_0^T \varphi^t(v(t)) dt - \int_0^T \varphi^t(u'_\varepsilon(t)) dt \geq \int_0^T (f - \varepsilon u''_\varepsilon - A u_\varepsilon, v - u'_\varepsilon) dt.$$

Letting $\varepsilon \rightarrow 0$ we obtain, by using the lower-semicontinuity of φ^t ,

$$\int_0^T \varphi^t(v(t)) dt - \int_0^T \varphi^t(u'(t)) dt \geq \int_0^T (f - A u, v - u') dt = \int_0^T (u^*, v - u') dt.$$

Therefore, we obtain $u^*(t) \in \partial \varphi^t(u'(t))$ for a.e. $t \in (0, T)$ with the aid of [12, Lemma 3.4] (or Attouch [1]).

Finally, the property $\text{ess. sup}_{0 \leq t \leq T} \varphi^t(u'(t)) \leq M_0 < \infty$ can be derived from the following inequality, which can be shown by the lower-semicontinuity of Φ restricted on $L^2(E; H)$ for an arbitrary measurable $E \subset (0, T)$

$$\int_E \varphi^t(u'(t)) \leq \liminf_{\varepsilon \rightarrow 0} \int_E \varphi^t(u'_\varepsilon(t)) \leq |E| M_0.$$

Thus we have proved that the limit function u is a solution of (E) .

The proof of the theorem is now complete, excluding the proof of Proposition 3.2, which is given in Section 4.

4. Proof of Proposition 3.2

In this section, for simplicity, we write u for the approximate solution $u_{\varepsilon, \lambda}$. The assertion (a) of the following lemma is a simple consequence of (A)(ii) and the definition of the Yosida approximation φ_λ^t . The assertion (b) can be proved in the same way as [13, Lemma 4.1].

Lemma 4.1.

(a) For all $t \in [0, T]$, $\lambda \in (0, 1]$, and $z \in H$, it holds that

$$\varphi_\lambda^t(z) \geq \frac{C_1}{1 + 2C_1} |z|_H^2.$$

(b) There exists a function $k \in W^{2,2}(0, T; H) \cap W^{1,\infty}(0, T; D(A^{1/2})) \cap W^{1,2}(0, T; D(A))$ such that $\sup_{0 \leq t \leq T} \varphi^t(k'(t)) < \infty$.

Now let us begin the derivation of the estimate given in Proposition 3.2. To derive the first and auxiliary estimate, multiply the approximate equation

$$\varepsilon u'' + \partial \varphi_\lambda^t(u') + A_\lambda u = f \quad (4.1)$$

by $u' - k'$. Then, we have

$$\begin{aligned} \frac{\varepsilon}{2} \frac{d}{dt} |u' - k'|_H^2 + \varphi_\lambda^t(u') + \frac{1}{2} \frac{d}{dt} |A_\lambda^{1/2}(u - k)|_H^2 &\leq (F, u' - k') + \varphi_\lambda^t(k'), \\ F &:= f - \varepsilon k'' - A_\lambda k. \end{aligned}$$

Applying Gronwall's lemma and Lemma 4.1 to this inequality, we can derive the following proposition (see [13, Section 4.1]).

Proposition 4.2. There exists a constant $M_1 > 0$ independent of $\varepsilon \in [0, \varepsilon_0]$ (with a fixed $\varepsilon_0 > 0$) and of $\lambda \in (0, 1]$, such that

$$\sqrt{\varepsilon} |u'|_{L^\infty(0,T;H)} + |u'|_{L^2(0,T;H)} + |A_\lambda^{1/2} u|_{L^\infty(0,T;H)} + |\varphi_\lambda^{(\cdot)}(u'(\cdot))|_{L^1(0,T)} \leq M_1.$$

The derivation of the main estimate is based on the following lemma, which is proved at the end of this section.

Lemma 4.3. For all $\lambda \in (0, 1]$ and $v \in W^{1,1}(0, T; H)$ the function $t \mapsto \varphi_\lambda^t(v(t))$ is of bounded variation, its positive variation is absolutely continuous on $[0, T]$, and for a.e. $t \in (0, T)$, it holds that

$$\frac{d}{dt} \varphi_\lambda^t(v(t)) - (\partial \varphi_\lambda^t(v(t)), v'(t)) \leq a(t) |\partial \varphi_\lambda^t(v(t))|_H + b(t) + c(t) \varphi_\lambda^t(v(t)) + d(\varphi_\lambda^t(v(t)))^2.$$

Now we derive the estimate in Proposition 3.2. First, multiply the approximate equation (4.1) by $A_\lambda u'$. Then, using (B)(i), we have

$$\frac{d}{dt} \left\{ \frac{\varepsilon}{2} |A_\lambda^{1/2} u'|_H^2 + \frac{1}{2} |A_\lambda u|_H^2 - (f, A_\lambda u) \right\} + C_2 |A_\lambda^{1/2} u'|_H^2 \leq e(t) |\partial \varphi_\lambda^t(u')|_H - (f', A_\lambda u). \quad (4.2)$$

Second, by Lemma 4.3, we have

$$\frac{d}{dt} \varphi_\lambda^t(u') - (\partial \varphi_\lambda^t(u'), u'') \leq a(t) |\partial \varphi_\lambda^t(u')|_H + b(t) + (c(t) + d\varphi_\lambda^t(u')) \varphi_\lambda^t(u').$$

Hence, by (4.1), we have

$$\frac{d}{dt} \varphi_\lambda^t(u') + \varepsilon |u''|_H^2 + (A_\lambda u, u'') \leq (f, u'') + a(t) |\partial \varphi_\lambda^t(u')|_H^2 + b(t) + (c(t) + d\varphi_\lambda^t(u')) \varphi_\lambda^t(u').$$

Note that here

$$(A_\lambda u, u'') = \frac{d}{dt} (A_\lambda^{1/2} u, A_\lambda^{1/2} u') - |A_\lambda^{1/2} u'|_H^2 = \frac{d}{dt} (A_\lambda u, u') - |A_\lambda^{1/2} u'|_H^2$$

and

$$(f, u'') = \frac{d}{dt} (f, u') - (f', u').$$

Then, we obtain

$$\begin{aligned} & \frac{d}{dt} \{ \varphi_\lambda^t(u') + (A_\lambda u, u') - (f, u') \} + \varepsilon |u''|_H^2 \\ & \leq |A_\lambda^{1/2} u'|_H^2 - (f', u') + a(t) |\partial \varphi_\lambda^t(u')|_H + b(t) + (c(t) + d\varphi_\lambda^t(u')) \varphi_\lambda^t(u'). \end{aligned} \quad (4.3)$$

Now calculate $\delta \times (4.2) + (4.3)$ with the constant $\delta > 0$ to be determined later, and define

$$\begin{aligned} U(t) &:= \delta \left\{ \frac{\varepsilon}{2} |A_\lambda^{1/2} u'|_H^2 + \frac{1}{2} |A_\lambda u|_H^2 - (f, A_\lambda u) \right\} + \{ \varphi_\lambda^t(u') + (A_\lambda u, u') - (f, u') \}, \\ \alpha(t) &:= \delta e(t) + a(t), \quad \beta(t) := -(f', u') + b(t), \quad \gamma(t) := c(t) + d\varphi_\lambda^t(u'). \end{aligned}$$

Then, using (4.1) and the Schwarz inequality, we obtain

$$\begin{aligned} \frac{d}{dt} U(t) + (\delta C_2 - 1) |A_\lambda^{1/2} u'|_H^2 + \varepsilon |u''|_H^2 &\leq \alpha(t) |u''|_H + A_\lambda u - f|_H + \beta(t) + \gamma(t) \varphi_\lambda^t(u') - \delta (f', A_\lambda u) \\ &\leq \frac{\varepsilon}{2} |u''|_H^2 + \frac{1}{2} |A_\lambda u|_H^2 + \tilde{\beta}(t) + \gamma(t) \varphi_\lambda^t(u') + \frac{\delta}{2} |A_\lambda u|_H^2. \end{aligned}$$

Here, we have defined

$$\tilde{\beta}(t) := \left(\frac{\varepsilon}{2} + \frac{1}{2} \right) \alpha(t)^2 + \alpha(t) |f|_H + \beta(t) + \frac{\delta}{2} |f'|_H^2.$$

Therefore, we obtain

$$\frac{d}{dt} U(t) + (\delta C_2 - 1) |A_\lambda^{1/2} u'|_H^2 + \frac{\varepsilon}{2} |u''|_H^2 \leq \tilde{\beta}(t) + \gamma(t) \varphi_\lambda^t(u') + \frac{1+\delta}{2} |A_\lambda u|_H^2. \quad (4.4)$$

Now note that by Proposition 4.2 and conditions (A)–(C)

$$|\tilde{\beta}|_{L^1(0,T)} + |\gamma|_{L^1(0,T)} \leq M'_1$$

with the constant $M'_1 > 0$ independent of $\varepsilon \in [0, \varepsilon_0]$ and $\lambda \in (0, 1]$. Hence, we can obtain the desired estimate by choosing $\delta > \max\{C_2^{-1}, 8C_3^{-1}\}$, where the constant $C_3 > 0$ is defined by

$$C_3 := \frac{1}{2} \frac{C_1}{1 + 2C_1} \quad (\text{see Lemma 4.1(a)}),$$

by noting (from the Schwarz inequality and Lemma 4.1(a)) that for $\lambda \in (0, 1]$

$$\begin{aligned} U(t) &\geq \frac{\delta\varepsilon}{2} |A_\lambda^{1/2} u'|_H^2 + \left\{ \delta \left(\frac{1}{2} - \frac{1}{4} \right) - 2C_3^{-1} \right\} |A_\lambda u|_H^2 + \frac{1}{2} \varphi_\lambda^t(u') + C_3 \left(\frac{1}{2} - \frac{1}{8} - \frac{1}{8} \right) |u'|_H^2 - (\delta + 2C_3^{-1}) |f|_H^2 \\ &\geq \frac{\delta\varepsilon}{2} |A_\lambda^{1/2} u'|_H^2 + \left(\frac{\delta}{4} - 2C_3^{-1} \right) |A_\lambda u|_H^2 + \frac{1}{2} \varphi_\lambda^t(u') + \frac{C_3}{4} |u'|_H^2 - (\delta + 2C_3^{-1}) |f|_H^2, \end{aligned}$$

and by applying Gronwall's lemma to (4.4). Thus the proof of Proposition 3.2 will be complete once we prove Lemma 4.3.

Proof of Lemma 4.3. Taking $z = v(t)$ in (A), we have

$$\begin{aligned} \varphi_\lambda^{t+h}(v(t+h)) - \varphi_\lambda^t(v(t)) &= \varphi_\lambda^{t+h}(v(t+h)) - \varphi_\lambda^{t+h}(v(t)) + \varphi_\lambda^{t+h}(v(t)) - \varphi_\lambda^t(v(t)) \\ &\leq (\partial \varphi_\lambda^{t+h}(v(t+h)), v(t+h) - v(t)) + \int_t^{t+h} \frac{d}{d\tau} \varphi_\lambda^\tau(v(t)) d\tau \\ &\leq (\partial \varphi_\lambda^{t+h}(v(t+h)), v(t+h) - v(t)) \\ &\quad + \int_t^{t+h} \{ a(\tau) |\partial \varphi_\lambda^\tau(v(t))|_H + b(\tau) + c(\tau) \varphi_\lambda^\tau(v(t)) + d(\varphi_\lambda^\tau(v(t)))^2 \} d\tau. \end{aligned} \quad (4.5)$$

From this, we see that

$$\varphi_\lambda^{t+h}(v(t+h)) - \varphi_\lambda^t(v(t)) \leq \int_t^{t+h} \rho(\tau) d\tau$$

for some function $\rho \in L^1(0, T)$. Hence, the function $t \mapsto \varphi_\lambda^t(v(t))$ is of bounded variation with an absolutely continuous positive variation. The desired inequality is obtained by dividing (4.5) by $h > 0$, letting $h \downarrow 0$ and using the continuity of $\tau \mapsto \varphi_\lambda^\tau(v(t))$ and $\tau \mapsto \partial \varphi_\lambda^\tau(v(t))$ as well as the Lipschitz continuity of $z \mapsto \partial \varphi_\lambda^t(z)$. \square

5. Applications

5.1. Sufficient conditions for (A) and (B)

Here, we provide sufficient conditions for (A) and (B)(i), which will be useful in applications of Theorem 2.2 and Corollary 2.3.

Let $\{K(t); 0 \leq t \leq T\}$ be a family of nonempty closed convex sets in H , and $I_{K(t)}$ be the indicator function of $K(t)$. Denote the projection operator onto $K(t)$ by $\text{Proj}_{K(t)}$. Note that $\text{Proj}_{K(t)} = (I + \lambda \partial I_{K(t)})^{-1}$ for all $\lambda > 0$.

For a constant $\nu > 0$, define $\varphi^t : H \rightarrow \mathbf{R} \cup \{\infty\}$ by

$$\varphi^t(z) := \frac{\nu}{2} |z|_H^2 + I_{K(t)}(z) = \begin{cases} \frac{\nu}{2} |z|_H^2, & \text{if } z \in K(t), \\ \infty, & \text{otherwise.} \end{cases}$$

Then we have

$$\partial \varphi^t = \nu I + \partial I_{K(t)}.$$

The following lemma gives a sufficient condition for condition (A) to hold.

Lemma 5.1. Assume that there exists $a \in L^2(0, T)$ such that for all $z \in H$, the function $t \mapsto \text{Proj}_{K(t)} z$ is absolutely continuous and

$$\left| \frac{d}{dt} \text{Proj}_{K(t)} z \right|_H \leq a(t) \quad \text{a.e. } t \in (0, T).$$

Then, the family $\{\varphi^t; 0 \leq t \leq T\}$ satisfies (A).

Proof. We can show by a standard argument (see [1,2]) that the mappings $t \mapsto I_{K(t)}$ and $t \mapsto \varphi^t$ are continuous in the sense of Mosco, if the H -valued function $t \mapsto \text{Proj}_{K(t)} z$ is continuous for all $z \in H$. Thus (A)(i) is satisfied. Also, it is clear that (A)(ii) is satisfied with $C_1 = \frac{\nu}{2}$.

Let us verify (A)(iii). Note first that for all $\lambda > 0$ and $z \in H$,

$$(I + \lambda \partial \varphi^t)^{-1}(z) = \text{Proj}_{K(t)} \left(\frac{z}{1 + \lambda \nu} \right)$$

and

$$\partial \varphi_\lambda^t(z) = \frac{1}{\lambda} \left(z - \text{Proj}_{K(t)} \left(\frac{z}{1 + \lambda \nu} \right) \right).$$

Therefore, we have

$$\begin{aligned} \varphi_\lambda^t(z) &= \frac{1}{2\lambda} \left| z - \text{Proj}_{K(t)} \left(\frac{z}{1 + \lambda \nu} \right) \right|_H^2 + \frac{1}{2} \left| \text{Proj}_{K(t)} \left(\frac{z}{1 + \lambda \nu} \right) \right|_H^2 \\ &= \frac{\lambda}{2} |\partial \varphi_\lambda^t(z)|^2 + \frac{1}{2} \left| \text{Proj}_{K(t)} \left(\frac{z}{1 + \lambda \nu} \right) \right|_H^2 \\ &\geq \frac{1}{2} \left| \text{Proj}_{K(t)} \left(\frac{z}{1 + \lambda \nu} \right) \right|_H^2. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \frac{d}{dt} \varphi_\lambda^t(z) &= - \left(\partial \varphi_\lambda^t(z), \frac{d}{dt} \text{Proj}_{K(t)} \left(\frac{z}{1 + \lambda \nu} \right) \right) + \left(\text{Proj}_{K(t)} \left(\frac{z}{1 + \lambda \nu} \right), \frac{d}{dt} \text{Proj}_{K(t)} \left(\frac{z}{1 + \lambda \nu} \right) \right) \\ &\leq |\partial \varphi_\lambda^t(z)|_H a(t) + \varphi_\lambda^t(z) + \frac{1}{2} a(t)^2. \end{aligned}$$

Therefore, we have (A)(iii). \square

Next, we give conditions which are sufficient for condition (B)(i) to hold.

Lemma 5.2. Assume that the following conditions (a) and (b) are satisfied.

(a) There exists $g \in L^2(0, T; H)$ such that for all $t \in [0, T]$, $z \in K(t)$, and $\lambda \in (0, 1]$, it holds that

$$(I + \lambda A)^{-1}(z + \lambda g(t)) \in K(t).$$

(b) $0 \in K(t)$ for all $t \in [0, T]$.

Then, the family $\{\varphi^t; 0 \leq t \leq T\}$ satisfies (B)(i).

Proof. First, note that by [13, Lemma 5.1] assumption (a) implies

$$((\partial I_{K(t)})_\lambda(z), A_\lambda z) \geq ((\partial I_{K(t)})_\lambda(z), (I + \lambda A)^{-1}g(t)).$$

In addition, we have (cf. proof of Lemma 5.1)

$$\partial \varphi_\lambda^t(z) = \frac{vz}{1 + \lambda v} + (\partial I_{K(t)})_\lambda \left(\frac{z}{1 + \lambda v} \right),$$

and therefore

$$|\partial \varphi_\lambda^t(z)|_H^2 \geq \left| (\partial I_{K(t)})_\lambda \left(\frac{z}{1 + \lambda v} \right) \right|_H^2,$$

since assumption (b) implies $(\partial I_{K(t)})_\lambda(0) = 0$, and hence

$$\left(\frac{z}{1 + \lambda v}, (\partial I_{K(t)})_\lambda \left(\frac{z}{1 + \lambda v} \right) \right) \geq 0.$$

Combining these relations leads to the inequalities

$$\begin{aligned} (\partial \varphi_\lambda^t(z), A_\lambda z) &= \left(\frac{vz}{1 + \lambda v}, A_\lambda z \right) + (1 + \lambda v) \left((\partial I_{K(t)})_\lambda \left(\frac{z}{1 + \lambda v} \right), A_\lambda \left(\frac{z}{1 + \lambda v} \right) \right) \\ &\geq \frac{v}{1 + \lambda v} |A_\lambda^{1/2} z|_H^2 + (1 + \lambda v) \left((\partial I_{K(t)})_\lambda \left(\frac{z}{1 + \lambda v} \right), (I + \lambda A)^{-1}g(t) \right) \\ &\geq \frac{v}{1 + \lambda v} |A_\lambda^{1/2} z|_H^2 - (1 + \lambda v) |\partial \varphi_\lambda^t(z)|_H |g(t)|_H, \end{aligned}$$

in which we have used the relation $|(I + \lambda A)^{-1}g(t)|_H \leq |g(t)|_H$. From this, we see that (B)(i) is satisfied with $C_2 = v/(1 + v)$ and $e(t) = (1 + v)|g(t)|_H$. \square

5.2. A unilateral problem

Here we apply the abstract Theorem 2.2 and Corollary 2.3, with the help of the lemmas in Section 5.1, to a unilateral obstacle problem in a bounded domain $\Omega \subset \mathbf{R}^N$ ($N \geq 1$) with a smooth boundary $\partial\Omega$.

Given a function

$$g \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)), \quad g \leq 0, \quad (5.1)$$

we define

$$K(t) := \{z \in L^2(\Omega); z \geq g(t) \text{ in } \Omega\}.$$

Furthermore, given that

$$f \in W^{1,2}(0, T; L^2(\Omega)), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad u_1 \in H_0^1(\Omega) \cap K(0), \quad (5.2)$$

we study the following variational inequality.

Definition 5.3. A function u is called a *solution of (VI) $_\varepsilon$* for $\varepsilon > 0$ (resp. (VI)) if items (a)–(d) are satisfied.

(a) $u \in W^{2,2}(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$ (resp. $u \in W^{1,\infty}(0, T; L^2(\Omega)) \cap W^{1,2}(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$).

(b) $u'(t) \in K(t)$ for all (resp. a.e.) $t \in [0, T]$.

(c) The following inequality (resp. the one in which $\varepsilon = 0$) holds for a.e. $t \in [0, T]$ and all $z \in K(t)$

$$\int_{\Omega} (\varepsilon u'' + u' - \Delta u)(z - u') dx \geq \int_{\Omega} f(z - u') dx.$$

(d) $u(0) = u_0$ and $u'(0) = u_1$ (resp. $u(0) = u_0$).

We notice that problems of type (VI) $_\varepsilon$ arise in weak formulations of the hyperbolic Stefan problem (cf. [21,6,7]) whereas problems of type (VI) arise in weak formulations of the usual parabolic Stefan problem (cf. [8,11]). By a transformation due originally to Duvaut [8]

$$u := \int_0^t \theta dt - \int_0^t h dt,$$

where θ is the temperature (see (1.5)) and h its boundary value, the hyperbolic and parabolic Stefan problems are formulated as $(VI)_\varepsilon$ and (VI), respectively, with the obstacle g defined by

$$g(t) := -h(t).$$

We have the following result for the singular limit of problems $(VI)_\varepsilon$ to (VI).

Theorem 5.4. Assume (5.1) and (5.2). Then, there is a unique solution u_ε of $(VI)_\varepsilon$ for each $\varepsilon > 0$ and as $\varepsilon \rightarrow 0$

$$u_\varepsilon \rightarrow u \quad \text{weakly-}^* \text{ in } W^{1,\infty}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^2(\Omega)) \text{ and weakly in } W^{1,2}(0, T; H_0^1(\Omega))$$

for a function u , which is a unique solution of (VI). Furthermore, we have

$$\int_0^T |u'_\varepsilon(t) - u'(t)|_{L^2(\Omega)}^2 dt + \sup_{0 \leq t \leq T} |u_\varepsilon(t) - u(t)|_{H_0^1(\Omega)}^2 \leq M_2 \sqrt{\varepsilon},$$

where $M_2 > 0$ is a constant independent of ε .

Proof. To apply Theorem 2.2 and Corollary 2.3 in proving Theorem 5.4, we set $H := L^2(\Omega)$, $A := -\Delta$ with $D(A) := H_0^1(\Omega) \cap H^2(\Omega)$ and

$$\varphi^t(z) := \frac{1}{2} |z|_H^2 + I_{K(t)}(z).$$

We can verify assumptions (A)–(D) with the aid of Lemmas 5.1 and 5.2 by noting that

$$\text{Proj}_{K(t)} z = \max\{z, g\} = z + [g - z]^+$$

and by taking Δg as the function g in Lemma 5.2 and using the maximum principle. For (B)(ii) we take $h = 0$. Hence, we have Theorem 5.4 by applying Theorem 2.2 and Corollary 2.3. \square

Remark 5.5. In applying the result of Senba [19] to a parabolic variational inequality with a bilateral constraint, one needs to assume a condition such as

$$\Delta g(t) \geq 0;$$

this is not required in our approach.

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