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Meromorphic functions in the unit disc that share values in an angular domain ☆

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ABSTRACT

In this paper, we investigate the uniqueness of meromorphic functions in the unit disc and consider the relation between the Borel points and shared-values of meromorphic functions in an angular domain.

1. Introduction and main results

Let $f(z)$ be a meromorphic function in $\mathbb{D}_R = \{z \mid |z| < R\}$, where $0 < R \leqslant \infty$. We adopt the standard notations of Nevanlinna's value distribution theory (see [1] or [2]), such as $T(r, f)$, $N(r, f)$ and $m(r, f)$. Suppose that $f(z)$ and $g(z)$ are two nonconstant meromorphic functions in $\mathbb{D}_R$, $a \in \mathbb{C}_\infty$ ($\mathbb{C}_\infty$ denotes the extended complex plane) and $\mathbb{X} \subseteq \mathbb{D}_R$. We say that $f(z)$ and $g(z)$ share a CM (counting multiplicities) in $\mathbb{X}$ provided that $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities in $\mathbb{X}$. Similarly, we say that $f(z)$ and $g(z)$ share a IM (ignoring multiplicities) in $\mathbb{X}$ provided that $f(z) - a$ and $g(z) - a$ have the same zeros in $\mathbb{X}$. When $a = \infty$ the zeros of $f(z) - a$ mean the poles of $f(z)$.

In 1929, R. Nevanlinna [3] proved that for two nonconstant meromorphic functions $f(z)$ and $g(z)$ in the complex plane $\mathbb{C}$, if they share five distinct values IM in $\mathbb{C}$, then $f(z) \equiv g(z)$ (five-value theorem); if they share four distinct values CM in $\mathbb{C}$, then f is a Möbius transformation of g (four-value theorem). After his very work, many results on uniqueness of meromorphic functions concerning shared values in the complex plane have been obtained (see [4]). In 2003, J.H. Zheng firstly took into account the uniqueness of meromorphic functions sharing values in an angular domain and extended five-value theorem and four-value theorem in the complex plane to an angular domain (see [5,6]). Since then, the uniqueness of meromorphic functions in an angular domain attracted many investigations (see [7–9], etc.). In fact, he proved the following theorems on the basis of the relation between the Pólya peaks and deficiencies of meromorphic functions.

Theorem A. *(See [5].) Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions in $\mathbb{C}$ and for some $a \in \mathbb{C}_\infty$ and an integer $p \geqslant 0$, $\delta = \delta(a, f^{(p)}) > 0$. Assume that for q radii $\arg z = \alpha_j$ $(1 \leqslant j \leqslant q)$ satisfying*

$$-\pi \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_q < \pi, \quad \alpha_{q+1} = \alpha_1 + 2\pi,$$

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$f(z)$ and $g(z)$ share five distinct values IM in $X=\mathbb{C}\setminus\bigcup_{j=1}^{q}\{z\mid \arg z=\alpha_j\}$. If

$$\max\left\{\frac{\pi}{\alpha_{j+1}-\alpha_j}\colon 1\leqslant j\leqslant q\right\}<\rho,$$

where ρ, and in the sequel, denotes the order of f, then $f(z)\equiv g(z)$.

Theorem B. *(See [6].) Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions in $\mathbb{C}$ and let $f(z)$ be of the finite lower order λ and for some $a\in\mathbb{C}_\infty$, $\delta=\delta(a,f)>0$.*

Given one angular domain $X=\{z\mid \alpha<\arg z<\beta\}$ with $0\leqslant\alpha<\beta\leqslant 2\pi$ and

$$\beta-\alpha>\max\left\{\frac{\pi}{\sigma},2\pi-\frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}}\right\},$$

where $\lambda\leqslant\sigma\leqslant\rho$ and $\sigma<\infty$, we assume that $f(z)$ and $g(z)$ share four distinct values a_j $(j=1,2,3,4)$ IM in X and $a_j\neq a$ $(j=1,2,3,4)$, then $f(z)\equiv g(z)$.

In 2009, Q.C. Zhang [9] proved the following theorems under giving some restrictions on the angular characteristic functions of meromorphic functions.

Theorem C. *(See [9].) Let $f(z)$ and $g(z)$ be two meromorphic functions of finite order in $\mathbb{C}$, $a_j\in\mathbb{C}_\infty$ $(j=1,2,\dots,5)$ be five distinct values, and let $\Delta_\delta=\{z\mid |\arg z-\theta_0|\leqslant\delta\}$ $(0<\delta<\pi)$ be an angular domain satisfying*

$$\varlimsup_{\varepsilon\to 0^+}\varlimsup_{r\to+\infty}\frac{\log T(r,\Delta_{\delta-\varepsilon},f)}{\log r}>\omega,\tag{1.1}$$

where $\omega=\frac{\pi}{2\delta}$, $T(r,\Delta_{\delta-\varepsilon},f)$ denotes the Ahlfors characteristic function of f in $\Delta_{\delta-\varepsilon}$. If $f(z)$ and $g(z)$ share a_j $(j=1,2,\dots,5)$ IM in Δ_δ, then $f(z)\equiv g(z)$.

Theorem D. *(See [9].) Let $f(z)$ and $g(z)$ be two meromorphic functions of finite order in $\mathbb{C}$, $a_j\in\mathbb{C}_\infty$ $(j=1,2,3,4)$ be four distinct values, and let $\Delta_\delta=\{z\mid |\arg z-\theta_0|\leqslant\delta\}$ $(0<\delta<\pi)$ be an angular domain satisfying (1.1). If $f(z)$ and $g(z)$ share a_j $(j=1,2,3,4)$ CM in Δ_δ, then $f(z)$ is a linear fractional transformation of $g(z)$.*

In this paper, we investigate uniqueness of meromorphic functions in the unit disc and consider the relation between the Borel points and shared-values of meromorphic functions in an angular domain.

Let $f(z)$ be meromorphic in the unit disc $\mathbb{D}_1$ and $\Delta(\theta_0,\delta)$ denote the domain $\{z\mid |z|<1\}\cap\{z\mid |\arg z-\theta_0|<\delta\}$, where $0\leqslant\theta_0\leqslant 2\pi$, $0<\delta<\pi$. We use $n(r,\Delta(\theta_0,\delta),f(z)=a)$ to denote the number of zeros of $f(z)-a$ in $\Delta(\theta_0,\delta)\cap\{z\mid |z|<r\}$ counting multiplicities.

In the proof of theorems in [5–9], Nevanlinna theory in an angular domain plays a key role (see [10]). In this paper, we adopt a proof method which is different from that of [5–9], and obtain

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two meromorphic functions in $\mathbb{D}_1$, $a_j\in\mathbb{C}_\infty$ $(j=1,2,\dots,5)$ be five distinct values, and $\Delta(\theta_0,\delta)$ $(0<\delta<\pi)$ be an angular domain such that for some $a\in\mathbb{C}_\infty$,*

$$\varlimsup_{r\to 1}\frac{\log n(r,\Delta(\theta_0,\delta/2),f(z)=a)}{\log\frac{1}{1-r}}:=\tau>1.\tag{1.2}$$

If $f(z)$ and $g(z)$ share a_j $(j=1,2,\dots,5)$ IM in $\Delta(\theta_0,\delta)$, then $f(z)\equiv g(z)$.

Remark 1.1. Let $f(z)$ be a meromorphic function in the unit disc. If for arbitrary small $\varepsilon>0$, we have

$$\varlimsup_{r\to 1}\frac{\log n(r,\Delta(\theta_0,\varepsilon),f(z)=a)}{\log\frac{1}{1-r}}=\tau$$

for all but at most two $a\in\mathbb{C}_\infty$, then $e^{i\theta_0}$ is called a Borel point of order τ of $f(z)$. In [11], G. Valiron proved that every meromorphic function of finite order ρ in the unit disc must have at least one Borel point of order $\rho+1$.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two meromorphic functions in $\mathbb{D}_1$, $a_j\in\mathbb{C}_\infty$ $(j=1,2,3,4)$ be four distinct values, and $\Delta(\theta_0,\delta)$ $(0<\delta<\pi)$ be an angular domain satisfying (1.2). If $f(z)$ and $g(z)$ share a_j $(j=1,2,3,4)$ CM in $\Delta(\theta_0,\delta)$, then $f(z)$ is a linear fractional transformation of $g(z)$.*

2. Lemmas

A meromorphic function $f(z)$ in the unit disc $\mathbb{D}_1$ is called admissible if and only if

$$\overline{\lim_{r\to 1}} \frac{T(r,f)}{\log\frac{1}{1-r}} = \infty. \tag{2.1}$$

Furthermore, $f(z)$ is called non-admissible if and only if (2.1) is not valid. To shorten the notations, it is often convenient to use a quantity $S(r,f)$ satisfying

$$S(r,f) = O\left\{\log\frac{1}{1-r}\right\} + O\left\{\log^+ T(r,f)\right\}$$

as $r \to 1$ possibly outside a set E such that $\int_E \frac{dr}{1-r} < \infty$.

Lemma 2.1. *(See [12].) Let $f(z)$ be meromorphic in $\mathbb{D}_1$ and k be a positive integer. Then*

$$m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) = S(r,f).$$

If $f(z)$ is of finite order, then

$$m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) = O\left\{\log\frac{1}{1-r}\right\} \quad (r \to 1).$$

Lemma 2.2. *(See [12].) Let $h_1(r)$ and $h_2(r)$ be monotonically increasing and real valued functions on $[0,1)$ such that $h_1(r) \leqslant h_2(r)$ possibly outside an exceptional set $E \subset [0,1)$, for which $\int_E \frac{dr}{1-r} < \infty$. Then there exists a constant $b \in (0,1)$ such that if $s(r) = 1 - b(1-r)$, then $h_1(r) \leqslant h_2(s(r))$ for all $r \in [0,1)$.*

Lemma 2.3. *(See [1].) Let $f(z)$ and $g(z)$ be two admissible meromorphic functions in $\mathbb{D}_1$, and $a_j \in \mathbb{C}_\infty$ $(j = 1,2,\dots,5)$ be five distinct values. If $f(z)$ and $g(z)$ share a_j $(j = 1,2,\dots,5)$ IM in $\mathbb{D}_1$, then $f(z) \equiv g(z)$.*

Lemma 2.4. *Suppose that $f(z)$ and $g(z)$ are distinct meromorphic functions in $\mathbb{D}_1$ and share four distinct values $a_j \in \mathbb{C}_\infty$ $(j = 1,2,3,4)$ IM in $\mathbb{D}_1$. If $f(z)$ is admissible, then $g(z)$ is also admissible.*

Proof. By the assumption of Lemma 2.4 and the second fundamental theorem in $\mathbb{D}_1$ (see Theorems 2.1 and 2.2 in [1]), we get

$$\begin{aligned} 2T(r,f) &\leqslant \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f) \\ &= \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{g-a_j}\right) + S(r,f) \\ &\leqslant \sum_{j=1}^{4} T\left(r, \frac{1}{g-a_j}\right) + S(r,f) \\ &\leqslant 4T(r,g) + S(r,f). \end{aligned}$$

Hence

$$T(r,f) \leqslant 4T(r,g) + O\left\{\log\frac{1}{1-r}\right\}$$

as $r \to 1$ possibly outside a set E such that $\int_E \frac{dr}{1-r} < \infty$. Then by Lemma 2.2 we get $g(z)$ is admissible. $\square$

Lemma 2.5. *Suppose that $f(z)$ and $g(z)$ are distinct admissible meromorphic functions in $\mathbb{D}_1$. If $f(z)$ and $g(z)$ share four distinct values $a_j \in \mathbb{C}_\infty$ $(j = 1,2,3,4)$ CM in $\mathbb{D}_1$, then $f(z)$ is a linear fractional transformation of $g(z)$.*

Proof. We divide our proof into two steps.

Step 1. We prove that $S(r,g)=S(r,f)$ and at least two among $\overline{N}(r,\frac{1}{f-a_j})$ $(j=1,2,3,4)$ are not $S(r,f)$.

Without loss of generality, we may assume that all a_j $(j=1,2,3,4)$ are finite. The second fundamental theorem in $\mathbb{D}_1$ implies

$$2T(r,f)\leqslant\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{f-a_j}\right)+S(r,f),\tag{2.2}$$

$$2T(r,g)\leqslant\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{g-a_j}\right)+S(r,g).\tag{2.3}$$

Since $f(z)$ and $g(z)$ share a_j $(j=1,2,3,4)$ IM in $\mathbb{D}_1$,

$$\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{f-a_j}\right)=\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{g-a_j}\right)\leqslant N\left(r,\frac{1}{f-g}\right)\leqslant T(r,f)+T(r,g)+O(1).\tag{2.4}$$

Then by (2.2)–(2.4) we get

$$T(r,f)\leqslant T(r,g)+S(r,f),$$
$$T(r,g)\leqslant T(r,f)+S(r,g).$$

Hence $S(r,g)=S(r,f)$. From (2.2) we see that at least two among $\overline{N}(r,\frac{1}{f-a_j})$ $(j=1,2,3,4)$ are not $S(r,f)$. Otherwise, we have $T(r,f)\leqslant S(r,f)$. Then by Lemma 2.2 we get f is non-admissible. This is impossible.

Step 2. We prove that $f(z)$ is a linear fractional transformation of $g(z)$.

By Step 1, without loss of generality, we may assume that

$$\overline{N}\left(r,\frac{1}{f-a_1}\right)\neq S(r,f),\qquad \overline{N}\left(r,\frac{1}{f-a_2}\right)\neq S(r,f).\tag{2.5}$$

Firstly, we assume that $a_1=\infty$. Set

$$H=\frac{f'}{(f-a_2)(f-a_3)(f-a_4)}-\frac{g'}{(g-a_2)(g-a_3)(g-a_4)}.\tag{2.6}$$

If $H\not\equiv 0$, then by Lemma 2.1 we get

$$m(r,H)\leqslant m\left(r,\sum_{j=2}^{4}\frac{f'}{f-a_j}\right)+m\left(r,\sum_{j=2}^{4}\frac{g'}{g-a_j}\right)+O(1)=S(r,f).$$

Since $f(z)$ and $g(z)$ share a_j $(j=1,2,3,4)$ CM in $\mathbb{D}_1$, H has no poles in $\mathbb{D}_1$. Hence H is analytic in $\mathbb{D}_1$ and $T(r,H)=m(r,H)=S(r,f)$. Suppose z_0 is a pole of $f(z)$ and $g(z)$ with multiplicity p. It follows from (2.6) that z_0 is a zero of H with multiplicity at least $3p-(p+1)=2p-1$. Therefore

$$\overline{N}(r,f)\leqslant\overline{N}\left(r,\frac{1}{H}\right)\leqslant T(r,H)+O(1)=S(r,f),$$

which contradicts (2.5). Hence $H\equiv 0$.

Set

$$Q=\frac{f'(f-a_2)}{(f-a_3)(f-a_4)}-\frac{g'(g-a_2)}{(g-a_3)(g-a_4)}.\tag{2.7}$$

If $Q\not\equiv 0$, then by Lemma 2.1 we get

$$m(r,Q)\leqslant m\left(r,\sum_{j=3}^{4}\frac{f'}{f-a_j}\right)+m\left(r,\sum_{j=3}^{4}\frac{g'}{g-a_j}\right)+O(1)=S(r,f).$$

Since $f(z)$ and $g(z)$ share a_1, a_3, a_4 CM in $\mathbb{D}_1$, Q has no poles in $\mathbb{D}_1$. Hence Q is analytic in $\mathbb{D}_1$ and $T(r,Q)=m(r,Q)=S(r,f)$. Suppose z_0 is a zero of $f(z)-a_2$ and $g(z)-a_2$ with multiplicity p. It follows from (2.7) that z_0 is a zero of Q with multiplicity at least $p+(p-1)=2p-1$. Therefore

$$\overline{N}\left(r, \frac{1}{f-a_2}\right) \leqslant \overline{N}\left(r, \frac{1}{Q}\right) \leqslant T(r, Q) + O(1) = S(r, f),$$

which contradicts (2.5). Hence $Q \equiv 0$.

From $H \equiv 0$ and $Q \equiv 0$ we get $(f-a_2)^2 = (g-a_2)^2$. Hence $f = -g + 2a_2$ and the conclusion is proved.

If $a_1 \neq \infty$, set $F = \frac{1}{f-a_1}$, $G = \frac{1}{g-a_1}$ and $b_j = \frac{1}{a_j - a_1}$ $(j = 2, 3, 4)$. Then F and G share ∞, b_2, b_3, b_4 CM in $\mathbb{D}_1$. By the above proof we get $F = -G + 2b_2$. Hence $f(z)$ is also a linear fractional transformation of $g(z)$. □

Lemma 2.6. *Set*

$$u = u(z) = \frac{z^{\frac{\pi}{\delta}} + 2z^{\frac{\pi}{2\delta}} - 1}{z^{\frac{\pi}{\delta}} - 2z^{\frac{\pi}{2\delta}} - 1}.$$

Then $u(z)$ maps conformally $\{z \mid |\arg z| < \delta,\ |z| < 1\}$ onto the unit disc $\{u \mid |u| < 1\}$, where $0 < \delta < \pi$.

Proof. Note that

$$\phi(z) = \left(\frac{z+1}{z-1}\right)^2 \tag{2.8}$$

maps conformally $\{z \mid \operatorname{Im} z > 0,\ |z| < 1\}$ onto the upper half-plane and

$$\varphi(z) = i \cdot \left(\frac{z-i}{z+i}\right) \tag{2.9}$$

maps conformally the upper half-plane onto the unit disc. Since $\omega(z) = (ze^{i\delta})^{\frac{\pi}{2\delta}}$ maps $\{z \mid |\arg z| < \delta, |z| < 1\}$ onto $\{\omega \mid \operatorname{Im} \omega > 0,\ |\omega| < 1\}$, then combining (2.8) and (2.9) we can get the conclusion. □

3. Proofs of Theorems 1.1 and 1.2

Proof. We want to prove Theorem 1.2. Without loss of generality, we may assume $\theta_0 = 0$. Set

$$u = u(z) = \frac{z^{\frac{\pi}{\delta}} + 2z^{\frac{\pi}{2\delta}} - 1}{z^{\frac{\pi}{\delta}} - 2z^{\frac{\pi}{2\delta}} - 1}. \tag{3.1}$$

Let $z = z(u)$ denote its inverse function. Then by Lemma 2.6 we know that u maps conformally $\Delta(0, \delta)$ onto the unit disc $\{u \mid |u| < 1\}$.

Set $z_0 = pe^{i\varphi} \in \Delta(0, \delta)$, by (3.1) we get

$$\begin{aligned}
1 - \left|u(z_0)\right| &= 1 - \sqrt{\frac{A^2 + B^2}{C^2 + D^2}} \\
&= \frac{C^2 + D^2 - A^2 - B^2}{C^2 + D^2 + \sqrt{(A^2 + B^2)(C^2 + D^2)}} \\
&= \frac{8p^{\frac{\pi}{2\delta}}(1 - p^{\frac{\pi}{\delta}})\cos\frac{\pi\varphi}{2\delta}}{C^2 + D^2 + \sqrt{(A^2 + B^2)(C^2 + D^2)}},
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
A &= p^{\frac{\pi}{\delta}} \cos\frac{\pi\varphi}{\delta} + 2p^{\frac{\pi}{2\delta}} \cos\frac{\pi\varphi}{2\delta} - 1, \\
B &= p^{\frac{\pi}{\delta}} \sin\frac{\pi\varphi}{\delta} + 2p^{\frac{\pi}{2\delta}} \sin\frac{\pi\varphi}{2\delta}, \\
C &= p^{\frac{\pi}{\delta}} \cos\frac{\pi\varphi}{\delta} - 2p^{\frac{\pi}{2\delta}} \cos\frac{\pi\varphi}{2\delta} - 1, \\
D &= p^{\frac{\pi}{\delta}} \sin\frac{\pi\varphi}{\delta} - 2p^{\frac{\pi}{2\delta}} \sin\frac{\pi\varphi}{2\delta}.
\end{aligned}$$

Since

$$C^2 + D^2 = p^{\frac{2\pi}{\delta}} + 2p^{\frac{\pi}{\delta}} + 1 + 4p^{\frac{\pi}{2\delta}}\left(1 - p^{\frac{\pi}{\delta}}\right)\cos\frac{\pi\varphi}{2\delta} + 2p^{\frac{\pi}{\delta}}\left(1 - \cos\frac{\pi\varphi}{\delta}\right),$$

we get

$$1 \leqslant C^2 + D^2 \leqslant C^2 + D^2 + \sqrt{(A^2 + B^2)(C^2 + D^2)} \leqslant 2(C^2 + D^2) \leqslant 20. \tag{3.3}$$

Note that $\lim_{p \to 1} \frac{1 - p^{\frac{\pi}{\delta}}}{1-p} = \frac{\pi}{\delta}$, so there exists $b \in ((\frac{1}{2})^{\frac{2\delta}{\pi}}, 1)$ such that for all p satisfying $b < p < 1$, we have

$$\frac{1}{2} < p^{\frac{\pi}{2\delta}} < 1, \tag{3.4}$$

$$\frac{\pi}{2\delta}(1-p) < 1 - p^{\frac{\pi}{\delta}} < \frac{3\pi}{2\delta}(1-p). \tag{3.5}$$

By (3.2)–(3.5) we get

$$\min\left\{1 - \left|u\left(pe^{i\varphi}\right)\right| \;\middle|\; b < p < r,\ |\varphi| < \frac{\delta}{2}\right\} > \frac{\pi}{20\delta}(1-r) \tag{3.6}$$

for all $r \in (b, 1)$.

Note that we have got two meromorphic functions $f(z(u))$ and $g(z(u))$ in the unit disc $\{u \mid |u| < 1\}$. Now we prove that $f(z(u))$ is admissible in the unit disc $\{u \mid |u| < 1\}$. Let τ_1 and τ_2 satisfy $\tau > \tau_1 > \tau_2 > 1$, by (1.2), there exists a sequence $\{r_n\}$ of positive numbers such that $r_n \to 1$ for $n \to \infty$ and such that for τ_1 we have

$$n\left(r_n, \Delta\left(0, \frac{\delta}{2}\right), f(z) = a\right) > \left(\frac{1}{1 - r_n}\right)^{\tau_1} \tag{3.7}$$

for n sufficiently large.

Set $t_n = 1 - \frac{\pi}{20\delta}(1 - r_n)$, (3.6) and (3.7) yield

$$\begin{aligned} n\big(t_n, f\big(z(u)\big) = a\big) &> n\left(r_n, \Delta\left(0, \frac{\delta}{2}\right), f(z) = a\right) - O(1) \\ &> \left(\frac{1}{1 - r_n}\right)^{\tau_2} \\ &> A\left(\frac{1}{1 - t_n}\right)^{\tau_2} \end{aligned} \tag{3.8}$$

for n sufficiently large.

Set $t'_n = t_n + \frac{1}{2}(1 - t_n)$, then by (3.8) we get

$$\begin{aligned} T\big(t'_n, f\big(z(u)\big)\big) &> N\big(t'_n, f\big(z(u)\big) = a\big) - B \\ &> \int_{t_n}^{t'_n} \frac{n(t, f(z(u)) = a)}{t}\, dt - B \\ &> n\big(t_n, f\big(z(u)\big) = a\big) \log \frac{t'_n}{t_n} - B \\ &> A(1 - t_n)\left(\frac{1}{1 - t_n}\right)^{\tau_2} \\ &= A\left(\frac{1}{1 - t_n}\right)^{\tau_2 - 1}, \end{aligned} \tag{3.9}$$

where A and B are two positive constants, they are not necessarily the same at each occurrence. Hence by (3.9) we get

$$\varlimsup_{t \to 1} \frac{T(t, f(z(u)))}{\log \frac{1}{1-t}} \geqslant \varlimsup_{t'_n \to 1} \frac{T(t'_n, f(z(u)))}{\log \frac{1}{1 - t'_n}} \geqslant \varlimsup_{t_n \to 1} \frac{A(\frac{1}{1-t_n})^{\tau_2 - 1}}{\log \frac{2}{1 - t_n}} = \infty.$$

So $f(z(u))$ is admissible in the unit disc $\{u \mid |u| < 1\}$.

Since $f(z)$ and $g(z)$ share a_j $(j = 1, 2, 3, 4)$ CM in $\Delta(0, \delta)$, $f(z(u))$ and $g(z(u))$ share a_j $(j = 1, 2, 3, 4)$ CM in $\{u \mid |u| < 1\}$. Then by Lemmas 2.4 and 2.5, we get $f(z(u))$ is a linear fractional transformation of $g(z(u))$. Hence the restriction $f|_{\Delta(0,\delta)}$ is a linear fractional transformation of $g|_{\Delta(0,\delta)}$. Then by the identity principle, we prove Theorem 1.2.

The proof of Theorem 1.1 is similar to that of Theorem 1.2, we only need to replace Lemma 2.5 with Lemma 2.3. □

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