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The continuous coagulation equation with multiple fragmentation

Ankik Kumar Giri *, Jitendra Kumar, Gerald Warnecke

Institute for Analysis and Numerics, Otto-von-Guericke University Magdeburg, Universitätsplatz 2, D-39106 Magdeburg, Germany

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ABSTRACT

We present a proof of the existence of solutions to the continuous coagulation equation with multiple fragmentation whenever the kernels satisfy certain growth conditions. The proof relies on weak L^1 compactness methods applied to suitably chosen approximating equations. The question of uniqueness is also considered.

1. Introduction

The coagulation-fragmentation equation describes the kinetics of particle growth in which particles can coagulate via binary interaction to form larger particles or fragment to form smaller ones. These models arise in many applications such as cluster formation in galaxies, kinetics of phase transformations in binary alloys, aggregation of red blood cells, fluidized bed granulation processes etc. The non-linear continuous coagulation and multiple fragmentation equation is given by

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)\,dy - \int_0^\infty K(x,y)f(x,t)f(y,t)\,dy + \int_x^\infty b(x,y)S(y)f(y,t)\,dy - S(x)f(x,t), \tag{1}$$

with

$$f(x,0) = f_0(x) \geqslant 0 \quad \text{a.e.} \tag{2}$$

where the variables $x \geqslant 0$ and $t \geqslant 0$ denote the size of the particles and time respectively. The number density of particles of size x at time t is denoted by $f(x,t)$. The coagulation kernel $K(x,y)$ represents the rate at which particles of size x coalesce with particles of size y. The fragmentation terms have a similar interpretation. The breakage function $b(x,y)$ is the probability density function for the formation of particles of size x from the particles of size y. It is non-zero only for $x < y$. The selection function $S(x)$ describes the rate at which particles of size x are selected to break. The selection function S and breakage function b are defined in terms of the multiple-fragmentation kernel Γ as

* Corresponding author. Fax: +49 391 6718073.
E-mail address: ankik.giri@ovgu.de (A. Kumar Giri).

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$$S(x) = \int_0^x \frac{y}{x} \Gamma(x, y)\, dy, \qquad b(x, y) = \Gamma(y, x)/S(y). \tag{3}$$

The breakage function has the following properties

$$\int_0^y b(x, y)\, dx = N < \infty, \quad \text{for all } y \quad \text{and} \quad b(x, y) = 0, \quad \text{for } x \geqslant y, \tag{4}$$

and

$$\int_0^y x b(x, y)\, dx = y \quad \text{for all } y > 0. \tag{5}$$

The quantity N represents the number of fragments obtained from the breakage of particles of size y. In this work, we assume that this quantity is size independent, a more general case is not treated here is to let N be a function of y. For the total mass in the system to remain conserved during fragmentation events, b must satisfy Eq. (5). It states that the total mass of the fragments equals the original mass when a particle of mass y breaks.

Eq. (1) is usually referred to as the continuous coagulation and multiple-fragmentation equation, or generalized coagulation-fragmentation equation, as fragmenting particles can split into more than two pieces. However, the continuous coagulation and binary-fragmentation equation has been investigated, for example in References [22,8]. It can be obtained as a special case of (1) by setting

$$S(x) = \frac{1}{2} \int_0^x F(y, x - y)\, dy, \qquad b(x, y) = F(x, y - x)/S(y) \tag{6}$$

where F is assumed to be symmetric. In this binary-fragmentation model, the function F represents the rate at which particles of size $x - y$ and y are produced from a fragmenting particle of size x.

Many results on the existence and uniqueness of solutions to the various forms of the coagulation-fragmentation equation have already been established using a number of different methods [22,8,1,18,9,4,5,10,11,6,3,19]. However, the case of multiple-fragmentation is not discussed too much. The first study of the coagulation equation with multiple fragmentation has been done by Melzak [18] where the first existence and uniqueness result was proved for bounded coefficients. McLaughlin et al. [16] established the existence and uniqueness of solutions to the multiple-fragmentation equation under the condition that

$$S(x) = \int_0^x \frac{y}{x} \Gamma(x, y)\, dy \leqslant C_n < \infty \quad \text{for all } x \in \,]0, n],\ n > 0$$

where the sequence C_n may be unbounded. This was extended in Reference [17] to the combined coagulation and multiple-fragmentation Eq. (1) under the assumptions that K is constant and

$$\Gamma \in L^\infty\big(]0, \infty[\, \times \,]0, \infty[\big).$$

Using similar arguments, Lamb [13] discussed the existence and uniqueness of solutions to (1) under the less restrictive conditions that K is bounded, S satisfies a linear growth condition, and $b(x, y)$ is such that the break-up of a particle of size y is a mass-conserving process that produces an average number of smaller particles that is finite and independent of y. But all of them used one particular method that involves the application of the theory of semigroup of operators.

Unlike most previous authors we prove the existence of solutions to (1) which is based on weak L^1 compactness methods applied to suitably chosen approximating equations. This approach originated in the work of Stewart [22] who investigated the case when both the coagulation kernel K and binary-fragmentation kernel F satisfy growth conditions almost up to linearity. Existence results for the continuous coagulation equation with multiple fragmentation were also established by Laurençot [14] by the approach of Stewart, the class of kernels being different but with a non-empty intersection. A more complete result is available for the discrete coagulation equation with multiple fragmentation in [15].

Here, our aim is to prove the existence of solutions to (1) under the much less restrictive conditions that K is unbounded and satisfies a certain growth condition as well as that S satisfies almost a linear growth condition. However, to investigate uniqueness we need some further restrictions on the kernels. The present paper improves [17,13] by relaxing the assumption of boundedness of the coagulation coefficient, the latter condition being crucial for the use of the semigroup approach.

Let X be the following Banach space with norm $\|\cdot\|$

$$X = \big\{ f \in L^1(0, \infty):\ \|f\| < \infty \big\} \quad \text{where } \|f\| = \int_0^\infty (1 + x) \big| f(x) \big|\, dx.$$

We also write

$$\|f\|_x = \int_0^\infty x\left|f(x)\right|dx \quad \text{and} \quad \|f\|_1 = \int_0^\infty \left|f(x)\right|dx$$

and set

$$X^+ = \{f \in X\colon\ f \geqslant 0 \text{ a.e.}\}.$$

Hypotheses 1.1.

(H1) K is a continuous non-negative function on $[0,\infty[\times[0,\infty[$ and Γ is a non-negative locally bounded function,
(H2) K is symmetric, i.e. $K(x,y) = K(y,x)$ for all $x, y \in]0,\infty[$,
(H3) $K(x,y) \leqslant \phi(x)\phi(y)$ for all $x, y \in]0,\infty[$ where $\phi(x) \leqslant k_1(1+x)^\mu$ for some $0 \leqslant \mu < 1$ and constant k_1.
(H4) $S:]0,\infty[\mapsto [0,\infty[$ is continuous and satisfies the bound $S(x) \leqslant k_2(1+x)^\gamma$ for all $x \in]0,\infty[$ where $0 \leqslant \gamma < 1$ and k_2 is a constant.

Definition 1.2. Let $T \in]0,\infty]$. A solution f of (1)–(2) is a function $f: [0,T[\to X^+$ such that for a.e. $x \in]0,\infty[$ and all $t \in [0,T[$ the following hold

(i) $f(x,t) \geqslant 0$,
(ii) $f(x,.)$ is continuous on $[0,T[$,
(iii) the following integrals are bounded

$$\int_0^t \int_0^\infty K(x,y) f(y,s)\,dy\,ds < \infty \quad \text{and} \quad \int_0^t \int_x^\infty b(x,y) S(y) f(y,s)\,dy\,ds < \infty,$$

(iv) the function f satisfies the following weak formulation of (1)

$$f(x,t) = f_0(x) + \int_0^t \left\{ \frac{1}{2}\int_0^x K(x-y,y) f(x-y,s) f(y,s)\,dy - \int_0^\infty K(x,y) f(x,s) f(y,s)\,dy \right.$$
$$\left. + \int_x^\infty b(x,y) S(y) f(y,s)\,dy - S(x) f(x,s) \right\} ds.$$

We know a few specific coagulation kernels which satisfy the hypotheses mentioned above. However, they do not satisfy the assumptions of previously existing results on coagulation together with multiple fragmentation given in Lamb [13]. These kernels are the following:

(1) Shear kernel (non-linear velocity profile) Aldous [2] or Smit et al. [21] who use the length coordinate $\lambda = x^{\frac{1}{3}}$

$$K(x,y) = k_0\big(x^{1/3} + y^{1/3}\big)^{7/3}.$$

(2) The modified Smoluchowski kernel, see Koch et al. [12], is given as

$$K(x,y) = k_0 \frac{(x^{1/3} + y^{1/3})^2}{x^{1/3} \cdot y^{1/3} + c}$$

with some fixed constant $c > 0$.

(3) Ding et al. [7] used the following kernel in the application of population balance models to activated sludge flocculation

$$K(x,y) = k_0 \frac{(x^{1/3} + y^{1/3})^q}{1 + \big(\frac{x^{1/3}+y^{1/3}}{2y_c^{1/3}}\big)^3}, \quad 0 \leqslant q < 3.$$

Here q is the order of the kernel.

Further we point out that the modified Smoluchowski kernel was derived from the Smoluchowski kernel (or Brownian motion kernel) given as, see Aldous [2] or Smit et al. [21],

$$K(x,y) = k_0\big(x^{1/3} + y^{1/3}\big)\big(x^{-1/3} + y^{-1/3}\big)$$

which can be rewritten as

$$K(x,y)=k_0\frac{(x^{1/3}+y^{1/3})^2}{x^{1/3}\cdot y^{1/3}}.$$

The modification eliminates the singular behavior of this kernel. The original Smoluchowski kernel does not satisfy (H3) in contrast to the modified one by Koch et al. [12].

Now we take the following type of fragmentation kernels which also satisfy the hypotheses mentioned above

$$S(x)=x^{\sigma}(1+x)^{\gamma-\sigma}\quad\text{and}\quad b(x,y)=\frac{\alpha+2}{y}\left(\frac{x}{y}\right)^{\alpha},\quad 0<x<y,$$

where $\sigma\geqslant 1>\gamma\geqslant 0$ and $\alpha\geqslant 0$. This is the cut-off version of the classical selection functions $S(x)=x^{\gamma}$ which have been studied in Peterson [20] and also in Ziff [25]. If we write it rather as

$$S(x)=x^{\sigma}(\delta+x)^{\gamma-\sigma}.$$

In the limit $\delta\to 0$ one recovers the classical kernel.

The outline of our paper is as follows. In Section 2, we extract a weakly convergent subsequence in L^1 from a sequence of unique solutions for truncated equations to (1)–(2). Then we prove in Theorem 2.3 that the limit function obtained from weakly convergent subsequence is indeed a solution to (1)–(2). In Section 3, we investigate the uniqueness, motivated by Stewart [23], of the solutions to (1)–(2) under the following further restrictions on the kernels.

(H3′) $K(x,y)\leqslant\phi(x)\phi(y)$ for all x, y where $\phi(x)\leqslant k(1+x)^{\frac{1}{2}}$ for some constant k.
(H4′) For all $x>0$, there exist $m_1, m_2>0$ such that

$$S(x)\leqslant m_1(1+x)^a$$

and

$$\int_0^x (1+y)^{\frac{1}{2}}b(y,x)\,dy\leqslant m_2(1+x)^b$$

where $a+b\leqslant\frac{1}{2}$.

2. Existence

2.1. The truncated problem

We prove the existence of solutions to (1)–(2) by taking the limit of a sequence of approximating equations obtained by replacing the kernel K and selection function S by the ‘cut-off’ kernels K_n and S_n, motivated by Stewart [22], where

$$K_n(x,y):=\begin{cases}K(x,y) & \text{if } x+y<n,\\ 0 & \text{if } x+y\geqslant n,\end{cases}$$

$$S_n(x):=\begin{cases}S(x) & \text{if } 0<x<n,\\ 0 & \text{if } x\geqslant n.\end{cases}$$

The resulting equations, with solutions denoted by f^n, are written as

$$\begin{aligned}\frac{\partial f^n(x,t)}{\partial t}&=\frac{1}{2}\int_0^x K_n(x-y,y)f^n(x-y,t)f^n(y,t)\,dy-\int_0^{n-x}K_n(x,y)f^n(x,t)f^n(y,t)\,dy\\&\quad+\int_x^n b(x,y)S_n(y)f^n(y,t)\,dy-S_n(x)f^n(x,t),\end{aligned}\tag{7}$$

with

$$f_0^n(x):=\begin{cases}f_0(x) & \text{if } 0<x<n,\\ 0 & \text{if } x\geqslant n.\end{cases}\tag{8}$$

Choose $T>0$. Proceeding as in [22, Theorem 3.1] we obtain the following result. For each $n=1,2,3\ldots$, (7)–(8) has a unique solution $f^n\in X^+$ with $f^n(x,t)\geqslant 0$ for a.e. $x\in]0,n[$ and $t\in[0,\infty[$, see Walker [24] also. Moreover, the total mass remains conserved, for all $t\in[0,\infty[$, i.e.

$$\int_0^n x f^n(x,t)\,dx = \int_0^n x f_0^n(x)\,dx. \tag{9}$$

From now on we consider the 'zero extension' of each f^n on $\mathbb{R}$, i.e.

$$\widehat{f}^n(x,t) := \begin{cases} f^n(x,t) & \text{if } 0 < x < n,\ t \in [0,T], \\ 0 & \text{if } x \leqslant 0 \text{ or } x \geqslant n. \end{cases}$$

For the simplicity we drop the $\widehat{\cdot}$ notation for the remainder of the work and the suffixes on the coagulation kernels and the selection functions.

Lemma 2.1. *Assume that* (H1), (H2), *and* (H4) *hold. Then the following are true*:

(i) $\displaystyle\int_0^\infty (1+x) f^n(x,t)\,dx \leqslant L$ *for* $n = 1, 2, 3 \ldots$ *and all* $t \in [0,T]$,

(ii) *given* $\epsilon > 0$ *there exists an* $R > 0$ *such that for all* $t \in [0,T]$

$$\sup_n \left\{ \int_R^\infty f^n(x,t)\,dx \right\} \leqslant \epsilon,$$

(iii) *given* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for all* $n = 1, 2, 3 \ldots$ *and* $t \in [0,T]$

$$\int_E f^n(x,t)\,dx < \epsilon \quad \text{whenever } \lambda(E) < \delta.$$

Proof. (i) From (7) and Fubini's Theorem, for each $n \geqslant 1$ we have by integration with respect to x and t

$$\begin{aligned} \int_0^1 f^n(x,t)\,dx = {} & -\frac{1}{2} \int_0^t \int_0^1 \int_0^{1-x} K(x,y) f^n(x,s) f^n(y,s)\,dy\,dx\,ds - \int_0^t \int_0^1 \int_{1-x}^{n-x} K(x,y) f^n(x,s) f^n(y,s)\,dy\,dx\,ds \\ & + \int_0^t \int_0^1 \int_x^n b(x,y) S(y) f^n(y,s)\,dy\,dx\,ds - \int_0^t \int_0^1 S(x) f^n(x,s)\,dx\,ds + \int_0^1 f^n(x,0)\,dx. \end{aligned}$$

Since the integrands are all non-negative, we may estimate

$$\begin{aligned} \int_0^1 f^n(x,t)\,dx & \leqslant \int_0^t \int_0^1 \int_x^n b(x,y) S(y) f^n(y,s)\,dy\,dx\,ds + \int_0^1 f^n(x,0)\,dx \\ & = \int_0^t \int_0^1 \int_x^1 b(x,y) S(y) f^n(y,s)\,dy\,dx\,ds + \int_0^t \int_0^1 \int_1^n b(x,y) S(y) f^n(y,s)\,dy\,dx\,ds + \int_0^1 f^n(x,0)\,dx. \end{aligned}$$

Using Fubini's Theorem, (H4) and (4) in the size independent case, we obtain

$$\begin{aligned} \int_0^1 f^n(x,t)\,dx & \leqslant \int_0^t \int_0^1 \int_0^y b(x,y) S(y) f^n(y,s)\,dx\,dy\,ds + \int_0^t \int_1^n \int_0^1 b(x,y) S(y) f^n(y,s)\,dx\,dy\,ds + \int_0^1 f^n(x,0)\,dx \\ & \leqslant k_2 N \int_0^t \int_0^1 (1+y)^\gamma f^n(y,s)\,dy\,ds + k_2 N \int_0^t \int_1^n (1+y)^\gamma f^n(y,s)\,dy\,ds + \int_0^1 f^n(x,0)\,dx \\ & \leqslant k_2 N \int_0^t \int_0^1 (1+y) f^n(y,s)\,dy\,ds + 2k_2 N \int_0^t \int_1^n y f^n(y,s)\,dy\,ds + \int_0^1 f^n(x,0)\,dx. \end{aligned} \tag{10}$$

From Eq. (9), for $s \in [0, T]$

$$\left\| f^n(s) \right\|_x = \left\| f^n(0) \right\|_x \leqslant \left\| f(0) \right\|. \tag{11}$$

Using (10) and (11) we obtain

$$\int_0^1 f^n(x,t)\,dx \leqslant k_2 N \int_0^t \int_0^1 f^n(y,s)\,dy\,ds + 3k_2 NT \|f_0\| + \|f_0\| = k_2 N \int_0^t \int_0^1 f^n(y,s)\,dy\,ds + \|f_0\|\{3k_2 NT + 1\}.$$

Applying Gronwall's Lemma we obtain

$$\int_0^1 f^n(x,t)\,dx \leqslant \|f_0\|\{3k_2 NT + 1\}\exp\{k_2 NT\}.$$

Thus, by using (9) again we may estimate

$$\begin{aligned}\int_0^\infty (1+x) f^n(x,t)\,dx &= \int_0^1 f^n(x,t)\,dx + \int_1^n f^n(x,t)\,dx + \int_0^n x f^n(x,t)\,dx \\ &\leqslant \int_0^1 f^n(x,t)\,dx + \int_1^n x f^n(x,t)\,dx + \|f_0\| \\ &\leqslant \|f_0\|\big\{(3k_2 NT + 1)\exp(k_2 NT) + 2\big\} := L.\end{aligned}$$

(ii) For $\epsilon > 0$, let $R > 0$ be such that $R > \|f_0\|/\epsilon$. Then, by (11), for each $n = 1, 2, 3, \ldots$ and for all $t \in [0, T]$ we have

$$\int_R^\infty f^n(x,t)\,dx = \int_R^\infty (x/x) f^n(x,t)\,dx \leqslant \frac{1}{R}\int_R^\infty x f^n(x,t)\,dx \leqslant \frac{1}{R}\|f_0\| < \epsilon.$$

(iii) Choose $\epsilon > 0$ and let $E \subset \mathbb{R}_{>0} :=]0, \infty[$. By part (ii) we can choose $m > 1$ such that for all $n = 1, 2, 3, \ldots$ and $t \in [0, T]$

$$\int_m^\infty f^n(x,t)\,dx < \epsilon/2. \tag{12}$$

Let χ denotes the characteristic function, i.e.

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E, \end{cases}$$

and for $n = 1, 2, 3, \ldots$ and $t \in [0, T]$, define

$$p^n(E,t) = \sup_{0 \leqslant z \leqslant m} \int_{\mathbb{R}_{>0}} \chi_{E \cap]0,m]}(x+z) f^n(x,t)\,dx.$$

Set

$$K_0 = \sup_{\substack{0 \leqslant x \leqslant m \\ 0 \leqslant y \leqslant m}} \frac{1}{2} K(x,y).$$

Consider $\gamma \in [0, 1[$ and k_2 as in (H4), N as given by (4). Then one can choose $r > m$ such that

$$k_2 NTL(1+r)^{\gamma-1} < \epsilon/\big\{8\exp(TLK_0)\big\} \tag{13}$$

and set

$$F_0 = \sup_{\substack{0 \leqslant y \leqslant r \\ 0 \leqslant x \leqslant m}} \Gamma(y,x).$$

By the absolute continuity of integral there exists a $\delta_1 > 0$ such that

$$p^n(E,0) \leqslant \sup_{0\leqslant z\leqslant m} \int_{\mathbb{R}_{>0}} \chi_{E\cap]0,m]}(x+z) f_0(x)\,dx < \epsilon/\big\{4\exp(TLK_0)\big\} \tag{14}$$

for all n whenever $\lambda(E) \leqslant \delta_1$ for the Lebesgue measure λ. Also, there exists a $\delta_2 > 0$ such that

$$\sup_{0\leqslant z\leqslant m} \int_{\mathbb{R}_{>0}} \chi_{E\cap]0,m]}(x+z)\,dx < \epsilon/\big\{8TF_0L\exp(TLK_0)\big\} \tag{15}$$

whenever $\lambda(E) \leqslant \delta_2$. Set $\delta = \min\{\delta_1, \delta_2\}$. Using the non-negativity of each f^n we can use (7)–(8) to prove that for $0 < z < m$ and $\lambda(E) < \delta$

$$\begin{aligned}
&\int_{\mathbb{R}_{>0}} \chi_{E\cap]0,m]}(x+z) f^n(x,t)\,dx \\
&\quad\leqslant \frac{1}{2}\int_0^t \int_{\mathbb{R}_{>0}}\int_{\mathbb{R}_{>0}} \chi_{E\cap]0,m]}(x+z)\chi_{]0,x]\cap]0,m]}(y) K(x-y,y) f^n(x-y,s) f^n(y,s)\,dy\,dx\,ds \\
&\qquad + \int_0^t\int_0^m \chi_{E\cap]0,m]}(x+z)\int_x^n b(x,y)S(y)f^n(y,s)\,dy\,dx\,ds + p^n(E,0).
\end{aligned}$$

Using the substitution $x' = x-y$, $y' = y$ and Fubini's Theorem in the first and second integrals on the right-hand side respectively we find that

$$\begin{aligned}
\int_{\mathbb{R}_{>0}} \chi_{E\cap]0,m]}(x+z) f^n(x,t)\,dx &\leqslant \frac{1}{2}\int_0^t \int_{\mathbb{R}_{>0}}\int_{\mathbb{R}_{>0}} \chi_{E\cap]0,m]}(x+y+z)\chi_{]0,m]}(y) K(x,y) f^n(x,s) f^n(y,s)\,dy\,dx\,ds \\
&\quad + \int_0^t\int_0^m\int_0^y \chi_{E\cap]0,m]}(x+z) b(x,y)S(y)f^n(y,s)\,dx\,dy\,ds \\
&\quad + \int_0^t\int_m^n\int_0^m \chi_{E\cap]0,m]}(x+z) b(x,y)S(y)f^n(y,s)\,dx\,dy\,ds + p^n(E,0).
\end{aligned}$$

By the definition of $p^n(E,t)$ and Lemma 2.1(i), this can be rewritten as

$$\begin{aligned}
\int_{\mathbb{R}_{>0}} \chi_{E\cap]0,m]}(x+z) f^n(x,t)\,dx &\leqslant K_0\int_0^t\int_0^m f^n(y,s) \sup_{0\leqslant v\leqslant m}\int_{\mathbb{R}_{>0}} \chi_{E\cap]0,m]}(x+v) f^n(x,s)\,dx\,dy\,ds \\
&\quad + \int_0^t\int_0^m\int_0^m \chi_{E\cap]0,m]}(x+z) b(x,y)S(y)f^n(y,s)\,dx\,dy\,ds \\
&\quad + \int_0^t\int_m^r\int_0^m \chi_{E\cap]0,m]}(x+z) b(x,y)S(y)f^n(y,s)\,dx\,dy\,ds \\
&\quad + \int_0^t\int_r^\infty\int_0^m \chi_{E\cap]0,m]}(x+z) b(x,y)S(y)f^n(y,s)\,dx\,dy\,ds + p^n(E,0), \\
&\leqslant K_0L\int_0^t p^n(E,s)\,ds + \int_0^t\int_0^r\int_0^m \chi_{E\cap]0,m]}(x+z)\Gamma(y,x) f^n(y,s)\,dx\,dy\,ds \\
&\quad + \int_0^t\int_r^\infty\int_0^y \chi_{E\cap]0,m]}(x+z) b(x,y)S(y)f^n(y,s)\,dx\,dy\,ds + p^n(E,0).
\end{aligned} \tag{16}$$

We use (15) and Lemma 2.1(i) to obtain the following estimate

$$\int_0^t \int_0^r \int_0^m \chi_{E\cap]0,m]}(x+z)\Gamma(y,x) f^n(y,s)\,dx\,dy\,ds \leqslant F_0 \int_0^t \int_0^r f^n(y,s)\,dy\,ds \cdot \epsilon / \{8TF_0 L \exp(TLK_0)\}$$
$$\leqslant \epsilon / \{8 \exp(TLK_0)\}. \tag{17}$$

By using (4), (H4), Lemma 2.1(i) and (13) we treat the other integral

$$\int_0^t \int_r^\infty \int_0^y \chi_{E\cap]0,m]}(x+z) b(x,y) S(y) f^n(y,s)\,dx\,dy\,ds \leqslant k_2 N \int_0^t \int_r^\infty (1+y)^\gamma f^n(y,s)\,dy\,ds$$
$$\leqslant k_2 N T L (1+r)^{\gamma-1} < \epsilon / \{8 \exp(TLK_0)\}. \tag{18}$$

It can be deduced from (14), (16), (17) and (18) that

$$p^n(E,t) \leqslant K_0 L \int_0^t p^n(E,s)\,ds + \epsilon / \{2 \exp(TLK_0)\}.$$

By using Gronwall's inequality, we obtain

$$p^n(E,t) \leqslant \exp(TLK_0)\epsilon / \{2 \exp(TLK_0)\} = \epsilon/2. \tag{19}$$

By (12) and (19), we obtain for $n = 1, 2, 3 \ldots$ and $t \in [0, T]$

$$\begin{aligned} \int_E f^n(x,t)\,dx &= \int \chi_{E\cap]0,m]}(x) f^n(x,t)\,dx + \int \chi_{E\cap [m,\infty[}(x) f^n(x,t)\,dx \\ &\leqslant p^n(E,t) + \int_m^\infty f^n(x,t)\,dx \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

whenever $\lambda(E) < \delta$. □

The above Lemma 2.1 implies that for each $t \in [0, T]$, the sequence of functions $(f^n(t))_{n\in\mathbb{N}}$ lies in a weakly relatively compact set in $L^1]0,\infty[$ by the Dunford–Pettis Theorem.

2.2. Equicontinuity in time

Now we proceed in this section to show equicontinuity of the sequence $(f^n)_{n\in\mathbb{N}}$ in time. It should be mentioned that (H3) is now assumed to be satisfied. Choose $\epsilon > 0$ and $\phi \in L^\infty]0,\infty[$. Let $s, t \in [0, T]$ and assume $t \geqslant s$. Choose $m > 1$ such that

$$\|\phi\|_{L^\infty} 2L/m < \epsilon/2. \tag{20}$$

For each n, by Lemma 2.1(i),

$$\int_m^\infty \left| f^n(x,t) - f^n(x,s) \right| dx \leqslant \frac{1}{m} \int_m^\infty x \{ f^n(x,t) + f^n(x,s) \}\,dx \leqslant 2L/m. \tag{21}$$

By using (7), (20) and (21), we get using $t \geqslant s$

$$\left| \int_0^\infty \phi(x) \{ f^n(x,t) - f^n(x,s) \}\,dx \right| \leqslant \left| \int_0^m \phi(x) \{ f^n(x,t) - f^n(x,s) \}\,dx \right| + \int_m^\infty |\phi(x)| \{ |f^n(x,t) - f^n(x,s)| \}\,dx$$
$$\leqslant \|\phi\|_{L^\infty} \int_s^t \left[\frac{1}{2} \int_0^m \int_0^x K(x-y,y) f^n(x-y,\tau) f^n(y,\tau)\,dy\,dx \right.$$

$$+\int_0^m\int_0^{n-x} K(x,y)f^n(x,\tau)f^n(y,\tau)\,dy\,dx+\int_0^m\int_x^n b(x,y)S(y)f^n(y,\tau)\,dy\,dx$$
$$\left.+\int_0^m S(x)f^n(x,\tau)\,dx\right]d\tau+\epsilon/2. \tag{22}$$

Now we consider the first term on the right-hand side of (22), by Fubini's Theorem, (H1)–(H4) and Lemma 2.1(i)

$$\begin{aligned}
\frac{1}{2}\int_0^m\int_0^x K(x-y,y)f^n(x-y,\tau)f^n(y,\tau)\,dy\,dx &= \frac{1}{2}\int_0^m\int_y^m K(x-y,y)f^n(x-y,\tau)f^n(y,\tau)\,dx\,dy\\
&= \frac{1}{2}\int_0^m\int_0^{m-y} K(x,y)f^n(x,\tau)f^n(y,\tau)\,dx\,dy\\
&= \frac{1}{2}\int_0^m\int_0^{m-x} K(y,x)f^n(y,\tau)f^n(x,\tau)\,dy\,dx\\
&= \frac{1}{2}\int_0^m\int_0^{m-x} K(x,y)f^n(x,\tau)f^n(y,\tau)\,dy\,dx\\
&\leqslant k_1^2\frac{1}{2}\int_0^m\int_0^{m-x}(1+x)^\mu(1+y)^\mu f^n(x,\tau)f^n(y,\tau)\,dy\,dx\\
&\leqslant \frac{1}{2}k_1^2L^2.
\end{aligned}$$

For the second term we may estimate

$$\int_0^m\int_0^{n-x} K(x,y)f^n(x,\tau)f^n(y,\tau)\,dy\,dx\leqslant k_1^2\int_0^m\int_0^{n-x}(1+x)^\mu(1+y)^\mu f^n(x,\tau)f^n(y,\tau)\,dy\,dx\leqslant k_1^2L^2.$$

For $n>m$, the third term using (4) gives that

$$\begin{aligned}
\int_0^m\int_x^n b(x,y)S(y)f^n(y,\tau)\,dy\,dx &\leqslant k_2\int_0^m\int_0^y b(x,y)(1+y)^\gamma f^n(y,\tau)\,dx\,dy+k_2\int_m^n\int_0^m b(x,y)(1+y)^\gamma f^n(y,\tau)\,dx\,dy\\
&\leqslant k_2N\int_0^n(1+y)^\gamma f^n(y,\tau)\,dy\leqslant k_2NL.
\end{aligned}$$

Similarly we can obtain the above inequality for $m>n$.

For the fourth term we have

$$\int_0^m S(x)f^n(x,t)\,dx\leqslant k_2L.$$

By using the above inequalities, Eq. (22) reduces to

$$\left|\int_0^\infty \phi(x)\left\{f^n(x,t)-f^n(x,s)\right\}dx\right|\leqslant\|\phi\|_{L^\infty]0,\infty[}(t-s)\left\{\frac{3}{2}k_1^2L^2+k_2(N+1)L\right\}+\epsilon/2<\epsilon \tag{23}$$

whenever $t-s<\delta$ for some $\delta>0$. The argument given above similarly holds if $s>t$. Hence (23) is true for all n and $|t-s|<\delta$. This implies the time equicontinuity of the family $\{f^n(t),\ t\in[0,T]\}$ in $L^1(\mathbb{R}_{>0})$. Thus, $\{f^n(t),\ t\in[0,T]\}$ lies in a relatively compact subset of the gauge space Ω. Details of the gauge space can be found in [22]. So, we may apply refined version of Arzelà–Ascoli theorem, see [22, Theorem 2.1] to conclude that there exists a subsequence f^{n_k} such that

$$f^{n_k}(t) \to f(t) \quad \text{in } \Omega \text{ as } n_k \to \infty$$

uniformly for $t \in [0, T]$ and for some $f \in C([0, T]; \Omega)$.

2.3. Convergence of the approximations of the integrals

For simplicity of notation we mostly suppress the dependence on arbitrary but fixed $t \in [0, T]$ when it is not explicitly needed. Now we have to show that the limit function which we obtained above is indeed a solution to (1)–(2). Define the operators Q_i^n, Q_i, $i = 1$ to 4, to be

$$Q_1^n(f^n)(x) = \frac{1}{2}\int_0^x K(x-y, y) f^n(x-y) f^n(y)\,dy, \qquad Q_1(f)(x) = \frac{1}{2}\int_0^x K(x-y, y) f(x-y) f(y)\,dy,$$

$$Q_2^n(f^n)(x) = \int_0^{n-x} K(x, y) f^n(x) f^n(y)\,dy, \qquad Q_2(f)(x) = \int_0^\infty K(x, y) f(x) f(y)\,dy,$$

$$Q_3^n(f^n)(x) = S(x) f^n(x), \qquad Q_3(f)(x) = S(x) f(x),$$

$$Q_4^n(f^n)(x) = \int_x^n b(x, y) S(y) f^n(y)\,dy, \qquad Q_4(f)(x) = \int_x^\infty b(x, y) S(y) f(y)\,dy,$$

where $f \in L^1]0, \infty[$, $x \in]0, \infty[$ and $n = 1, 2, 3, \ldots$. Set $Q^n = Q_1^n - Q_2^n - Q_3^n + Q_4^n$ and $Q = Q_1 - Q_2 - Q_3 + Q_4$.

Lemma 2.2. *Suppose $(f^n)_{n\in\mathbb{N}} \subset X^+$, $f \in X^+$, where $\|f^n\| \leqslant L$, and $f^n \rightharpoonup f$ in $L^1]0, \infty[$ as $n \to \infty$. Then for each $m > 0$*

$$Q^n(f^n) \rightharpoonup Q(f) \quad \text{in } L^1]0, m[\text{ as } n \to \infty.$$

Proof. Let χ denotes the characteristic function. Choose $m > 0$ and let $\phi \in L^\infty]0, \infty[$. We show that $Q_i^n(f^n) \rightharpoonup Q_i(f)$ in $L^1]0, m[$ as $n \to \infty$ for $i = 1, 2, 3, 4$.

Case $i = 1, 2$: By proceeding the same computation as in [22, Lemma 4.1], we can easily obtain

$$Q_i^n(f^n) \rightharpoonup Q_i(f) \quad \text{in } L^1]0, m[\text{ as } n \to \infty. \tag{24}$$

Case $i = 3$: For a.e. $x \in]0, m]$, by using (H4) we find that

$$\left|\phi(x) S(x)\right| \leqslant k_2 \|\phi\|_{L^\infty]0,m[} (1+m)^\gamma.$$

Then

$$\chi_{]0,m[}\phi S \in L^\infty]0, \infty[. \tag{25}$$

Thus by (25) and since $f^n \rightharpoonup f$ in $L^1]0, \infty[$ as $n \to \infty$,

$$\left|\int_0^m \phi(x)\left\{Q_3^n(f^n)(x) - Q_3(f)(x)\right\}dx\right| = \left|\int_0^m \phi(x) S(x)\left\{f^n(x) - f(x)\right\}dx\right| \to 0 \quad \text{as } n \to \infty.$$

Since ϕ is arbitrary

$$Q_3^n(f^n) \rightharpoonup Q_3(f) \quad \text{in } L^1]0, m[\text{ as } n \to \infty. \tag{26}$$

Case $i = 4$: Choose $\epsilon > 0$. By (H4) we have $0 \leqslant \gamma < 1$ and we can therefore choose $r > m$ such that for N given by (4)

$$2k_2 N \|\phi\|_{L^\infty]0,m[} L (1+r)^{\gamma-1} < \epsilon. \tag{27}$$

Then by Fubini's Theorem, (H4) and (27)

$$\begin{aligned}
\left|\int_0^m \int_r^\infty \phi(x) b(x, y) S(y)\left\{f^n(y) - f(y)\right\}dy\,dx\right| &= \left|\int_r^\infty \int_0^m \phi(x) b(x, y) S(y)\left\{f^n(y) - f(y)\right\}dx\,dy\right| \\
&\leqslant k_2 N \|\phi\|_{L^\infty]0,m[} \int_r^\infty (1+y)^\gamma \left\{f^n(y) + f(y)\right\}dy \\
&\leqslant 2\,k_2 N \|\phi\|_{L^\infty]0,m[} L (1+r)^{\gamma-1} < \epsilon.
\end{aligned} \tag{28}$$

Also, for a.e. $x \in]0, m]$ the function

$$\chi_{]x,r]}(\cdot)\phi(x)S(\cdot)b(x,\cdot) = \chi_{]x,r]}(\cdot)\phi(x)\Gamma(\cdot,x) \in L^{\infty}]0,\infty[.$$

Thus, since $f^n \rightharpoonup f$ in $L^1]0,\infty[$, for a.e. $x \in]0, m]$

$$\phi(x)\int_x^r S(y)b(x,y)\{f^n(y) - f(y)\}\,dy \to 0 \quad \text{as } n \to \infty. \tag{29}$$

For a.e. $x \in]0, m]$ we take $k_3 = \sup_{\substack{x<y\leqslant r\\ 0<x\leqslant m}} \Gamma(y,x)$ and by using Lemma 2.1(i)

$$\begin{aligned}
\left|\phi(x)\right|\left|\int_x^r S(y)b(x,y)\{f^n(y) - f(y)\}\,dy\right| &= \left|\phi(x)\right|\left|\int_x^r \Gamma(y,x)\{f^n(y) - f(y)\}\,dy\right| \\
&\leqslant k_3\|\phi\|_{L^\infty]0,m[}\int_x^r \{\left|f^n(y)\right| + \left|f(y)\right|\}\,dy \\
&\leqslant k_3\|\phi\|_{L^\infty]0,m[}\cdot 2L.
\end{aligned} \tag{30}$$

As a function of x this belongs to $L^1]0,m]$. Hence by (29), (30) and the dominated convergence theorem

$$\left|\int_0^m\int_x^r \phi(x)S(y)b(x,y)\{f^n(y) - f(y)\}\,dy\,dx\right| \to 0 \quad \text{as } n \to \infty.$$

Thus, by using Lemma 2.1(i), (4), and (27) in the third integral on right-hand side, we obtain for $n \geqslant m$

$$\begin{aligned}
&\left|\int_0^m \phi(x)\{Q_4^n(f^n)(x) - Q_4(f)(x)\}\,dx\right| \\
&\quad= \left|\int_0^m\int_x^r \phi(x)S(y)b(x,y)\{f^n(y) - f(y)\}\,dy\,dx\right. \\
&\qquad\left.+\int_0^m\int_r^\infty \phi(x)b(x,y)S(y)\{f^n(y) - f(y)\}\,dy\,dx - \int_0^m\int_n^\infty \phi(x)S(y)b(x,y)f^n(y)\,dy\,dx\right| \\
&\quad\leqslant \left|\int_0^m\int_x^r \phi(x)S(y)b(x,y)\{f^n(y) - f(y)\}\,dy\,dx\right| + \epsilon \\
&\qquad+ k_2N\|\phi\|_{L^\infty]0,m[}L(1+n)^{\gamma-1} \to \epsilon \quad \text{as } n \to \infty.
\end{aligned}$$

By the arbitrariness of ϕ and ϵ, we obtain from above inequality

$$Q_4^n(f^n) \rightharpoonup Q_4(f) \quad \text{in } L^1]0,m[\text{ as } n \to \infty. \tag{31}$$

Lemma 2.2 follows from (24), (26) and (31). □

2.4. The existence theorem

Now we are in a position to state and prove the main result.

Theorem 2.3. *Suppose that* (H1), (H2), (H3) *and* (H4) *hold and assume that* $f_0 \in X^+$. *Then* (1) *has a solution* f *on* $]0,\infty[$.

Proof. Choose $m > 0$, $T > 0$, and let $(f^n)_{n\in\mathbb{N}}$ be the subsequence of approximating solutions obtained above. We have from Section 2.1, for $t \in [0, T]$

$$f^n(t) \rightharpoonup f(t) \quad \text{in } L^1]0,m[\text{ as } n \to \infty. \tag{32}$$

For any $l > 0$, since we know $f^n \rightharpoonup f$ in $L^1]0,\infty[$, we obtain

$$\int_0^l xf(x,t)\,dx = \lim_{n\to\infty}\int_0^l xf^n(x,t)\,dx \leqslant \|f_0\|_x < \infty \tag{33}$$

using (9), the non-negativity of each f^n and f, and then $l \to \infty$ implies that $f \in X^+$. Let $\phi \in L^\infty]0,m[$. From Lemma 2.2 we have for each $s \in [0,t]$

$$\int_0^m \phi(x)\bigl\{Q^n\bigl(f^n(s)\bigr)(x) - Q\bigl(f(s)\bigr)(x)\bigr\}\,dx \to 0 \quad \text{as } n \to \infty. \tag{34}$$

Also, for $s \in [0,t]$, using Young's Theorem for convolutions and Lemma 2.1(i)

$$\begin{aligned}
&\int_0^m \bigl|\phi(x)\bigr|\bigl|Q^n\bigl(f^n(s)\bigr)(x) - Q\bigl(f(s)\bigr)(x)\bigr|\,dx \\
&\quad\leqslant \|\phi\|_{L^\infty]0,m[}\Biggl\{\frac{1}{2}\int_0^m\int_0^x K(x-y,y)\bigl\{f^n(x-y,s)f^n(y,s) + f(x-y,s)f(y,s)\bigr\}\,dy\,dx \\
&\qquad + \int_0^m\int_0^{n-x} K(x,y)f^n(x,s)f^n(y,s)\,dy\,dx + \int_0^m\int_0^\infty K(x,y)f(x,s)f(y,s)\,dy\,dx \\
&\qquad + \int_0^m S(x)\bigl\{f^n(x,s) + f(x,s)\bigr\}\,dx \\
&\qquad + \int_0^m\int_x^n S(y)b(x,y)f^n(y,s)\,dy\,dx + \int_0^m\int_x^\infty S(y)b(x,y)f(y,s)\,dy\,dx\Biggr\} \\
&\quad\leqslant \|\phi\|_{L^\infty]0,m[}\bigl\{3k_1^2L^2 + 2k_2(N+1)L\bigr\}.
\end{aligned} \tag{35}$$

Since the left-hand side of (35) is in $L^1]0,t[$ we have by (34), (35) and the dominated convergence theorem

$$\left|\int_0^t\int_0^m \phi(x)\bigl\{Q^n\bigl(f^n(s)\bigr)(x) - Q\bigl(f(s)\bigr)(x)\bigr\}\,dx\,ds\right| \to 0 \quad \text{as } n \to \infty. \tag{36}$$

Since ϕ is arbitrary, and Eq. (36) holds for all $\phi \in L^\infty]0,m[$, by the application of Fubini's Theorem we obtain

$$\int_0^t Q^n\bigl(f^n(s)\bigr)\,ds \rightharpoonup \int_0^t Q\bigl(f(s)\bigr)\,ds \quad \text{in } L^1]0,m[\text{ as } n \to \infty. \tag{37}$$

From the definition of Q^n and Eq. (7) we have for $t \in [0,T]$

$$f^n(x,t) = \int_0^t Q^n\bigl(f^n(s)\bigr)(x)\,ds + f^n(x,0),$$

and thus it follows from (37) and (32) that

$$\int_0^m \phi(x)f(x,t)\,dx = \int_0^t\int_0^m \phi(x)Q\bigl(f(s)\bigr)(x)\,dx\,ds + \int_0^m \phi(x)f(x,0)\,dx, \tag{38}$$

for any $\phi \in L^\infty]0,m]$. Therefore it holds for all $\phi \in C_0^\infty(]0,m])$. This implies for almost any x in $]0,m]$ we have

$$f(x,t) = \int_0^t Q\bigl(f(s)\bigr)(x)\,ds + f(x,0). \tag{39}$$

It now follows from the arbitrariness of T and m that f is a solution to (1) on $[0,\infty[$. This completes the proof of Theorem 2.3. □

3. Uniqueness

Theorem 3.1. *If* (H1), (H2), (H3′) *and* (H4′) *hold then solutions to* (1)–(2) *are unique.*

Proof. Let f and g be two solutions to (1)–(2) on $[0, T[$ where $T > 0$, with $f(0) = g(0)$, and set $Y = f - g$. For $n = 1, 2, 3\ldots,$ we define

$$u^n(t) = \int_0^n (1+x)^{\frac{1}{2}} \big|Y(x,t)\big|\, dx.$$

Multiplying $|Y|$ by $(1+x)^{\frac{1}{2}}$ and applying Fubini's Theorem to Definition 1.2(iv) above, we obtain for each n and $0 < t < T$,

$$\begin{aligned} u^n(t) = \int_0^t \int_0^n (1+x)^{\frac{1}{2}} \operatorname{sgn}\big(Y(x,s)\big)\Bigg[\frac{1}{2}\int_0^x K(x-y,y)\big\{f(x-y,s)f(y,s) - g(x-y,s)g(y,s)\big\}\,dy \\ - \int_0^\infty K(x,y)\big\{f(x,s)f(y,s) - g(x,s)g(y,s)\big\}\,dy \\ + \int_x^\infty b(x,y)S(y)\big\{f(y,s) - g(y,s)\big\}\,dy - S(x)\big\{f(x,s) - g(x,s)\big\}\Bigg]\,dx\,ds. \end{aligned} \tag{40}$$

Using the substitution $x' = x - y$, $y' = y$ in the first integral on the right-hand side of (40) we find that

$$\begin{aligned} u^n(t) = \int_0^t \int_0^n \int_0^{n-x} \bigg[\frac{1}{2}(1+x+y)^{\frac{1}{2}} \operatorname{sgn}\big(Y(x+y,s)\big) - (1+x)^{\frac{1}{2}} \operatorname{sgn}\big(Y(x,s)\big)\bigg] \\ \times K(x,y)\big\{f(x,s)f(y,s) - g(x,s)g(y,s)\big\}\,dy\,dx\,ds \\ - \int_0^t \int_0^n \int_{n-x}^\infty (1+x)^{\frac{1}{2}} \operatorname{sgn}\big(Y(x,s)\big)K(x,y)\big\{f(x,s)f(y,s) - g(x,s)g(y,s)\big\}\,dy\,dx\,ds \\ + \int_0^t \int_0^n \int_x^\infty (1+x)^{\frac{1}{2}} \operatorname{sgn}\big(Y(x,s)\big)b(x,y)S(y)\big\{f(y,s) - g(y,s)\big\}\,dy\,dx\,ds \\ - \int_0^t \int_0^n (1+x)^{\frac{1}{2}} \operatorname{sgn}\big(Y(x,s)\big)S(x)\big\{f(x,s) - g(x,s)\big\}\,dx\,ds. \end{aligned} \tag{41}$$

By interchanging the order of integration and interchanging the roles of x and y, the symmetry of K yields the identity

$$\begin{aligned} \int_0^n \int_0^{n-x} (1+x)^{\frac{1}{2}} \operatorname{sgn}\big(Y(x,s)\big)K(x,y)\big\{f(x,s)f(y,s) - g(x,s)g(y,s)\big\}\,dy\,dx \\ = \int_0^n \int_0^{n-x} (1+y)^{\frac{1}{2}} \operatorname{sgn}\big(Y(y,s)\big)K(x,y)\big\{f(x,s)f(y,s) - g(x,s)g(y,s)\big\}\,dy\,dx. \end{aligned} \tag{42}$$

For $x, y > 0$ and $t \in [0, T[$ we define the function r by

$$r(x,y,t) = (1+x+y)^{\frac{1}{2}} \operatorname{sgn}\big(Y(x+y,t)\big) - (1+x)^{\frac{1}{2}} \operatorname{sgn}\big(Y(x,t)\big) - (1+y)^{\frac{1}{2}} \operatorname{sgn}\big(Y(y,t)\big).$$

Using (42) we can show that (41) can be rewritten as

$$u^n(t) = \frac{1}{2}\int_0^t\int_0^n\int_0^{n-x} r(x,y,s)K(x,y)f(x,s)Y(y,s)\,dy\,dx\,ds + \frac{1}{2}\int_0^t\int_0^n\int_0^{n-x} r(x,y,s)K(x,y)g(y,s)Y(x,s)\,dy\,dx\,ds$$
$$+ \int_0^t\int_0^n\int_x^\infty (1+x)^{\frac{1}{2}}\operatorname{sgn}\big(Y(x,s)\big)b(x,y)S(y)Y(y,s)\,dy\,dx\,ds - \int_0^t\int_0^n (1+x)^{\frac{1}{2}}\operatorname{sgn}\big(Y(x,s)\big)S(x)Y(x,s)\,dx\,ds$$
$$- \int_0^t\int_0^n\int_{n-x}^\infty (1+x)^{\frac{1}{2}}\operatorname{sgn}\big(Y(x,s)\big)K(x,y)\big\{f(x,s)Y(y,s)+g(y,s)Y(x,s)\big\}\,dy\,dx\,ds. \tag{43}$$

Since the fourth integral and the last term in the fifth integral on the right-hand side of (43) are non-negative. We may omit them. Thus we obtain, by interchanging the order of integration for the third integral,

$$u^n(t) \leqslant \frac{1}{2}\int_0^t\int_0^n\int_0^{n-x} r(x,y,s)K(x,y)f(x,s)Y(y,s)\,dy\,dx\,ds + \frac{1}{2}\int_0^t\int_0^n\int_0^{n-x} r(x,y,s)K(x,y)g(y,s)Y(x,s)\,dy\,dx\,ds$$
$$+ \int_0^t\int_0^n\int_0^y (1+x)^{\frac{1}{2}}b(x,y)S(y)\big|Y(y,s)\big|\,dx\,dy\,ds + \int_0^t\int_0^n\int_n^\infty (1+x)^{\frac{1}{2}}b(x,y)S(y)\big|Y(y,s)\big|\,dy\,dx\,ds$$
$$- \int_0^t\int_0^n\int_{n-x}^\infty (1+x)^{\frac{1}{2}}\operatorname{sgn}\big(Y(x,s)\big)K(x,y)f(x,s)Y(y,s)\,dy\,dx\,ds$$
$$=: \int_0^t \sum_{i=1}^5 S_i^n(s)\,ds. \tag{44}$$

Here S_i^n, for $i = 1, \ldots, 5$, are the corresponding integrands in the preceding lines.

We now consider each S_i^n individually. Noting that for all $q, q_1, q_2 \in \mathbb{R}$, we have $\operatorname{sgn}(q_1)\operatorname{sgn}(q_2) = \operatorname{sgn}(q_1 q_2)$ and $|q| = q\operatorname{sgn}(q)$. We find that

$$\begin{aligned} r(x,y,s)Y(y,s) &\leqslant \big[(1+x+y)^{\frac{1}{2}} + (1+x)^{\frac{1}{2}} - (1+y)^{\frac{1}{2}}\big]\big|Y(y,s)\big| \\ &\leqslant \big[(1+x)^{\frac{1}{2}} + (1+y)^{\frac{1}{2}} + (1+x)^{\frac{1}{2}} - (1+y)^{\frac{1}{2}}\big]\big|Y(y,s)\big| \\ &\leqslant 2(1+x)^{\frac{1}{2}}\big|Y(y,s)\big|. \end{aligned} \tag{45}$$

Now, by using (H3′) we consider

$$\begin{aligned} \int_0^t S_1^n(s)\,ds &= \frac{1}{2}\int_0^t\int_0^n\int_0^{n-x} r(x,y,s)K(x,y)f(x,s)Y(y,s)\,dy\,dx\,ds \\ &\leqslant \int_0^t\int_0^n\int_0^{n-x} (1+x)^{\frac{1}{2}}K(x,y)f(x,s)\big|Y(y,s)\big|\,dy\,dx\,ds \\ &\leqslant k^2\int_0^t\int_0^n\int_0^{n-x} (1+x)(1+y)^{\frac{1}{2}}f(x,s)\big|Y(y,s)\big|\,dy\,dx\,ds \\ &\leqslant R_1\int_0^t u^n(s)\,ds, \quad \text{where } R_1 = k^2\sup_{s\in[0,t]}\big\|f(s)\big\|. \end{aligned} \tag{46}$$

Similarly, there is a constant R_2 such that

$$\int_0^t S_2^n(s)\,ds \leqslant R_2\int_0^t u^n(s)\,ds. \tag{47}$$

Now, we consider

$$\int_0^t S_3^n(s)\,ds = \int_0^t \int_0^n \int_0^y (1+x)^{\frac{1}{2}} b(x,y) S(y) \big|Y(y,s)\big|\,dx\,dy\,ds. \tag{48}$$

By interchanging the role of x and y in (48) and using (H4′) we obtain

$$\begin{aligned}\int_0^t S_3^n(s)\,ds &= \int_0^t \int_0^n \int_0^x (1+y)^{\frac{1}{2}} b(y,x) S(x) \big|Y(x,s)\big|\,dy\,dx\,ds \\ &\leqslant m_1 m_2 \int_0^t \int_0^n (1+x)^{a+b} \big|Y(x,s)\big|\,dx\,ds \\ &\leqslant R_3 \int_0^t u^n(s)\,ds, \quad \text{where } R_3 = m_1 m_2. \end{aligned} \tag{49}$$

Next, using Fubini's Theorem and hypothesis (H4′) we have for each $s \in [0,t]$

$$\begin{aligned}\int_0^n \int_n^\infty (1+x)^{\frac{1}{2}} b(x,y) S(y) \big|Y(y,s)\big|\,dy\,dx &= \int_n^\infty \int_0^n (1+y)^{\frac{1}{2}} b(y,x) S(x) \big|Y(x,s)\big|\,dy\,dx \\ &\leqslant \int_n^\infty \int_0^x (1+y)^{\frac{1}{2}} b(y,x) S(x) \big[f(x,s)+g(x,s)\big]\,dy\,dx \\ &\leqslant m_1 m_2 \int_n^\infty (1+x)^{a+b} \big[f(x,s)+g(x,s)\big]\,dy\,dx. \end{aligned} \tag{50}$$

The right-hand side of (50) is always bounded by the constant $m_1 m_2 \sup_{s\in[0,t]}[\|f(s)\| + \|g(s)\|]$ and therefore the dominated convergence theorem leads to

$$\int_0^t S_4^n(s)\,ds \to 0 \quad \text{as } n \to \infty. \tag{51}$$

To consider S_5^n we first observe that

$$\left| \int_0^\infty \int_0^\infty (1+x)^{\frac{1}{2}} \operatorname{sgn}\big(Y(x,s)\big) K(x,y) f(x,s) Y(y,s)\,dy\,dx \right| \leqslant k^2 \int_0^\infty \int_0^\infty (1+x)(1+y)^{\frac{1}{2}} f(x,s) \big|Y(y,s)\big|\,dy\,dx < \infty.$$

Thus, we obtain

$$\int_0^t S_5^n(s)\,ds \to 0 \quad \text{as } n \to \infty. \tag{52}$$

The sequence u^n is bounded and monotone. Thus, from (44), (46), (47), (49), (51), (52) and taking $R = R_1 + R_2 + R_3$ we obtain

$$\begin{aligned} u(t) := \int_0^\infty (1+x)^{\frac{1}{2}} \big|Y(x,t)\big|\,dx &= \lim_{n\to\infty} u^n(t) \\ &\leqslant \lim_{n\to\infty} \int_0^t \sum_{i=1}^5 S_i^n(s)\,ds \end{aligned}$$

$$\leqslant \lim_{n\to\infty} R \int_0^t u^n(s)\,ds + \lim_{n\to\infty} \int_0^t \big[S_4^n(s) + S_5^n(s)\big]\,ds$$
$$= R \int_0^t \int_0^\infty (1+x)^{\frac{1}{2}} \big|Y(x,s)\big|\,dx\,ds.$$

This gives the inequality

$$u(t) \leqslant R \int_0^t u(s)\,ds. \tag{53}$$

Then by applying Gronwall's inequality to (53), we obtain

$$u(t) = \int_0^\infty (1+x)^{\frac{1}{2}} \big|Y(x,t)\big|\,dx = 0 \quad \text{for all } t \in [0,T[.$$

Therefore,

$$f(x,t) = g(x,t) \quad \text{for a.e. } x \in\,]0,\infty[. \qquad \square$$

4. Conclusions

A detailed study on the existence of weak solutions to the continuous coagulation equation with multiple fragmentation has been given for a large class of kernels. The uniqueness of the weak solutions has also been established under more stringent assumptions on the coagulation and fragmentation kernels. An interesting open question is how one can include $\mu, \gamma = 1$ in the hypotheses made in this paper to improve the existence result. Furthermore, it would also be of great interest to enlarge the classes of kernels for the uniqueness of solutions.

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