J. Math. Anal. Appl. 376 (2011) 385–392

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Sharp Turán inequalities via very hyperbolic polynomials ☆

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ARTICLE INFO

Article history:
Received 1 August 2009
Available online 10 December 2010
Submitted by Steven G. Krantz

Keywords:
Laguerre–Pólya class
Maclaurin coefficients
Turán inequalities
Hyperbolic polynomials
Very hyperbolic polynomials

ABSTRACT

We present new sharp inequalities for the Maclaurin coefficients of an entire function from the Laguerre–Pólya class. They are obtained by a new technique involving the so-called very hyperbolic polynomials. The results may be considered as extensions of the classical Turán inequalities.

1. Introduction

The real entire function $\psi(x)$ is said to belong to the Laguerre–Pólya class $\mathcal{LP}$ if it can be represented as

$$\psi(x) = cx^m e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} (1 + x/x_k) e^{-x/x_k},$$

where c, β, x_k are real, $\alpha \geqslant 0$, m is a nonnegative integer and $\sum x_k^{-2} < \infty$. Similarly, the real entire function $\psi_1(x)$ is a function of type I in the Laguerre–Pólya class, written $\psi_1 \in \mathcal{LPI}$, if $\psi_1(x)$ or $\psi_1(-x)$ can be represented in the form

$$\psi_1(x) = cx^m e^{\sigma x} \prod_{k=1}^{\infty} (1 + x/x_k), \tag{1}$$

where c and σ are real, $\sigma \geqslant 0$, m is a nonnegative integer, $x_k > 0$, and $\sum 1/x_k < \infty$. It is clear that $\mathcal{LPI} \subset \mathcal{LP}$. The functions in $\mathcal{LP}$, and only these, are uniform limits on compact subsets of $\mathbb{C}$ of polynomials with only real zeros (see, for example, Levin [10, Chapter 8]). Similarly, $\psi \in \mathcal{LPI}$ if and only if it is a uniform limit on the compact sets of the complex plane of polynomials whose zeros are real and are either all positive, or all negative. Thus, the classes $\mathcal{LP}$ and $\mathcal{LPI}$ are closed under differentiation; that is, if $\psi \in \mathcal{LP}$, then $\psi^{(\nu)} \in \mathcal{LP}$ for every $\nu \in \mathbb{N}$ and similarly, if $\psi \in \mathcal{LPI}$, then $\psi^{(\nu)} \in \mathcal{LPI}$. Pólya and Schur [13] proved that if the function

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{2}$$

☆ Research supported by the Brazilian Science Foundations FAPESP under Grant 03/01874-2, CNPq under Grant 304830/2006-2 and the French Foundation CNRS under Project 20682.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.12.014

belongs to $\mathcal{LP}$ and its Maclaurin coefficients $\gamma_k = \psi^k(0)$ are all nonnegative, then $\psi \in \mathcal{LPI}$. It is worth mentioning that the sequences $\{\gamma_k\}$ of Maclaurin coefficients of $\mathcal{LPI}$-functions are called *multiplier sequences* and these are the sequences with the property that, for any $n \in \mathbb{N}$ and every hyperbolic polynomial $\sum_{k=0}^n a_k\gamma_k x^k$, the polynomial $\sum_{k=0}^n a_k\gamma_k x^k$ is also a hyperbolic one.

The main reason for the interest in the Laguerre–Pólya class is the fact that it is closely related to the celebrated Riemann hypothesis. Recall that the Riemann ξ-function is defined by

$$\xi(iz) = \frac{1}{2}\left(z^2 - 1/4\right)\pi^{-z/2-1/4}\Gamma(z/2+1/4)\zeta(z+1/2),$$

where $\zeta(z)$ is the Riemann zeta-function and $\Gamma(z)$ is the gamma-function. It is known that $\xi(z)$ is an entire function of order one. Moreover, it can be represented in the form

$$\xi(x/2) = 8\int_0^\infty \Phi(t)\cos xt\,dt,$$

where

$$\Phi(t) = \sum_{n=1}^\infty \left(2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t}\right)\exp\left(-n^2\pi e^{4t}\right).$$

Then

$$\frac{1}{8}\xi(x/2) = \sum_{k=0}^\infty (-1)^k\hat{\beta}_k\frac{x^{2k}}{(2k)!} \quad \text{with } \hat{\beta}_k = \int_0^\infty t^{2k}\Phi(t)\,dt,\ k = 0, 1, \ldots.$$

Setting $z = -x^2$ we obtain the entire function

$$\xi_1(z) = \sum_{k=0}^\infty \hat{\beta}_k\frac{z^k}{(2k)!} = \sum_{k=0}^\infty \hat{\gamma}_k\frac{z^k}{k!}, \quad \hat{\gamma}_k = \frac{k!}{(2k)!}\hat{\beta}_k,$$

of order $1/2$. Thus, the Riemann hypothesis is equivalent to the statement that $\xi \in \mathcal{LP}$, or equivalently, that $\xi_1 \in \mathcal{LPI}$. Then any new necessary and/or sufficient conditions for a function ψ to belong to $\mathcal{LP}$ are of interest.

The simplest necessary conditions for a function (2) to be in $\mathcal{LP}$ is that the so-called Turán inequalities

$$\gamma_{k+1}^2 - \gamma_k\gamma_{k+2} \geqslant 0, \quad k \geqslant 0, \tag{3}$$

hold. Proofs of this fact can be found in [13,11,12]. The most straightforward proof is based on the relation between functions from $\mathcal{LP}$ of the form (2) and their generalized Jensen polynomials, defined by

$$g_{n,k}(x) = g_{n,k}(\psi; x) := \sum_{j=0}^n \binom{n}{j}\gamma_{k+j}x^j, \quad n, k = 0, 1, \ldots. \tag{4}$$

It is known (see [3,11]) that $\psi \in \mathcal{LP}$ if and only if the corresponding Jensen polynomials $g_{n,k}(\psi; x)$ possess real zeros only. The real polynomials that obey this property are usually called *hyperbolic* ones. Then the fact that the Jensen polynomials $g_{2,k}(x)$ must be hyperbolic immediately yields Turán's inequalities (3). Further extensions and generalizations of (3) were obtained in [5,2,6] and most of those results were derived as consequences of the fact that $g_{n,k}(\psi; x)$ must be hyperbolic for certain small values of n.

In this paper we obtain a new sharp extension of the Turán inequalities involving five consecutive coefficients of a function in the Laguerre–Pólya class:

Theorem 1. *If the real entire function $\psi(x)$ of the form* (2)*, with nonzero Maclaurin coefficients γ_k, belongs to $\mathcal{LP}$, then the following inequalities hold*

$$\begin{aligned} &3\left(\gamma_{k+1}^2\gamma_{k+2}^2 - \gamma_k^2\gamma_{k+3}^2\right) - 6\gamma_k\gamma_{k+2}^3 - 4\gamma_{k+1}^3\gamma_{k+3} + 10\gamma_k\gamma_{k+1}\gamma_{k+2}\gamma_{k+3} \\ &\quad - 2\gamma_k\gamma_{k+4}\left(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}\right) \geqslant 0, \quad k = 0, 1, \ldots, \end{aligned} \tag{5}$$

$$6\gamma_{k+1}^4 - 12\gamma_k\gamma_{k+1}^2\gamma_{k+2} + 3\gamma_k^2\gamma_{k+2}^2 + 4\gamma_k^2\gamma_{k+1}\gamma_{k+3} - \gamma_k^3\gamma_{k+4} \geqslant 0, \quad k = 0, 1, \ldots. \tag{6}$$

It is worth mentioning that the inequalities (6) are weaker than (5) and we give geometric interpretation of this fact in the proof of the theorem.

These are new type inequalities, different from those known in the literature because they reflect the fact that all Appell polynomials (see the definitions in the next section) associated with $\psi(x)$ are not only hyperbolic, but also that they are *very hyperbolic*. Very hyperbolic polynomials are those hyperbolic ones which possess hyperbolic primitives of any integer order. More precisely, the hyperbolic polynomial $p_n(z)$ of degree n is a very hyperbolic one if, for every $k \in \mathbb{N}$, there exists a hyperbolic polynomial $P_{n+k}(z)$ of degree $n+k$, such that $P_{n+k}^{(k)}(z) = p_n(z)$. Therefore, though the inequalities (5) involve only five consecutive Maclaurin coefficients of $\psi(x)$, they contain information not only about hyperbolicity of the corresponding fourth degree Appell polynomial, but also about the fact that the latter can be integrated arbitrarily many times to obtain hyperbolic polynomials again. This means that the inequalities (5) are much sharper than those obtained in [2,5], because the former implicitly contain the structure of the entire function as a whole and not only portions of its power series expansion (2).

We mention that inequalities (6) were obtained in a different way in a recent paper by Cardon [1]. Indeed, set $f(z) = \psi^{(k)}(z)$, where $\psi(x)$ is defined by (2), and $z = 0$, in the expression that represents the polynomial $(3/2)A_4(x)$, in [1, Example 1]. Thus we obtain exactly the quantities which appear on the left-hand side of (6).

In Section 2 we provide a brief information about Jensen and Appell polynomials and in Section 3 we describe the very hyperbolic polynomials of degree four and an algorithm which allows us to verify if a given fourth degree polynomial is a very hyperbolic one. Though the algorithm is fundamental for obtaining the inequalities (5), the reader may skip it at first reading and concentrate directly on Remark 2. The proof of Theorem 1 is furnished in Section 4 and finally, in Section 5, we show some numerical experiments which illustrate the main result for the function $\xi_1(z)$ and make some interesting observations concerning the behaviour of its Maclaurin coefficients.

2. Jensen, Appell and very hyperbolic polynomials

Consider the real entire function $\psi(x)$ defined by (2). Its Jensen polynomials are

$$g_n(x) = g_n(\psi; x) := \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^j, \quad n = 0, 1, \ldots.$$

Jensen himself proved that $\psi \in \mathcal{LP}$ if and only if the corresponding Jensen polynomials $g_n(\psi; x)$ possess real zeros only and that the sequence $\{g_n(\psi; x/n)\}$ converges locally uniformly to $\psi(x)$. Observe that, for any fixed $k \in \mathbb{N}$, the generalized Jensen polynomials $g_{n,k}(\psi; x)$, $k = 0, 1, \ldots,$ are the Jensen polynomials associated with $\psi^{(k)}(x)$; that is,

$$g_{n,k}(\psi; x) = g_n\big(\psi^{(k)}; x\big).$$

It is easy to check that the identities

$$g_n^{(\nu)}(x) = \big(n!/(n-\nu)!\big) g_{n-\nu,\nu}(x)$$

and

$$g_{n,k}^{(\nu)}(x) = \big(n!/(n-\nu)!\big) g_{n-\nu,k+\nu}(x) \tag{7}$$

hold. Hence, if $\psi \in \mathcal{LP}$, then all generalized Jensen polynomials (4) are hyperbolic, and by (7), we conclude that $\mathcal{LP}$ is indeed closed under differentiation.

Consider also the corresponding Appell polynomials of $\psi(x)$, defined by

$$A_n(x) = A_n(\psi; x) = x^n g_n(1/x) = \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^{n-j}$$

as well as its generalized Appell polynomials

$$A_{n,k}(x) = A_{n,k}(\psi; x) = x^n g_{n,k}(1/x) = \sum_{j=0}^{n} \binom{n}{j} \gamma_{k+j} x^{n-j}, \quad n = 0, 1, \ldots.$$

Again, we may easily verify the identities

$$A_n^{(\nu)}(x) = \big(n!/(n-\nu)!\big) A_{n-\nu}(x)$$

and

$$A_{n,k}^{(\nu)}(x) = \big(n!/(n-\nu)!\big) A_{n-\nu,k}(x). \tag{8}$$

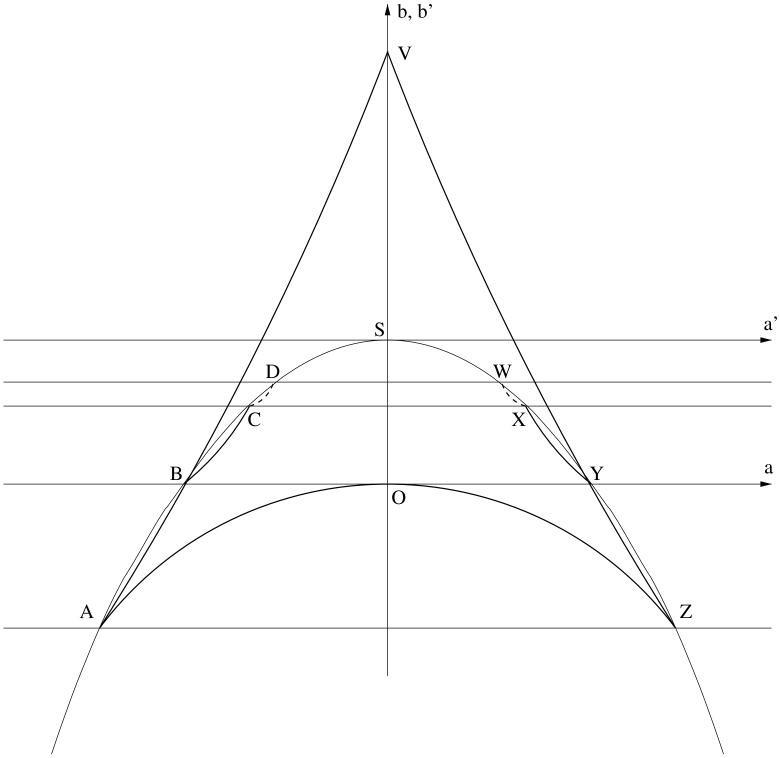

Fig. 1. The domain of very hyperbolic degree four polynomials.

Then, by Jensen's result, the entire function $\psi(x)$, with nonzero Maclaurin coefficients, belongs to $\mathcal{LP}$ if and only if the corresponding Appell polynomials $A_n(\psi; x)$ are hyperbolic. Moreover, the above differential identities imply the following simple, but important for our purposes, result:

Proposition 1. *If the real entire function $\psi(x)$ with nonzero Maclaurin coefficients belongs to the Laguerre–Pólya class, then for every $n \in \mathbb{N}$ and $k = 0, 1, \ldots,$ the generalized Appell polynomials $A_{n,k}(\psi; x)$ are very hyperbolic.*

It follows immediately from (8) that

$$A_{n,k}(x) = \big(n!/(n+\nu)!\big) A^{(\nu)}_{n+\nu,k}$$

which means that Appell polynomials possess hyperbolic primitives of any order. Unfortunately, the corresponding relations (7) imply only that the Jensen polynomial $g_{n,k}(x)$ has hyperbolic primitives of order up to k, but not necessarily of higher one. There are simple examples which show that, in general, Jensen polynomials are not very hyperbolic.

3. The domain of very hyperbolic polynomials of degree 4

In the present section we first recall some properties of the domain of very hyperbolic polynomials of degree 4 and then give an algorithm testing whether a point belongs to that domain. The reader can find all necessary details in [7–9]. Up to an affine transformation of the x-coordinate and a multiplication, a hyperbolic degree 4 polynomial takes the form $x^4 - x^2 + ax + b$, so that we consider the space

$$\mathcal{P}(a,b) := \big\{p(x) = x^4 - x^2 + ax + b\big\}.$$

The domain of values of the parameters $(a, b) \in \mathbb{R}^2$ for which the polynomials are hyperbolic, is the curvilinear triangle AZV having cusps at the vertices A and Z and a finite nonzero angle between the tangent lines at the vertex V with coordinates $(0, 1/4)$, see Fig. 1. This vertex represents the polynomial $(x - 1/\sqrt{2})^2(x + 1/\sqrt{2})^2$.

Very hyperbolic polynomials are represented by the subdomain $\Delta \subset AZV$ resembling a moustache and bounded by the curve $\partial\Delta := ABCD\ldots S\ldots WXYZOA$. Consider alongside with the coordinates (a, b) the coordinates (a', b'), where

$$a' = \frac{3\sqrt{6}a}{4}, \qquad b' = 6b - \frac{1}{2}.$$

Then we consider the family

$$\mathcal{Q}(a',b') := \left\{ p(x) = x^4 - x^2 + \frac{4}{3\sqrt{6}}a'x + \frac{2b'+1}{12} \right\}.$$

The domain Δ has the following properties:

1. It is symmetric with respect to the b-axis.

2. From below it is bounded by the algebraic arc AZ which can be parametrized as follows

$$a = \frac{p^3}{2} - p, \qquad b = \frac{3p^4}{16} - \frac{p^2}{4} \quad \text{or} \quad a' = \frac{3\sqrt{6}p^3}{8} - \frac{3\sqrt{6}p}{4}, \qquad b' = \frac{9p^4}{8} - \frac{3p^2}{2} - \frac{1}{2}, \tag{9}$$

where $p \in [-\sqrt{6}/3, \sqrt{6}/3]$. Indeed, this (open) arc represents all hyperbolic degree 4 polynomials with a double root in the middle (see the details in [8]). Hence, the (closed) arc can be parametrized by the roots $x_1 \geqslant x_2 \geqslant x_3$ of multiplicity respectively 1, 2, 1 like this:

$$x_1 + 2x_2 + x_3 = 0, \qquad x_1^2 + 2x_2^2 + x_3^2 = 2, \qquad a = -\left(2x_1x_2x_3 + x_1x_2^2 + x_2^2x_3\right), \qquad b = x_1x_2^2x_3.$$

From the first equation of this system one eliminates x_2, then one sets $x_1 + x_3 = p$, $x_1x_3 = q$. The second equation becomes $3p^2 - 4q = 4$ which allows to express q via p. The third and fourth equations then become the first two of Eqs. (9). For $p = \pm\sqrt{6}/3$ the polynomial has a triple root $\mp 1/\sqrt{6}$ and a simple one $\pm\sqrt{3/2}$. Thus the points A and Z represent respectively the polynomials $u(x) := (x - 1/\sqrt{6})^3(x + 3/\sqrt{6})$ and $u(-x)$.

3. The domain Δ is bounded from above by countably many algebraic arcs $AB, BC, CD, \dots, WX, XY, YZ$. We call the arcs AB, BC, etc. (respectively YZ, XY, etc.) the first, the second, etc. arcs from the left (respectively from the right). The sth arc from the left is parametrized as follows

$$-a' = \theta^3 + s\psi^3, \qquad -b' = \theta^4 + s\psi^4, \quad \text{where } 0 \leqslant \theta \leqslant \psi,\ \theta^2 + s\psi^2 = 1. \tag{10}$$

For the arcs from the right just change a' to $-a'$.

4. The (a',b')-coordinates of the left endpoint of the sth arc from the left are equal to $(-1/\sqrt{s}, -1/s)$. They belong to the parabola $\mathcal{P}$: $b' = -a'^2$.

This leads to the following result on which inequalities (5) and (6) are based:

Proposition 2. *For a very hyperbolic polynomial from the family $\mathcal{Q}(a',b')$ one has $b' \leqslant -(a')^2$ and, in particular, $b' \leqslant 0$.*

5. The points B and Y belong to the line $b = 0$. They represent respectively the polynomials $u(x) := x(x - 1/\sqrt{3})^2 \times (x + 2/\sqrt{3})$ and $u(-x)$. The curvilinear triangle AOB (respectively ZOY) consists of all polynomials with one negative and three nonnegative roots (respectively with one positive and three nonpositive roots).

6. The concavities of all arcs building up $\partial\Delta$ are towards the interior of the domain Δ.

7. The tangent lines to all these arcs are nowhere vertical, this applies also to their left and right limits at their endpoints $A, B, \dots, Y, Z$.

8. The point S is the common limit of the left and right endpoints of the arcs when $s \to \infty$. Its coordinates are $a = a' = 0$, $b = 1/12$, $b' = 0$.

9. When a point from $\partial\Delta$ tends to the point S, then in the limit the tangent lines become horizontal.

10. The slopes of the tangent lines and their limits at the endpoints are everywhere positive for $a < 0$ and everywhere negative for $a > 0$. This means that each arc $AB, BC, \dots, XY, YZ$ and the left and right halves of the arc AZ (i.e. the open arcs AO and OZ) can be considered at the same time as graphs of C^1-functions $b = b(a)$ or $b' = b'(a')$ and as graphs of C^1-functions $a = a(b)$ or $a' = a'(b')$.

The last property implies that one can find out whether a point with (a',b')-coordinates (a'_0, b'_0) belongs to the domain Δ using the algorithm given below. It is based on the idea to check whether the point belongs to the intersection of Δ with the horizontal strips $b'_0 \in [-1/s, -1(s+1))$, $s \in \mathbb{N}^*$, defined by the b'_0-coordinates of two consecutive endpoints of arcs.

Algorithm.

Step 1. If $b'_0 \in [-1, 0]$, then go to Step 2. If $b'_0 \notin [-1, 0]$, then $(a'_0, b'_0) \notin \Delta$. Stop.
Step 2. If $a'_0 = 0$, then go to Step 3. If not, then go to Step 4.
Step 3. $((a'_0, b'_0) \in \Delta) \Leftrightarrow (b'_0 \in [-1/2, 0])$. Stop.
Step 4. Find the unique positive integer s such that $b'_0 \in [-1/s, -1/(s+1))$. Go to Step 5.
Step 5. If $s = 1$, then go to Step 9. If $s > 1$, then find the unique solution (θ, ψ) to the system

$$\theta^2 + s\psi^2 = 1, \qquad \theta^4 + s\psi^4 = -b'_0, \quad 0 \leqslant \theta \leqslant \psi. \tag{11}$$

Go to Step 6.

Step 6. If $a_0' < 0$, then go to Step 7. If $a_0' > 0$, then go to Step 8.
Step 7. Set $a_1' := -(\theta^3 + s\psi^3)$ where (θ, ψ) is the solution to system (11). One has $((a_0', b_0') \in \Delta) \Leftrightarrow (a_0' \geqslant a_1')$. Stop.
Step 8. Set $a_1' := \theta^3 + s\psi^3$ where (θ, ψ) is the solution to system (11). One has $((a_0', b_0') \in \Delta) \Leftrightarrow (a_0' \leqslant a_1')$. Stop.
Step 9. If $a_0' < 0$, then set $a_2' := 3\sqrt{6}p(p^2 - 2)/8$, where p is the unique solution to the system

$$\frac{9p^4}{8} - \frac{3p^2}{2} - \frac{1}{2} - b' = 0, \quad a_2'(p) < 0 \tag{12}$$

belonging to the interval $[-\sqrt{6}/3, \sqrt{6}/3]$. One has $((a_0', b_0') \in \Delta) \Leftrightarrow (a_1' \leqslant a_0' \leqslant a_2')$. Stop.

If $a_0' > 0$, then set $a_2' := 3\sqrt{6}p(p^2 - 2)/8$, where p is the unique solution to the system

$$\frac{9p^4}{8} - \frac{3p^2}{2} - \frac{1}{2} - b' = 0, \quad a_2'(p) > 0 \tag{13}$$

belonging to the interval $[-\sqrt{6}/3, \sqrt{6}/3]$. One has $((a_0', b_0') \in \Delta) \Leftrightarrow (a_2' \leqslant a_0' \leqslant a_1')$. Stop.

The algorithm is justified by the above properties of Δ and by the following two lemmas:

Lemma 1. *For each positive integer s system* (11) *in which*

$$b_0' \in \left[-\frac{1}{s}, -\frac{1}{s+1}\right) \tag{14}$$

has a unique solution (θ, ψ).

Indeed, set $\sigma := \theta^2$, $\tau := \psi^2$. One has $\sigma + s\tau = 1$, $\sigma^2 + s\tau^2 = -b_0'$, $0 \leqslant \sigma \leqslant \tau$, hence, τ is solution to the equation $(1 - s\tau)^2 + s\tau^2 = -b_0'$, i.e.

$$\tau = \frac{s \pm \sqrt{s^2 - (s^2 + s)(1 + b_0')}}{s^2 + s}, \qquad \sigma = \frac{1 \mp \sqrt{s^2 - (s^2 + s)(1 + b_0')}}{s + 1}.$$

One has $s^2 - (s^2 + s)(1 + b_0') \geqslant 0$ and (for both possible choices of the signs) $\sigma \geqslant 0$; this follows from (14). One has to choose the upper signs in order to have $\sigma \leqslant \tau$. Hence, the solution exists and is unique.

Lemma 2. *For $b' \in [-1, -1/2)$ each of the two systems* (12) *and* (13) *has a unique solution p belonging to the interval $I := [-\sqrt{6}/3, \sqrt{6}/3]$.*

Indeed, the equations from systems (12) and (13) are the same. Their solutions are $p = \pm\sqrt{2(1 \pm \sqrt{2 + 2b'})/3}$. Inside the external square root one has to choose the sign "$-$" in order to have $p \in I$. Outside one chooses the sign "$+$" (respectively "$-$") to obtain the solution to system (12) (respectively (13)) because a_2' and p have opposite signs.

Remark 1. One could also consider an algorithm based on the slicing of Δ not horizontally, but vertically, the borders of the vertical strips being defined by the a'-coordinates of the endpoints $A, B, C, \ldots, X, Y, Z$, see property 10. We prefer horizontal slicing for two reasons:

- for horizontal slicing only for $s = 1$ (see Steps 5 and 9 of the algorithm) does one have to consider the parametrization of the arc AZ;
- for vertical slicing in the analog of Step 5 one would have to consider the system $\theta^2 + s\psi^2 = 1$, $\theta^3 + s\psi^3 = -b_0'$, $0 \leqslant \theta \leqslant \psi$ instead of system (11), and solving the latter involves more cumbersome formulas.

Remark 2. In the present paper we approximate Δ by the domain between the parabola $\mathcal{P}$ and the arc AZ. By doing so we gain the simplicity of the description of the domain. Our loss in the precision of this description is small – we add to Δ the union of domains defined by the couples of arcs $AB, BC, \ldots, WX, XY, YZ$. In each couple there is an arc from $\mathcal{P}$ and an arc from $\partial\Delta$. From these domains only the ones defined by the couples of arcs AB and YZ consist of points not from the hyperbolicity domain, i.e. not from the curvilinear triangle AZV.

One could use the concavity of the arcs from $\partial\Delta$ (see property 6) to approximate Δ by a domain bounded by a piecewise-linear curve (say, $YZOAB$) and by a part of the parabola $\mathcal{P}$ (say, the one between the points B and Y). Such an approximation is more precise from above, less precise and simpler from below and less elegant as a whole.

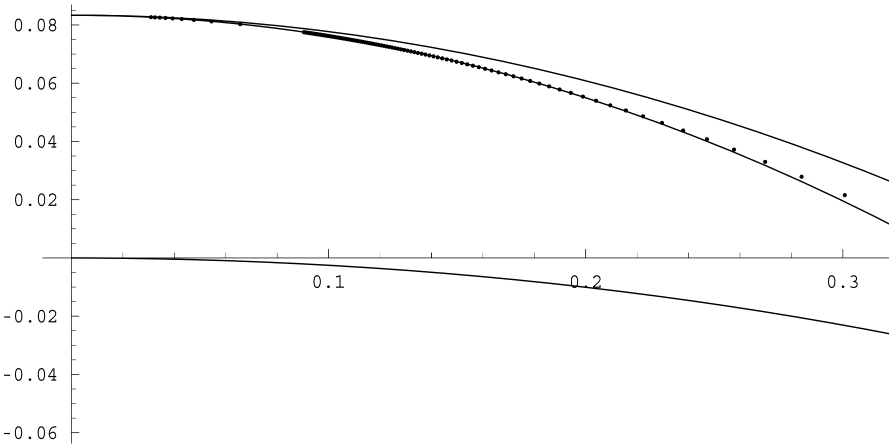

Fig. 2. The points (a_k, b_k) corresponding to the function $\xi_1(z)$ and the parabola $\mathcal{B}$.

4. Proof of Theorem 1

Let

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$$

be an entire function in the Laguerre–Pólya class with nonzero Maclaurin coefficients γ_k. Consider its generalized Appell polynomials

$$A_{4,k}(x) = \gamma_k x^4 + 4\gamma_{k+1}x^3 + 6\gamma_{k+2}x^2 + 4\gamma_{k+3}x + \gamma_{k+4}. \tag{15}$$

We perform an affine transformation of the independent variable x which is a superposition of the following three consecutive changes:

$$Q_k(x) = A_{4,k}(x - \gamma_{k+1}/\gamma_k)/\gamma_k,$$
$$H_k(x) = Q_k\left(\frac{\sqrt{6(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2})}}{\gamma_k}x\right),$$
$$R_k(x) = \left(6\left(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}\right)/\gamma_k^2\right)^{-2} H_k(x).$$

We obtain

$$R_k(x) = x^4 - x^2 + a_k x + b_k,$$

where

$$a_k = \sqrt{\frac{2}{3}}\frac{2\gamma_{k+1}^3 - 3\gamma_k\gamma_{k+1}\gamma_{k+2} + \gamma_k^2\gamma_{k+3}}{3(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2})^{3/2}},$$
$$b_k = \frac{\gamma_k^3\gamma_{k+4} - 4\gamma_k^2\gamma_{k+1}\gamma_{k+3} + 6\gamma_k\gamma_{k+1}^2\gamma_{k+2} - 3\gamma_{k+1}^4}{36(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2})^2}.$$

Recall that $a'_k = 3\sqrt{6}a_k/4$ and $b'_k = 6b_k - 1/2$. Thus

$$a'_k = \frac{2\gamma_{k+1}^3 - 3\gamma_k\gamma_{k+1}\gamma_{k+2} + \gamma_k^2\gamma_{k+3}}{2(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2})^{3/2}},$$
$$b'_k = \frac{\gamma_k^3\gamma_{k+4} - 4\gamma_k^2\gamma_{k+1}\gamma_{k+3} - 3\gamma_k^2\gamma_{k+2}^2 + 12\gamma_k\gamma_{k+1}^2\gamma_{k+2} - 6\gamma_{k+1}^4}{6(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2})^2}.$$

By Proposition 1, if $\psi(x)$ is a real entire function in the Laguerre–Pólya class with nonzero Maclaurin coefficients, then for every $k \in \mathbb{N}$ the polynomial $R_k(x)$ is very hyperbolic. On the other hand, by Proposition 1, the parameters a'_k and b'_k must satisfy the inequalities $b'_k \leqslant -(a'_k)^2$ and $b'_k \leqslant 0$. These are equivalent to (5) and (6), respectively.

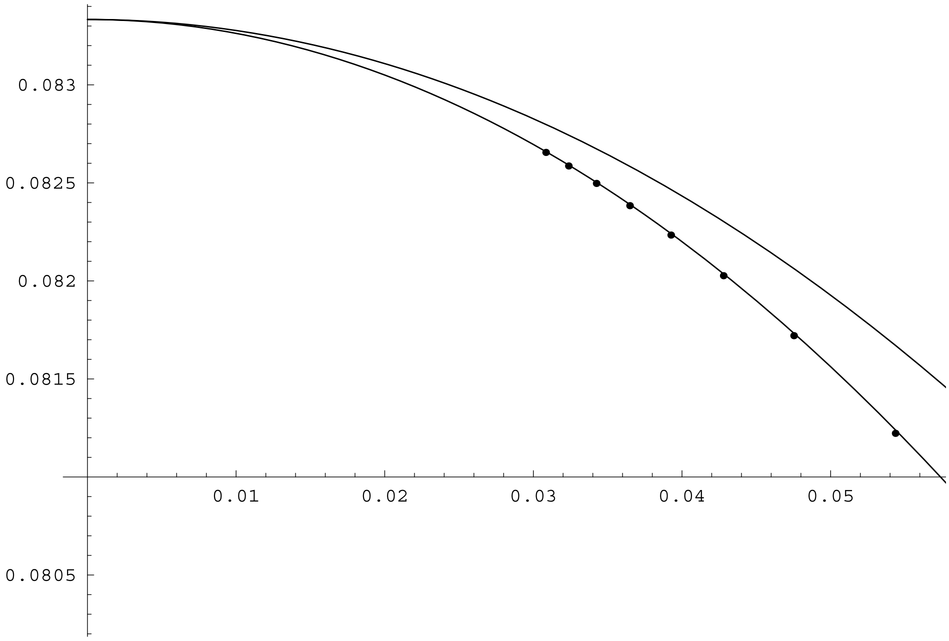

Fig. 3. The points (a_k, b_k) for $k = 300, \ldots, 900, 996$, a detailed view.

5. Numerical experiments about the Riemann ξ function

Observe that the proof of the main result is based on the following idea. With every fourth degree polynomial $A_{4,k}(x)$ of the form (15) we associate a point (a_k, b_k). If $A_{4,k}(x)$ is very hyperbolic, then (a_k, b_k) must belong to the moustache region described in Section 3. Moreover, the corresponding point (a_k, b_k) lies on the upper curve which bounds this region if and only if the inequality (5) reduces to an equality for the coefficients of $A_{4,k}(x)$. Similarly, (a_k, b_k) lies on the lower curve which bounds the moustache region if and only if the inequality (6) reduces to an equality for the coefficients of $A_{4,k}(x)$. In this section we show the results of some numerical experiments which we have performed. Based on precise numerical calculations of the moments $\hat{\beta}_k$ due to Norfolk, Ruttan and Varga [4], we compute the Maclaurin coefficients $\hat{\gamma}_k$ of the function $\xi_1(z)$. Then the coefficients a_k and b_k of the corresponding polynomial $R_k(x)$ are found for $k = 3, \ldots, 100$, and also for $k = 200, 300, \ldots, 900$ and 996. We have observed empirically that the points (a_k, b_k) fit very well the parabola $\mathcal{B}$ with equation $b = \frac{1}{12} - \frac{17}{24}a^2$. Fig. 2 represents the right half of the moustache region described in Section 3 together with the aforementioned points (a_k, b_k) and the parabola $\mathcal{B}$.

It seems that the corresponding points converge (very slowly) to $(0, 1/12)$. Needless to say, it would be of interest to see if such a convergence holds indeed. Fig. 3 represents a detailed part of the previous one where only the coefficients (a_k, b_k) are shown for $k = 300, 400, \ldots, 900$ and 996.

Acknowledgment

The authors thank the referee whose suggestions helped us to improve the presentation of the paper.

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