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On radial solutions of inhomogeneous nonlinear scalar field equations

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ABSTRACT

We study the existence of radially symmetric solutions $u \in H^1(\Omega)$ of the following nonlinear scalar field equation $-\Delta u = g(|x|, u)$ in Ω. Here $\Omega = \mathbf{R}^N$ or $\{x \in \mathbf{R}^N \mid |x| > R\}$, $N \geqslant 2$. We generalize the results of Li and Li (1993) [13] and Li (1990) [14] in which they studied the problem in $\mathbf{R}^N$ and $\{|x| > R\}$ with the Dirichlet boundary condition. Furthermore, we extend it to the Neumann boundary problem and we also consider the nonlinear Schrödinger equation that is the case $g(r,s) = -V(r)s + \tilde{g}(s)$.

1. Introduction

In this paper, we are concerned with the following nonlinear scalar field equation:

$$\begin{cases} -\Delta u = g\big(|x|, u\big) & \text{in } \Omega, \\ u \in H^1(\Omega). \end{cases} \tag{1}$$

Here $\Omega \subset \mathbf{R}^N$ is either the whole space $\Omega = \mathbf{R}^N$ or the exterior domain of the ball $B_R(0)$ with radius $R > 0$ ($\Omega = \{x \in \mathbf{R}^N \mid |x| > R\}$) and the function $g(r,s) : [R, \infty) \times \mathbf{R} \to \mathbf{R}$ is continuous in both variables and odd with respect to $s \in \mathbf{R}$. In the case where Ω is the exterior domain, we consider (1) under the homogeneous Dirichlet or Neumann boundary condition:

$$u = 0 \quad \text{on } \partial\Omega \quad \text{or} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where ν is the outward normal vector of $\partial\Omega$. Namely, we consider the following equations:

$$-\Delta u = g\big(|x|, u\big) \quad \text{in } \mathbf{R}^N,\ u \in H^1\big(\mathbf{R}^N\big), \tag{2a}$$

$$-\Delta u = g\big(|x|, u\big) \quad \text{in } \big\{|x| > R\big\}, \qquad u = 0 \quad \text{on } |x| = R,\ u \in H^1\big(\big\{|x| > R\big\}\big), \tag{2b}$$

$$-\Delta u = g\big(|x|, u\big) \quad \text{in } \big\{|x| > R\big\}, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } |x| = R,\ u \in H^1\big(\big\{|x| > R\big\}\big). \tag{2c}$$

When $\Omega = \mathbf{R}^N$ and $g(r,s)$ does not depend on r, that is $g(r,s) = g(s)$, (2a) has been studied by many researchers. For example, we refer to [3–7,9,11,18] and references therein.

On the other hand, when $g(r,s)$ depends on r in a monotone decreasing way, Li and Li [13] and Li [14] studied (2a) and (2b). They showed the existence of a positive radial solution and infinitely many possibly sign-changing radial solutions for a suitable class of nonlinearities (see Remark 2.3 for a precise statement).

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One of the aims of this paper is to deal with the Neumann boundary problem (2c) as well as (2a) and (2b), and give a generalization of the results of [13,14]. Especially, we relax the condition on the behavior of $g(r,s)$ near $s=0$. In [13,14], the authors assumed $\lim_{s\to 0} g(r,s)/s = -1$ uniformly with respect to r (see Remark 2.3). However, our main results (Theorems 2.1 and 2.2 below) enable us to deal with the following case: $-\infty < \liminf_{s\to 0}\inf_{r\geqslant R} g(r,s)/s \leqslant \limsup_{s\to 0}\sup_{r\geqslant R} g(r,s)/s < 0$. Therefore we can treat the following example: $-\Delta u = -(V(|x|) + a(|x|)\sin^2(1/u))u + b(|x|)f(u)$ in Ω where $V(r)$, $a(r)$, $b(r)$ are monotone functions and $f(s)$ is superlinear near $s=0$.

Another aim of this paper is to deal with nonlinear Schrödinger type problems without the monotonicity assumption on $g(r,s)$ with respect to r, namely the case $g(r,s) = -V(r)s + \tilde{g}(s)$.

When $\Omega = \mathbf{R}^N$, Azzollini and Pomponio [2] studied (2a) and obtained the existence of at least one positive radial solution. See also Remark 2.5. We give an extension of their result to the exterior problems (2b) and (2c). Moreover, we show the existence of infinitely many solutions. See Theorem 2.4 for a precise statement.

We will prove our theorems by variational methods and use the monotonicity method due to Struwe [19], and developed by Jeanjean [10] and Rabier [16]. With the monotonicity method, a newly developed Pohozaev type inequality (see Propositions 5.5 and 5.7) will play important roles in our argument.

This paper is organized as follows. We state our main results in Section 2. In Section 3, we introduce an auxiliary functional J and prepare some lemmas. Proofs of lemmas in Section 3 will be given in Appendix A. In Section 4, we define minimax values based on the symmetric mountain pass arguments. Section 5 is devoted to proving Theorems 2.1, 2.2, 2.4. We shall state some open problems in Section 6 and we prove some lemmas in Appendix A.

2. Statement of main results

In this section, we state our main results of this paper.

2.1. Results for Eqs. (2a)–(2c)

First we consider Eq. (1). We assume that $g(r,s) : [R,\infty)\times\mathbf{R}\to\mathbf{R}$ satisfies the following conditions. In what follows, we regard $R=0$ if $\Omega=\mathbf{R}^N$.

$$g \in C\big([R,\infty)\times\mathbf{R},\mathbf{R}\big) \quad\text{and}\quad g(r,-s) = -g(r,s). \tag{3a}$$

$$\text{If } R \leqslant r_1 \leqslant r_2 < \infty \text{ and } s \geqslant 0, \quad \text{then } g(r_1,s) \leqslant g(r_2,s). \tag{3b}$$

$$g(r,s) \to g_\infty(s) \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbf{R}) \text{ as } r\to\infty. \tag{3c}$$

$$\text{There exists an } m_1 > 0 \text{ such that } \infty < \liminf_{s\to 0}\inf_{r\geqslant R}\frac{g(r,s)}{s} \leqslant \limsup_{s\to 0}\sup_{r\geqslant R}\frac{g(r,s)}{s} \leqslant -m_1. \tag{3d}$$

$$\begin{cases} \text{(i) } (N=2) \quad \displaystyle\lim_{s\to\infty}\sup_{r\geqslant R}\frac{|g(r,s)|}{\exp(\alpha s^2)} = 0 \quad \text{for any } \alpha > 0, \\ \text{(ii) } (N\geqslant 3) \quad \displaystyle\lim_{s\to\infty}\sup_{r\geqslant R}\frac{|g(r,s)|}{s^{2^*-1}} = 0 \quad \text{where } 2^* = 2N/(N-2). \end{cases} \tag{3e}$$

$$\text{There exist } \zeta_0 > 0,\ R_0 \geqslant R \text{ such that } \inf_{r\geqslant R_0} G(r,\zeta_0) > 0 \text{ where } G(r,s) = \int_0^s g(r,\tau)\,d\tau. \tag{3f}$$

Except for (3c) and (3d), the above conditions are the same as the ones in [13,14]. As for (3d), this type of condition is used in [4,5,9,18] when $g(r,s)$ does not depend on r, i.e., $g(r,s) = g(s)$ (cf. (4b) below). We remark that in [13,14], the authors suppose $\lim_{s\to 0} g(r,s)/s = -1$ uniformly with respect to r, which is stronger than (3d).

For the Neumann problem (2c), in addition to (3a)–(3f), we assume

$$-\infty < \inf_{s\in\mathbf{R}} G(R,s). \tag{3g}$$

Our main results are as follows. First we state a result for (2c).

Theorem 2.1. *Suppose that* $\Omega = \{|x| > R\}$ *and* (3a)–(3g) *are satisfied. Then* (2c) *has at least one positive radial solution and infinitely many possibly sign-changing radial solutions.*

For (2a) and (2b), we assume (3a)–(3f) and we do not need (3g).

Theorem 2.2. *Suppose that* $\Omega = \mathbf{R}^N$ (*resp.* $\Omega = \{|x| > R\}$) *and* (3a)–(3f) *are satisfied. Then* (2a) (*resp.* (2b)) *has at least one positive radial solution and infinitely many possibly sign-changing radial solutions.*

Remark 2.3. In [13,14], in addition to (3a), (3b), (3d)–(3f), the authors suppose that the function g has a form $g(r,s) = -s + f(r,s)$ where $f(r,s) = o(1)$ uniformly with respect to r as $s \to 0$ (cf. (3d)). Under these conditions, they proved the existence of one positive radial solution and infinitely many possibly sign-changing radial solutions to (2a) and (2b). However Theorem 2.2 enables us to deal with the following type of equations: $-\Delta u = -(V(|x|) + a(|x|)\sin^2(1/u))u + b(|x|)f(u)$ where V, a, b are monotone functions and $f(s)$ is superlinear near $s = 0$.

2.2. Results for the equation of Schrödinger type

Next we consider (1) with $g(r,s) = -V(r)s + \tilde{g}(s)$ for $N \geqslant 3$ and assume the following conditions:

$$\tilde{g} \in C(\mathbf{R}, \mathbf{R}), \quad \tilde{g}(-s) = -\tilde{g}(s). \tag{4a}$$

$$\text{There exists } \tilde{m}_1 > 0 \text{ such that } -\infty < \liminf_{s\to 0} \frac{\tilde{g}(s)}{s} \leqslant \limsup_{s\to 0} \frac{\tilde{g}(s)}{s} \leqslant -\tilde{m}_1. \tag{4b}$$

$$\limsup_{s\to\infty} \frac{\tilde{g}(s)}{s^{2^*-1}} \leqslant 0. \tag{4c}$$

$$\text{There exists a } \tilde{\zeta}_0 > 0 \text{ such that } \tilde{G}(\tilde{\zeta}_0) > 0 \text{ where } G(s) = \int_0^s \tilde{g}(\tau)\,d\tau. \tag{4d}$$

$$-\infty < \inf_{s\in\mathbf{R}} \left(-\frac{1}{2} V(R)s^2 + \tilde{G}(s) \right). \tag{4e}$$

The conditions (4a)–(4d) are the same as the ones in [4,5,9]. The condition (4e) is corresponding to (3g) above and is only needed for (2c). For V, we assume the following:

$$V \in C^1\big([R,\infty)\big) \quad \text{and} \quad V(r) \geqslant 0 \quad \text{for all } r \geqslant R, \tag{5a}$$

$$\lim_{r\to\infty} V(r) = 0, \tag{5b}$$

$$\big\| \big(x \cdot \nabla V(|x|)\big)^+ \big\|_{L^{\frac{N}{2}}(|x|>R)} < 2S_N \tag{5c}$$

where

$$\big(x \cdot \nabla V(|x|)\big)^+ = \max\big\{0, x \cdot \nabla V(|x|)\big\} \quad \text{and} \quad S_N = \inf_{u\in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\mathbf{R}^N)}}{\|u\|^2_{L^{2^*}(\mathbf{R}^N)}}.$$

When $\Omega = \mathbf{R}^N$, the above conditions (4a)–(4d), (5a)–(5c) are the same as the ones in [2]. Next we give a remark about (5c). If $g(r,s) = -V(r)s + \tilde{g}(s)$ satisfies (3b), then we can see $x \cdot \nabla V(|x|) \leqslant 0$, which implies (5c). Therefore, we can relax the monotonicity condition (3b) by (5c) for the nonlinear Schrödigner type equation.

Now we state a result for the equation of Schrödinger type.

Theorem 2.4. *Suppose that $N \geqslant 3$ and $g(r,s) = -V(r)s + \tilde{g}(s)$ satisfies* (4a)–(4d) *and* (5a)–(5c)*. Then the following hold*:

(i) (2a) (*resp.* (2b)) *admits at least one positive radial solution and infinitely many possibly sign-changing radial solutions.*
(ii) *Assume* (4e) *in addition to* (4a)–(4d) *and* (5a)–(5c)*. Then* (2c) *admits at least one positive radial solution and infinitely many possibly sign-changing radial solutions.*

Remark 2.5. In [2], the authors showed the existence of one positive radial solution to (2a) with $g(r,s) = -V(r)s + \tilde{g}(s)$ under the conditions (4a)–(4d) and (5a)–(5c).

In the following, we give an idea of proofs of Theorems 2.1, 2.2, 2.4.

We will prove Theorems 2.1, 2.2, 2.4 by variational methods and find critical points of

$$I(u) \equiv \frac{1}{2} \int_\Omega |\nabla u|^2\,dx - \int_\Omega G\big(|x|, u\big)\,dx.$$

One of difficulties is to show the boundedness of Palais–Smale (for short (PS)) sequences.

In [13,14], the authors introduced the following parametrized functional in order to obtain bounded (PS) sequences: (cf. Remark 2.3)

$$\hat{I}_\lambda(u) \equiv \frac{1}{2} \int_\Omega |\nabla u|^2 + u^2\,dx - \int_\Omega F\big(|x|, u\big)\,dx - \lambda \int_\Omega q\big(|x|\big)B(u)\,dx, \quad \lambda \in [0,1].$$

Here $F(r,s)=\int_0^s f(r,t)\,dt$, and $B(s)$ and $q(r)$ are suitable penalty functions. The virtue of their penalty functions is that $\hat{I}_\lambda$ satisfies the (PS) condition. However, the construction is rather complicated.

In our proofs, we consider another parametrized functional to obtain bounded (PS) sequences:

$$I_\lambda(u)\equiv\frac{1}{2}\int_\Omega|\nabla u|^2\,dx-\int_\Omega G\bigl(|x|,u\bigr)\,dx-\lambda\int_\Omega H(u)\,dx\quad\lambda\in[0,1].$$

Here $H(s)$ is also a penalty function which is different from $B(s)$ in $\hat{I}_\lambda$ and we can construct the function $H(s)$ in a simply way (see the definition of $H(s)$ in Section 3). To obtain critical points of I, we will apply the monotonicity method to I_λ. In this paper, we apply a version of Rabier [16] (see Propositions 5.1 and 5.2) and obtain sequences (λ_k), (u_k) such that

$$\lambda_k\to 0,\quad -\Delta u_k=g\bigl(|x|,u_k\bigr)+\lambda_k h(u_k)\quad\text{in }\Omega,$$

where $h(s)=H'(s)$. To show that (u_k) has a strongly convergent subsequence, we use the Pohozaev type inequality (15), (21), (22). Here we remark that in [13,14] the authors used the Pohozaev Identity (for instance, see (16), (23), (24)) which includes the term $x\cdot\nabla G(|x|,u)$ and they need to approximate $g(r,s)$ with a function of class C^1 in r. However, in this paper, we introduce a new Pohozaev type inequality, which enables us to argue without introducing approximations.

Our proofs can also be applied for the equation of Schrödinger type, namely $g(r,s)=-V(r)s+\tilde{g}(s)$ in (1). By virtue of our proofs of Theorems 2.1 and 2.2, we will be able to show that not only (2a) but also (2b) and (2c) admit at least one positive radial solution and infinitely many possibly sign-changing radial solutions under the conditions (4a)–(4d), (5a)–(5c) or (4a)–(4e), (5a)–(5c).

3. Preliminaries

In this section, we introduce an auxiliary functional J and state some lemmas. Proofs of lemmas in this section will be given in Appendix A.

First, we remark that when we consider (2b) or (2c) under the assumptions of Theorems 2.1, 2.2 or 2.4 we may assume $R=1$ without loss of generality. Indeed, set $\Omega=\{|x|>R\}$, $v(x)\equiv u(Rx)$ and $g_R(r,s)\equiv R^2g(Rr,s)$. Then (1) is equivalent to the following equation:

$$-\Delta v=g_R(r,v)\quad\text{in }\bigl\{|x|>1\bigr\}.$$

Moreover, it is easily seen that g satisfies (3a)–(3g) in $\{|x|>R\}$ if and only if g_R satisfies (3a)–(3g) in $\{|x|>1\}$. In the case where $g(r,s)=-V(r)s+\tilde{g}(s)$, set $V_R(r)\equiv R^2V(Rr)$ and $\tilde{g}_R(s)\equiv R^2\tilde{g}(s)$. Then it is also clear that $\tilde{g}$ and V satisfy (4a)–(4e), (5a)–(5c), in $\{|x|>R\}$ if and only if $\tilde{g}_R$ and V_R satisfy (4a)–(4e), (5a)–(5c) in $\{|x|>1\}$. Therefore to prove Theorems 2.1, 2.2 and 2.4, we may assume $R=1$ without loss of generality.

Hereafter we mainly consider (2c) and let $\Omega=\{x\in\mathbf{R}^N\mid|x|>1\}$. Furthermore we assume the following condition in this section:

$$\text{The conditions (3a) and (3c)–(3e) are satisfied.}\tag{6}$$

In order to obtain radial solutions, we consider the following function space:

$$E\equiv H^1_r(\Omega)=\bigl\{u\in H^1(\Omega)\bigm| u\text{ is a radial function}\bigr\}.$$

The following properties hold (for (i) and (ii), see Berestycki and Lions [4], Strauss [18]):

(i) *There exists a $C>0$ such that for all $u\in E$ and $|x|\geqslant 1$,*

$$\bigl|u(x)\bigr|\leqslant C|x|^{-\frac{N-1}{2}}\|u\|_{H^1(\Omega)}.\tag{7}$$

(ii) *The embedding $E\subset L^q(\Omega)$ is continuous for $2\leqslant q\leqslant 2^*$ if $N\geqslant 3$ and $2\leqslant q<\infty$ if $N=2$ and it is compact for $2<q<2^*$ if $N\geqslant 3$ and $2<q<\infty$ if $N=2$.*

(iii) *For each $s\in(0,1]$, we define the extension operator $T_s:H^1_r(\{|x|>s\})\to H^1_r(\mathbf{R}^N)$ by*

$$(T_su)(x)=(T_su)\bigl(|x|\bigr)=\begin{cases}u(|x|) & \text{if }|x|\geqslant s,\\ u(2s-|x|) & \text{if }|x|<s.\end{cases}\tag{8}$$

Then, for each $s\in(0,1]$ and $u\in H^1_r(\{|x|>s\})$, it holds that

$$\|T_su\|_{L^2(\mathbf{R}^N)}\leqslant\sqrt{2}\|u\|_{L^2(\{|x|>s\})},\qquad\|\nabla T_su\|_{L^2(\mathbf{R}^N)}\leqslant\sqrt{2}\|\nabla u\|_{L^2(\{|x|>s\})}.\tag{9}$$

Using (9), we have that the following Sobolev inequality holds for $N\geqslant 3$:

$$\|u\|_{L^{2^*}(\{|x|>s\})}\leqslant C\|\nabla u\|_{L^2(\{|x|>s\})}\quad\text{for all }u\in H^1_r\bigl(\bigl\{|x|>s\bigr\}\bigr),\ s\in(0,1].\tag{10}$$

We define the following functional:

$$I(u) \equiv \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G\big(|x|, u\big)\,dx : E \to \mathbf{R}.$$

We note that $I \in C^1(E, \mathbf{R})$ under the condition (6) and the functional I corresponds to (2c). So, we will find critical points of I.

We prepare a penalty function to construct an auxiliary functional. For $s \geqslant 0$, we define $f(s)$ and $h(s)$ as follows:

$$f(s) \equiv \max\left\{0, \frac{1}{2}m_1 s + \sup_{r\geqslant 1} g(r,s)\right\}, \qquad h(s) \equiv s^p \sup_{0<\tau\leqslant s} \frac{f(\tau)}{\tau^p}.$$

Here m_1 is a constant appearing in (3d) and p is a positive number satisfying $1 < p < (N+2)/(N-2)$ if $N \geqslant 3$ and $1 < p < \infty$ if $N = 2$. Note that by (3c) and (3d), f and h are well defined. We extend h as an odd function on $\mathbf{R}$ and set

$$H(s) \equiv \int_0^s h(t)\,dt.$$

Then h and H have the following properties.

Lemma 3.1. *(See Lemma 2.1 and Corollary 2.2 in [9].)*

(i) $h \in C(\mathbf{R})$, $0 \leqslant h(s)$ *and* $h(-s) = -h(s)$ *for all* $s \in [0, \infty)$.
(ii) *There exists an* $s_0 > 0$ *such that* $h = H = 0$ *on* $[-s_0, s_0]$.
(iii) *For all* $s \in \mathbf{R}$, *it follows that*

$$\frac{1}{2}m_1 s^2 + \sup_{r\geqslant 1} g(r,s)s \leqslant h(s)s, \qquad \frac{1}{4}m_1 s^2 + \sup_{r\geqslant 1} G(r,s) \leqslant H(s).$$

(iv) *The following hold*:

$$\lim_{s\to\infty} \frac{h(s)}{\exp(\alpha s^2)} = 0 \quad \text{for all } \alpha > 0 \text{ if } N = 2,$$
$$\lim_{s\to\infty} \frac{h(s)}{s^{2^*-1}} = 0 \quad \text{if } N \geqslant 3.$$

(v) *h satisfies a global Ambrosetti–Rabinowitz condition*:

$$0 \leqslant (p+1)H(s) \leqslant h(s)s \quad \text{for all } s \in \mathbf{R}.$$

Here p appears in the definition of h.

Next we rewrite the functional I as follows:

$$I(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \int_\Omega G\big(|x|, u\big)\,dx = \frac{1}{2}\|u\|_E^2 - \int_\Omega \frac{m_1}{4}u^2 + G\big(|x|, u\big)\,dx$$

where

$$\|u\|_E^2 \equiv \|\nabla u\|_{L^2}^2 + \frac{m_1}{2}\|u\|_{L^2}^2.$$

We remark that $\|\cdot\|_E$ and the standard H^1-norm are equivalent.

Next, we define a parametrized functional I_λ ($\lambda \in [0,1]$) and an auxiliary functional J which gives us lower bounds of minimax values $b_n(\lambda)$ defined in Section 4:

$$I_\lambda(u) \equiv \frac{1}{2}\|u\|_E^2 - \int_\Omega \frac{m_1}{4}u^2 + G\big(|x|, u\big) + \lambda H(u)\,dx \in C^1(E, \mathbf{R}),$$

$$J(u) \equiv \frac{1}{2}\|u\|_E^2 - 2\int_\Omega H(u)\,dx \in C^1(E, \mathbf{R}).$$

Note that if $\lambda = 0$, then $I_0(u) = I(u)$. Furthermore, by Lemma 3.1, I_λ and J satisfy the following: for any $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant 1$ and $u \in E$,

$$J(u) \leqslant I_1(u) \leqslant I_{\lambda_2}(u) \leqslant I_{\lambda_1}(u) \leqslant I_0(u). \tag{11}$$

Now we state properties of I_λ, J. Similar properties are obtained in [2,9].

Lemma 3.2. *(See Lemma 3.5 in [2], Lemmas 2.3, 2.5, Proposition 5.3 in [9].) Set $K(u) \equiv \int_\Omega H(u)\,dx$. Then,*

(i) *$K : E \to \mathbf{R}$ and $K' : E \to E^*$ are weakly continuous.*
(ii) *Any bounded* (PS) *sequence $(u_k) \subset E$ for I_λ has a strongly convergent subsequence.*
(iii) *J satisfies the* (PS) *condition.*

4. Minimax arguments

In this section, we define minimax values $b_n(\lambda)$ of I_λ based on the arguments of symmetric mountain pass theorem (cf. [9] and Rabinowitz [17]). In this section, we assume the following conditions:

$$\text{The conditions (3a) and (3c)–(3f) are satisfied.} \tag{12}$$

First of all, we prove that I_λ and J have a symmetric mountain pass geometry under the condition (12). More precisely, we have

Lemma 4.1.

(i) *There exist $\delta > 0$ and $\rho > 0$ such that*

$$0 < \delta \leqslant J(u) \quad \text{for } \|u\|_E = \rho, \qquad 0 \leqslant J(u) \quad \text{for } \|u\|_E \leqslant \rho.$$

(ii) *For each $n \in \mathbf{N}$, there exists an odd continuous map $\gamma_n : S^{n-1} = \{\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbf{R}^n \mid |\sigma| = 1\} \to H^1_{0,r}(\Omega)$ such that*

$$I_0\big(\gamma_n(\sigma)\big) < 0 \quad \text{for all } \sigma \in S^{n-1}.$$

Here $H^1_{0,r}(\Omega) = \{u \in E \mid u(1) = 0\}$.

Remark 4.2. By (11), we see that I_λ and J have a symmetric mountain pass geometry.

Proof of Lemma 4.1. We only prove (i). (ii) will be proven in Appendix A.
First, we show for $N \geqslant 3$. By Lemma 3.1, there exists a $C > 0$ such that

$$H(s) \leqslant C|s|^{2^*} \quad \text{for all } s \in \mathbf{R}.$$

Using Sobolev's embedding, we obtain

$$J(u) \geqslant \|u\|_E^2 - C\|u\|_{L^{2^*}(\Omega)}^{2^*} \geqslant \|u\|_E^2\big(1 - C\|u\|_E^{2^*-2}\big).$$

Thus (i) holds for $N \geqslant 3$.
Next we consider the case $N = 2$. By Lemma 3.1, there exists a $C_1 > 0$ such that

$$H(s) \leqslant C_1\Phi\big(s^2/2\big) \quad \text{where } \Phi(s) = \exp(s) - 1 - s.$$

By Lemma A.2(iii), we have

$$\int_\Omega H(u)\,dx \leqslant C_2\|u\|_E^4 \quad \text{for all } u \in E \text{ with } \|u\|_E \leqslant 1.$$

Thus it follows that if $\|u\|_E \leqslant 1$, then

$$J(u) \geqslant \|u\|_E^2 - C_2\|u\|_E^4,$$

which completes the proof of (i). □

Next, we define minimax values of I_λ and J using mappings (γ_n) in Lemma 4.1.

Definition 4.3. For each $n \in \mathbf{N}$ and $\lambda \in [0, 1]$, we define $b_n(\lambda)$ and c_n as follows:

$$b_n(\lambda) \equiv \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} I_\lambda\big(\gamma(\sigma)\big), \qquad c_n \equiv \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} J\big(\gamma(\sigma)\big),$$

where $D_n = \{\sigma \in \mathbf{R}^n \mid |\sigma| \leqslant 1\}$ and

$$\Gamma_n \equiv \big\{\gamma \in C(D_n, E) \;\big|\; \gamma \text{ is odd and } \gamma = \gamma_n \text{ on } S^{n-1}\big\}.$$

The values $b_n(\lambda)$ and c_n have the following properties.

Lemma 4.4.

(i) *$\Gamma_n \neq \emptyset$ for each $n \in \mathbf{N}$.*
(ii) *$b_n(\lambda_2) \leqslant b_n(\lambda_1)$ for each $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant 1$.*
(iii) *$0 < \delta \leqslant c_n \leqslant b_n(\lambda)$ for each $n \in \mathbf{N}$ and $\lambda \in [0, 1]$, where δ appears in Lemma 4.1(i).*

Proof. (i) We define $\tilde{\gamma}_n$ as follows: for $\sigma \in D^n$, $\tilde{\gamma}_n(\sigma) = |\sigma|\gamma_n(\sigma/|\sigma|)$. Then $\tilde{\gamma}_n \in \Gamma_n$.

(ii) By (11), (ii) holds.

(iii) By (11) and (i), it holds $c_n \leqslant b_n(\lambda)$ for each $\lambda \in [0, 1]$. The property $\delta \leqslant c_n$ follows from the fact

$$\left\{u \in E \,\middle|\, \|u\|_E = \rho\right\} \cap \gamma(D_n) \neq \emptyset \quad \text{for all } \gamma \in \Gamma_n. \qquad \square$$

Since J satisfies the (PS) condition by Lemma 3.2, we can show the following lemma as in [9].

Lemma 4.5. *(See Lemma 3.2 in [9].)*

(i) *The value c_n is a critical value of J.*
(ii) *As $n \to \infty$, $c_n \to \infty$.*

5. Proofs of Theorems 2.1, 2.2 and 2.4

In this section, we prove Theorems 2.1, 2.2 and 2.4 by using the monotonicity method and the Pohozaev type inequality (Propositions 5.5 and 5.7).

5.1. Monotonicity method

First, we will recall Rabier's result [16]. Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{A} : X \to \mathbf{R}$, $\mathcal{B} : [0, 1] \times X \to \mathbf{R}$ be C^1 functionals and set $\mathcal{I}_\lambda(u) \equiv \mathcal{A}(u) - \mathcal{B}(\lambda, u)$. We assume that $\mathcal{A}$ and $\mathcal{B}$ satisfy the following:

$$\mathcal{B}(\cdot, u) \text{ is nondecreasing on } [0, 1] \text{ for every } u \in X, \tag{13a}$$

$$\lim_{\mathcal{B}(\lambda,u)\to\infty} \frac{\partial \mathcal{B}}{\partial \lambda}(\lambda, u) = \infty, \tag{13b}$$

$$\lim_{\|u\|\to\infty} \mathcal{A}(u) = \infty. \tag{13c}$$

Moreover, we suppose that there exist $e_1, e_2 \in X$ such that

$$\max\left\{\mathcal{I}_\lambda(e_1), \mathcal{I}_\lambda(e_2)\right\} < c_\lambda \quad \text{for all } \lambda \in [0, 1]. \tag{13d}$$

Here

$$c_\lambda \equiv \inf_{\gamma \in \Gamma^*} \max_{0 \leqslant t \leqslant 1} \mathcal{I}_\lambda\big(\gamma(t)\big) \quad \text{and} \quad \Gamma^* \equiv \left\{\gamma \in C\big([0, 1], X\big) \,\middle|\, \gamma(0) = e_1,\ \gamma(1) = e_2\right\}.$$

Then the following proposition holds.

Proposition 5.1. *(See Rabier [16].) Under the conditions* (13a)–(13d)*, for almost every $\lambda \in [0, 1]$, $\mathcal{I}_\lambda$ has a bounded* (PS) *sequence at level c_λ.*

We will apply the above proposition for the functional which satisfies the symmetric mountain pass structure. Assume the following conditions in addition to (13a)–(13c):

$$A(-u) = A(u) \quad \text{and} \quad B(\lambda, -u) = B(\lambda, u) \quad \text{for all } u \in X \text{ and } \lambda \in [0, 1]. \tag{13e}$$

$$\begin{cases} \text{For all } n \in \mathbf{N}, \text{ there exists an odd map } \gamma_n^* \in C(S^{n-1}, E) \text{ such that } \max_{\sigma \in S^{n-1}} \mathcal{I}_\lambda(\gamma_n^*(\sigma)) < d_n(\lambda) \\ \quad \text{where } d_n(\lambda) \equiv \inf_{\gamma \in \Gamma_n^*} \max_{\sigma \in D^n} \mathcal{I}_\lambda(\gamma(\sigma)) \text{ and } \Gamma_n^* \equiv \{\gamma \in C(D_n, E) \mid \gamma \text{ is odd and } \gamma = \gamma_n^* \text{ on } S^{n-1}\}. \end{cases} \tag{13f}$$

The following proposition holds from the arguments in [16].

Proposition 5.2. *Suppose* (13a)–(13c) *and* (13e)–(13f)*. Then, for almost every $\lambda \in [0, 1]$, there exists a bounded* (PS) *sequence of $\mathcal{I}_\lambda$ at level $d_n(\lambda)$ for all $n \in \mathbf{N}$.*

Next, we show that we can apply Proposition 5.2 for I_λ to obtain a bounded (PS) sequence of I_λ.

Lemma 5.3. *Under the assumption* (12)*, for almost every* $\lambda \in [0,1]$*,* I_λ *has a bounded* (PS) *sequence at level* $b_n(\lambda)$ *for all* $n \in \mathbf{N}$*.*

Proof. Set $X = E$, $\gamma_n^* = \gamma_n$,

$$\mathcal{A}(u) = \frac{1}{2}\|u\|_E^2, \qquad \mathcal{B}(\lambda,u) = \int_\Omega \frac{m_1}{4}u^2 + G\big(|x|,u\big) + \lambda H(u)\,dx.$$

It is easily seen that (13a), (13c) and (13e) are satisfied. Moreover, by Lemmas 4.1 and 4.4, (13f) holds. As to (13b), by Lemma 3.1, we have

$$\mathcal{B}(\lambda,u) \leqslant (1+\lambda)\int_\Omega H(u)\,dx.$$

On the other hand, it follows that

$$\frac{\partial \mathcal{B}}{\partial \lambda}(\lambda,u) = \int_\Omega H(u)\,dx,$$

which implies (13b). Then by Proposition 5.2, for almost every $\lambda \in [0,1]$, I_λ has a bounded (PS) sequence at level $b_n(\lambda)$ for all $n \in \mathbf{N}$. $\square$

Combining Lemmas 3.2 and 5.3, we have the following:

Proposition 5.4. *Suppose that* (12) *is satisfied. Then for almost every* $\lambda \in (0,1]$*, there is a critical point* $u_{\lambda,n} \in E$ *such that* $I_\lambda(u_{\lambda,n}) = b_n(\lambda)$ *for all* $n \in \mathbf{N}$*.*

From Proposition 5.4, it follows that for each $n \in \mathbf{N}$, there exist $(\lambda_{n,k}) \subset [0,1]$, $(u_{n,k}) \subset E$ such that $\lambda_{n,k} \to 0$ and

$$I_{\lambda_{n,k}}(u_{n,k}) = b_n(\lambda_{n,k}), \qquad I'_{\lambda_{n,k}}(u_{n,k}) = 0. \tag{14}$$

5.2. Pohozaev type inequality

To show that $(u_{n,k})$ in (14) is bounded, we introduce the following Pohozaev type inequality.

Proposition 5.5. *Assume that the conditions* (3a)–(3f) *are satisfied. Let* $u_N \in E$ *be a solution of*

$$-\Delta u = g\big(|x|,u\big) + \lambda h(u) \quad \text{in } \Omega, \qquad \frac{\partial u_N}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where ν *is the outward normal vector of* $\partial\Omega$*. Then* u_N *satisfies the following*:

$$\frac{N-2}{2}\|\nabla u_N\|_{L^2}^2 - N\int_\Omega \hat{G}_\lambda\big(|x|,u_N\big)\,dx \geqslant \int_{\partial\Omega} \hat{G}_\lambda\big(|x|,u_N\big)\,dS. \tag{15}$$

Here $\hat{G}_\lambda(|x|,s) \equiv G(|x|,s) + \lambda H(s)$.

Remark 5.6. If we suppose that g is of class C^1 with respect to r in addition to (3a)–(3f), then the Pohozaev Identity holds:

$$\frac{N-2}{2}\|\nabla u_N\|_{L^2(\Omega)}^2 - N\int_\Omega \hat{G}_\lambda\big(|x|,u_N\big)\,dx = \int_\Omega x\cdot\nabla G\big(|x|,u_N\big)\,dx + \int_{\partial\Omega} \hat{G}_\lambda\big(|x|,u_N\big)\,dS. \tag{16}$$

Thus from (16) and (3b), we can see that (15) holds. For a proof of (16), see the end of a proof below or Lemma 1.4 in Chapter III of Struwe [20].

Proof of Proposition 5.5. Note that under the conditions (3a)–(3f), u_N has an exponential decay:

$$\big|u_N(r)\big| + \big|u_N'(r)\big| + \big|u_N''(r)\big| \leqslant C_1\exp(-C_2 r) \quad \text{for all } r \geqslant 1.$$

Therefore $x\cdot\nabla u_N \in H^1(\Omega)$ and the curve $\eta(t) \equiv u_N(tx) : [1,2] \to H^1(\Omega)$ is of class C^1. Since $I'_\lambda(u_N) = 0$, we have

$$\frac{d}{dt}I_\lambda\big(\eta(t)\big)\Big|_{t=1} = I'_\lambda\big(u_N(x)\big)\big(x\cdot\nabla u_N(x)\big) = 0. \tag{17}$$

On the other hand, it holds that

$$I_\lambda\big(\eta(t)\big) = \frac{t^{-N+2}}{2} \int_{|x|\geqslant t} \big|\nabla u_N(x)\big|^2\,dx - t^{-N} \int_{|x|\geqslant t} G\left(\frac{|x|}{t}, u_N\big(|x|\big)\right) + \lambda H\big(u_N\big(|x|\big)\big)\,dx. \tag{18}$$

By (3b), it follows that

$$I_\lambda\big(\eta(t)\big) \geqslant \hat{I}_\lambda(t) \equiv \frac{t^{-N+2}}{2} \int_{|x|\geqslant t} \big|\nabla u_N(x)\big|^2\,dx - t^{-N} \int_{|x|\geqslant t} G\big(|x|, u_N\big(|x|\big)\big) + \lambda H\big(u_N\big(|x|\big)\big)\,dx. \tag{19}$$

Noting that $I(\eta(1)) = \hat{I}_\lambda(1)$, from (19), we infer

$$\frac{I_\lambda(\eta(t)) - I_\lambda(\eta(1))}{t-1} \geqslant \frac{\hat{I}_\lambda(t) - \hat{I}_\lambda(1)}{t-1} \quad \text{for all } t \in (1,2]. \tag{20}$$

By (17),

$$\frac{I_\lambda(\eta(t)) - I_\lambda(\eta(1))}{t-1} \to 0 \quad \text{as } t \to 1+0.$$

On the other hand, since $\partial u_N/\partial\nu = 0$ on $\partial\Omega$, it is easily seen that as $t \to 1+0$,

$$\frac{\hat{I}_\lambda(t) - \hat{I}_\lambda(1)}{t-1} \to -\frac{N-2}{2}\|\nabla u_N\|_{L^2}^2 + N\int_\Omega G\big(|x|, u_N\big) + \lambda H(u_N)\,dx + \int_{\partial\Omega} G\big(|x|, u_N\big) + \lambda H(u_N)\,dS.$$

Thus, from (20), we conclude that

$$\int_{\partial\Omega} G\big(|x|, u_N\big) + \lambda H(u_N)\,dS \leqslant \frac{N-2}{2}\|\nabla u_N\|_{L^2}^2 - N\int_\Omega G\big(|x|, u_N\big) + \lambda H(u_N)\,dx.$$

If $g(r,s)$ is of class C^1 in r, then we note that the right hand side of (18) below is differentiable with respect to t. Combining (17), we can obtain (16). □

Here we also state the Pohozaev type inequality for (2a) and (2b).

Proposition 5.7. *Assume that* (3a)–(3f) *are satisfied. Let $u_D \in H^1_{0,r}(\Omega)$ (resp. $u_{\mathbf{R}^N} \in H^1_r(\mathbf{R}^N)$) be a solution of*

$$-\Delta u = g\big(|x|, u\big) + \lambda h(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \qquad \big(\text{resp. } -\Delta u = g\big(|x|, u\big) + \lambda h(u) \quad \text{in } \mathbf{R}^N\big).$$

Then u_D (resp. $u_{\mathbf{R}^N} \in H^1_r(\mathbf{R}^N)$) satisfies the following:

$$\frac{N-2}{2}\|\nabla u_D\|_{L^2(\Omega)}^2 - N\int_\Omega \hat{G}_\lambda\big(|x|, u_D\big)\,dx \geqslant \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u_D}{\partial\nu}\right)^2 dS \tag{21}$$

$$\left(\text{resp. } \frac{N-2}{2}\|\nabla u_{\mathbf{R}^N}\|_{L^2(\mathbf{R}^N)}^2 - N\int_{\mathbf{R}^N} \hat{G}_\lambda\big(|x|, u_{\mathbf{R}^N}\big)\,dx \geqslant 0\right). \tag{22}$$

Remark 5.8. As in Remark 5.6, if $g(r,s)$ is of class C^1 with respect to r, then the following Pohozaev identity holds:

$$\frac{N-2}{2}\|\nabla u_D\|_{L^2}^2 - N\int_\Omega \hat{G}_\lambda\big(|x|, u_D\big)\,dx = \int_\Omega x\cdot\nabla G\big(|x|, u_D\big)\,dx + \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u_D}{\partial\nu}\right)^2 dS \tag{23}$$

$$\left(\text{resp. } \frac{N-2}{2}\|\nabla u_{\mathbf{R}^N}\|_{L^2(\mathbf{R}^N)}^2 - N\int_{\mathbf{R}^N} \hat{G}_\lambda\big(|x|, u_{\mathbf{R}^N}\big)\,dx = \int_{\mathbf{R}^N} x\cdot\nabla G\big(|x|, u_{\mathbf{R}^N}\big)\,dx\right). \tag{24}$$

By (3b), we can show (21) and (22) from (23) and (24).

Proof of Proposition 5.7. We only show for u_D since a proof for $u_{\mathbf{R}^N}$ is similar to the one of Proposition 5.5.

For the Dirichlet problem, critical points of $I_\lambda \in C^1(H_0^1(\Omega), \mathbf{R})$ correspond to solutions. However, for technical reasons, we regard $I_\lambda \in C^1(H^1(\Omega), \mathbf{R})$ in this proof. We set $\tilde{\eta}(t) \equiv u_D(tx) \in C^1([1,2], H^1(\Omega))$ and as in the proof of Proposition 5.5, we shall calculate

$$\frac{d}{dt} I_\lambda\big(\tilde{\eta}(t)\big)\big|_{t=1}.$$

Since u_D satisfies $-\Delta u_D = g(|x|, u_D) + \lambda h(u_D)$ in Ω, $u_D = 0$ on $\partial\Omega$, using integration by parts, for any $\varphi \in H^1(\Omega) \cap C^1(\overline{\Omega})$, we have

$$I'_\lambda(u_D)\varphi = \int_\Omega \nabla u_D \cdot \nabla\varphi\,dx - \int_\Omega \big(g\big(|x|, u_D\big) + \lambda h(u_D)\big)\varphi\,dx = -\int_{\partial\Omega} \nabla u_D \cdot x\varphi\,dS.$$

Noting $\tilde{\eta}'(1) = x \cdot \nabla u_D(x) \in H^1(\Omega) \cap C^1(\overline{\Omega})$, it follows that

$$\frac{d}{dt} I_\lambda\big(\tilde{\eta}(t)\big)\big|_{t=1} = -\int_{\partial\Omega} \left(\frac{\partial u_D}{\partial\nu}\right)^2 dS. \tag{25}$$

On the other hand, set

$$\tilde{I}_\lambda(t) \equiv \frac{t^{-N+2}}{2} \int_{|x|\geqslant t} \big|\nabla u_D(x)\big|^2\,dx - t^{-N} \int_{|x|\geqslant t} G\big(|x|, u_D\big) + \lambda H(u_D)\,dx,$$

then we have

$$\tilde{I}'_\lambda(1) = -\frac{N-2}{2}\|\nabla u_D\|_{L^2}^2 - \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u_D}{\partial\nu}\right)^2 dS + N\int_\Omega G\big(|x|, u_D\big) + \lambda H(u_D)\,dx. \tag{26}$$

Since $I(\tilde{\eta}(t)) \geqslant \tilde{I}_\lambda(t)$ and $I(\tilde{\eta}(1)) = \tilde{I}_\lambda(1)$, by (25) and (26), it follows that

$$\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u_D}{\partial\nu}\right)^2 dS \leqslant \frac{N-2}{2}\|\nabla u_D\|_{L^2}^2 - N\int_\Omega G\big(|x|, u_D\big) + \lambda H(u_D)\,dx. \qquad \square$$

5.3. Proof of Theorem 2.1

Now we prove Theorem 2.1. Suppose that the conditions (3a)–(3g) are satisfied. Let $(u_{n,k})$ be a sequence satisfying (14) and set

$$b_{n,0} \equiv \lim_{\lambda\to 0} b_n(\lambda) = \lim_{k\to\infty} I_{\lambda_{n,k}}(u_{n,k}) \in \big[b_n(1), b_n(0)\big].$$

Proposition 5.9. *There exists a $C_n > 0$ such that $\|u_{n,k}\|_E \leqslant C_n$ for all $k \in \mathbf{N}$.*

Proof. First, we prove that $(\nabla u_{n,k})_{k=1}^\infty$ is bounded in $L^2(\Omega)$. Since $I'_{\lambda_{n,k}}(u_{n,k}) = 0$, by Proposition 5.5, we have

$$-\int_\Omega G\big(|x|, u_{n,k}\big) + \lambda_{n,k}H(u_{n,k})\,dx \geqslant -\frac{N-2}{2N}\|\nabla u_{n,k}\|_{L^2(\Omega)}^2 + \frac{1}{N}\int_{\partial\Omega} G(1, u_{n,k}) + \lambda_{n,k}H(u_{n,k})\,dx. \tag{27}$$

From (27), we obtain

$$\begin{aligned} b_n(\lambda_{n,k}) &= \frac{1}{2}\|\nabla u_{n,k}\|_{L^2(\Omega)}^2 - \int_\Omega G\big(|x|, u_{n,k}\big) + \lambda_{n,k}H(u_{n,k})\,dx \\ &\geqslant \frac{1}{N}\|\nabla u_{n,k}\|_{L^2(\Omega)}^2 + \frac{1}{N}\int_{\partial\Omega} G(1, u_{n,k}) + \lambda_{n,k}H(u_{n,k})\,dx. \end{aligned} \tag{28}$$

Noting that $H(s) \geqslant 0$ for all $s \in \mathbf{R}$, $\lim_{k\to\infty} b_n(\lambda_{n,k}) = b_{n,0} \leqslant b_n(0)$ and (3g), we deduce from (28) that there exists a $C_n > 0$ such that $\|\nabla u_{n,k}\|_{L^2(\Omega)} \leqslant C_n$ for all $k \in \mathbf{N}$.

Next, we show $\|u_{n,k}\|_E \leqslant C_n$ for all $k \in \mathbf{N}$. First, we consider the case $N \geqslant 3$. By Lemma 3.1, it holds

$$b_n(\lambda_{n,k}) = I_{\lambda_{n,k}}(u_{n,k}) = \frac{1}{2}\|u_{n,k}\|_E^2 - \int_\Omega \frac{m_1}{4}u_{n,k}^2 + G\big(|x|, u_{n,k}\big) + \lambda_{n,k}H(u_{n,k})\,dx$$
$$\geqslant \frac{1}{2}\|u_{n,k}\|_E^2 - (1+\lambda_{n,k})\int_\Omega H(u_k)\,dx \geqslant \frac{1}{2}\|u_{n,k}\|_E^2 - C\|u_{n,k}\|_{L^{2^*}(\Omega)}^{2^*}. \tag{29}$$

From (10) and (29), it holds that

$$b_n(\lambda_{n,k}) \geqslant \frac{1}{2}\|u_{n,k}\|_E^2 - C\|\nabla u_{n,k}\|_{L^2(\Omega)}^{2^*}. \tag{30}$$

Since $b_n(\lambda_{n,k})$ and $(\|\nabla u_{n,k}\|_{L^2(\Omega)})_{k=1}^\infty$ are bounded, taking C_n sufficiently large, $\|u_{n,k}\|_E \leqslant C_n$ follows from (30).

Next we consider the case $N = 2$. Following the arguments in [13] (cf. proof of Proposition 5.5 in Jeanjean and Tanaka [12]), we prove indirectly. Assume that $r_k \equiv \|u_{n,k}\|_{L^2(\Omega)}^{-1} \to 0$. Set

$$v_k(x) \equiv (T_{r_k}\tilde{v}_k)(x), \qquad \tilde{v}_k(x) \equiv u_{n,k}\left(\frac{x}{r_k}\right), \qquad \Omega_k \equiv \big\{x \bigm| |x| > r_k\big\},$$

where T_{r_k} defined by (8). From $\|\nabla\tilde{v}_k\|_{L^2(\Omega_k)} = \|\nabla u_k\|_{L^2(\Omega)}$, $\|\tilde{v}_k\|_{L^2(\Omega_k)} = 1$ and (9), (v_k) is bounded in $H^1(\mathbf{R}^2)$. Therefore, we may assume

$$v_k \rightharpoonup v_0 \quad \text{weakly in } H^1\big(\mathbf{R}^2\big) \quad \text{and} \quad v_k(x) \to v_0(x) \quad \text{a.a. } x \in \mathbf{R}^2.$$

Next, we show $v_0 = 0$. We remark that since $v_k(x) = \tilde{v}_k(x)$ in Ω_k, v_k satisfies

$$\begin{cases} -r_k^2\Delta v_k = g\left(\dfrac{|x|}{r_k}, v_k\right) + \lambda_{n,k}h(v_k) & \text{in } \Omega_k, \\ v_k'(r_k) = 0. \end{cases} \tag{31}$$

By the boundedness of (v_k) in $H^1(\mathbf{R}^2)$, for any $\varphi \in C_0^\infty(\mathbf{R}^2)$ with $\operatorname{supp}\varphi \subset \mathbf{R}^2\backslash\{0\}$, we can show

$$\int_{\Omega_k} h(v_k)\varphi\,dx \to \int_{\mathbf{R}^2} h(v_0)\varphi\,dx, \qquad \int_{\Omega_k} g\left(\frac{|x|}{r_k}, v_k\right)\varphi\,dx \to \int_{\mathbf{R}^2} g_\infty(v_0)\varphi\,dx. \tag{32}$$

By (31) and (32), we obtain

$$\int_{\mathbf{R}^2} g_\infty(v_0)\varphi\,dx = 0 \quad \text{for any } \varphi \in C_0^\infty\big(\mathbf{R}^2\big) \text{ with } \operatorname{supp}\varphi \subset \mathbf{R}^2\backslash\{0\},$$

which implies

$$g_\infty\big(v_0(x)\big) = 0 \quad \text{a.a. } x \in \mathbf{R}^2. \tag{33}$$

Since $v_0 \in H_r^1(\mathbf{R}^2) \subset C(\mathbf{R}^2\backslash\{0\})$, (3d) and (33), we infer that $v_0 \equiv 0$.

On the other hand, by (31), we have

$$r_k^2\int_{\Omega_k} |\nabla v_k|^2\,dx = \int_{\Omega_k} g\left(\frac{|x|}{r_k}, v_k\right)v_k + \lambda_{n,k}h(v_k)v_k\,dx. \tag{34}$$

Therefore it follows from (34), $1 = \|\tilde{v}_k\|_{L^2(\Omega_k)} = \|v_k\|_{L^2(\Omega_k)}$ and Lemma 3.1 that

$$0 < \frac{m_1}{2} = \frac{m_1}{2}\|v_k\|_{L^2(\Omega_k)}^2 \leqslant r_k^2\|\nabla v_k\|_{L^2(\Omega_k)}^2 + \frac{m_1}{2}\|v_k\|_{L^2(\Omega_k)}^2$$
$$= \int_{\Omega_k} \frac{m_1}{2}v_k^2 + g\left(\frac{|x|}{r_k}, v_k\right)v_k + \lambda_k h(v_k)v_k\,dx \leqslant (1+\lambda_{n,k})\int_{\Omega_k} h(v_k)v_k\,dx.$$

Since $\lambda_{n,k} \leqslant 1$ and $h(s)s \geqslant 0$ for all $s \in \mathbf{R}$, we obtain

$$\frac{m_1}{2} \leqslant 2\int_{\mathbf{R}^2} h(v_k)v_k\,dx. \tag{35}$$

On the other hand, since $v_k \rightharpoonup 0$ weakly in $H^1(\mathbf{R}^2)$, by Lemma 3.2(i), we have

$$\int_{\mathbf{R}^2} h(v_k)v_k\,dx \to 0.$$

This contradicts (35), therefore it holds that $\|u_{n,k}\|_{L^2(\Omega)} \leqslant C_n$, which completes the proof. $\square$

By virtue of Proposition 5.9, we have

Corollary 5.10. *The sequence* $(u_{n,k})_{k=1}^{\infty}$ *is a bounded* (PS) *sequence at level* $b_{n,0}$ *for* I_0.

Proof. We remark that it holds that

$$\bigl|I_0(u_{n,k}) - I_{\lambda_{n,k}}(u_{n,k})\bigr| \leqslant \lambda_{n,k} K(u_{n,k}), \qquad \bigl|I'(u_{n,k})\varphi - I'_{\lambda_{n,k}}(u_{n,k})\varphi\bigr| \leqslant \lambda_{n,k} \bigl\|K'(u_{n,k})\bigr\|_{E^*} \|\varphi\|_E.$$

By Lemma 3.2 and $\lambda_{n,k} \to 0$, we can prove $I_0(u_{n,k}) \to b_{n,0}$ and $I'_0(u_{n,k}) \to 0$ as $k \to \infty$. Thus $(u_{n,k})_{k=1}^{\infty}$ is a bounded (PS) sequence at level $b_{n,0}$ for I. □

Now we complete the proof of Theorem 2.1.

Proof of Theorem 2.1. For each $n \in \mathbf{N}$, by Corollary 5.10, there exists a bounded sequence $(u_{n,k})_{k=1}^{\infty} \subset E$

$$I_0(u_{n,k}) \to b_{n,0}, \qquad I'_0(u_{n,k}) \to 0 \quad \text{as } k \to \infty.$$

Thus by Lemma 3.2(ii), there exists a $u_{n,0} \in E$ such that

$$I_0(u_{n,0}) = b_{n,0}, \qquad I'_0(u_{n,0}) = 0.$$

On the other hand, by Lemmas 4.4 and 4.5, $b_{n,0} \to \infty$ as $n \to \infty$. Therefore we show the existence of infinitely many radial solutions.

In order to obtain positive solutions, we modify $g(r,s)$ as follows:

$$g_+(r,s) = \begin{cases} g(r,s) & \text{if } s \geqslant 0, \\ 0 & \text{if } s \leqslant 0. \end{cases}$$

Then any nontrivial radial solution of

$$-\Delta u = g_+\bigl(|x|, u\bigr) \quad \text{in } \Omega, \qquad u'(1) = 0$$

is positive on $\{|x| \geqslant 1\}$ by the maximum principle. Thus we will find a critical point of

$$I_+(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \int_{\Omega} G_+\bigl(|x|, u\bigr)\,dx.$$

We can prove that I_+ has a mountain pass geometry as in Lemma 4.1. Moreover, using the monotonicity method as before, we can show that I_+ has a nontrivial critical point. Thus we complete the proof. □

5.4. Outline of proof of Theorem 2.2

In this subsection, we give an outline of proof of Theorem 2.2. Throughout this subsection, we assume the conditions (3a)–(3f).

As in the Neumann case, we define the following functionals: for each $\lambda \in [0,1]$,

$$I_{D,\lambda}(v) \equiv \frac{1}{2}\|\nabla v\|_{L^2(\Omega)}^2 - \int_{\Omega} G\bigl(|x|, v\bigr) + \lambda H(v)\,dx \in C^1\bigl(H^1_{0,r}(\Omega), \mathbf{R}\bigr),$$

$$I_{\mathbf{R}^N,\lambda}(w) \equiv \frac{1}{2}\|\nabla w\|_{L^2(\mathbf{R}^N)}^2 - \int_{\mathbf{R}^N} G\bigl(|x|, w\bigr) + \lambda H(w)\,dx \in C^1\bigl(H^1_r\bigl(\mathbf{R}^N\bigr), \mathbf{R}\bigr),$$

$$J_{\mathbf{R}^N}(w) \equiv \frac{1}{2}\|\nabla w\|_{L^2(\mathbf{R}^N)}^2 - 2\int_{\mathbf{R}^N} H(w)\,dx \in C^1\bigl(H^1_r\bigl(\mathbf{R}^N\bigr), \mathbf{R}\bigr).$$

Then, noting $H^1_{0,r}(\Omega) \subset H^1_r(\mathbf{R}^N)$, we can see that

$$J_{\mathbf{R}^N}(v) \leqslant I_{D,\lambda}(v), \qquad J_{\mathbf{R}^N}(w) \leqslant I_{\mathbf{R}^N,\lambda}(w)$$

for all $\lambda \in [0,1]$, $v \in H^1_{0,r}(\Omega)$, $w \in H^1_r(\mathbf{R}^N)$. Furthermore $I_{D,\lambda}$, $I_{\mathbf{R}^N,\lambda}$ satisfy (11).

Let $\gamma_n \in C(S^{n-1}, H^1_{0,r}(\Omega))$ appear in Lemma 4.1. Then $\gamma_n \in C(S^{n-1}, H^1_r(\mathbf{R}^N))$ and we can define minimax values for $I_{D,\lambda}$, $I_{\mathbf{R}^N,\lambda}$ and $J_{\mathbf{R}^N}$:

$$b_{n,D}(\lambda) \equiv \inf_{\gamma \in \Gamma_{n,D}} \max_{\sigma \in D_n} I_{D,\lambda}\big(\gamma(\sigma)\big), \qquad b_{n,\mathbf{R}^N}(\lambda) \equiv \inf_{\gamma \in \Gamma_{n,\mathbf{R}^N}} \max_{\sigma \in D_n} I_{\mathbf{R}^N,\lambda}\big(\gamma(\sigma)\big),$$
$$c_{n,\mathbf{R}^N} \equiv \inf_{\gamma \in \Gamma_{n,\mathbf{R}^N}} \max_{\sigma \in D_n} J_{\mathbf{R}^N}\big(\gamma(\sigma)\big),$$

where

$$\Gamma_{n,D} \equiv \big\{\gamma \in C\big(D_n, H^1_{0,r}(\Omega)\big) \bigm| \gamma = \gamma_n \text{ on } S^{n-1}\big\},$$
$$\Gamma_{n,\mathbf{R}^N} \equiv \big\{\gamma \in C\big(D_n, H^1_r\big(\mathbf{R}^N\big)\big) \bigm| \gamma = \gamma_n \text{ on } S^{n-1}\big\}.$$

It is easily seen that all lemmas in Sections 3 and 4 hold if we replace I_λ, J, $b_n(\lambda)$, c_n by $I_{D,\lambda}$, $I_{\mathbf{R}^N,\lambda}$, $b_{n,D}(\lambda)$, $b_{n,\mathbf{R}^N}(\lambda)$, $c_{n,\mathbf{R}^N}$. Moreover, we can apply the monotonicity method for $I_{D,\lambda}$ and $I_{\mathbf{R}^N,\lambda}$ (cf. Lemma 5.3). Therefore for each $n \in \mathbf{N}$ there are sequences $(\lambda_{n,k}) \subset [0,1]$, $(v_{n,k}) \subset H^1_{0,r}(\Omega)$, $(w_{n,k}) \subset H^1_r(\mathbf{R}^N)$ such that $\lambda_{n,k} \to 0$ and

$$I_{D,\lambda_{n,k}}(v_{n,k}) = b_{n,D}(\lambda_{n,k}), \qquad I'_{D,\lambda_{n,k}}(v_{n,k}) = 0,$$
$$I_{\mathbf{R}^N,\lambda_{n,k}}(w_{n,k}) = b_{n,\mathbf{R}^N}(\lambda_{n,k}), \qquad I'_{\mathbf{R}^N,\lambda_{n,k}}(w_{n,k}) = 0.$$

As in the Neumann case, it is sufficient to show that $(v_{n,k})_{k=1}^\infty$ (resp. $(w_{n,k})_{k=1}^\infty$) is bounded in $H^1_{0,r}(\Omega)$ (resp. $H^1_r(\mathbf{R}^N)$). Using (21) and (22) instead of (15), it is easily seen that $(v_{n,k})_{k=1}^\infty$ (resp. $(w_{n,k})_{k=1}^\infty$) is bounded in $H^1_{0,r}(\Omega)$ (resp. $H^1_r(\mathbf{R}^N)$) in a similar way as in the proof of Proposition 5.9.

The remaining part of the proof of Theorem 2.2 is the same as the proof of Theorem 2.1, so we omit it.

5.5. Proof of Theorem 2.4

In this subsection, we prove Theorem 2.4 and let $g(r,s) = -V(r)s + \tilde{g}(s)$. We only consider (3c), since proofs in other cases are similar. As mentioned in Section 3, we can suppose $\Omega = \{x \in \mathbf{R}^N \mid |x| > 1\}$. Furthermore, as in [2,4,5,9], instead of (4c), we can assume

$$\lim_{s \to \infty} \frac{\tilde{g}(s)}{s^{2^*-1}} = 0. \tag{4c$'$}$$

Indeed, set $\tilde{\zeta}_1 \equiv \inf\{s \in [\tilde{\zeta}_0, \infty) \mid \tilde{g}(s) = 0\}$ where $\tilde{\zeta}_0 > 0$ appearing in (4d). If $\tilde{g}(s) > 0$ for all $s \geqslant \tilde{\zeta}_0$, then we set $\tilde{\zeta}_1 = \infty$. We define $\bar{g}(s)$ as follows:

$$\bar{g}(s) = \begin{cases} \tilde{g}(s) & \text{if } |s| \leqslant \tilde{\zeta}_1, \\ 0 & \text{if } |s| > \tilde{\zeta}_1. \end{cases}$$

Then $\bar{g}$ satisfies (4a), (4b), (4c$'$) and (4d). Moreover, any solution of

$$-\Delta u + V\big(|x|\big)u = \bar{g}(u) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{36}$$

satisfies $\|u\|_{L^\infty(\Omega)} \leqslant \tilde{\zeta}_1$ by the maximum principle. Therefore any solution of (36) satisfies (4c) with $g(r,s) = -V(r)s + \tilde{g}(s)$, which implies that we can assume (4c$'$) instead of (4c) without loss of generality.

As stated in the above, we prove Theorem 2.4 under

$$N \geqslant 3, \text{ the conditions (4a), (4b), (4c}'\text{), (4d), (4e), (5a)–(5c) are satisfied.} \tag{37}$$

Under the condition (37) we will find infinitely many critical points of

$$\begin{aligned} \tilde{I}(u) &\equiv \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega \left(V\big(|x|\big) + \frac{\tilde{m}_1}{2}\right)u^2\,dx - \int_\Omega \frac{\tilde{m}_1}{4}u^2 + \tilde{G}(u)\,dx \\ &= \frac{1}{2}\|u\|^2 - \int_\Omega \frac{\tilde{m}_1}{4}u^2 + \tilde{G}(u)\,dx, \end{aligned}$$

where $\tilde{m}_1$ appears in (4b) and $\tilde{G}(s) = \int_0^s \tilde{g}(t)\,dt$.

In this case, we can define $h \in C(\mathbf{R})$ satisfying Lemma 3.1. Thus we define an auxiliary functional $\tilde{J}$ and parametrized functional $\tilde{I}_\lambda$ for each $\lambda \in [0,1]$. We note that all lemmas and propositions in Section 4 hold for these functionals. Moreover, noting $V(r) \to 0$ as $r \to \infty$ and a proof of Proposition A.1 (in Appendix A), we can also prove that $\tilde{I}_\lambda$, $\tilde{J}$ have a symmetric mountain pass structure and define $\tilde{b}_n(\lambda)$ and $\tilde{c}_n$ as in Definition 4.3. Furthermore, we see that all lemmas in Section 3 hold. By Proposition 5.2, for each $n \in \mathbf{N}$ there exist $(\tilde{\lambda}_{n,k})_{k=1}^\infty$ and $(\tilde{u}_{n,k})_{k=1}^\infty \subset H^1_r(\Omega)$ such that $\tilde{\lambda}_{n,k} \to 0$ as $k \to \infty$ and

$$\tilde{I}_{\tilde{\lambda}_{n,k}}(\tilde{u}_{n,k}) = \tilde{b}_n(\tilde{\lambda}_{n,k}), \qquad \tilde{I}'_{\tilde{\lambda}_{n,k}}(\tilde{u}_{n,k}) = 0.$$

Next, we show that $(\tilde{u}_{n,k})_{k=1}^\infty$ is bounded in $H^1_r(\Omega)$.

Lemma 5.11. *There exists a $C_n > 0$ such that $\|\tilde{u}_{n,k}\| \leqslant C_n$ for all $k \geqslant 1$.*

Proof. As in Proposition 5.9, firstly we show that $(\nabla \tilde{u}_{n,k})_{k=1}^{\infty}$ is bounded in $L^2(\Omega)$. By Remark 5.6, $\tilde{u}_{n,k}$ satisfies

$$\frac{1}{2}\int_{\Omega} V\bigl(|x|\bigr)\tilde{u}_{n,k}^2\,dx - \int_{\Omega} \tilde{G}(\tilde{u}_{n,k}) + \tilde{\lambda}_{n,k}H(\tilde{u}_{n,k})\,dx = -\frac{1}{2^*}\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2 - \frac{1}{2N}\int_{\Omega} x\cdot\nabla V\bigl(|x|\bigr)\tilde{u}_{n,k}^2\,dx + \int_{\partial\Omega} -\frac{1}{2}V(1)\tilde{u}_{n,k}^2 + \tilde{G}(\tilde{u}_{n,k}) + \tilde{\lambda}_{n,k}H(\tilde{u}_{n,k})\,dS.$$

By (4e) and Hölder's inequality, there exists a $C > 0$ such that

$$\begin{aligned}\tilde{I}_{\tilde{\lambda}_{n,k}}(\tilde{u}_{n,k}) &= \frac{1}{2}\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_{\Omega} V\bigl(|x|\bigr)\tilde{u}_{n,k}^2\,dx - \int_{\Omega}\tilde{G}(\tilde{u}_{n,k}) + \tilde{\lambda}_{n,k}H(\tilde{u}_{n,k})\,dx \\ &= \frac{1}{N}\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2 - \frac{1}{2N}\int_{\Omega} x\cdot\nabla V\bigl(|x|\bigr)\tilde{u}_{n,k}^2\,dx + \int_{\partial\Omega} -\frac{1}{2}V(1)\hat{u}_{n,k}^2 + \tilde{G}(\tilde{u}_{n,k}) + \tilde{\lambda}_{n,k}H(\tilde{u}_{n,k})\,dS \\ &\geqslant \frac{1}{N}\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2 - \frac{1}{2N}\bigl\|\bigl(x\cdot\nabla V\bigl(|x|\bigr)\bigr)^+\bigr\|_{L^{\frac{N}{2}}(\Omega)}\|\tilde{u}_{n,k}\|_{L^{2^*}(\Omega)}^2 - C.\end{aligned}$$

We extend $\tilde{u}_{n,k}$ as follows:

$$\hat{u}_{n,k}(x) = \begin{cases}\tilde{u}_{n,k}(|x|) & \text{if } |x| \geqslant 1,\\ \tilde{u}_{n,k}(1) & \text{if } |x| < 1.\end{cases}$$

Then it is clear that $\hat{u}_{n,k} \in H_r^1(\mathbf{R}^N)$, $\|\nabla\hat{u}_{n,k}\|_{L^2(\mathbf{R}^N)} = \|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}$ and $\|\tilde{u}_{n,k}\|_{L^{2^*}(\Omega)} \leqslant \|\hat{u}_{n,k}\|_{L^{2^*}(\mathbf{R}^N)}$. Furthermore, since $\|\hat{u}_{n,k}\|_{L^{2^*}(\mathbf{R}^N)}^2 \leqslant \|\nabla\hat{u}_{n,k}\|_{L^2(\mathbf{R}^N)}^2/S_N$ holds, we obtain $\|\tilde{u}_{n,k}\|_{L^{2^*}(\Omega)}^2 \leqslant \|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2/S_N$. Here, from (5c), we can take an $\varepsilon_0 > 0$ such that

$$\bigl\|\bigl(x\cdot\nabla V\bigl(|x|\bigr)\bigr)^+\bigr\|_{L^{\frac{N}{2}}(\Omega)} < 2S_N - \varepsilon_0.$$

Then we have

$$\tilde{I}_{\tilde{\lambda}_{n,k}}(\tilde{u}_{n,k}) \geqslant \frac{1}{N}\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2 - \frac{1}{N}\frac{2S_N - \varepsilon_0}{2S_N}\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2 - C \geqslant \varepsilon_1\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)}^2 - C$$

for some $\varepsilon_1 > 0$. Thus there exists a $C_n > 0$ such that $\|\nabla\tilde{u}_{n,k}\|_{L^2(\Omega)} \leqslant C_n$ for all $k \in \mathbf{N}$.

Since a proof of the boundedness of $(\tilde{u}_{n,k})_{k=1}^{\infty}$ in $L^2(\Omega)$ is similar to the one of Proposition 5.9, we omit it. □

Now we complete the proof of Theorem 2.4.

Proof of Theorem 2.4. From Lemma 5.11, we see that $(\tilde{u}_{n,k})_{k=1}^{\infty}$ is a bounded (PS) sequence for $\tilde{I}$ as in Corollary 5.10. Therefore we can show the existence of infinitely many solutions and at least one positive solution as in Theorem 2.1, which completes the proof. □

6. Open problems

Lastly, we state some open problems concerning (1):

(i) Can we relax the monotonicity condition (3b)?
(ii) Does the sequences $(b_n(\lambda))$ (resp. $(b_{D,n(\lambda)}), (b_{\mathbf{R}^N,n}(\lambda))$) converge to $b_n(0)$ (resp. $b_{D,n}(0), b_{\mathbf{R}^N,n}(0)$)?

(i) In the case where $g(r,s) = -V(r)s + \tilde{g}(s)$ namely the Schrödinger type equation, we could replace (3b) by the weaker condition (5c). Can we replace (3b) by a weaker condition in general? We note that by Proposition 5.4, we can construct a sequence of approximate solutions to (2c) without (3b) (see (12)). Hence the problem is whether we can prove the boundedness of the approximate solutions without the monotonicity condition.

(ii) In the previous section, we showed that $\lim_{\lambda\to 0} b_n(\lambda) =: b_{n,0} \leqslant b_n(0)$ is a critical value of I_0 for each $n \in \mathbf{N}$. In particular, $b_{1,0}$ corresponds to a positive solution of (2c). If the uniqueness of positive radial solution to (2c) holds or $b_1(0)$ is equal to the least energy value of (2c), then we have $b_1(0) \leqslant b_{1,0}$, which implies $b_1(\lambda) \to b_1(0)$. Hence, the natural question is that we can prove whether $b_n(\lambda) \to b_n(0)$ holds or not in general.

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Appendix A

In this appendix, we prove Lemma 4.1(ii), Lemmas 3.1 and 3.2. Moreover, we state a useful lemma. Firstly, we give a proof of Lemma 4.1(ii).

A.1. Proof of Lemma 4.1(ii)

In this subsection, we prove the following proposition.

Proposition A.1. *Let $\Omega = \{x \in \mathbf{R}^N \mid |x| > 1\}$ and (12) be satisfied. Then for each $n \in \mathbf{N}$, there exists a continuous odd map $\gamma_n : S^{n-1} \to H^1_{0,r}(\Omega)$ such that*

$$I(\gamma_n(\sigma)) < 0 \quad \textit{for all } \sigma \in S^{n-1}.$$

Before proving Proposition A.1, we introduce some notations. Firstly, we define $\underline{G}(s)$ for $s \geqslant 0$ as follows:

$$\underline{G}(s) \equiv \inf_{r \geqslant R_0} G(r,s),$$

where R_0 appears in (3f). By (3c) and (3f), $\underline{G}(s)$ is well defined and satisfies $\underline{G}(\zeta_0) > 0$. We also set

$$\underline{I}(u) \equiv \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbf{R}^N)} - \int_{\mathbf{R}^N} \underline{G}(u)\,dx \in C\big(H^1_r\big(\mathbf{R}^N\big), \mathbf{R}\big).$$

Note that if $u \in H^1_r(\Omega)$ and $\operatorname{supp} u \subset \{|x| > R_0\}$, then $I(u) \leqslant \underline{I}(u)$. Therefore it is sufficient to prove that there exists a continuous odd map $\gamma_n : S^{n-1} \to H^1_r(\Omega)$ such that

$$\underline{I}(\gamma_n(\sigma)) < 0, \qquad \operatorname{supp} \gamma_n(\sigma) \subset \big\{|x| > R_0\big\} \quad \text{for all } \sigma \in S^{n-1}. \tag{38}$$

Proof of Proposition A.1. By the arguments of Theorem 10 in [5], for each $n \in \mathbf{N}$, there exists a $\pi_n \in C(S^{n-1}, H^1_r(\mathbf{R}^N))$ such that

$$\pi_n(-\sigma) = -\pi_n(\sigma), \qquad \big\|\pi_n(\sigma)\big\|_{L^\infty(\mathbf{R}^N)} = \zeta_0, \qquad \int_{\mathbf{R}^N} \underline{G}\big(\pi_n(\sigma)\big)\,dx \geqslant 1 \quad \text{for all } \sigma \in S^{n-1}.$$

We modify π_n to obtain γ_n satisfying the property (38). Let $\varphi \in C^\infty([0,\infty))$ be a cut-off function such that

$$0 \leqslant \varphi(t) \leqslant 1, \quad \varphi(t) = \begin{cases} 0 & \text{if } t \leqslant 1, \\ 1 & \text{if } t \geqslant 2, \end{cases}$$

and set $\varphi_k(t) \equiv \varphi(kt)$ and $\eta_k(\sigma)(x) \equiv \varphi_k(|x|)\pi_n(\sigma)(x)$ for $k \in \mathbf{N}$. Then it holds $\operatorname{supp} \eta_k(\sigma) \subset \{|x| \geqslant 1/k\}$ for all $\sigma \in S^{n-1}$ and

$$\int_{\mathbf{R}^N} \underline{G}\big(\eta_k(\sigma)\big)\,dx \to \int_{\mathbf{R}^N} \underline{G}\big(\pi_n(\sigma)\big)\,dx \quad \text{as } k \to \infty \text{ uniformly w.r.t. } \sigma \in S^{n-1},$$

since $\pi_n(S^{n-1})$ is uniformly bounded in $L^\infty(\mathbf{R}^N)$. Therefore for a large $k_0 \in \mathbf{N}$, we have

$$\int_{\mathbf{R}^N} \underline{G}(\eta_{k_0})\,dx \geqslant \frac{1}{2} \quad \text{for all } \sigma \in S^{n-1}. \tag{39}$$

We consider $\eta_{k_0}(\sigma)(x/t)$ for $t \geqslant 1$. By (39), we see that $\operatorname{supp} \eta_{k_0}(\sigma)(\cdot/t) \subset \{|x| \geqslant t/k_0\}$ and

$$\begin{aligned} \underline{I}\big(\eta_{k_0}(\sigma)(\cdot/t)\big) &= t^{N-2}\left(\frac{1}{2}\big\|\nabla \eta_{k_0}(\sigma)\big\|^2_{L^2(\mathbf{R}^N)} - t^2 \int_{\mathbf{R}^N} \underline{G}(\eta_{k_0})\,dx\right) \\ &\leqslant t^{N-2}\left(\frac{1}{2}\big\|\nabla \eta_{k_0}(\sigma)\big\|^2_{L^2(\mathbf{R}^N)} - \frac{t^2}{2}\right). \end{aligned}$$

Since $\|\nabla \eta_{k_0}(\sigma)\|_{L^2(\mathbf{R}^N)}$ is uniformly bounded with respect to $\sigma \in S^{n-1}$, we can choose a $t_0 \geqslant 1$ satisfying $t_0/k_0 > R_0$ and

$$\underline{I}\big(\eta_{k_0}(\sigma)(\cdot/t_0)\big) < 0 \quad \text{for all } \sigma \in S^{n-1}.$$

Set $\gamma_n(\sigma)(x) = \eta_{k_0}(\sigma)(x/t_0)$, then γ_n satisfies (38). The oddness and continuity of γ_n follows from the ones of η_{k_0}, which completes the proof. $\square$

A.2. Proofs of Lemmas 3.1 and 3.2

Next we give proofs of Lemmas 3.1 and 3.2. Firstly we show Lemma 3.1.

Proof of Lemma 3.1. By (3c) and (3d), it is clear that (i)–(iii) hold.

We prove (iv) for $N \geqslant 3$. The case $N = 2$ can be proven in a similar way. From (3e), for any $\varepsilon > 0$, we can choose $s_0 > 0$ such that if $s \geqslant s_0$, then $f(s)/s^{2^*-1} \leqslant \varepsilon$. By the definition of h, if $s \geqslant s_0$, then

$$\frac{h(s)}{s^{2^*-1}} = \frac{1}{s^{2^*-p-1}} \sup_{0\leqslant\tau\leqslant s} \frac{f(\tau)}{\tau^p} \leqslant \frac{1}{s^{2^*-p-1}}\left(\sup_{0\leqslant\tau\leqslant s_0} \frac{f(\tau)}{\tau^p} + \sup_{s_0\leqslant s\leqslant s} \frac{f(\tau)}{\tau^{2^*-1}}\right).$$

Thus we can show that there exists an $s_1 \geqslant s_0$ such that if $s \geqslant s_1$, then

$$\frac{1}{s^{2^*-p-1}} \sup_{0\leqslant\tau\leqslant s_0} \frac{f(\tau)}{\tau^p} \leqslant \varepsilon,$$

which implies that $h(s)/s^{2^{*-1}} \leqslant 2\varepsilon$. Therefore (iv) holds.

(v) By the definition of h and H, we have

$$\begin{aligned}
(p+1)H(s) - h(s)s &= \int_0^s (p+1)h(t) - h(s)\,dt \\
&= \int_0^s \left((p+1)t^p \sup_{0\leqslant\tau\leqslant t} \frac{f(\tau)}{\tau^p} - s^p \sup_{0\leqslant\tau\leqslant s} \frac{f(\tau)}{\tau^p}\right) dt \\
&\leqslant \sup_{0\leqslant\tau\leqslant s} \frac{f(\tau)}{\tau^p} \int_0^s (p+1)t^p - s^p\,dt = 0.
\end{aligned}$$

Thus (v) holds. $\square$

Next we show Lemma 3.2.

Proof of Lemma 3.2. (i) Firstly we show that K is weakly continuous. Let u_k satisfy $u_k \rightharpoonup u_0$ weakly in E. Without loss of generality, we may assume

$$u_k(x) \to u_0(x) \quad \text{a.a. } x \in \Omega, \qquad \|u_k\|_E \leqslant M.$$

Since (u_k) is bounded, by (7) and Lemma 3.1, there exists an $R_1 > 0$ such that if $|x| \geqslant R_1$, then $H(u_k(x)) = H(u_0(x)) = 0$ for all $k \geqslant 1$. Therefore, it is sufficient to show

$$\int_{\Omega\cap B_{R_1}} \big|H(u_k) - H(u_0)\big|\,dx \to 0.$$

We set $Q(s) = |s|^{2^*}$ $(N \geqslant 3)$, $Q(s) = \exp(s^2/(2M^2)) - 1 - s^2/(2M^2)$ $(N = 2)$. Then by Lemma 3.1, for each $\varepsilon > 0$ there exists an $s_\varepsilon \geqslant 0$ such that if $|s| \geqslant s_\varepsilon$, then $H(s) \leqslant \varepsilon Q(s)$. Then we define $\hat{H}(s)$ as follows:

$$\hat{H}(s) = \begin{cases} H(s) & \text{if } |s| \leqslant s_\varepsilon, \\ H(s_\varepsilon) & \text{if } |s| > s_\varepsilon. \end{cases}$$

Since $\hat{H}$ is bounded, it is easy to see that

$$\hat{H}(u_k) \to \hat{H}(u_0) \quad \text{in } L^1(\Omega \cap B_{R_1}).$$

On the other hand, since $|\hat{H}(s) - H(s)| \leqslant \varepsilon Q(s)$ we have

$$\int_{\Omega\cap B_{R_1}} \left|H(u_k)-H(u_0)\right| dx \leqslant \int_{\Omega\cap B_{R_1}} \left|H(u_k)-\hat{H}(u_k)\right| + \left|\hat{H}(u_k)-\hat{H}(u_0)\right| + \left|\hat{H}(u_0)-H(u_0)\right| dx$$
$$\leqslant \varepsilon \int_{\Omega\cap B_{R_1}} Q(u_k)+Q(u_0)\,dx + \left\|\hat{H}(u_k)-\hat{H}(u_0)\right\|_{L^1(\Omega\cap B_{R_1})}.$$

Thus to prove the weak continuity of K, it is sufficient to prove

$$\sup_{k\geqslant 1}\int_\Omega Q(u_k)\,dx<\infty. \tag{40}$$

In the case $N\geqslant 3$, (40) follows from Sobolev's inequality and in the case $N=2$, (40) holds by Lemma A.2(iii). Therefore K is weakly continuous.

Next we prove that $K'(u_k)\to K'(u_0)$ strongly in E^*. Since

$$K'(u_k)\varphi=\int_\Omega h(u_k)\varphi\,dx \quad \text{for all } \varphi\in E,$$

if we can show

$$h(u_k)\to h(u_0) \quad \text{strongly in } L^{p_N}(\Omega), \qquad p_N=\begin{cases}2 & \text{if } N=2,\\ 2N/(N+2) & \text{if } N\geqslant 3,\end{cases} \tag{41}$$

then $K'(u_k)\to K'(u_0)$ strongly in E^*.

We prove (41). As in the above, there exists an $R_1\geqslant 1$ such that if $|x|\geqslant R_1$, then $h(u_k(x))=h(u_0(x))=0$ for all $k\geqslant 1$. Therefore we only show

$$h(u_k)\to h(u_0) \quad \text{strongly in } L^{p_N}(\Omega\cap B_{R_1}).$$

Set $Q(s)=\exp(s^2/(8M^2))-1-s^2/(8M^2)$ if $N=2$ and $Q(s)=|s|^{(N+2)/(N-2)}$ if $N\geqslant 3$. By Lemma 3.1, for each $\varepsilon>0$, there exists an $s_\varepsilon\geqslant 0$ such that if $|s|\geqslant s_\varepsilon$, then $|h(s)|\leqslant \varepsilon Q(s)$. Then define $\hat{h}(s)$ as follows:

$$\hat{h}(s)=\begin{cases}h(s) & \text{if } |s|\leqslant s_\varepsilon,\\ h(s_\varepsilon) & \text{if } s>s_\varepsilon,\\ h(-s_\varepsilon) & \text{if } s<-s_\varepsilon.\end{cases}$$

Then we have $\hat{h}(u_k)\to\hat{h}(u_0)$ strongly in $L^{p_N}(\Omega\cap B_{R_1})$. Therefore, to prove (41), it is sufficient to show

$$\sup_{k\geqslant 1}\int_\Omega Q(u_k)^{p_N}\,dx<\infty. \tag{42}$$

In the case $N\geqslant 3$, by Sobolev's inequality and $p_N(2^*-1)=2^*$, (42) holds. In the case $N=2$, we remark that $Q(s)^2\leqslant Q(2s)$ for all $s\in\mathbf{R}$. By Lemma A.2(iii), we have

$$\sup_{k\geqslant 1}\int_\Omega Q(u_k)^2\,dx\leqslant \sup_{k\geqslant 1}\int_\Omega Q(2u_k)\,dx\leqslant C\sup_{k\geqslant 1}\|u_k\|_E^4<\infty,$$

which implies (42). Therefore $K'(u_k)\to K'(u_0)$ strongly in E^*.

(ii) Let $(u_k)\subset E$ be a (PS) sequence at level c for I_λ and $\|u_k\|_E\leqslant M$. Since (u_k) is bounded, there exist $u_0\in E$ and subsequence (u_{k_ℓ}) such that

$$u_{k_\ell}\rightharpoonup u_0 \quad \text{weakly in } E, \qquad u_{k_\ell}(x)\to u_0(x) \quad \text{a.a. } x\in\Omega.$$

Let $\varphi\in C^\infty_{0,r}(\{|x|\geqslant 1\})=\{\phi\in C^\infty(\{|x|\geqslant 1\}) \mid \phi(x)=\phi(|x|) \text{ and } \operatorname{supp}\phi \text{ is compact}\}$. Set $p_N=2$ if $N=2$ and $p_N=2N/(N+2)$ if $N\geqslant 3$. Applying similar arguments as in the above, we can show

$$g\big(|x|,u_{k_\ell}\big)\to g\big(|x|,u_0(x)\big) \quad \text{strongly in } L^{p_N}(\Omega\cap B_{\hat{R}}),$$
$$h(u_{k_\ell})\to h(u_0) \quad \text{strongly in } L^{p_N}(\Omega)$$

for all $\hat{R}>1$. Therefore we obtain

$$\int_\Omega g\big(|x|,u_{k_\ell}\big)\varphi\,dx\to\int_\Omega g\big(|x|,u_0\big)\varphi\,dx, \qquad \int_\Omega h(u_{k_\ell})u_{k_\ell}\,dx\to\int_\Omega h(u_0)u_0\,dx. \tag{43}$$

Noting that $I'_\lambda(u_{k_\ell}) \to 0$, by (43), we see that $I'_\lambda(u_0)\varphi = 0$ for all $\varphi \in C^\infty_{0,r}(\{|x| \geqslant 1\})$. Since $C^\infty_{0,r}(\{|x| \geqslant 1\})$ is dense in E, it holds $I'_\lambda(u_0)u_0 = 0$, that is,

$$\|u_0\|_E^2 = \int_\Omega \frac{m_1}{2}u_0^2 + g\big(|x|, u_0\big)u_0 + \lambda h(u_0)u_0\,dx. \tag{44}$$

On the other hand, since (u_{k_ℓ}) is bounded, we have $I'_\lambda(u_{k_\ell})u_{k_\ell} \to 0$, which implies

$$\|u_{k_\ell}\|_E^2 - \int_\Omega \frac{m_1}{2}u_{k_\ell}^2 + g\big(|x|, u_{k_\ell}\big)u_{k_\ell} + \lambda h(u_{k_\ell})u_{k_\ell}\,dx \to 0. \tag{45}$$

Next, we rewrite

$$\int_\Omega \frac{m_1}{2}u_{k_\ell}^2 + g\big(|x|, u_{k_\ell}\big)u_{k_\ell} + \lambda h(u_{k_\ell})u_{k_\ell}\,dx = (1+\lambda)\int_\Omega h(u_{k_\ell})u_{k_\ell}\,dx - \int_\Omega h(u_{k_\ell})u_{k_\ell} - \frac{m_1}{2}u_{k_\ell}^2 - g\big(|x|, u_{k_\ell}\big)u_{k_\ell}\,dx.$$

By Lemma 3.1 and Fatou's lemma, we have

$$\liminf_{\ell\to\infty}\int_\Omega h(u_{k_\ell})u_{k_\ell} - \frac{m_1}{2}u_{k_\ell}^2 - g\big(|x|, u_{k_\ell}\big)u_{k_\ell}\,dx \geqslant \int_\Omega h(u_0)u_0 - \frac{m_1}{2}u_0^2 - g\big(|x|, u_0\big)u_0\,dx. \tag{46}$$

By (43)–(46), we obtain

$$\begin{aligned}\limsup_{\ell\to\infty}\|u_{k_\ell}\|_E^2 &\leqslant (1+\lambda)\int_\Omega h(u_0)u_0\,dx - \int_\Omega h(u_0)u_0 - \frac{m_1}{2}u_0^2 - g\big(|x|, u_0\big)u_0\,dx \\ &= \int_\Omega \frac{m_1}{2}u_0^2 + g\big(|x|, u_0\big)u_0 + \lambda h(u_0)u_0\,dx = \|u_0\|_E^2.\end{aligned}$$

Thus u_{k_ℓ} converges to u_0 strongly in E, which completes the proof.

(iii) Next we prove that J satisfies the (PS) condition. Let $(u_k) \subset E$ be a (PS) sequence at level c of J, i.e., $J(u_k) \to c$ and $J'(u_k) \to 0$ strongly in E^*. Since $h(s)$ satisfies a global Ambrosetti–Rabinowitz condition, we can infer that (u_k) is bounded. Indeed, the boundedness of (u_k) in E comes from

$$\begin{aligned}J(u_k) - \frac{J'(u_k)u_k}{p+1} &= \left(1 - \frac{2}{p+1}\right)\|u_k\|_E^2 - \int_{|x|>1} H(u_k) - \frac{1}{p+1}h(u_k)u_k\,dx \\ &\geqslant \left(1 - \frac{2}{p+1}\right)\|u_k\|_E^2.\end{aligned}$$

Thus we may assume that taking a subsequence if necessary,

$$u_k \rightharpoonup u_0 \quad \text{weakly in } E.$$

By (i), we have $K'(u_k) \to K'(u_0)$ strongly in E^*. Therefore by standard arguments we can conclude that (u_k) has a strongly convergent subsequence and this completes the proof. □

A.3. A technical lemma

The following lemma is useful and we use it in proofs of Lemmas 3.2 and 4.1.

Lemma A.2. *(See Adachi and Tanaka [1], Byeon, Jeanjean and Tanaka [8], Ogawa [15].)*

(i) *Let $\Phi(s) = \exp(s) - 1 - s$ and $\beta \in (0, 4\pi)$. Then there exists a $\tilde{C}_\beta > 0$ such that*

$$\int_{\mathbf{R}^2} \Phi\left(\beta\frac{u^2}{\|\nabla u\|^2_{L^2(\mathbf{R}^2)}}\right)dx \leqslant \tilde{C}_\beta \frac{\|u\|^4_{L^4(\mathbf{R}^2)}}{\|\nabla u\|^4_{L^2(\mathbf{R}^2)}} \quad \textit{for all } u \in H^1\big(\mathbf{R}^2\big)\setminus\{0\}.$$

(ii) *For any $M > 0$ and $\beta \in (0, 4\pi)$, there exists a $\tilde{C}_{\beta,1} > 0$ such that*

$$\int_{\mathbf{R}^2} \Phi\left(\frac{\beta u^2}{M^2}\right)dx \leqslant \tilde{C}_{\beta,1}\frac{\|u\|^4_{L^4(\mathbf{R}^2)}}{M^4} \quad \textit{for all } u \in H^1\big(\mathbf{R}^2\big) \textit{ with } \|\nabla u\|_{L^2(\mathbf{R}^2)} \leqslant M.$$

(iii) *For any $M > 0$ and $\beta \in (0, 4\pi)$, there exists a $\tilde{C}_{\beta,2} > 0$ such that*

$$\int_\Omega \Phi\left(\frac{\beta u^2}{2M^2}\right) dx \leqslant \tilde{C}_{\beta,2} \frac{\|u\|_E^4}{M^4} \quad \text{for all } u \in H_r^1(\Omega) \text{ with } \|\nabla u\|_{L^2(\Omega)} \leqslant M,$$

where $\Omega = \{x \in \mathbf{R}^2 \mid |x| > 1\}$.

Proof. The inequality in (i) can be proven in the same way as in [1]. (ii) is a direct consequence of (i). Indeed, since for each $x \in \mathbf{R}^N$ it follows that

$$M^4 \Phi\left(\frac{\beta u^2(x)}{M^2}\right) = M^4 \sum_{j=2}^\infty \frac{(\beta u^2(x))^j}{j! M^{2j}} = \sum_{j=2}^\infty \frac{(\beta u^2(x))^j}{j! M^{2j-4}}$$
$$\leqslant \sum_{j=2}^\infty \frac{(\beta u^2(x))^j}{j! \|\nabla u\|_{L^2}^{2j-4}} = \|\nabla u\|_{L^2}^4 \Phi\left(\frac{\beta u^2(x)}{\|\nabla u\|_{L^2}^2}\right),$$

(ii) holds by (i). As to (iii), using the operator T_1 (see (8)), by (ii) and Sobolev's inequality, we can easily obtain (iii). □

References

[1] S. Adachi, K. Tanaka, Trudinger type inequalities in $\mathbf{R}^N$ and their best exponents, Proc. Amer. Math. Soc. 128 (7) (2000) 2051–2057.
[2] A. Azzollini, A. Pomponio, On the Schrödinger equation in $\mathbb{R}^N$ under the effect of a general nonlinear term, Indiana Univ. Math. J. 58 (3) (2009) 1361–1378.
[3] H. Berestycki, T. Gallouët, O. Kavian, Équations de champs scalaires euclidiens non linéaires dans le plan, C. R. Acad. Sci. Paris Sér. I Math. 297 (5) (1983) 307–310.
[4] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Ration. Mech. Anal. 82 (4) (1983) 313–345.
[5] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. II. Existence of infinitely many solutions, Arch. Ration. Mech. Anal. 82 (4) (1983) 347–375.
[6] H. Brezis, E.H. Lieb, Minimum action solutions of some vector field equations, Comm. Math. Phys. 96 (1) (1984) 97–113.
[7] E. Brüning, A note on solutions of two-dimensional semilinear elliptic vector-field equations with strong nonlinearity, in: New Methods and Results in Nonlinear Field Equations, Bielefeld, 1987, in: Lecture Notes in Phys., vol. 347, Springer, Berlin, 1989, pp. 37–57.
[8] J. Byeon, L. Jeanjean, K. Tanaka, Standing waves for nonlinear Schrödinger equations with a general nonlinearity: one and two dimensional cases, Comm. Partial Differential Equations 33 (6) (2008) 1113–1136.
[9] J. Hirata, N. Ikoma, K. Tanaka, Nonlinear scalar field equations in $\mathbf{R}^N$: mountain pass and symmetric mountain pass approaches, Topol. Methods Nonlinear Anal. 35 (2) (2010) 253–276.
[10] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem set on $\mathbf{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A 129 (4) (1999) 787–809.
[11] L. Jeanjean, K. Tanaka, A remark on least energy solutions in $\mathbf{R}^N$, Proc. Amer. Math. Soc. 131 (8) (2003) 2399–2408.
[12] L. Jeanjean, K. Tanaka, Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities, Calc. Var. Partial Differential Equations 21 (3) (2004) 287–318.
[13] C. Li, Y. Li, Nonautonomous nonlinear scalar field equations in $\mathbf{R}^2$, J. Differential Equations 103 (2) (1993) 421–436.
[14] Y. Li, Nonautonomous nonlinear scalar field equations, Indiana Univ. Math. J. 39 (2) (1990) 283–301.
[15] T. Ogawa, A proof of Trudinger's inequality and its application to nonlinear Schrödinger equations, Nonlinear Anal. 14 (9) (1990) 765–769.
[16] P.J. Rabier, Bounded Palais–Smale sequences for functionals with a mountain pass geometry, Arch. Math. (Basel) 88 (2) (2007) 143–152.
[17] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1986.
[18] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (2) (1977) 149–162.
[19] M. Struwe, The existence of surfaces of constant mean curvature with free boundaries, Acta Math. 160 (1) (1988) 19–64.
[20] M. Struwe, Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, fourth edition, Ergeb. Math. Grenzgeb. (3) (A Series of Modern Surveys in Mathematics), vol. 34, Springer-Verlag, Berlin, 2008.