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Properties of the voice transform of the Blaschke group and connections with atomic decomposition results in the weighted Bergman spaces

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ABSTRACT

Feichtinger and Gröchenig described a unified approach to atomic decomposition through integrable group representations in Banach spaces. Studying the properties of a special voice transform of the Blascke group, outlined by the general theory developed by Feichtinger and Gröchenig, we obtain that every function from the minimal Möbius invariant space will generate an atomic decomposition in the weighted Bergman spaces. In the unified approach of the atomic decomposition a useful tool is the Q-density, the V-separated property and the bounded uniform partitions of the unity of the locally compact group. Using the hyperbolic metric we can describe the Q-density from right, and the separation from right in the Blaschke group. Using this we can give an example of bounded uniform partitions of the unity from right. In the general theory of atomic decomposition it is used the Q-density from the left, this is the reason why we will make a small modification in the discretizing operator which corresponds to the Q-density from the right in order to obtain atomic decomposition in the weighted Bergman spaces.

1. The voice transform and the atomic decomposition

In signal processing and image reconstruction the wavelet and Gábor transforms play an important role. Feichtinger and Gröchenig unified the Gábor and wavelet transforms into a single theory. The common generalization of these transforms is the so-called *voice transform* (see [1–3]).

In this section we summarize the basic notations and notions used in the definition of voice transform, the most important properties of this transform and we also present a short description of the Feichtinger and Gröchenig theory which produces atomic decomposition of a large class of Banach spaces (see [1–5]).

In the construction of voice transform the starting point will be a locally compact topological group $(G,\cdot)$. Let m be a left invariant Haar measure of G. Let $f : G \to \mathbb{C}$ be a Borel-measurable function which is integrable regarding to the left invariant Haar measure m, the integral of f will be denoted by $\int_G f\,dm = \int_G f(x)\,dm(x)$. Because of left-translation invariance of the measure m it follows that

$$\int_G f(x)\,dm(x) = \int_G f\left(a^{-1}\cdot x\right)dm(x) \quad (a \in G).$$

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If the left invariant Haar measure of G is at the same time right invariant then G is *unimodular group*. Such measure will be called Haar measure of G. It can be proved that if the left Haar measure is invariant under the inverse transformation $G \ni x \to x^{-1} \in G$, then G is also unimodular.

In the definition of voice transform a *unitary representation of the group* $(G,\cdot)$ is used. Let us consider a Hilbert space $(H,\langle\cdot,\cdot\rangle)$ and let $\mathcal{U}$ denote the set of unitary bijections $U : H \to H$. Namely, the elements of $\mathcal{U}$ are bounded linear operators which satisfy $\langle Uf, Ug\rangle = \langle f, g\rangle$ $(f, g \in H)$. The set $\mathcal{U}$ with the composition operation $(U \circ V)f := U(Vf)$ $(f \in H)$ is a group, the neutral element of which is I, the identity operator on H and the inverse element of $U \in \mathcal{U}$ is the operator U^{-1} which is equal to the adjoint operator of U: $U^{-1} = U^*$. The homomorphism of the group $(G,\cdot)$ on the group $(\mathcal{U},\circ)$ satisfying

- $U_{x\cdot y} = U_x \circ U_y \quad (x, y \in G)$,
- $G \ni x \to U_x f \in H$ is continuous for all $f \in H$ (1.1)

is called the unitary representation of $(G,\cdot)$ on H. The *voice transform* of $f \in H$ generated by the representation U and by the parameter $\rho \in H$ is the (complex-valued) function on G defined by

$$(V_\rho f)(x) := \langle f, U_x\rho\rangle \quad (x \in G,\ f, \rho \in H). \tag{1.2}$$

For any representation $U : G \to \mathcal{U}$ and for each $f, \rho \in H$ the voice transform $V_\rho f$ is a continuous and bounded function on G and $V_\rho : H \to C(G)$ is a bounded linear operator.

The set of continuous bounded functions defined on the group G with the norm defined by $\|F\| := \sup\{|F(x)|:\ x \in G\}$ form a Banach space. From the unitarity of $U_x : H \to H$ it follows that, for all $x \in G$,

$$\left|(V_\rho f)(x)\right| = \left|\langle f, U_x\rho\rangle\right| \leqslant \|f\|\|U_x\rho\| = \|f\|\|\rho\|,$$

consequently $\|V_\rho\| \leqslant \|\rho\|$.

Taking as starting point not necessarily commutative locally compact groups we can construct in this way important transformations in signal processing and control theory. For example, the affine wavelet transform and the Gábor-transform are all special voice transforms (see [4,5,1]).

The invertibility of V_ρ it is connected to the irreducibility of the representation U. A representation U is called *irreducible* if the only closed invariant subspaces of H, i.e., closed subspaces H_0 which satisfy $U_x H_0 \subset H_0$, are $\{0\}$ and H. Since the closure of the linear span of the set

$$\{U_x\rho:\ x \in G\} \tag{1.3}$$

is always a closed invariant subspace of H, it follows that U is irreducible if and only if the collection (1.3) is a closed system for any $\rho \in H$, $\rho \neq 0$.

The property of irreducibility gives a simple criterion for deciding when a voice transform is 1–1 (see for example [5]).

A voice transform V_ρ generated by a unitary representation U is 1–1 for all $\rho \in H \setminus \{0\}$ if and only if U is irreducible.

The function $V_\rho f$ is continuous on G but in general is not square integrable. If there exist $\rho \in H$, $\rho \neq 0$ such that $V_\rho\rho \in L^2_m(G)$, then the representation U is *square integrable* and the ρ is called admissible for U. For a fixed square integrable U the collection of admissible elements of H will be denoted by $\mathcal{H}^2$. Choosing a convenient $\rho \in \mathcal{H}^2$ the voice transform $V_\rho : H \to L^2_m(G)$ will be unitary. This is a consequence of the following theorem (see [4,5]):

Theorem A. *Let $U_x \in \mathcal{U}$ $(x \in G)$ be an irreducible square integrable representation of G in H. Then the collection of admissible elements $\mathcal{H}^2$ is a linear subspace of H and for every $\rho \in \mathcal{H}^2$ the voice transform of the function f is square integrable on G, namely $V_\rho f \in L^2_m(G)$ if $f \in H$. Moreover there is a symmetric, positive bilinear map $B : \mathcal{H}^2 \times \mathcal{H}^2 \to \mathbb{R}$ such that*

$$[V_{\rho_1} f, V_{\rho_2} g] = B(\rho_1, \rho_2)\langle f, g\rangle, \tag{1.4}$$

for all $f, g \in H$ and $\rho_1, \rho_2 \in \mathcal{H}^2$, where $[\cdot,\cdot]$ is the usual inner product in $L^2_m(G)$. If the group G is unimodular then $B(\rho, \rho) = \|C\rho\|^2$ $(\rho \in \mathcal{H}^2)$, where $C > 0$ is a constant. In this case if we choose ρ so that $\langle C\rho, C\rho\rangle = 1$ then

$$[V_\rho f, V_\rho g] = \langle f, g\rangle \quad (f, g \in H). \tag{1.5}$$

An important consequence of this theorem is the following reproducing formula: if we choose a non-zero $g \in \mathcal{H}^2$ such that $\|Cg\|^2 = 1$, then

$$V_g f = V_g f * V_g g. \tag{1.6}$$

Suppose that the set of analyzing vectors:

$$\mathcal{A} = \left\{g \in H:\ V_g g \in L^1(G)\right\} \neq \{0\}, \tag{1.7}$$

and let define

$$\mathcal{H}^1 := \left\{f \in H:\ V_g f \in L^1(G)\right\}. \tag{1.8}$$

Denote by $\mathcal{H}^{1*}$ the dual of $\mathcal{H}^1$. Then the reproducing formula (1.6) can be extended for $f \in \mathcal{H}^{1*}$ and $g \in \mathcal{A}$ with $\|Cg\|^2 = 1$:

$$\int_G V_g f(x) V_g g(x^{-1}y)\, dm(x) = V_g f(y). \tag{1.9}$$

Note that the integral operator on the left-hand side is a convolution operator on G.

Feichtinger and Gröchenig in [1,6,2,3] described a unified approach to atomic decomposition through integrable group representations. In what follows we will outline how it can be obtained atomic decomposition results in $\mathcal{H}^1$ following the exposition published in [1] for the case when the weight function $w = 1$. Assume that U is an irreducible unitary representation of G on H which is integrable, i.e., there is a $g \in H \setminus \{0\}$ such that $\int_G |V_g g(a)|\, dm(a) < \infty$, and which is continuous, i.e., $U_a g$ is a continuous map of G into H for all $a \in G$. For certain spaces Y of functions on G for which the convolution operator is defined and is continuous for $g \in \mathcal{A}$ the coorbit spaces are defined in the following way:

$$Co(Y) = \{ f \in \mathcal{H}^{1*}\colon\ V_g f \in Y \}, \tag{1.10}$$

and this is independent of the choice of $g \in \mathcal{A}$. Place on $Co(Y)$ the norm $\|f\|_{Co(Y)} = \|V_g f\|_Y$. For example

$$H = Co(L^2(G)), \qquad \mathcal{H}^1 = Co(L^1(G)).$$

At the same time it is defined an appropriate sequence space Y_d corresponding to Y (for example if $Y = L^p(G)$ then $Y_d = \ell^p(\mathbb{Z})$). Let

$$S = \{ F \in Y\colon\ F = V_g f \text{ for some } f \in Co(Y) \}. \tag{1.11}$$

The above convolution operator (which is the identity on S) can be approximated by a discrete operator, similar to a Riemann sum using the so-called *bounded uniform partition of unity*.

Definition 1.1. Given a compact set Q with non-void interior, a countable family $X = (x_i)$ in G is said to be Q-dense if $\bigcup x_i Q = G$. It is separated, if for some compact neighborhood V of the unity we have $x_j V \cap x_j V = \emptyset$, $j \neq i$. We say that $\Psi = \{\psi_k\}_{k\in\mathbb{N}}$ is a bounded uniform partition of unity of size Q (Q-BUPU) if for an open neighborhood Q of unity in G with compact closure there exist points in x_i in G such that

- $0 \leqslant \psi_i(x) \leqslant 1$,
- $\operatorname{supp} \psi_i \subset x_i Q$,
- $\sum_i \psi_i(x) = 1$,
- $\sup_{z\in\mathbb{B}} \#\{ i \in \mathbb{N}\colon\ z \in x_i Q' \} < \infty \quad$ for any $Q' \subset G$ compact. (1.12)

In order to approximate by discrete sum $V_g f$ let write the reproducing formula

$$\int_G V_g f(x) V_g g(x^{-1}y)\, dm(x) = V_g f(y),$$

as convolution operator on G as follows $F = V_g f$, and $F = F * V_g g$. Define the operators $TF = F * V_g g$ and T_Ψ on Y associated to a particular bounded uniform partition of unity Ψ, by

$$T_\Psi(y) = \sum_i \langle F, \psi_i \rangle V_g g(x_i^{-1} y). \tag{1.13}$$

From Lemma 4.3 of [1] it follows that if $F \in L^1(G)$ the sequence of coefficients $\Lambda = (\lambda_i)_{i\in\mathbb{N}}$, given by $\lambda_i = \langle F, \psi_i \rangle$ belongs to ℓ^1, more precisely, given a fixed compact neighborhood Q of unity there exists a constant C_0 such that the norms of the linear operators $F \to \Lambda$ are uniformly bounded by C_0 for all Q-BUPUs. Conversely if $g \in \mathcal{A}$ and $\Lambda = (\lambda_i)_{i\in\mathbb{N}} \in \ell^1$ then $F := \sum_i \lambda_1 V_g g(x_i^{-1} y) \in L^1(G)$, the sum being absolutely convergent in $L^1(G)$ and there is a universal constant C_1 such that $\|F\|_1 \leqslant C_1 \|\Lambda\|_{\ell^1}$. As a consequence the set of operators $\{T_\Psi\}$, where Ψ runs through the family of Q-BUPUs acts uniformly bounded on $L^1(G)$.

Lemma 4.5 of [1] says that the net $\{T_\Psi\}$ of Q-BUPUs directed according to inclusions of the neighborhoods Q of unity is norm convergent to T as operators on $L^1(G)$. As consequence it can be obtained the following atomic decomposition result for $\mathcal{H}^1$.

Theorem B. *(See [1].) For any $g \in \mathcal{A} \setminus \{0\}$, normalized by $\|Cg\|^2 = 1$, there exist a small neighborhood Q of identity and a constant C_0 (both only dependent of g), such that for any collection of points $\{x_i\} \subset G$ which is Q-dense and V-separated and any bounded uniform partition of unity Ψ associated to $\{x_i\}$ any $f \in \mathcal{H}^1$ can be written*

$$f = \sum \lambda_i(f) U_{x_i} g, \quad \text{with } \sum_i \left|\lambda_i(f)\right| \leqslant C_0 \|f\|_{\mathcal{H}^1} \tag{1.14}$$

where the sum is absolutely convergent in $\mathcal{H}^1$. The coefficients $\lambda_i(f) = \langle T_\Psi^{-1} V_g f, \psi_i\rangle$ depend linearly from f.

Thus this gives an atomic decomposition of $f \in \mathcal{H}^1$ with atoms $U_{x_i} g$ which can be seen as generalizations of the frames to Banach spaces, other than Hilbert spaces.

2. The voice transform generated by a representation of the Blaschke group on the weighted Bergman spaces

2.1. The weighted Bergman spaces A_α^p

In this section we summarize the basic results connected to the weighted Bergman spaces (see [7–9]). Let denote by $\mathcal{A}$ the set of functions $f : \mathbb{D} \to \mathbb{C}$ which are analytic in $\mathbb{D}$, denote by

$$dA_\alpha(z) := \frac{\alpha+1}{\pi}\left(1 - |z|^2\right)^\alpha dx\,dy, \quad z = x + iy$$

the weighted area measure on $\mathbb{D}$. For all $\alpha > -1$ let consider the following subset of analytic functions:

$$A_\alpha^p := \left\{ f \in \mathcal{A}: \int_{\mathbb{D}} \left|f(z)\right|^p dA_\alpha(z) < \infty \right\}. \tag{2.1.1}$$

The set $H = A_\alpha^2$ is a Hilbert space with the scalar product

$$\langle f, g\rangle_\alpha := \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA_\alpha(z). \tag{2.1.2}$$

In the special case when $\alpha = 0$, $A^2 = A_0^2$ is the so-called *Bergman space* (see [7,8]). For $0 < p < \infty$ and $-1 < \alpha < \infty$ the weighted Bergman space A_α^p is a closed subspace of $L^p(\mathbb{D}, dA_\alpha) = L^p$.

For a function $f \in A_\alpha^p$ and for a compact subset K of $\mathbb{D}$ there exists a positive constant $C = C(n, K, p, \alpha)$ such that

$$\sup\left\{\left|f^{(n)}(z)\right|: z \in K\right\} \leqq C\|f\|_{A_\alpha^p}.$$

From this it follows that the point-evaluation map is a bounded linear functional on A_α^p, and the norm convergence in A_α^p implies the locally uniform convergence on $\mathbb{D}$.

The weighted Bergman kernel is given by

$$K_\alpha(\xi, z) = \frac{1}{(1 - \bar{z}\xi)^{\alpha+2}}, \tag{2.1.3}$$

and the corresponding weighted Bergman projection is defined by

$$P_\alpha f(z) = \int_{\mathbb{D}} f(\xi)\frac{1}{(1 - \bar{\xi}z)^{\alpha+2}}\, dA_\alpha(\xi). \tag{2.1.4}$$

For $-1 < \alpha < +\infty$ the weighted Bergman projection

$$P_\alpha : L^2(\mathbb{D}, dA_\alpha) \to A_\alpha^2$$

is an orthogonal projection operator, which satisfies $P_\alpha f = f$ for $f \in A_\alpha^2$ and is a pointwise formula. The projection operator can be extended to $L^1(\mathbb{D}, dA_\alpha)$ by mapping each $f \in L^1(\mathbb{D}, dA_\alpha)$ to a analytic function in $\mathbb{D}$, and

$$f(z) = \frac{\alpha+1}{\pi}\int_{\mathbb{D}} f(\xi)\frac{1}{(1 - \bar{\xi}z)^{\alpha+2}}\left(1 - |\xi|^2\right)^\alpha d\xi_1\, d\xi_2 \quad \left(f \in A_\alpha^1,\ z, \xi \in \mathbb{D},\ \xi = \xi_1 + i\xi_2\right) \tag{2.1.5}$$

and the integral converges uniformly in z in every compact subset of $\mathbb{D}$ (see [8, p. 6]).

Theorem C. *(See [8].) For any $-1 < \alpha < +\infty$ and any real β, let*

$$I_{\alpha,\beta}(z) = \int\limits_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-z\bar{w}|^{2+\alpha+\beta}}\, dA(w), \quad z \in \mathbb{D}. \tag{2.1.6}$$

Then we have the following estimates:

$$I_{\alpha,\beta}(z) \sim \begin{cases} 1, & \beta < 0, \\ \log\frac{1}{1-|z|^2}, & \beta = 0, \\ \frac{1}{(1-|z|^2)^\beta}, & \beta > 0 \end{cases} \tag{2.1.7}$$

as $|z| \to 1^-$.

Theorem D. *(See [8].) Suppose $-1 < \alpha, \beta < +\infty$ and $1 \leqslant p < +\infty$. Then P_α is a bounded projection from $L^p(\mathbb{D}, dA_\beta)$ onto A^p_β if and only if $(\beta+1) < (\alpha+1)p$.*

2.2. The Blaschke group

Let us denote by

$$B_a(z) := \epsilon\frac{z-b}{1-\bar{b}z} \quad \big(z \in \mathbb{C},\ a = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T},\ \bar{b}z \neq 1\big) \tag{2.2.1}$$

the so-called *Blaschke functions*, where

$$\mathbb{D} := \big\{z \in \mathbb{C}\colon\ |z| < 1\big\}, \qquad \mathbb{T} := \big\{z \in \mathbb{C}\colon\ |z| = 1\big\}. \tag{2.2.2}$$

If $a \in \mathbb{B}$, then B_a is an 1–1 map on $\mathbb{T}$, $\mathbb{D}$, respectively. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{a_1} \circ B_{a_2})(z) := B_{a_1}(B_{a_2}(z))$ form a group. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way $B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2}$. The group $(\mathbb{B}, \circ)$ will be isomorphic with the group $(\{B_a, a \in \mathbb{B}\}, \circ)$. If we use the notations $a_j := (b_j, \epsilon_j)$, $j \in \{1, 2\}$ and $a := (b, \epsilon) =: a_1 \circ a_2$ then

$$b = \frac{b_1\bar{\epsilon}_2 + b_2}{1 + b_1\bar{b}_2\bar{\epsilon}_2} = B_{(-b_2\epsilon_2, \bar{\epsilon}_2)}(b_1), \qquad \epsilon = \epsilon_1\frac{\epsilon_2 + b_1\bar{b}_2}{1 + \epsilon_2\bar{b}_1 b_2} = B_{(-b_1\bar{b}_2, \epsilon_1)}(\epsilon_2). \tag{2.2.3}$$

The neutral element of the group $(\mathbb{B}, \circ)$ is $e := (0, 1) \in \mathbb{B}$ and the inverse element of $a = (b, \epsilon) \in \mathbb{B}$ is $a^{-1} = (-b\epsilon, \bar{\epsilon})$.

The integral of the function $f : \mathbb{B} \to \mathbb{C}$, with respect to this left invariant Haar measure m of the group $(\mathbb{B}, \circ)$, is given by

$$\int\limits_{\mathbb{B}} f(a)\, dm(a) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\int\limits_{\mathbb{D}} \frac{f(b, e^{it})}{(1-|b|^2)^2}\, db_1\, db_2\, dt, \tag{2.2.4}$$

where $a = (b, e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{D} \times \mathbb{T}$.

It can be shown that this integral is invariant under the inverse transformation $a \to a^{-1}$, so this group is unimodular.

2.3. The representation of Blaschke group on the Hilbert space A^2_α

In [10,11] the voice transform induced by a representation of the Blaschke group on the weighted Bergman spaces was studied. We summarize the basic properties of this special voice transform which we will need. Let consider the following set of functions

$$F_a(z) := \frac{\sqrt{\epsilon(1-|b|^2)}}{1-\bar{b}z} \quad \big(a = (b, \epsilon) \in \mathbb{B},\ z \in \bar{\mathbb{D}}\big). \tag{2.3.1}$$

For every power α ($\alpha \geqslant 0$), F_a induce a unitary representation of Blaschke group on the space A^2_α. Namely, let define

$$U^\alpha_a f := [F_{a^{-1}}]^{\alpha+2} f \circ B_a^{-1} \quad \big(a \in \mathbb{B},\ \alpha \geqslant 0,\ f \in A^2_\alpha\big). \tag{2.3.2}$$

It can be proved that for all $\alpha \geqslant 0$, U^α_a $(a \in \mathbb{B})$ is a unitary representation of the group $\mathbb{B}$ on the Hilbert space A^2_α which is irreducible.

The representation has the following explicit form

$$\big(U^\alpha_{a^{-1}} f\big)(z) := e^{i\frac{\alpha+2}{2}\psi}\frac{(1-|b|^2)^{\frac{\alpha+2}{2}}}{(1-\bar{b}z)^{\alpha+2}} f\left(e^{i\psi}\frac{z-b}{1-\bar{b}z}\right) \quad \big(a = \big(b, e^{i\psi}\big) \in \mathbb{B}\big) \tag{2.3.3}$$

and the unitarity means that

$$\langle f, g\rangle = \langle f, g\rangle_\alpha := \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA_\alpha(z) = \langle U_a^\alpha f, U_a^\alpha g\rangle_\alpha. \tag{2.3.4}$$

It is simpler to take the expression of the representation for $a^{-1} \in \mathbb{B}$, correspondingly it is easier to study the voice transform in $a^{-1} \in \mathbb{B}$ ($a = (b, e^{i\psi}) \in \mathbb{B}$, $f, \rho \in A_\alpha^2$):

$$(V_g f)(a^{-1}) = (V_g f)(-b\epsilon, \bar{\epsilon}) := \langle f, U_{a^{-1}} g\rangle_\alpha. \tag{2.3.5}$$

From the general theory (see [5,4]) it follows that: the voice transform generated by representation U_a ($a \in \mathbb{B}$) is one to one. The function $V_\rho f$ is continuous and bounded on $\mathbb{B}$. It can be shown that every element from A_α^2 is admissible. Taking in consideration that the Blaschke group is unimodular Theorem A implies that for $f, g \in A_\alpha^2$ such that $g \neq 0$ and $\|Cg\| = 1$ the following reproducing formula is valid:

$$V_g f = V_g f * V_g g, \quad \text{i.e., } V_g f(y^{-1}) = \int_{\mathbb{B}} V_g f(x^{-1}) V_g g(x \circ y^{-1})\, dm(x). \tag{2.3.6}$$

3. New results

In this section we show that in the Blaschke group there exist right bounded uniform partitions of the unity, we will study the integrability of the voice transform given by (2.3.5). It turns out that the constant function $f = 1$ and every function from the minimal Möbius invariant space B_1 satisfies the integrability condition. It is shown that in the case of the weighted Bergman spaces, where the weight is generated by $\alpha > 0$, the general theory of atomic decomposition can be applied and in this way we can fide new atoms for these spaces.

3.1. Bounded uniform partition on Blaschke group

As we have seen before in the unified approach of the atomic decomposition the Q-density, the V-separated property and the bounded uniform partitions of the unity are the basic starting points.

Our aim is to show that in the Blaschke group there exist Q-dense V-separated sequences. As we will see it is easier to show the Q-density from right i.e., there is a sequence $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{B}$ such that $\bigcup Qx_i = \mathbb{B}$, and separated from right (for some compact neighborhood V of the unity we have $Vx_j \cap Vx_j = \emptyset$, $j \neq i$) and there exist also bounded uniform partitions of the unity.

The description of the Q-density from the left (as it is given in Definition 1.1) in general it is not same with the Q-density from right, which is the case when a group is IN-group.

A group G is an IN-group if there exists a compact neighborhood of the unity which is invariant under all inner automorphisms. Our conjecture is that the Blaschke group is not an IN-group.

Recall that the hyperbolic distance of two points from the unit disc is given by

$$\beta(z, w) = \frac{1}{2}\log\frac{1+\rho(z, w)}{1-\rho(z, w)}, \quad \rho(z, w) = \left|\frac{z-w}{1-\bar{w}z}\right| = \left|B_{(w,1)}(z)\right|, \tag{3.1.1}$$

and the hyperbolic disc or Bergman disc of radius $r > 0$ and center b is

$$D(b, r) = \{z \in \mathbb{D}\colon \beta(z, b) < r\}. \tag{3.1.2}$$

Lemma 3.1.1. *Let consider $r > 0$ and $Q = Q_1 \times \mathbb{T}$, where $Q_1 = \{z \in \mathbb{D}\colon |z| < \tanh r\}$. Then there exists a sequence $x_n = (b_n, -1) \in \mathbb{B}$ which is Q-dense from the right, i.e. $\bigcup Qx_n = \mathbb{B}$ and V-separated from right, i.e. $Vx_n \cap Vx_m = \emptyset$, and there is also a corresponding right bounded uniform partition of the unity corresponding to $\{x_n\}$.*

Proof. Due to Lemma 2.13 from [8, p. 39], for every fix r, $0 < r < +\infty$, and N positive integer there exists a sequence $\{b_n\}_{n\in\mathbb{N}} \in \mathbb{D}$ such that the disc is covered by the hyperbolic discs $\{D(b_n, r)\}_{n\in\mathbb{N}}$, and if $m \neq n$ then $\beta(b_n, b_m) \geqslant \frac{r}{2}$ and every $z \in \mathbb{D}$ belongs to at most N hyperbolic discs $D(b_n, r)$. We observe that $z \in D(b, r)$ is equivalent with $z \in \{z \in \mathbb{D}\colon \rho(z, b) < \tanh r < 1\} = B_{(b,-1)}(\{z \in \mathbb{D}\colon |z| < \tanh r\}) = B_{(b,-1)}(Q_1)$ (see [7, p. 40]). Then for

$$Qx_n = \{x \circ x_n\colon x = (b, \epsilon) \in Q\} = \{(B_{(b_n,-1)}(b), B_{(-b\overline{b_n},\epsilon)}(-1))\colon b \in Q_1,\ \epsilon \in \mathbb{T}\} = \{D(b_n, r)\} \times \mathbb{T}, \tag{3.1.3}$$

from this we obtain that $\bigcup Qx_n = \mathbb{B}$. If we take $V = V_1 \times \mathbb{T}$ with $V_1 = \{z \in \mathbb{D}\colon |z| < \tanh\frac{r}{4}\}$, then $Vx_n \cap Vx_m = \phi$ for $m \neq n$. Now we are ready to give an example of right bounded uniform partition of unity. Due to Lemma 2.28 from [9, p. 63], there exists a Borel set D_k satisfying the following conditions:

- $D\left(b_k, \frac{r}{4}\right) \subset D_k \subset D(b_k, r)$,
- $D_m \cap D_n = \phi$,
- $\mathbb{D} = \bigcup D_k$. (3.1.4)

Then $B_{(b_k,-1)}(\{z \in \mathbb{D}\colon\ |z| < \tanh\frac{r}{4}\}) \subset D_k \subset B_{(b_k,-1)}(\{z \in \mathbb{D}\colon\ |z| < \tanh r\})$. Let consider $\psi_k = \chi_{D_k \times \mathbb{T}}$ the characteristic function of the set $D_k \times \mathbb{T}$ then $\Psi = \{\psi_k\}_{k\in\mathbb{N}}$ is a bounded uniform partition of unity from right of size Q. Indeed, for all $i \in \mathbb{N}$

- $0 \leqslant \psi_i(x) \leqslant 1$,
- $\operatorname{supp}\psi_i \subset Q x_i$,
- $\sum_i \psi_i(x) = 1, \quad x \in \mathbb{B}$,
- $\sup_{z\in\mathbb{B}} \#\{i \in \mathbb{N}\colon\ z \in Q' x_i\} < \infty \quad \text{for any } Q' \subset \mathbb{B} \text{ compact.} \qquad \Box$ (3.1.5)

We shall consider the set of Q-bounded uniform partitions of unity from right (Q-RBUPUs) as a net directed by inclusion of the associated neighborhoods, and write $\Psi \to \infty$ if these neighborhoods run trough a neighborhood base of identity. In the general theory of atomic decomposition it is used the Q-density from the left, this is the reason why in Section 3.3 we will make a small modification in the discretizing operator which corresponds to the Q-density from the right in order to obtain atomic decomposition in the weighted Bergman spaces.

3.2. New properties of the voice transform of the Blaschke group

We observe that the voice transform of the Blaschke group generated by the representation of this group on the weighted Bergman space A^2_α given by formula (2.3.5) can be expressed by the weighted Bergman projection operator in the following way:

$$V_g f\left(a^{-1}\right) = \langle f, U_{a^{-1}} g\rangle_\alpha = e^{\frac{\alpha+2}{2}\psi}\left(1-|b|^2\right)^{\frac{\alpha+2}{2}} P_\alpha\left(f \cdot \overline{g(B_a)}\right) \quad \left(a = \left(b, e^{i\psi}\right) \in \mathbb{B},\ f, g \in A^2_\alpha\right). \tag{3.2.1}$$

First we will study the integrability of the voice transform, i.e. we show that there exist an element $g \in A^2_\alpha$, $g \neq 0$ such that

$$\int_{\mathbb{B}} \left|V_g g\left(a^{-1}\right)\right| dm(a) < \infty. \tag{3.2.2}$$

Theorem 3.2.1. *If $\alpha > 0$, then the representation $U^\alpha_{a^{-1}}$ is integrable.*

Proof. Let consider $g = 1 \in A^2_\alpha$, using (3.2.1) we get:

$$\begin{aligned} V_g g\left(a^{-1}\right) &= e^{\frac{\alpha+2}{2}\psi}\left(1-|b|^2\right)^{\frac{\alpha+2}{2}} P_\alpha\left(g \cdot \overline{g(B_a)}\right) \\ &= e^{\frac{\alpha+2}{2}\psi}\left(1-|b|^2\right)^{\frac{\alpha+2}{2}} \int_{\mathbb{D}} \frac{1}{(1-\bar{z}b)^{\alpha+2}}\, dA_\alpha(z) = e^{\frac{\alpha+2}{2}\psi}\left(1-|b|^2\right)^{\frac{\alpha+2}{2}}. \end{aligned} \tag{3.2.3}$$

Then

$$\int_{\mathbb{B}} \left|V_g g\left(a^{-1}\right)\right| dm(a) = \int_{\mathbb{D}} \left(1-|b|^2\right)^{\frac{\alpha+2}{2}} \frac{1}{(1-|b|^2)^2}\, dA(b) = \int_0^1 (1-r)^{\frac{\alpha}{2}-1}\, dr = \frac{2}{\alpha} < \infty. \tag{3.2.4}$$

Thus we have that

$$\mathcal{A} = \left\{g \in A^2_\alpha\colon\ V_g g \in L^1(\mathbb{B})\right\} \neq \{0\}. \qquad \Box \tag{3.2.5}$$

From Theorem A and (3.2.3) it follows that the value of the constant C in Theorem A for the voice transform of the Blaschke group given by (3.2.1) is $\sqrt{\pi/(\alpha+1)}$.

We will show that the integrability condition is also satisfied by every g from the minimal Möbius invariant space of analytic functions (see [12,13]), denoted by B_1, which contains exactly the analytic functions on the unit disc which admit the representation

$$g(z) = \sum_{j=0}^{\infty} \lambda_j \frac{z-b_j}{1-\overline{b_j}z}, \quad |b_j| \leqslant 1, \qquad \sum_{j=0}^{\infty} |\lambda_j| < \infty. \tag{3.2.6}$$

It is easy to prove that for $1 \leqslant p$ and $-1 < \alpha$ the space B_1 is included in A^p_α.

Theorem 3.2.2. *For $\alpha > 0$ the space B_1 is a subset of $\mathcal{A}$.*

Proof. For $g \in B_1$ we have the following estimate

$$\begin{aligned}\left|V_g g\left(a^{-1}\right)\right| &= \left|e^{\frac{\alpha+2}{2}\psi}\left(1-|b|^2\right)^{\frac{\alpha+2}{2}} P_\alpha\left(g \cdot \overline{g(B_a)}\right)\right| \leqslant \left(1-|b|^2\right)^{\frac{\alpha+2}{2}} \int_{\mathbb{D}} \left(\sum_j^\infty |\lambda_j|\right)^2 \frac{1}{|1-\bar{z}b|^{\alpha+2}}\, dA_\alpha(z) \\ &= \left(1-|b|^2\right)^{\frac{\alpha+2}{2}} \left(\sum_j^\infty |\lambda_j|\right)^2 I_{\alpha,0}(b).\end{aligned} \tag{3.2.7}$$

Due to Theorem C, when $|b| \to 1^-$ we have $I_{\alpha,0}(b) \sim \log \frac{1}{1-|b|^2}$. For $\alpha > 0$

$$\begin{aligned}&\int_{\mathbb{D}} \left(1-|b|^2\right)^{\frac{\alpha+2}{2}} \log \frac{1}{1-|b|^2} \frac{1}{(1-|b|^2)^2}\, dA(b) \\ &\quad = -\int_0^1 (1-r)^{\frac{\alpha-2}{2}} \log(1-r)\, dr = -\left[\frac{2}{\alpha}(1-y)^{\frac{\alpha}{2}} \log(1-y) - \left(\frac{2}{\alpha}\right)^2 (1-y)^{\frac{\alpha}{2}}\right]_0^1 = \frac{4}{\alpha^2}.\end{aligned} \tag{3.2.8}$$

From this it follows that

$$\int_{\mathbb{B}} \left|V_g g\left(a^{-1}\right)\right| dm(a) < +\infty. \qquad \square$$

From now on we choose the parameter function g always from the space $B_1 \cup \{1\}$, we also restrict the domain of the definition of the voice transform for $a = (b, 1) \in \mathbb{B}$. We show that the voice transform $V_g f$ can be defined not only for f belonging to A^2_α but under some assumptions on the parameters $V_g f$ has sense for $f \in A^p_\beta$, and we will study some growth properties of the voice transform.

Theorem 3.2.3. *Let fix the function g from $B_1 \cup \{1\}$. If $-1 < \alpha, \beta < +\infty$, $1 \leqslant p$, $(\beta+1) < (\alpha+1)p$, then for every $f \in A^p_\beta$ the voice transform is well defined. If $a = (b, 1) \in \mathbb{B}$, then*

$$V_g f\left(a^{-1}\right) = V_g f(-b, 1) = \left(1-|b|^2\right)^{\frac{\alpha+2}{2}} F_1(b), \tag{3.2.9}$$

where $F_1(b) \in A^p_\beta$, and

$$\lim_{|b| \to 1^-} \left(1-|b|^2\right)^{\frac{\beta+2}{p} - \frac{\alpha+2}{2}} \left|V_g f(b)\right| = 0. \tag{3.2.10}$$

Proof. The proof for $g = 1$ is trivial. If $g \in B_1$, then

$$g(z) = \sum_{j=0}^\infty \lambda_j \frac{z - b_j}{1 - \overline{b_j} z} = \sum_{j=0}^\infty \lambda_j B_{(b_j,1)}(z), \quad |b_j| \leqslant 1, \qquad \sum_{j=0}^\infty |\lambda_j| < \infty. \tag{3.2.11}$$

This implies that

$$g\left(B_a(z)\right) = \sum_{j=0}^\infty \lambda_j B_{(b_j,1)\circ a}(z) \in B_1, \qquad \sum_{j=0}^\infty |\lambda_j| < \infty. \tag{3.2.12}$$

We show that if $f \in A^p_\beta$, then $f \cdot \overline{g(B_a)} \in L^p(\mathbb{D}, dA_\beta)$. This follows immediately from the following inequality:

$$\left|f(z)\overline{g\left(B_a(z)\right)}\right|^p \leqslant \left|f(z)\right|^p \left(\sum_{j=1}^{+\infty} \left|\lambda_j B_{(b_j,1)\circ a}(z)\right|\right)^p \leqslant \left|f(z)\right|^p \left(\sum_{j=1}^{+\infty} |\lambda_j|\right)^p.$$

Using Theorem D, we obtain that if $-1 < \alpha, \beta < +\infty$, $1 \leqslant p$ and $(\beta+1) < (\alpha+1)p$, then P_α is a bounded projection from $L^p(\mathbb{D}, dA_\beta)$ onto A^p_β, which implies that, for every $g \in B_1$ and $f \in A^p_\beta$ the voice transform

$$V_g f\big(a^{-1}\big) = e^{\frac{\alpha+2}{2}\psi}\big(1-|b|^2\big)^{\frac{\alpha+2}{2}} P_\alpha\big(f \cdot \overline{g(B_a)}\big),$$

is well defined. If we consider $a = (b, 1)$ and denote by

$$F_1(b) = P_\alpha\big(f \cdot \overline{g(B_{a^{-1}})}\big),$$

then $F_1 \in A^p_\beta$. For all $F_1 \in A^p_\beta$, if $-1 < \beta < +\infty$, $p > 0$, then we have (see [9])

$$\big|F_1(b)\big| \leqslant \frac{\|F_1\|_{A^p_\beta}}{(1-|b|^2)^{\frac{\beta+2}{p}}}, \quad b \in \mathbb{D}, \tag{3.2.13}$$

the exponent of $(1-|b|^2)$ is best possible, and it can be obtained the following improved behavior of F_1 near the boundary:

$$\lim_{|b|\to 1^-} \big|F_1(b)\big|\big(1-|b|^2\big)^{\frac{\beta+2}{p}} = 0. \tag{3.2.14}$$

This implies that

$$\lim_{|b|\to 1^-} \big(1-|b|^2\big)^{\frac{\beta+2}{p}-\frac{\alpha+2}{2}} \big|V_g f(b)\big| = 0. \qquad \square$$

For $\alpha = \beta$ and $p = 2$ it follows that, if $f \in A^2_\alpha$, then

$$\lim_{|b|\to 1^-} \big|V_g f(b)\big| = 0. \tag{3.2.15}$$

The next theorem gives information about the set

$$\mathcal{H}^1 = \big\{f \in A^2_\alpha\colon\ V_g f \in L^1(\mathbb{B})\big\}. \tag{3.2.16}$$

Theorem 3.2.4. *Let $g \in B_1 \cup \{1\}$, $\alpha > 0$, $p \geqslant 1$ and $p > \max\{\frac{\beta+1}{\alpha+1}, \frac{4+2\beta}{\alpha}\}$, then for every $f \in A^p_\beta$ the voice transform $V_g f$ is integrable, i.e., $V_g f \in L^1(\mathbb{B})$.*

As an immediate consequence of this theorem we get that for $\alpha = \beta > 0$, $p > 2 + \frac{4}{\alpha}$ we have that $A^p_\alpha \subset \mathcal{H}^1$.

Proof. We have to show that if the assumptions of the theorem are satisfied, then

$$\int_{\mathbb{B}} \big|V_g f\big(a^{-1}\big)\big|\, dm(a) < +\infty.$$

Using Theorem 3.2.3 and (3.2.13) we obtain that

$$\begin{aligned}\int_{\mathbb{B}} \big|V_g f\big(a^{-1}\big)\big|\, dm(a) &= \int_{\mathbb{D}} \big(1-|b|^2\big)^{\frac{\alpha-2}{2}} \big|F_1(b)\big|\, dA(b) \leqslant \|F_1\|_{A^p_\beta} \int_{\mathbb{D}} \big(1-|b|^2\big)^{\frac{\alpha-2}{2}-\frac{2+\beta}{p}}\, dA(b)\\ &= \|F_1\|_{A^p_\beta} \int_{\mathbb{D}} \big(1-r^2\big)^{\frac{\alpha-2}{2}-\frac{2+\beta}{p}} 2r\, dr = \|F_1\|_{A^p_\beta} \frac{1}{\frac{\alpha}{2}-\frac{2+\beta}{p}} < +\infty. \qquad \square\end{aligned}$$

3.3. Application of the Feichtinger–Gröchenig theory

Now we are ready to apply the general theory of Feichtinger and Gröchenig to obtain atomic decompositions in weighted Bergman spaces. From this result, as a special case, we reobtain some well-known atomic decompositions in the weighted Bergman spaces, but also we get new atomic decompositions for this spaces. As we have mentioned earlier in the Blaschke group it is easier to give Q-RBUPU, it is more convenient to compute the voice transform given by (2.3.5) in $a^{-1} \in \mathbb{B}$, also the reproducing formula (2.3.7), taking into account that the Blaschke group is unimodular, can be written as follows

$$V_g f\big(y^{-1}\big) = \int_{\mathbb{B}} V_g f\big(x^{-1}\big) V_g g\big(x \circ y^{-1}\big)\, dm(x), \quad f, g \in A^2_\alpha,\ g \neq 0, \qquad \|Cg\| = 1. \tag{3.3.1}$$

From Theorem 3.2.4 for $\alpha=\beta>0$, $p>2+\frac{4}{\alpha}$, $g\in B_1\cup\{1\}$ we have the following inclusion $A^p_\alpha\subset\mathcal{H}^1$, where

$$\mathcal{H}^1=\{f\in A^2_\alpha:\ V_gf\in L^1(\mathbb{B})\},$$

and $\|f\|_{\mathcal{H}^1}=\|V_gf\|_{L^1(\mathbb{B})}\leqslant C_2\|F_1\|_{A^p_\beta}$. Let denote $F(y^{-1})=V_gf(y^{-1})$, $G(y^{-1})=V_gg(y^{-1})$, then the reproducing formula (3.3.1) is a convolution operator T, $TF=F\star G$, to discretize this for $F,G\in L^1(\mathbb{B})$ by means of Q-RBUPU we will use the modified version of the operator (1.13) given by

$$T_\Psi F(y^{-1})=\sum_i\langle F,\psi_i\rangle L_{x_i^{-1}}G(y^{-1}),\quad F,G\in L^1(\mathbb{B}),\tag{3.3.2}$$

which is composed of a coefficients mapping $F\to(\lambda_i)_{i\in\mathbb{N}}$ with $\lambda_i=\langle F,\psi_i\rangle=\int_{\mathbb{B}}F(y^{-1})\psi_i(y)\,dm(y)$ and a convolution operator $(\lambda_i)_{i\in\mathbb{N}}\to\sum_i\lambda_iL_{x_i^{-1}}G=(\sum_i\lambda_i\delta_{x_i^{-1}})\star G$. Our aim is to approximate the convolution operator $TF=F\star G$ by the modified operator (3.3.2). Analogous to Lemma 4.3 from [1] it can be proved that:

i) For $F\in L^1(\mathbb{B})$ the sequence of coefficients $(\lambda_i)_{i\in\mathbb{N}}$ given by $\lambda_i=\langle F,\psi_i\rangle=\int_{\mathbb{B}}F(y^{-1})\psi_i(y)\,dm(y)$ belongs to ℓ^1, and the norms of the linear operators $F\to(\lambda_i)_{i\in\mathbb{N}}$ are uniformly bounded.

ii) Given $G\in L^1(\mathbb{B})$, $(\lambda_i)_{i\in\mathbb{N}}\in\ell^1$ and any family $X=(x_i)_{i\in\mathbb{N}}$ in the group one has

$$F(y^{-1})=\sum_i\lambda_iL_{x_i^{-1}}G(y^{-1})\in L^1(\mathbb{B}),\tag{3.3.3}$$

the sum being absolutely convergent in $L^1(\mathbb{B})$, and there is a universal constant C_1 such that $\|F\|_1\leqslant C_1\|(\lambda_i)_{i\in\mathbb{N}}\|_1$.

There is valid also the analogue of Lemma 4.5 from [1] the only differences in the proof arise because of Q-RBUPU.

Lemma 3.3.1. *The net set $\{T_\Psi\}$ of Q-RBUPU, directed according to inclusions of the neighborhoods Q to $\{e=(0,1)\}$, is norm convergent as operators on $L^1(\mathbb{B})$: $\lim_{\Psi\to0}|||T_\Psi-T|||_1=0$.*

Proof. The proof follows the steps of the proof of Lemma 4.5 from [1], the only difference occurs when we decompose the integral over the group using the R-BUPUs. For a given $F\in L^1(\mathbb{B})$ we can give the following estimate:

$$\begin{aligned}\|TF-T_\Psi F\|_1&=\Big\|\Big(\sum_i\big(F\psi_i-\langle F,\psi_i\rangle\delta_{x_i^{-1}}\big)\star G\Big)\Big\|_1\leqslant\sum_i\Big\|\int_{Qx_i}F(y^{-1})\psi_i(L_{y^{-1}}G-L_{x_i^{-1}}G)\,dy\Big\|_1\\&\leqslant\sum_i\int_{Qx_i}|F(y^{-1})|\psi_i\|(L_{y^{-1}}G-L_{x_i^{-1}}G)\|\,dy\leqslant\sum_i\sup_{u\in Q}\|(L_{x_i^{-1}u^{-1}}G-L_{x_i^{-1}}G)\|\langle|F|,\psi_i\rangle\\&\leqslant\sup_{u\in Q}\|(L_{u^{-1}}G-G)\|_1\sum_i\langle|F|,\psi_i\rangle\leqslant\omega_Q(G)C_0\|F\|_1,\end{aligned}\tag{3.3.4}$$

where $\omega_Q(G)=\sup_{u\in Q}\|(L_{u^{-1}}G-G)\|_1$. Since $Q=Q_1\times\mathbb{T}$ is invariant under the inverse operation i.e., $u\in Q$ if and only if $u^{-1}\in Q$, we have that $\omega_Q(G)=\sup_{u\in Q}\|(L_{u^{-1}}G-G)\|_1=\sup_{u\in Q}\|(L_uG-G)\|_1$ is the modulus of continuity of G with respect to $\|.\|_1$. Thus from $G\in L^1(\mathbb{B})$ we have that

$$\big|\|T_\Psi-T\|\big|_1\leqslant C_0\omega_Q(G)\to0\quad\text{for }Q\to\{e\}.\qquad\square$$

Now, taking in consideration that

$$V_g(U^\alpha_{a^{-1}}f)=L_{a^{-1}}V_gf,$$

from Lemma 3.3.1 we get in analogous way as in [1, Theorem 4.7] that T_Ψ has an inverse and:

Consequence 3.3.1. *For any $g\in\mathcal{A}$, $g\neq0$ and $\|Cg\|=1$ there exists a neighborhood Q of the identity and a constant $C_1>0$ both depending only on g such that for every Q-dense family $(x_i)_{i\in\mathbb{N}}$ from right of the Blaschke group any $f\in\mathcal{H}^1$ can be written as*

$$f(z)=\sum_i\lambda_i(U^\alpha_{x_i^{-1}}g)(z)\quad\textit{with}\ \sum_i|\lambda_i|\leqslant C_1\|f\|_{\mathcal{H}^1},\tag{3.3.5}$$

the series is absolutely convergent in $\mathcal{H}^1$. The coefficients depend linearly on f, namely $\lambda_i=\int_{\mathbb{D}}T_\Psi^{-1}(V_gf(y^{-1}))\psi_i(y)\,dA(y)$.

Thus this gives an atomic decomposition of $f\in\mathcal{H}^1$ with atoms $U^\alpha_{x_i^{-1}}g$, $g\in B_1$. From Theorem 3.2.5 it follows that for $p>2+\frac{4}{\alpha}$ we have $A^p_\alpha\subset\mathcal{H}^1$, consequently the previous atomic decomposition is true also for A^p_α under the mentioned restrictions to the parameters.

The Q-density from right of the set $\{x_i = (b_i, -1)\}_{i\in\mathbb{N}}$ in the language of the complex analysis is equivalent to the ϵ-net property of $\{b_i\}_{i\in\mathbb{N}}$, with $\epsilon = \tanh r$ (see [8, p. 172]). From Lemma 8 [7, p. 188] we have that the lower density of the set $\{b_i\}$

$$D^-\big(\{b_i\}\big) \geqslant \frac{(1-\tanh r)^2}{2\tanh^2 r}.$$

Using Theorem 5.23 from [8, p. 161], we have that a separated sequence $\{b_i\}$ is a sampling sequence for A^p_α if and only if

$$D^-\big(\{b_i\}\big) > \frac{\alpha+1}{p}.$$

Let choose r so small that

$$\frac{(1-\tanh r)^2}{2\tanh^2 r} > \frac{\alpha+1}{p},$$

then $\{b_i\}$ is a sampling sequence for A^p_α.

Then for the special case $g = 1$ we obtain the following atomic decomposition: if $f \in A^p_\alpha$, $\alpha > 0$, and $p > 2 + \frac{4}{\alpha}$,

$$f = \sum \lambda_i(f) U^\alpha_{x_i^{-1}} 1 = \sum \lambda_i(f) \frac{(1-|b_i|^2)^{\frac{\alpha+2}{2}}}{(1-\overline{b_i}z)^{\alpha+2}}, \tag{3.3.6}$$

holds, which is very similar to the atomic decompositions obtained with complex analysis techniques (see [9, p. 69]), the difference is that in our case we have ℓ^1 information about the coefficients instead of ℓ^p information and the convergence is in $\mathcal{H}^1$ norm instead of A^p_α. Using the classical techniques of the complex analysis in the atomic decomposition of a function $f \in A^p_\beta$, the atoms are of form (see [9, p. 69])

$$\frac{(1-|x_i|^2)^a}{(1-\overline{x_i}z)^b}.$$

Applying the Feichtinger–Gröchenig theory we obtain more general atoms for the weighted Bergman spaces, i.e., every function $g \in B_1$ generates an atomic decomposition for $f \in A^p_\alpha$ with atoms of the form

$$U^\alpha_{x_i^{-1}} g.$$

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