



The Douglas property for multiplier algebras of operators[☆]

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ABSTRACT

For a collection of reproducing kernels k which includes those for the Hardy space of the polydisk and the ball and for the Bergman space, k is a complete Pick kernel if and only if the multiplier algebra of $H^2(k)$ has the Douglas property. Consequences for solving the operator equation $AX = Y$ are examined.

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1. Introduction

Let H denote a (complex) Hilbert space and let $B(H)$ denote the algebra of bounded operators on H . Given $A, B \in B(H)$, when does there exist $X \in B(H)$ such that $AX = B$ and, if such an X exists, what is the smallest possible norm? The solution to both questions is given by the well-known Douglas Lemma [4], which says that there is an X of norm at most one such that $AX = B$ if and only if $AA^* \succcurlyeq BB^*$.

Let E denote a Hilbert space. A theorem of Leech [6] says that the Douglas Lemma remains true if the algebra $B(H)$ is replaced by the algebra \mathcal{T}_E of E -valued Toeplitz operators on the unit circle; i.e., if T_A and T_B are bounded analytic Toeplitz operators with symbols A and B respectively acting on the Hardy space of Hilbert space of E -valued functions (denoted by $H_E^2(\mathbb{D})$), then there is a bounded analytic Toeplitz operator T_C with symbol C of norm at most one such that $T_A T_C = T_B$ if and only if $T_A T_A^* \succcurlyeq T_B T_B^*$.

If \mathcal{A} is an algebra of operators on a Hilbert space H , then $M_n(\mathcal{A})$, the $n \times n$ matrices with entries from \mathcal{A} is, in the natural way, an algebra of operators on $\bigoplus_1^n H$, the Hilbert space direct sum of H with itself n times. The algebra \mathcal{A} has the Douglas property if, given n and $A, B \in M_n(\mathcal{A})$, there exists a contraction, $C \in M_n(\mathcal{A})$, such that $AC = B$ if and only if $AA^* \succcurlyeq BB^*$ (a more flexible, but equivalent, definition is given later). The Douglas Lemma and Leech's Theorem say that $B(H)$ and the algebra of analytic Toeplitz operators respectively have the Douglas property. Fialkow and Salas considered the problem of which C^* -algebras, like $B(H)$, have the Douglas property [5]. This article considers the question of which multiplier algebras on reproducing kernel Hilbert spaces, like \mathcal{T}_E , have the Douglas property.

A main result of this article, Theorem 1.10, says, for a natural collection of reproducing kernels k , if the algebra of multipliers on the corresponding reproducing kernel Hilbert space has the Douglas property, then k is a complete Pick kernel. As a consequence it follows that the multiplier algebras of the Hardy spaces on the unit ball and the unit polydisk in dimension $n \geq 2$ and the Bergman spaces on the unit ball and the unit polydisk in all dimensions, do not have the Douglas property, since it is well known that the reproducing kernels of these spaces are not complete Pick kernels [7,9]. If \mathcal{M} is one

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of these multiplier algebras, then there exist $A, B \in \mathcal{M} \otimes_{\text{wot}} B(l^2)$ (details on the tensor product appear in Section 1.2 below) for which the equation $AX = B$ cannot necessarily be solved in $\mathcal{M} \otimes_{\text{wot}} B(l^2)$, even if $AA^* \succcurlyeq BB^*$. Stated as Theorem 1.12, this is the other main result of this paper. Examples and questions appear at the end of the article.

In the remainder of this introduction we state precisely the main results, first introducing the needed definitions and background. Section 1.1 discusses reproducing kernel Hilbert spaces and their multiplier algebras. The Douglas property is discussed in further detail in Section 1.2. The main results are stated in Section 1.3.

1.1. Reproducing kernels and multiplier algebras

Let Ω denote a set, which in applications is generally a bounded domain in \mathbb{C}^d . A positive semi-definite function, or kernel, $k : \Omega \times \Omega \rightarrow \mathbb{C}$, determines, by standard constructions, a Hilbert space $H^2(k)$ of functions $f : \Omega \rightarrow \mathbb{C}$. In particular, for each $w \in \Omega$ the function $k(\cdot, w) \in H^2(k)$ reproduces the value of an $f \in H^2(k)$ at w ; i.e.,

$$f(w) = \langle f, k(\cdot, w) \rangle.$$

Thus, $\langle k(\cdot, w), k(\cdot, z) \rangle = k(z, w)$ and the span of $\{k(\cdot, w) : w \in \Omega\}$ is dense in $H^2(k)$. There is little lost by assuming, as we generally will, that $k(z, z) > 0$ for all $z \in \Omega$.

The multipliers of $H^2(k)$ are those functions $\phi : \Omega \rightarrow \mathbb{C}$ such that $\phi h \in H^2(k)$ for every $h \in H^2(k)$. By the closed graph theorem ϕ then determines a bounded operator M_ϕ on $H^2(k)$ defined by $M_\phi h = \phi h$. Let $\mathcal{M}(k)$ denote the set of multipliers of $H^2(k)$ identified as the unital subalgebra $\{M_\phi : \phi \in \mathcal{M}(k)\}$ of $B(H^2(k))$. For example, the Hardy space $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space whose kernel s is the Szegő kernel

$$s(z, w) = \frac{1}{1 - z\bar{w}}.$$

In this case the multiplier algebra $\mathcal{M}(s)$ is $H^\infty(\mathbb{D})$, the algebra of bounded analytic functions on the unit disk.

Definition 1.1. More generally, given Hilbert spaces E and E_* , let $\mathcal{M}_{E_*, E}$ denote the corresponding multipliers; i.e., those functions $\Phi : \Omega \rightarrow B(E_*, E)$ such that $\Phi H \in H^2(k) \otimes E$ for every $H \in H^2(k) \otimes E_*$.

Observe, if e, e_* are in E and E_* respectively, then $\phi(w) = \langle \Phi(w)e_*, e \rangle$ is in $\mathcal{M}(k)$. Further, if $\Phi \in \mathcal{M}_{E, F}$ and $\Psi \in \mathcal{M}_{F, G}$, then $\Psi\Phi \in \mathcal{M}_{E, G}$.

Definition 1.2. We say that a reproducing kernel k is *nice* if the Hilbert space $H^2(k)$ is separable and there exist $p, q \in \mathcal{M}_{\ell^2, \mathbb{C}}$ such that

$$1 = k(z, w)[p(z)p(w)^* - q(z)q(w)^*] \tag{1.1}$$

for all $z, w \in \Omega$.

Of course, if k is nice, then $k(z, w)$ is never zero.

We close this subsection by recalling the notion of a complete Pick kernel [1].

Definition 1.3. Suppose k is a positive semi-definite function on Ω . The kernel k is *complete Pick kernel*, an *NP kernel* for short, if for each $\omega \in \Omega$ there exists a positive definite function $L_\omega : \Omega \times \Omega \rightarrow \mathbb{C}$ so that

$$k(y, x)k(\omega, \omega) - k(y, \omega)k(\omega, x) = L_\omega(y, x)k(y, x). \tag{1.2}$$

Remark 1.4. The reason for the names Pick and NP kernel can be found in [2]. See also [7] and [9].

Remark 1.5. If $k(z, w)$ never vanishes and if Eq. (1.2) holds for one ω , then it holds for all ω and thus k is an NP kernel. See [8] for details.

Remark 1.6. By standard reproducing kernel arguments, the positive semi-definite function L_ω can be factored as $B(\omega)^*B(\omega)$, where $B : \Omega \rightarrow \mathcal{E}$, for some auxiliary Hilbert space \mathcal{E} . When, as in all the examples in this article, $H^2(k)$ is separable, \mathcal{E} can be chosen separable. In that case, choosing a basis $\{e_j\}$ for \mathcal{E} and letting $b_j(z) = \langle B(z), e_j \rangle$ it follows that

$$L_\omega(y, x) = \sum b_j(y)b_j(x)^*.$$

1.2. The Douglas property

Given Hilbert spaces H and K and operators $A \in B(H)$ and $B \in B(K)$, the tensor product $A \otimes B$ is the operator on the Hilbert space $H \otimes K$ determined by its action on elementary tensors,

$$A \otimes B(h \otimes f) = Ah \otimes Bf.$$

It can be verified that $A \otimes B$ is bounded. In fact $\|A \otimes B\| = \|A\| \|B\|$.

As an example, if k is a kernel, $\varphi \in \mathcal{M}(k)$, and $B \in B(K)$, then $\Phi(z) = \varphi(z)B$ is in $\mathcal{M}_{K,K}$ and corresponds to the operator $M_\Phi = M_\varphi \otimes B$.

Definition 1.7. Given a unital subalgebra \mathcal{A} of $B(H)$, let $\mathcal{A} \otimes B(K)$ denote the algebraic tensor product; i.e., finite sums $\sum_1^n A_j \otimes B_j$. Let $\mathcal{A} \otimes_{\text{wot}} B(K)$ denote the closure, in the weak operator topology (wot) of $B(H \otimes K)$, of the algebraic tensor product.

Definition 1.8. A wot closed unital subalgebra \mathcal{A} of $B(H)$ has the Douglas property: if $A, B \in \mathcal{A} \otimes_{\text{wot}} B(\ell^2)$ and

$$AA^* \succcurlyeq BB^*,$$

then there exists

$$C \in \mathcal{A} \otimes_{\text{wot}} B(\ell^2)$$

such that $AC = B$ and $\|C\| \leq 1$.

Note that the Douglas property for \mathcal{A} is equivalent, by a compactness argument, to the property, if A and B are any finite matrices with entries in \mathcal{A} satisfying $AA^* \succcurlyeq BB^*$, then there exists a finite matrix C with entries in \mathcal{A} such that $AC = B$ and $\|C\| \leq 1$.

The following standard lemma says that it makes sense to ask if the multiplier algebra $\mathcal{M}(k)$ corresponding to a kernel k has the Douglas property. Note that

$$\mathcal{M}(k) \otimes B(\ell^2) \subset \mathcal{M}_{\ell^2, \ell^2} \subset B(H^2(k) \otimes \ell^2).$$

Lemma 1.9. If k is a reproducing kernel and $k(z, z) > 0$ for all $z \in \Omega$, then the algebra $\mathcal{M}(k) \subset B(H^2(k))$ is wot-closed and moreover $\mathcal{M}_{\ell^2, \ell^2} = \mathcal{M}(k) \otimes_{\text{wot}} B(\ell^2)$.

The proof appears in Section 2.

1.3. Main results

The following is our main result on multipliers algebras with the Douglas property.

Theorem 1.10. Suppose k is a nice reproducing kernel over the set Ω . If $\mathcal{M}(k)$ has the Douglas property, then k is a complete Pick kernel.

Conversely, if k is a non-vanishing complete Pick kernel, then $\mathcal{M}(k)$ has the Douglas property.

The converse direction in Theorem 1.10 is a result from [3].

Theorem 1.10 applies to some favorite examples.

Corollary 1.11. The multiplier algebras for each of the spaces $A^2(\mathbb{B}^m)$, $H^2(\mathbb{D}^n)$, and $H^2(\mathbb{B}^n)$, for $m \geq 1$ and $n \geq 2$ do not have the Douglas property.

Here $A^2(\mathbb{B}^m)$ is the Bergman space of the unit ball \mathbb{B}^m in \mathbb{C}^m ; $H^2(\mathbb{D}^n)$ is the Hardy space of the polydisk \mathbb{D}^n in \mathbb{C}^n ; and $H^2(\mathbb{B}^n)$ is the Hardy space of the ball.

Proof. It is clear that these are nice reproducing kernel Hilbert spaces. Further, it is well known, and easy to verify, that their respective kernels are not complete Pick kernels. \square

It turns out that without the Douglas property it is not always possible to factor, even dropping the norm constraint.

Theorem 1.12. Let \mathcal{A} denote the multiplier algebra on any of the Hilbert spaces $A^2(\mathbb{B}^m)$, $H^2(\mathbb{D}^n)$, and $H^2(\mathbb{B}^n)$ for $m \geq 1$ and $n \geq 2$. The equation $AX = B$ for $A, B \in \mathcal{A} \otimes B(\ell^2)$ and $AA^* \succcurlyeq BB^*$ cannot always be solved for X in $\mathcal{A} \otimes B(\ell^2)$.

The next section contains routine, but necessary, preliminary results. The proofs of Theorems 1.10 and 1.12 occupy Sections 3 and 4 respectively. The paper closes with examples and questions in Section 5.

2. Preliminary results

This section collects a few preliminary observations used in the proofs of Theorems 1.10 and 1.12.

Lemma 2.1. *If $\Phi \in \mathcal{M}_{E, E_*}$, $e \in E$ and $w \in \Omega$, then*

$$M_\Phi^* k(\cdot, w)e = k(\cdot, w)\Phi(w)^*e.$$

Proof. Given $F \in H_E^2(k)$,

$$\langle F, M_\Phi^* k(\cdot, w)e \rangle = \langle \Phi F, k(\cdot, w)e \rangle = \langle \Phi(w)F(w), e \rangle = \langle F(w), \Phi(w)^*e \rangle = \langle F, k(\cdot, w)\Phi(w)^*e \rangle. \quad \square$$

The following is a slight generalization of Lemma 1.9.

Lemma 2.2. *Given separable Hilbert spaces E and E_* , the space of multipliers \mathcal{M}_{E, E_*} is equal to $\mathcal{M}(k) \otimes_{\text{wot}} B(E_*, E)$.*

Proof. The proof, in outline, involves showing that \mathcal{M}_{E, E_*} is wot-closed and contains the algebraic tensor product $\mathcal{M}(k) \otimes B(E_*, E)$ and hence $\mathcal{M}(k) \otimes_{\text{wot}} B(E_*, E) \subset \mathcal{M}_{E, E_*}$. The reverse inclusion follows from the fact that, since E and E_* are separable, there exist sequences of finite rank projections P_n and Q_n which converge, in the strong operator topology, to the identities on E and E_* respectively.

For the details, suppose (ϕ_α) is a net from \mathcal{M}_{E, E_*} which converges, in the weak operator topology of $B(H^2(k) \otimes E_*, H^2(k) \otimes E)$, to some T . Fix $z \in \Omega$ and define $W_z : E \rightarrow H^2(k) \otimes E$ by $W_z e = k(\cdot, z)e$, and $V : E_* \rightarrow H^2(k) \otimes E_*$ by $V e_* = e_*$ (the constant function). Let $\Phi(z) = W_z^* T V : E \rightarrow E_*$. Then $V^* M_{\phi_\alpha}^* W_z$ converges WOT to $\Phi(z)^*$. For $f \in H^2(k)$, $e \in E$ and $e_* \in E_*$, using Lemma 2.1, compute

$$\langle f \otimes e_*, M_{\phi_\alpha}^* k(\cdot, z)e \rangle = f(z) \langle e_*, \phi_\alpha(z)^* e \rangle = f(z) \langle e_*, V^* M_{\phi_\alpha}^* W_z e \rangle. \tag{2.1}$$

The left-hand side of Eq. (2.1) converges to

$$\langle f \otimes e_*, T^* k(\cdot, z)e \rangle; \tag{2.2}$$

whereas the right-hand side of Eq. (2.1) converges to

$$f(z) \langle e_*, \Phi(z)^* e \rangle = \langle f \otimes e_*, k(\cdot, z)\Phi(z)^* e \rangle. \tag{2.3}$$

Combining Eqs. (2.2) and (2.3) gives,

$$T^* k(\cdot, z)e = k(\cdot, z)\Phi(z)^* e.$$

Thus, $T = M_\Phi$ and T is in \mathcal{M}_{E, E_*} .

Now let $\Phi \in \mathcal{M}_{E, E_*}$ be given. Note that $(I \otimes Q_n)M_\Phi(I \otimes P_n)$ is in the algebraic tensor product $\mathcal{M}(k) \otimes B(E_*, E)$ for each n and also converges wot to Φ . Hence $M_\Phi \in \mathcal{M}(k) \otimes_{\text{wot}} B(E_*, E)$. \square

3. The proof of Theorem 1.10

This section contains the proof of Theorem 1.10.

The assumption that k is nice implies there exist $p, q \in \mathcal{M}_{\ell^2, \mathbb{C}}$ satisfying Eq. (1.1). From Lemma 2.1,

$$\langle M_q^* k(\cdot, y), M_q^* k(\cdot, x) \rangle = \langle q(x)q^*(y)k(\cdot, y), k(\cdot, y) \rangle,$$

where $q^*(y) = q(y)^*$. Hence,

$$\langle [M_p M_p^* - M_q M_q^*] k(\cdot, y), k(\cdot, x) \rangle = 1. \tag{3.1}$$

Thus, $M_p M_p^* - M_q M_q^* \succcurlyeq 0$. Hence, if $M(k)$ has the Douglas property, then, using Lemma 2.2, there exists $C \in \mathcal{M}_{\ell^2, \ell^2}$ such that $q = pC$ ($M_q = M_p M_C$) and $\|M_C\| \leq 1$. The remainder of the proof involves exploiting the resulting identity,

$$M_p M_p^* - M_q M_q^* = M_p [I - M_C M_C^*] M_p^*.$$

Fix a point $\omega \in \Omega$ and let $\mathcal{H}_\omega^2(k)$ denote those $f \in H^2(k)$ which vanish at ω . Let P_ω denote the projection onto $H_\omega^2(k)$. The operator

$$D = P_\omega M_p [I \otimes P_\omega] (I - M_C [I \otimes P_\omega] M_C^*) [I \otimes P_\omega] M_p^* P_\omega$$

is positive semi-definite since $\|M_C\| \leq 1$. Thus the function $L_\omega(x, y)$ defined by

$$\Omega \times \Omega \ni (x, y) \mapsto L_\omega(x, y) := \langle Dk(\cdot, y), k(\cdot, x) \rangle$$

is positive semi-definite.

Observe that

$$P_\omega k(\cdot, y) = k(\cdot, y) - \frac{k(\omega, y)}{k(\omega, \omega)} k(\cdot, \omega). \tag{3.2}$$

Further $M_p^* k(\cdot, y) = p(y)^* k(\cdot, y)$ and similarly for M_C^* . Thus,

$$[I \otimes P_\omega] M_p^* P_\omega k(\cdot, w) = p^*(w) P_\omega k(\cdot, w) \tag{3.3}$$

and similarly

$$[I \otimes P_\omega] M_C^* [I \otimes P_\omega] p^*(y) k(\cdot, y) = C^*(y) p^*(y) P_\omega k(\cdot, y) = q(y)^* P_\omega k(\cdot, y). \tag{3.4}$$

Combining Eqs. (3.3), (3.4), and (3.1) gives

$$k(x, y) L_\omega(x, y) = \langle [(p(x)p(y)^* - q(x)q(y)^*)k(x, y)] P_\omega k(\cdot, y), P_\omega k(\cdot, x) \rangle = \langle P_\omega k(\cdot, y), P_\omega k(\cdot, x) \rangle. \tag{3.5}$$

From Eqs. (3.2) and (3.5) it follows that

$$k(x, y) - \frac{k(x, \omega)k(\omega, y)}{k(\omega, \omega)} = k(x, y) L_\omega(x, y)$$

and k is a complete Pick kernel.

4. The proof of Theorem 1.12

Theorem 1.12 is really three theorems, one each for the Bergman spaces of the ball \mathbb{B}^m in \mathbb{C}^m ; the Hardy spaces \mathbb{B}^m for $m \geq 2$; and the Hardy spaces of the polydisk \mathbb{D}^m for $m \geq 2$. Accordingly, this section starts with three lemmas – one about each of these collection of spaces – before turning to the proof of Theorem 1.12.

Given an indexed set $S = \{h_j; j \in J\}$ of vectors from a Hilbert space H , let $[h_j; j \in J]$ denote the closed linear span of S in H and let $\text{Proj}[h_j; j \in J]$ denote the orthogonal projection onto $[h_j; j \in J]$.

Lemma 4.1. *Let B denote M_z on $A^2(\mathbb{D})$, the Bergman space on the unit disk. For $N = 1, 2, \dots$*

$$I + NB^{N+1}B^{*(N+1)} - (N+1)B^N B^{*N} = \text{Proj}[0, 1, \dots, z^{N-1}] = \sum_{j=0}^{N-1} (j+1)z^j \otimes z^j.$$

Proof. Substituting into the inner product $\langle (\cdot)k_w, k_z \rangle_{A^2(D)}$, it suffices to show that

$$\frac{1 + N(\bar{w}z)^{N+1} - (N+1)(\bar{w}z)^N}{(1 - \bar{w}z)^2} = \sum_{j=0}^{N-1} (j+1)(\bar{w}z)^j, \quad \text{for } N = 1, 2, \dots$$

Fix $N \in \mathbb{N}$ and let $x = \bar{w}z$. Then

$$\begin{aligned} \frac{1 + Nx^{N+1} - (N+1)x^N}{(1-x)^2} &= \frac{(1-x^N) - Nx^N(1-x)}{(1-x)^2} \\ &= \frac{\sum_{j=0}^{N-1} x^j - Nx^N}{1-x} \\ &= \sum_{j=0}^{N-1} \frac{x^j(1-x^{N-j})}{1-x} \\ &= \sum_{j=0}^{N-1} x^j \sum_{k=0}^{N-1-j} x^k \\ &= \sum_{j=0}^{N-1} (j+1)x^j. \quad \square \end{aligned}$$

Lemma 4.2. Let S and W denote the operators of multiplication by z and w respectively on $H^2(\mathbb{D}^2)$, the Hardy space on the bidisk \mathbb{D}^2 in \mathbb{C}^2 . For each N ,

$$I + \sum_{j=1}^N S^j W^{N-j+1} W^{*(N-j+1)} S^{*j} - \sum_{j=0}^N S^j W^{N-j} W^{*(N-j)} S^{*j} = \text{Proj}[z^j w^k: 0 \leq j+k \leq N-1]$$

$$= \sum_{k=0}^{N-1} \sum_{p=0}^{N-1-j} z^p w^k \otimes z^p w^k.$$

Proof. Again, as in Lemma 4.1, we apply the above operators to the reproducing kernel k_{v_1, v_2} and take the inner product with k_{u_1, u_2} . Thus it suffices to show that

$$\frac{1 + \sum_{j=1}^N (u_1 \bar{v}_1)^{N-j+1} (u_2 \bar{v}_2)^j - \sum_{j=0}^N (u_1 \bar{v}_1)^{N-j} (u_2 \bar{v}_2)^j}{(1 - u_1 \bar{v}_1)(1 - u_2 \bar{v}_2)} = \sum_{j=0}^{N-1} \sum_{p=0}^{N-1-j} (u_1 \bar{v}_1)^j (u_2 \bar{v}_2)^p,$$

for each $N = 1, 2, \dots$. Fix $N \in \mathbb{N}$, let $x = u_1 \bar{v}_1$ and $y = u_2 \bar{v}_2$ and observe,

$$\begin{aligned} \frac{1 + \sum_{j=1}^N x^{N-j+1} y^j - \sum_{j=0}^N x^{N-j} y^j}{(1-x)(1-y)} &= \frac{(1-x^N) \sum_{j=1}^N x^{N-j} y^j (1-x)}{(1-x)(1-y)} \\ &= \frac{\sum_{j=0}^{N-1} x^j - \sum_{j=0}^{N-1} x^j y^{N-j}}{1-y} \\ &= \sum_{j=0}^{N-1} x^j \left(\sum_{p=0}^{N-1-j} y^p \right) \end{aligned}$$

to complete the proof. \square

Lemma 4.3. Let S and W denote multiplication by z and w on $H^2(\mathbb{B}^2)$, the Hardy space of the unit ball \mathbb{B}^2 in \mathbb{C}^2 . For $N = 1, 2, \dots$,

$$I + \sum_{j=0}^{N+1} N \binom{N+1}{j} S^{N+1-j} W^j W^{*j} S^{*(N+1-j)} \succcurlyeq \sum_{j=0}^N (N+1) \binom{N}{j} S^{N-j} W^j W^{*j} S^{(N-j)*}.$$

Proof. For $N = 1$,

$$I + S^2 S^{*2} + 2SWW^*S^* + W^2W^{*2} = 2SS^* + 2WW^* + 1 \otimes 1.$$

Let P_N denote the projection of $H^2(\mathbb{B}^N)$ onto the span of $\{z^j w^k: 0 \leq j+k < N\}$. An induction argument similar to that in the proof of Lemma 4.1 shows that

$$I + \sum_{j=0}^{N+1} N \binom{N+1}{j} S^{N+1-j} W^j W^{*j} S^{*(N+1-j)*} - \sum_{j=0}^N (N+1) \binom{N}{j} S^{N-j} W^j W^{*j} S^{(N-j)*} = P_N. \quad \square$$

Proof of Theorem 1.12. We let $\mathcal{H}(\Omega) = H^2(\mathbb{D}^2)$, the Hardy space on the bidisk. Note that $\mathcal{M}(\mathcal{H}(\mathbb{D}^2)) = H^\infty(\mathbb{D}^2)$. We will show that the equation $AX = B$ for $A, B \in H^\infty(\mathbb{D}^2) \otimes B(l^2)$ and $AA^* \succcurlyeq BB^*$ cannot always be solved for X in $H^\infty(\mathbb{D}^2) \otimes B(l^2)$.

To do this, we will use Lemma 4.2. The analogous proofs for Bergman spaces and Hardy space on the unit ball require Lemmas 4.1 and 4.3, respectively. Those proofs have a similar pattern to this one and will be omitted.

Suppose that whenever $A, B \in H^\infty(\mathbb{D}^2) \otimes B(l^2)$ with $AA^* \succcurlyeq BB^*$, then there exists $X \in H^\infty(\mathbb{D}^2) \otimes B(l^2)$ with $AX = B$. Then from Lemma 4.2,

$$I + S^N W W^* S^{*N} + \dots + S W^N W^{*N} S^* \succcurlyeq S^N S^{*N} + \dots + W^N W^{*N}.$$

So we are assuming that there exists an $N \times N$ matrix of $H^\infty(\mathbb{D}^2)$ functions, $[C_{ij}(z, w)]$ so that

$$[I, S^N W, \dots, S W^N][C_{ij}(S, W)] = [S^N, \dots, W^N].$$

Fix $1 \leq k \leq N$. We have for all $z, w \in \mathbb{D}$,

$$C_{1k}(z, w) + \sum_{j=1}^N z^{N-j+1} W^j C_{j+1,k}(z, w) = z^{N-k+1} w^{k-1}.$$

Thus, the $(N - k + 1, k - 1)$ th coefficient of $C_{1k}(z, w)$ is 1.

Estimating,

$$\begin{aligned} N + 1 &\leq \sum_{k=1}^{N+1} \|C_{1k}\|_{L^2(T^2)}^2 = \sum_{k=1}^{N+1} \int_{T^2} |C_{1k}|^2 d\sigma \\ &\leq \sup_{(z,w) \in \mathbb{D}^2} \sum_{k=1}^{N+1} |C_{1k}(z, w)|^2 \\ &\leq \sup_{(z,w) \in \mathbb{D}^2} \|[C_{jk}(z, w)]\|_{B(\mathbb{C}^N)} \\ &= \|[C_{jk}(S, W)]\|_{B(H^2(\mathbb{D}^2))}. \end{aligned}$$

Hence any $[C_{ij}(S, W)]$ solving

$$[I, S^N W, \dots, SW^N][C_{ij}(S, W)] = [S^N, \dots, W^N]$$

must have

$$\|[C_{jk}(S, W)]\| \geq N + 1.$$

Let A_N and B_N denote the $(N + 1) \times (N + 1)$ operator matrix

$$A_N = \begin{bmatrix} I & S^N W & \dots & SW^N \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix} \quad \text{and} \quad B_N = \begin{bmatrix} S^N & S^{N-1} W & \dots & W^N \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix}.$$

Define A, B by

$$A = \bigoplus_{N=1}^{\infty} \frac{A_N}{\|A_N\|} \quad \text{and} \quad B = \bigoplus_{N=1}^{\infty} \frac{B_N}{\|B_N\|} \quad \text{acting on} \quad \bigoplus_{N=1}^{\infty} \left(\bigoplus_{j=1}^N H^2(\mathbb{D}^2) \right).$$

By Lemma 4.2, $AA^* \succcurlyeq BB^*$. If there exists an analytic Toeplitz operator $X = [X_{jk}]_{j,k=1}^{\infty}$ with $AX = B$, then $A_N X_{NN} = B_N$, so $\|X\| \geq \sup_N \|X_{NN}\| \geq \sup(N + 1)$, a contradiction. \square

5. Examples and questions

It turns out that the multiplier algebra of an $H^2(k)$ can have the property that $AA^* \succcurlyeq BB^*$ implies the existence of a multiplier C such that $AC = B$, but not necessarily with C a contraction.

Example 5.1. For an example, let k denote the kernel over the unit disk given by

$$k(z, w) = 1 + 2 \frac{z\bar{w}}{1 - z\bar{w}}.$$

Choosing $\omega = 0$, gives,

$$k(\omega, \omega) - \frac{k(z, \omega)k(\omega, w)}{k(z, w)} = 2 \frac{z\bar{w}}{1 + z\bar{w}},$$

which is not a positive semi-definite function on $\mathbb{D} \times \mathbb{D}$. Hence k is not an NP kernel and it is not possible to factor (with the strict norm constraint) in $\mathcal{M}(k) \otimes_{\text{wot}} B(\ell^2)$.

On the other hand, it is easy to verify that, with $s = (1 - z\bar{w})^{-1}$ the Szegő kernel,

$$s(z, w) \preccurlyeq k(z, w) \preccurlyeq 2s(z, w),$$

where the inequalities are in the sense of positive semi-definite kernels (so in particular $k(z, w) - s(z, w)$ is positive semi-definite). It follows that $H^\infty(\mathbb{D}) = \mathcal{M}(s) = \mathcal{M}(k)$ as sets and moreover for $f \in \mathcal{M}(k) \otimes B(\ell^2)$, that

$$\frac{1}{2} \|f\|_{\mathcal{M}(s)} \leq \|f\|_{\mathcal{M}(k)} \leq 2 \|f\|_{\mathcal{M}(s)}.$$

Hence, it is possible to factor in $\mathcal{M}(k) \otimes_{\text{wot}} B(\ell^2)$, because it is possible to factor (with the strict norm constraint) in $\mathcal{M}(s) \otimes_{\text{wot}} B(\ell^2)$.

The example naturally leads to the following questions.

Problem 5.2. Say that an algebra \mathcal{A} has the *bounded Douglas property* if it satisfies the conditions of Definition 1.8, except for the norm constraint $\|C\| \leq 1$. In this case, there exists a constant γ , independent of C , such that $\|C\| \leq \gamma$. Characterize those nice reproducing kernels k for which $\mathcal{M}(k)$ has the bounded Douglas property.

Problem 5.3. Can B in Theorem 1.12 be chosen to be I ?

See Trent [10] for the relevance of Problem 5.3 to the corona problem for the bidisk. The following example shows that the hypothesis that k is nice is natural.

Example 5.4. Let $\Omega = \mathbb{C}$ and $k(z, w) = \exp(z\bar{w})$. In this case, $H^2(k)$ consists of those entire functions f such that

$$\int_{\mathbb{C}} |f(z)|^2 \exp(-|z|^2) dA$$

is finite. Hence, by Liouville's Theorem, the only multipliers of $H^2(k)$ are constant and thus $\mathcal{M}(k) = \mathbb{C}$. Thus, trivially, $\mathcal{M}(k)$ has the Douglas property. Of course, k is not nice.

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