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Lebesgue–Bochner spaces, decomposable sets and strong weakly compact generation ☆

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ABSTRACT

Let X be a Banach space and μ a probability measure. We prove that X is strongly reflexive (resp. super-reflexive) generated if, and only if, there exist a reflexive (resp. super-reflexive) Banach space Z and a bounded linear operator $S : Z \to L^1(\mu, X)$ such that for each weakly compact decomposable set $K \subset L^1(\mu, X)$ and each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nS(B_Z) + \varepsilon B_{L^1(\mu,X)}$. This answers partially a question posed by Schlüchtermann and Wheeler. Some applications are also given.

1. Introduction

Let X be a Banach space and let B_X be its closed unit ball. We say that X is *strongly generated* by a Banach space Y (or that Y strongly generates X) if there is a bounded linear operator $T : Y \to X$ such that for each weakly compact set $K \subset X$ and each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $K \subset nT(B_Y) + \varepsilon B_X$. The space X is said to be *strongly $\mathcal{P}$ generated* (for a given property $\mathcal{P}$ of Banach spaces) if there is a Banach space with property $\mathcal{P}$ that strongly generates X.

Of particular interest for us are the classes of strongly reflexive generated and strongly super-reflexive generated Banach spaces. An application of the factorization theorem of Davis, Figiel, Johnson and Pelczynski (see e.g. [10, Theorem 13.22]) yields that X is strongly reflexive generated if, and only if, X is strongly weakly compactly generated. The space X is called *strongly weakly compactly generated* if there is a weakly compact set $G \subset X$ such that for each weakly compact set $K \subset X$ and each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_X$. This class of spaces was introduced by Schlüchtermann and Wheeler [16] and has also been studied in [11,15]. Every strongly weakly compactly generated space is weakly compactly generated, but the converse does not hold in general (for instance, c_0 is not strongly weakly compactly generated). Typical examples of spaces in this class are the reflexive spaces, separable spaces with the Schur property (which are strongly generated by ℓ^2) and $L^1(\mu)$ for any probability measure μ (this space is strongly generated by $L^2(\mu)$, see [11, Proposition 12]).

In [1] it was proved that every weakly compact subset of $L^1(\mu)$ is *uniform Eberlein*, i.e. it is homeomorphic to a weakly compact subset of a Hilbert space. In fact, the same statement holds true for any strongly super-reflexive generated space (see [11, Corollary 11]). Such a space admits an equivalent uniformly weak Hadamard smooth norm, a property that lies between the classical notions of uniformly Gâteaux and uniformly Fréchet smoothness (see [11, Theorem 7 and Corollary 8]).

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There exist reflexive Banach spaces which do not admit an equivalent uniformly Gâteaux smooth norm (see e.g. [3, IV.6]), so the class of strongly reflexive generated spaces is strictly larger than the class of strongly super-reflexive generated spaces.

Given a probability space (Ω, Σ, μ), we denote by $L^1(\mu, X)$ the Banach space made up of all (equivalence classes of) Bochner integrable functions $f : \Omega \to X$, equipped with the norm $\|f\|_1 := \int_\Omega \|f(\omega)\| \, d\mu(\omega)$. A set $D \subset L^1(\mu, X)$ is called *decomposable* if $f 1_A + g 1_{\Omega \setminus A} \in D$ for every $f, g \in D$ and every $A \in \Sigma$ (where 1_E stands for the characteristic function of $E \in \Sigma$). This concept has proved to be a useful tool in control theory, optimization and mathematical economics (see e.g. [14]) and plays an important role in the study of weak compactness in Lebesgue–Bochner spaces. Decomposable sets arise as collections of Bochner integrable selectors of suitable set-valued functions [12,13]. In particular, if W is any subset of X, then the set

$$L(W) = \{ f \in L^1(\mu, X): \ f(\omega) \in W \text{ for } \mu\text{-a.e. } \omega \in \Omega \},$$

is decomposable. In [6] (see also [7, Corollary 2.6]) it was shown that if W is relatively weakly compact (resp. weakly compact and convex) then $L(W)$ is relatively weakly compact (resp. weakly compact).

In [16, Section 4] it was asked if the space $L^1(\mu, X)$ is strongly weakly compactly generated whenever X is strongly weakly compactly generated. We point out that the corresponding problem for weakly compact generation was solved affirmatively in [5]. Our main result provides a positive answer to that question (and the analogous problem for strongly super-reflexive generated spaces) when restricted to weakly compact decomposable subsets of $L^1(\mu, X)$, as follows.

Theorem 1. *Let X be a Banach space and (Ω, Σ, μ) a probability space. Then X is strongly reflexive (resp. super-reflexive) generated if, and only if, there exist a reflexive (resp. super-reflexive) Banach space Z and a bounded linear operator*

$$S : Z \to L^1(\mu, X)$$

such that for each weakly compact decomposable set $K \subset L^1(\mu, X)$ and each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nS(B_Z) + \varepsilon B_{L^1(\mu,X)}$.

As application we obtain a version of the aforementioned statements on uniform Eberlein weakly compact sets and uniformly weak Hadamard smooth renormings in the setting of Lebesgue–Bochner spaces (Corollaries 6 and 7).

We follow standard Banach space notation and terminology which can be found, for instance, in [10]. For the basics of Lebesgue–Bochner spaces, see [8].

2. Proof of Theorem 1 and consequences

The proof of Theorem 1 is based on the following result.

Lemma 2. *Let X be a Banach space and (Ω, Σ, μ) a probability space. Suppose X is strongly reflexive generated and fix a weakly compact absolutely convex set $G_0 \subset X$ such that for each weakly compact set $K_0 \subset X$ and each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K_0 \subset nG_0 + \varepsilon B_X$. Then for each weakly compact decomposable set $K \subset L^1(\mu, X)$ and each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nL(G_0) + \varepsilon B_{L^1(\mu,X)}$.*

In the proof of Lemma 2 we shall use the following easy fact.

Lemma 3. *Let X be a Banach space and (Ω, Σ, μ) a probability space. Let $V \subset X$ and $\varepsilon > 0$. If $f : \Omega \to X$ is a strongly measurable function such that $f(\omega) \in V + \varepsilon B_X$ for μ-a.e. $\omega \in \Omega$, then for every $\eta > 0$ there is a strongly measurable function $g : \Omega \to X$ taking values in V such that $\|f(\omega) - g(\omega)\| \leqslant \varepsilon + \eta$ for μ-a.e. $\omega \in \Omega$.*

Proof. Since f is strongly measurable, there is a countable disjoint family $\mathcal{A} \subset \Sigma$ with $\mu(\bigcup \mathcal{A}) = 1$ such that, for each $A \in \mathcal{A}$, we have $\mu(A) > 0$ and

$$\sup_{\omega, \omega' \in A} \| f(\omega) - f(\omega') \| \leqslant \eta$$

(see e.g. [8, p. 42, Corollary 3]). For every $A \in \mathcal{A}$ we can choose $\omega_A \in A$ and $x_A \in V$ such that $\|f(\omega_A) - x_A\| \leqslant \varepsilon$. Fix $x \in V$ arbitrary. Define $g : \Omega \to X$ by

$$g(\omega) := \begin{cases} x_A & \text{if } \omega \in A \text{ for some } A \in \mathcal{A}, \\ x & \text{if } \omega \notin \bigcup \mathcal{A}. \end{cases}$$

It is clear that g satisfies the required properties. □

Proof of Lemma 2. We can assume without loss of generality that (Ω, Σ, μ) is complete. We begin by proving the following particular case.

PARTICULAR CASE. *Suppose $K \subset L^1(\mu, Y)$ for some separable closed subspace $Y \subset X$.* Since K is weakly compact and decomposable, by [13, Corollary 3.10] there is a set-valued function $F : \Omega \to 2^Y \setminus \{\emptyset\}$ satisfying the following properties:

(a) $F(\omega)$ is relatively weakly compact for every $\omega \in \Omega$;
(b) $\{\omega \in \Omega\colon\ F(\omega) \cap U \neq \emptyset\} \in \Sigma$ for every norm open set $U \subset Y$;
(c) for every $f \in K$ we have $f(\omega) \in F(\omega)$ for μ-a.e. $\omega \in \Omega$.

For each $n \in \mathbb{N}$, define $A_n := \{\omega \in \Omega\colon\ F(\omega) \subset nG_0 + \frac{\varepsilon}{2} B_X\}$. Observe that $nG_0 + \frac{\varepsilon}{2} B_X$ is weakly closed (because G_0 is weakly compact) and convex, hence it is norm closed. Since

$$A_n = \Omega \setminus \left\{\omega \in \Omega\colon\ F(\omega) \cap \left(Y \setminus \left(nG_0 + \frac{\varepsilon}{2} B_X\right)\right) \neq \emptyset\right\},$$

property (b) ensures that $A_n \in \Sigma$. Observe that $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ (bear in mind that G_0 is balanced) and that, moreover, $\Omega = \bigcup_{n\in\mathbb{N}} A_n$ (because $F(\omega)$ is relatively weakly compact for every $\omega \in \Omega$). The weak compactness of K implies that K is uniformly integrable (see [8, p. 104, Theorem 4]), hence we can choose $n \in \mathbb{N}$ large enough such that

$$\sup_{f\in K} \int_{\Omega\setminus A_n} \|f(\omega)\|\, d\mu(\omega) \leqslant \frac{\varepsilon}{4}. \tag{1}$$

We claim that $K \subset nL(G_0) + \varepsilon B_{L^1(\mu,X)}$. Fix $f \in K$ and set $\tilde{f} := f 1_{A_n}$. By (c) we have $\tilde{f}(\omega) \in nG_0 + \frac{\varepsilon}{2} B_X$ for μ-a.e. $\omega \in \Omega$. Lemma 3 applied to $\tilde{f}$ guarantees the existence of $g \in nL(G_0)$ such that $\|\tilde{f}(\omega) - g(\omega)\| \leqslant \frac{3\varepsilon}{4}$ for μ-a.e. $\omega \in \Omega$, thus $\|\tilde{f} - g\|_1 \leqslant \frac{3\varepsilon}{4}$. This inequality and (1) yield $\|f - g\|_1 \leqslant \varepsilon$. This shows that $K \subset nL(G_0) + \varepsilon B_{L^1(\mu,X)}$, as claimed.

General case. We argue by contradiction. Suppose that for every $n \in \mathbb{N}$ there is $f_n \in K$ such that $f_n \notin nL(G_0) + \varepsilon B_{L^1(\mu,X)}$. Since the f_n's are strongly measurable, we can assume without loss of generality that $f_n \in L^1(\mu, Y)$ for every $n \in \mathbb{N}$, where Y is some *separable* closed subspace of X.

Let H be the decomposable hull of the sequence (f_n), i.e. H is the intersection of all decomposable subsets of $L^1(\mu, X)$ containing all f_n's. Clearly, $H \subset L^1(\mu, Y)$. Let $\tilde{H}$ be the weak closure of H in $L^1(\mu, Y)$. We claim that $\tilde{H}$ is decomposable and weakly compact. Indeed, given any $A \in \Sigma$, the mapping

$$\phi_A : L^1(\mu, Y) \times L^1(\mu, Y) \to L^1(\mu, Y), \qquad \phi_A(f, g) := f 1_A + g 1_{\Omega\setminus A},$$

is continuous when $L^1(\mu, Y)$ is equipped with its weak topology and so

$$\phi_A(\tilde{H} \times \tilde{H}) \subset \overline{\phi_A(H \times H)}^{\text{weak}} = \tilde{H}.$$

Thus, $\tilde{H}$ is decomposable. Since $H \subset K$ (because K is decomposable and $f_n \in K$ for all $n \in \mathbb{N}$) and K is weakly compact, $\tilde{H} \subset K$ is weakly compact as well.

Finally, the particular case applied to $\tilde{H}$ ensures the existence of some $n \in \mathbb{N}$ for which $\tilde{H} \subset nL(G_0) + \varepsilon B_{L^1(\mu,X)}$, thus contradicting the choice of $f_n \in \tilde{H}$. The proof is over. □

Now, we proceed with the proof of Theorem 1. As usual, we denote by $L^2(\mu, X)$ the Banach space made up of all (equivalence classes of) strongly measurable functions $f : \Omega \to X$ such that $\|f\|_2 := (\int_\Omega \|f(\omega)\|^2\, d\mu(\omega))^{1/2} < \infty$. By a result of Figiel and Pisier (cf. [3, Chapter IV, Corollary 4.5]) it follows that $L^2(\mu, X)$ is super-reflexive whenever X is super-reflexive. The analogous assertion for reflexive spaces also holds true (cf. [8, p. 100, Corollary 2]).

Proof of Theorem 1. *"Only if" part.* Let Y be a reflexive (resp. super-reflexive) Banach space and let $T : Y \to X$ be a bounded linear operator such that for every weakly compact set $K_0 \subset X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K_0 \subset nT(B_Y) + \varepsilon B_X$. Since reflexivity (resp. super-reflexivity) is inherited by quotients (cf. [3, Chapter IV, Corollary 4.6]) we can assume that T is one-to-one. It is clear that the formula

$$S(f) := T \circ f$$

defines a bounded linear operator S from $L^2(\mu, Y)$ into $L^1(\mu, X)$. We claim that $L(T(B_Y)) \subset S(B_{L^2(\mu,Y)})$. Indeed, let $g : \Omega \to X$ be a strongly measurable function such that $g(\Omega) \subset T(B_Y)$. We can write $g = T \circ f$ for some function $f : \Omega \to B_Y$. Since g is strongly measurable, Y is reflexive and T is one-to-one, an appeal to [4, Proposition 4.3] allows us to conclude that f is strongly measurable, hence $f \in B_{L^2(\mu,Y)}$ and so $g \in S(B_{L^2(\mu,Y)})$.

Now, Lemma 2 tells us that if $K \subset L^1(\mu, X)$ is any weakly compact decomposable set and $\varepsilon > 0$, then there is $n \in \mathbb{N}$ such that

$$K \subset nL\big(T(B_Y)\big) + \varepsilon B_{L^1(\mu,X)} \subset nS(B_{L^2(\mu,Y)}) + \varepsilon B_{L^1(\mu,X)}.$$

Since the space $L^2(\mu, Y)$ is reflexive (resp. super-reflexive) our assertion is satisfied.

"If" part. Suppose that there exist a reflexive (resp. super-reflexive) Banach space Z and a bounded linear operator $S : Z \to L^1(\mu, X)$ such that for each weakly compact decomposable set $K \subset L^1(\mu, X)$ and each $\varepsilon > 0$ there is $n \in \mathbb{N}$ satisfying $K \subset nS(B_Z) + \varepsilon B_{L^1(\mu,X)}$. Let us write, for each $f \in L^1(\mu, X)$,

$$P(f) := \int_\Omega f(\omega)\, d\mu(\omega),$$

and set $T := P \circ S$. Then T defines a bounded linear operator from Z into X.

Let K_0 be a weakly compact subset of X and pick $\varepsilon > 0$. Since $K := L(\overline{\mathrm{co}}(K_0))$ is a weakly compact decomposable subset of $L^1(\mu, X)$, there is $n \in \mathbb{N}$ such that $K \subset nS(B_Z) + \varepsilon B_{L^1(\mu,X)}$, and consequently

$$P(K) \subset nP\big(S(B_Z)\big) + \varepsilon P(B_{L^1(\mu,X)}) \subset nT(B_Z) + \varepsilon B_X.$$

It remains to show that $K_0 \subset P(K)$. Take $x \in K_0$ and set $f(\omega) := x$ for every $\omega \in \Omega$. Then $f \in L(K_0) \subset K$, and thus $x = P(f) \in P(K)$. So, $K_0 \subset P(K)$ and the proof is finished. $\square$

Remark 4. A careful examination of the proof of Theorem 1 shows that X is strongly weakly compactly generated if, and only if, there is a weakly compact decomposable set $G \subset L^1(\mu, X)$ such that for each weakly compact decomposable set $K \subset L^1(\mu, X)$ and each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_{L^1(\mu,X)}$.

Remark 5. The statement of Theorem 1 is also valid if "reflexivity" and "super-reflexivity" are replaced by the property of "being a Hilbert space". We stress that being strongly Hilbert generated is strictly stronger than being strongly super-reflexive generated. If $1 < p < \infty$ and $p \neq 2$ then none of the spaces ℓ^p and $L^p(\mu)$ is strongly Hilbert generated. Indeed, if X and Y are reflexive Banach spaces and Y strongly generates X, then X is a quotient of Y (see [11, Lemma 5]). In particular, every reflexive strongly Hilbert generated space is a Hilbert space.

As a consequence of Theorem 1 we have the following result.

Corollary 6. *Let X be a strongly super-reflexive generated Banach space and μ a probability measure. Then every weakly compact decomposable subset of $L^1(\mu, X)$ is uniform Eberlein.*

Proof. Let us consider a super-reflexive Banach space Z and a bounded linear operator $S : Z \to L^1(\mu, X)$ as in Theorem 1. The super-reflexivity of Z ensures that B_Z is uniform Eberlein [9]. Since the class of uniform Eberlein compact spaces is stable under continuous images (see [2, Theorem 3.1]), $S(B_Z)$ is uniform Eberlein as well. Take any weakly compact decomposable set $K \subset L^1(\mu, X)$. By the choice of S, for every $\varepsilon > 0$ there is a uniform Eberlein weakly compact set $K_\varepsilon \subset L^1(\mu, X)$ (of the form $n_\varepsilon S(B_Z)$ for some $n_\varepsilon \in \mathbb{N}$) such that $K \subset K_\varepsilon + \varepsilon B_{L^1(\mu,X)}$. This property implies that K is uniform Eberlein (see [11, Proposition 11]). $\square$

We end this note with an application of Theorem 1 to the study of smooth renormings in Lebesgue–Bochner spaces. Let $(X, \|\cdot\|)$ be a Banach space and let S_X be its unit sphere. Given a collection $\mathcal{B}$ of bounded subsets of X, we say that the norm $\|\cdot\|$ is *$\mathcal{B}$-smooth* if for every $x \in S_X$ there is $f_x \in X^*$ such that

$$\lim_{t\to 0} \sup_{h\in B} \left| \frac{\|x+th\| - \|x\|}{t} - f_x(h) \right| = 0 \quad \text{for every } B \in \mathcal{B}.$$

If in addition the above limit is uniform for $x \in S_X$, then we say that $\|\cdot\|$ is *uniformly $\mathcal{B}$-smooth*. In [11, Theorem 7] it is shown that if X is strongly super-reflexive generated, then X admits an equivalent norm which is uniformly smooth with respect to the collection of all weakly compact subsets of X (i.e. a *uniformly weak Hadamard smooth norm*). A variant of the argument employed there together with Theorem 1 yield the following result.

Corollary 7. *Let X be a strongly reflexive* (*resp. super-reflexive*) *generated Banach space and μ a probability measure. Let $\mathcal{B}$ be the collection of all weakly compact decomposable subsets of $L^1(\mu, X)$. Then $L^1(\mu, X)$ admits an equivalent $\mathcal{B}$-smooth* (*resp. uniformly $\mathcal{B}$-smooth*) *norm.*

Proof. By Theorem 1, there exist a reflexive Banach space Z and a bounded linear operator $S : Z^* \to L^1(\mu, X)$ with the property that for each $K \in \mathcal{B}$ and each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $K \subset nS(B_{Z^*}) + \varepsilon B_{L^1(\mu,X)}$. According to a result of Troyanski (see e.g. [3, Chapter VII, Corollary 1.13]), Z^* admits an equivalent dual locally uniformly rotund norm $|\cdot|$. Moreover, if X is strongly super-reflexive generated, then Z can be chosen super-reflexive and we can assume further that $|\cdot|$ is uniformly rotund. In both cases the formula

$$|||\varphi|||^2 := \|\varphi\|^2_{L^1(\mu,X)^*} + \left|S^*(\varphi)\right|^2, \quad \varphi \in L^1(\mu, X)^*,$$

defines an equivalent dual norm on $L^1(\mu, X)^*$ whose predual norm satisfies the required differentiability property (see the proof of [11, Theorem 7]). □

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