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Homogenization of a Reynolds equation describing compressible flow

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ABSTRACT

We homogenize a Reynolds equation with rapidly oscillating film thickness function h_ε, assuming a constant compressibility factor in the pressure–density relation. The oscillations are due to roughness on the bounding surfaces of the fluid film. As shown by previous studies, homogenization is an effective approach for analyzing the effects of surface roughness in hydrodynamic lubrication. By two-scale convergence theory we obtain the limit problem (homogenized equation) and strong convergence in L^2 for the unknown density ρ_ε. By adding a small corrector term, convergence is obtained also in the Sobolev norm.

1. Introduction

Lubrication problems involve surfaces in relative motion interacting through a thin film of viscous fluid (lubricant). Such examples include bearings, hip joints and gearboxes. In order to understand and optimize the effects of lubrication it is important to describe the flow in the lubricant film. To this end, Reynolds' lubrication equation, relating pressure and density, is widely used by engineers today. When the pressure distribution is known it is possible to compute other fundamental quantities such as the velocity field, friction forces and transversal loads carried by the surfaces. In many applications the distance between the surfaces is so small that the surface roughness has to be taken into account. The main focus of this paper is to model and analyze the effects of surface roughness under the assumption that the fluid has constant compressibility (see relation (2) below).

The fluid film is assumed to be confined between two rigid surfaces. At time $t=0$ we assume that the film is bounded by the surfaces $x_3 = h^+(x_1,x_2)$ (the upper surface contained in the region $x_3>0$) and $x_3 = h^-(x_1,x_2)$ (the lower surface contained in the region $x_3<0$), where h^+ and h^- are functions defined on $\mathbb{R}^2$. For simplicity it is assumed that the motions of the surfaces are translational with constant velocities and parallel to the plane $x_3=0$. The corresponding velocity vectors are denoted by $V^+ = (v_1^+, v_2^+)$ and $V^- = (v_1^-, v_2^-)$. Given a bounded domain Ω in $\mathbb{R}^2$, we assume that the region in space

$$F_t = \left\{(x,x_3)\colon\ x=(x_1,x_2)\in\Omega,\ h^-\left(x-V^-t\right) < x_3 < h^+\left(x-V^+t\right)\right\}$$

is filled with fluid for all $t\in[0,T]$. If the fluid film F_t is thin, the pressure p and density ρ in the fluid film are approximately governed by the Reynolds equation (named so after O. Reynolds [27]):

$$\frac{\partial}{\partial t}(h\rho) + \operatorname{div}\left(-\frac{h^3\rho}{12\mu}\nabla p + \frac{h\rho}{2}\left(V^+ + V^-\right)\right) = 0 \quad \text{in } \Omega\times(0,T], \tag{1}$$

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where μ is the viscosity of the fluid and h is the function defined by

$$h(x,t) = h^+\big(x - tV^+\big) - h^-\big(x - tV^-\big).$$

In this sense h describes the thickness of the film. It is assumed that $\alpha \leqslant h \leqslant \beta$ for positive constants α and β. For a derivation of the Reynolds equation see e.g. [19], where it is also shown how the velocity field of the fluid is recovered from p. Clearly Eq. (1) must be complemented with some relation between the unknowns p and ρ. The simplest case is obtained if one assumes that ρ is constant throughout the film. This leads to the incompressible Reynolds equation

$$\frac{\partial h}{\partial t} + \operatorname{div}\left(-\frac{h^3}{12\mu}\nabla p + \frac{h}{2}\big(V^+ + V^-\big)\right) = 0,$$

which has only one unknown p.

The present study pertains to a case where ρ is not constant. As a measure of the compressibility of a fluid one can introduce the compressibility factor β, which is defined as $d\rho/dp = \beta\rho$. In general β depends on both pressure and temperature, but if we assume β is constant we obtain the density–pressure relation

$$\rho = \rho_a e^{\beta p}, \tag{2}$$

where ρ_a is the density at ambient pressure. Assuming (2), we can write (1) as a linear equation for ρ:

$$\frac{\partial}{\partial t}(h\rho) + \operatorname{div}\big(-\lambda h^3 \nabla\rho + h\rho\Lambda\big) = 0, \tag{3}$$

where $\lambda = 1/(12\beta\mu)$ and $\Lambda = (V^+ + V^-)/2$ are constants. As initial–boundary conditions for p we take $p(x,t) = p_a$ on $\partial\Omega \times (0,T]$, where p_a (ambient pressure) is assumed constant, and $p(x,0) = p_I(x)$.

Since the functions h^+ and h^- satisfy transport equations, i.e.

$$\frac{\partial}{\partial t}\big(h^\pm\big(x - tV^\pm\big)\big) + V^\pm \cdot \nabla h^\pm\big(x - tV^\pm\big) = 0,$$

it follows that $\partial h/\partial t$ can be represented as

$$\frac{\partial h}{\partial t} = -\operatorname{div} G,$$

where G is the vector field $G(x,t) = h^+(x - tV^+)V^+ - h^-(x - tV^-)V^-$. This observation is important in the analysis.

In the case of rough surfaces we add to $h^\pm$ a rapidly oscillating function. More precisely, for $\varepsilon > 0$ we set

$$h_\varepsilon^\pm(x) = h^\pm(x) + r^\pm(x/\varepsilon)$$

where $r^\pm$ is $\square$-periodic in $\mathbb{R}^2$, $\square$ denoting the cell of periodicity. The film thickness function then becomes

$$h_\varepsilon(x,t) = h(x,t,x/\varepsilon,t/\varepsilon),$$

where h denotes the function

$$\begin{aligned} h(x,t,\xi,\tau) &= h^+\big(x - tV^+\big) - h^-\big(x - tV^-\big) + r^+\big(\xi - \tau V^+\big) - r^-\big(\xi - \tau V^-\big) \\ &= h^0(x,t) + r^0(\xi,\tau). \end{aligned} \tag{4}$$

In other words, ξ and τ correspond to the fast variables x/ε and t/ε. This means that $h^0(x,t) = h^+(x - tV^+) - h^-(x - tV^-)$ describes the global film thickness, the periodic functions $r^0(\xi,\tau) = r^+(\xi - \tau V^+) - r^-(\xi - \tau V^-)$ represent the roughness contribution and ε is related to the wavelength of the roughness. For simplicity, we assume that V^+ and V^- are such that r^0, and hence h, are periodic in τ and we denote by $\mathbb{T}$ its period (an interval in $\mathbb{R}$). Without loss of generality we assume that the Lebesgue measure of $\square$ and $\mathbb{T}$ equals one. It may happen that h_ε does not depend on t, e.g. if one of the surfaces is flat and the other one is stationary. We will refer to this a the stationary case. A particular consequence of (4) deserves special attention. Let G denote the vector field defined by

$$G(x,t,\xi,\tau) = h^+\big(x - tV^+\big)V^+ - h^-\big(x - tV^-\big)V^- + r^+\big(\xi - \tau V^+\big)V^+ - r^-\big(\xi - \tau V^-\big)V^-. \tag{5}$$

Then

$$\frac{\partial h_\varepsilon}{\partial t}(x,t) = -\operatorname{div} G_\varepsilon(x,t), \tag{6}$$

where $G_\varepsilon(x,t) = G(x,t,x/\varepsilon,t/\varepsilon)$.

On introducing the unknown $u_\varepsilon = \rho_\varepsilon - \rho_a$ we formulate the following initial–boundary value problem for u_ε:

$$\begin{cases} \dfrac{\partial}{\partial t}(h_\varepsilon u_\varepsilon) + \operatorname{div}\big(-\lambda h_\varepsilon^3 \nabla u_\varepsilon + h_\varepsilon u_\varepsilon \Lambda\big) = f_\varepsilon & \text{in } \Omega \times (0,T], \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times [0,T], \\ u_\varepsilon = \rho_0 - \rho_a := u_0 & \text{on } \Omega \times \{0\}, \end{cases} \tag{7}$$

where

$$f_\varepsilon = -\rho_a \frac{\partial h_\varepsilon}{\partial t} - \rho_a \operatorname{div}(h_\varepsilon \Lambda).$$

When the pressure is found it is possible to compute the stresses on the rigid surfaces. The friction force (due to shear stresses) gives information about the energy losses while the surfaces are kept in relative motion and the load carrying capacity (due to normal stresses) is the load carried by the surfaces. The friction force $F_\varepsilon = (F_1^\varepsilon, F_2^\varepsilon)$ on e.g. the lower surface $x_3 = h_\varepsilon^-(x,t)$ is given by (see e.g. [19, Theorem 9.1, p. 17])

$$F_\varepsilon(t) = \int_\Omega -\frac{h_\varepsilon(x,t)}{2}\nabla p_\varepsilon(x,t) + \frac{\mu}{h_\varepsilon(x,t)}\left(V^+ - V^-\right)dx,$$

or in terms of $u_\varepsilon = \rho_\varepsilon - \rho_a$

$$F_\varepsilon(t) = \int_\Omega -\frac{h_\varepsilon(x,t)}{2\beta(u_\varepsilon(x,t)+\rho_a)}\nabla u_\varepsilon(x,t) + \frac{\mu}{h_\varepsilon(x,t)}\left(V^+ - V^-\right)dx. \tag{8}$$

The load carrying capacity L_ε is given by

$$L_\varepsilon(t) = \int_\Omega p_\varepsilon(x,t)\,dx = \int_\Omega \frac{1}{\beta}\log\left(\frac{u_\varepsilon(x,t)}{\rho_a} + 1\right)dx. \tag{9}$$

1.1. Main result

For small values of ε (i.e. the roughness scale is much smaller than the global scale) the film thickness function h_ε is rapidly oscillating in both space and time. This means that a direct numerical treatment of Eq. (7) will require an extremely fine mesh to resolve the surface roughness. Hence some kind of averaging is required. The field of mathematics which handles this type of averaging is known as homogenization theory, see e.g. [17]. The main idea in homogenization is to prove that $u_\varepsilon \to u$ as $\varepsilon \to 0$ (in some sense) and that u solves the so-called homogenized equation. In the present paper it is proved that $u_\varepsilon \to u$ strongly in $L^2(Q)$, u being the solution of the homogenized equation

$$\frac{\partial}{\partial t}(\bar{h}u) + \operatorname{div}(-A\nabla u + ub) = -\rho_a \operatorname{div} b - \rho_a \frac{\partial \bar{h}}{\partial t}, \tag{10}$$

where A (matrix), b (vector) and $\bar{h}$ (scalar) do not involve any rapid oscillations. This means that it is much easier to find u numerically and that u can be used as a good approximation of u_ε for small values of ε.

1.2. Previous work

We conclude the Introduction by giving a short guide to the literature: In the case of an incompressible fluid there are numerous studies where homogenization has been used to analyze the effects of surface roughness in hydrodynamic lubrication. Indeed, the incompressible Reynolds equation (ρ constant) was homogenized in [28] by using two-scale convergence, by G-convergence in [16] and by the formal method of multiple scale expansions in [9] and [22]. The case of several different length scales (both roughness and texture) was analyzed in [2]. By introducing two parameters, one for the film thickness and one for the fineness of the roughness, the authors of [7] and [13] studied homogenization of the Stokes flow and its relation to Reynolds flow. These works rigorously verify that homogenization of Reynolds equation can be used when the film thickness is small compared to the wavelength of the roughness. If both surfaces are rough, then the distance will, in addition to the space variable, oscillate rapidly with respect to time. This situation was analyzed by multiple scale expansion in [4] and by two-scale convergence in [8]. Roughness effects, with one rough surface taking cavitation into account by the Elrod–Adams model [18], have been studied by two-scale convergence in [11] and [26] and by asymptotic expansions in [10] and [12]. In addition to cavitation, elasto-hydrodynamic phenomenon is taken into account in [12]. Cavitation in the case that both surfaces are rough was modeled by variational inequalities in [8] and the effects of surface roughness were analyzed by two-scale convergence.

In the case of a compressible fluid, where the fluid is assumed to have constant bulk modulus, the effects of surface roughness on one surface have been considered in [1] and with both surfaces rough in [5]. In both of these works the homogenization results were merely justified by using the formal method of multiple scale expansion. The present work rigorously proves the homogenization of ρ_ε (density) in the case of two rough surfaces. Some other works devoted to roughness effects in compressible thin film lubrication are [14,15,21].

There are several papers where the homogenization process is illustrated by numerical investigations, examples are [5,8, 22,26]. Recently a new idea, based on bounds related to the homogenized equation, has been used with success, see [3,6,24].

2. Weak maximum principle

The pressure–density relation (2) requires some justification, because p is defined only when ρ, the solution of Eq. (3), is positive. We prove here that p is always defined provided that ρ is sufficiently smooth and takes positive boundary values.

We shall assume that ρ is a solution of the Reynolds equation

$$\frac{\partial}{\partial t}(h\rho) + \operatorname{div}\bigl(-\lambda h^3 \nabla\rho + h\rho\Lambda\bigr) = 0 \quad \text{in } \Omega \times (0, T] \tag{11}$$

such that $\rho \in C^{2,1}(Q) \cap C(\overline{Q})$, that is ρ is two times continuously differentiable w.r.t. x, continuously differentiable w.r.t. t and continuous on $\overline{Q} = \overline{\Omega} \times [0, T]$. The parabolic boundary of the cylinder $Q = \Omega \times (0, T)$ is denoted by $\Gamma = \Omega \times \{0\} \cup \partial\Omega \times [0, T]$. Here it is more convenient to write (11) as $L\rho = 0$ where L is a differential operator (acting on some function u) of the form

$$Lu = \sum_{i,j} a_{ij} D_i D_j u + \sum_i b_i D_i u + cu - h\frac{\partial u}{\partial t}.$$

For $a_{ij} = \lambda h^3 \delta_{ij}$, $b_i = 3\lambda h^2 D_i h - h\Lambda_i$ and $c = -(\partial h/\partial t + \Lambda \cdot \nabla h) = (V^- - V^+) \cdot \nabla h/2$, it is clear that (11) is equivalent to $L\rho = 0$.

The following versions of the maximum principle are valid for any parabolic operator of the form L with smooth coefficients. The proofs are based on ideas from Lieberman [23, Ch. 2].

Lemma 2.1. *Suppose there exists a positive constant k such that $c \leqslant kh$ in Q and that $u \in C^{2,1}(Q) \cap C(\overline{Q})$. If*

$$\begin{cases} Lu \geqslant 0 & \text{in } Q, \\ u \leqslant 0 & \text{on } \Gamma, \end{cases}$$

then $u \leqslant 0$ in Q.

Proof. This lemma is a special case of [23, Lemma 2.3, p. 8], but we give a simplified proof for the sake clarity. Set $v = e^{-(k+1)t} u$. Then

$$Lv = e^{-(k+1)t} Lu + (k+1)hv \geqslant (k+1)hv \quad \text{in } Q \tag{12}$$

and $v \leqslant 0$ on Γ. If v has a positive maximum at some $q \in Q$, then $\partial v/\partial t(q) = 0$, $D_i v(q) = 0$ and $\sum_{i,j} a_{ij} D_i D_j v(q) \leqslant 0$, hence

$$Lv(q) \leqslant cv(q) \leqslant khv(q),$$

contradicting (12). There remains the possibility that v attains a positive maximum on $\Omega \times \{T\}$. Suppose this is the case and choose an increasing sequence of positive numbers $t_1, t_2, \ldots$ converging to T. Set $Q_i = \Omega \times (0, t_i)$ and $M_i = \sup_{Q_i} v$. By the continuity of v, $v(q_i) = M_i$ for some $q_i \in \overline{Q_i}$ with $M_i > 0$ for i sufficiently large. By reasoning as above (on Q_i) it follows that $q_i \in \Omega \times \{t_i\}$ with $\partial v/\partial t(q_i) \geqslant 0$, $D_j v(q_i) = 0$ and $\sum_{j,k} a_{jk} D_j D_k v(q_i) \leqslant 0$, so

$$Lv(q_i) \leqslant cv(q_i) \leqslant khv(q_i)$$

provided that i is large enough, again contradicting (12). □

The following theorem is also taken from [23, Theorem 2.4, p. 9].

Theorem 2.2. *Suppose c and u as in Lemma 2.1. If*

$$\begin{cases} Lu \geqslant 0 & \text{in } Q, \\ u \leqslant M & \text{on } \Gamma, \end{cases}$$

for some constant $M \geqslant 0$, then $u(x, t) \leqslant e^{kt} M$ for all $(x, t) \in Q$.

Proof. Set $v = u - e^{kt} M$. Then

$$Lv = Lu - L\bigl(e^{kt} M\bigr) = Lu + (kh - c)e^{kt} M \geqslant 0 \quad \text{in } Q$$

and

$$v = u - e^{kt} M \leqslant \bigl(1 - e^{kt}\bigr) M \leqslant 0 \quad \text{on } \Gamma,$$

since $e^{kt} \geqslant 1$ $(k, t \geqslant 0)$. In view of Lemma 2.1, $v \leqslant 0$ in Q. □

Theorem 2.3. *Suppose in addition to the hypotheses of Lemma* 2.1 *that* $-kh \leqslant c$ *in* Q. *If*

$$\begin{cases} Lu \leqslant 0 & \text{in } Q, \\ u \geqslant m & \text{on } \Gamma, \end{cases}$$

for some constant $m \geqslant 0$, *then* $u(x,t) \geqslant e^{-kt}m$ *for all* $(x,t) \in Q$.

Proof. Set $v = e^{-kt}m - u$. Then

$$Lv = L\big(e^{-kt}m\big) - Lu = (c + kh)e^{-kt}m - Lu \geqslant 0 \quad \text{in } Q$$

and

$$v = e^{-kt}m - u \leqslant \big(e^{-kt} - 1\big)m \leqslant 0 \quad \text{on } \Gamma,$$

since $e^{-kt} \leqslant 1$ $(k, t \geqslant 0)$. In view of Lemma 2.1, $v \leqslant 0$ in Q. □

Combining the preceding two theorems we obtain

Corollary 2.4 *(Weak maximum principle for the Reynolds equation). Suppose* $\rho \in C^{2,1}(Q) \cap C(\overline{Q})$ *is a solution of the compressible Reynolds equation* (1). *Set*

$$m = \inf_{\Gamma} \rho, \qquad M = \sup_{\Gamma} \rho, \qquad k = \sup_{Q} \frac{|(V^- - V^+) \cdot \nabla h|}{2h}.$$

If $m \geqslant 0$, *then*

$$e^{-kt}m \leqslant \rho(x,t) \leqslant e^{kt}M \quad \text{for all } (x,t) \in Q.$$

From Corollary 2.4 we deduce that $\rho > 0$ in Q provided that $\rho > 0$ on Γ. Hence p can be recovered from ρ by inverting (3). This shows that the change of variables is consistent provided the solutions are sufficiently regular. Similarly we obtain

Corollary 2.5 *(Weak maximum principle for the homogenized equation). Suppose* $\rho \in C^{2,1}(Q) \cap C(\overline{Q})$ *is a solution of the homogenized Reynolds equation, i.e.*

$$\frac{\partial}{\partial t}(\bar{h}\rho) + \operatorname{div}(-A\nabla\rho + \rho b) = 0,$$

where $\bar{h}$, A *and* b *are defined in Theorem* 5.3. *Set*

$$m = \inf_{\Gamma} \rho, \qquad M = \sup_{\Gamma} \rho, \qquad k = \sup_{Q} \frac{|\frac{\partial \bar{h}}{\partial t} + \operatorname{div} b|}{\bar{h}}.$$

If $m \geqslant 0$, *then*

$$e^{-kt}m \leqslant \rho(x,t) \leqslant e^{kt}M \quad \text{for all } (x,t) \in Q.$$

3. Existence and uniqueness

In this section it is proved that, under general assumptions on the data, an initial–boundary value problem of the form (7) has a unique solution provided that the film thickness function h is sufficiently smooth. We ignore here any dependence on ε, since this parameter is irrelevant in this regard.

The standard space for studying parabolic equations is

$$X = L^2\big(0, T; H_0^1(\Omega)\big) \cap H^1\big(0, T; H^{-1}(\Omega)\big).$$

It is well known that each $u \in X$ has a continuous representative in $C([0,T]; L^2(\Omega))$ in which sense the initial condition for $u(0)$ is understood. Assume $u_0 \in L^2(\Omega)$, $\varphi \in L^2(0,T; L^2(\Omega))$ and $F \in L^2(0,T; L^2(\Omega; \mathbb{R}^2))$ and consider the initial–boundary value problem for u

$$\begin{cases} \dfrac{\partial}{\partial t}(hu) + \operatorname{div}\big(-\lambda h^3 \nabla u + hu\Lambda\big) = \varphi - \operatorname{div} F & \text{in } \Omega \times (0,T], \\ u = 0 & \text{on } \partial\Omega \times (0,T], \\ u = u_0 & \text{on } \Omega \times \{0\}. \end{cases} \tag{13}$$

We want to define a notion of weak solution $u \in X$ for this problem. This can be done in several ways. For our approach to be successful we must impose some restrictions on the function h, namely

(i) $h \in W^{1,\infty}(Q)$, where $Q = \Omega \times (0,T)$,
(ii) there exist positive constants α and β such that $\alpha \leqslant h \leqslant \beta$.

Assume (temporarily) that $u = u(x,t)$ and $h = h(x,t)$ are smooth functions, for fixed $t \in (0,T]$ and multiply the equation

$$h\frac{\partial u}{\partial t} + \frac{\partial h}{\partial t}u + \operatorname{div}\bigl(-\lambda h^3\nabla u + hu\Lambda\bigr) = \varphi - \operatorname{div} F$$

with a test function, i.e. a smooth function $v = v(x)$ with compact support in Ω, and integrate by parts. Thus we obtain

$$\int_\Omega h\frac{\partial u}{\partial t}v + \frac{\partial h}{\partial t}uv + \bigl(\lambda h^3\nabla u - hu\Lambda\bigr)\cdot\nabla v\,dx = \int_\Omega \varphi v + F\cdot\nabla v\,dx \tag{14}$$

for all test functions v. Since h is bounded from below by a positive constant, $v(x)/h(x,t)$ (t fixed) is also admissible as a test function. Replacing v with v/h in (14), we obtain

$$\int_\Omega \frac{\partial u}{\partial t}v + \frac{\partial}{\partial t}(\log h)uv + \bigl(\lambda h^2\nabla u - u\Lambda\bigr)\cdot\bigl(\nabla v - v\nabla(\log h)\bigr)\,dx = \int_\Omega \tilde{\varphi}v + \widetilde{F}\cdot\nabla v\,dx \tag{15}$$

for all test functions v, where $\tilde{\varphi} = \varphi/h - (F\cdot\nabla h)/h^2$ and $\widetilde{F} = F/h$. The identity (15) leads us to define the bilinear form

$$\begin{aligned}\widetilde{R}(u,v;t) &= \int_\Omega \frac{\partial}{\partial t}(\log h)uv + \bigl(\lambda h^2\nabla u - u\Lambda\bigr)\cdot\bigl(\nabla v - v\nabla(\log h)\bigr)\,dx \\ &= \int_\Omega \lambda h^2\nabla u\cdot\nabla v - u(\Lambda\cdot\nabla v) - \lambda h^2\bigl(\nabla u\cdot\nabla(\log h)\bigr)v + \left(\Lambda\cdot\nabla(\log h) + \frac{\partial}{\partial t}(\log h)\right)uv\,dx \end{aligned}\tag{16}$$

which makes sense for all $u, v \in H^1(\Omega)$ provided that $h \in L^\infty(Q)$ and $\log h \in W^{1,\infty}(Q)$. (Since $\alpha \leqslant h \leqslant \beta$ and log is Lipschitz continuous on the interval $[\alpha,\beta]$, for $\alpha > 0$, we might as well assume $h \in W^{1,\infty}(Q)$.) This motivates the following abstract definition of weak solution for (13).

We say that $u \in X$ is a weak solution of (13) if

$$\bigl\langle u'(t), v\bigr\rangle + \widetilde{R}\bigl(u(t), v; t\bigr) = \bigl\langle \tilde{f}(t), v\bigr\rangle \tag{17}$$

for all $v \in H_0^1(\Omega)$ and a.e. $t \in (0,T]$, where $\tilde{f} \in L^2(0,T;H^{-1}(\Omega))$ is defined by

$$\bigl\langle \tilde{f}(t), v\bigr\rangle = \int_\Omega \tilde{\varphi}(t)v + \widetilde{F}(t)\cdot\nabla v\,dx$$

for all $v \in H_0^1(\Omega)$.

By employing standard estimates, one shows that

$$\widetilde{R}(v,v;t) \geqslant c\|\nabla v\|_{L^2(\Omega)} - d\|v\|_{L^2(\Omega)}$$

for all $v \in H_0^1(\Omega)$, where $c = c(\alpha,\lambda)$ and $d = d(\alpha,\lambda,\Lambda,\|h\|_{W^{1,\infty}(Q)})$, are positive constants. From standard existence theory for parabolic equations (see e.g. Zeidler [29, Corollary 23.26, p. 426]) it then follows that for each (φ, F) and u_0 (as above) there exists a unique $u \in X$ satisfying (17).

4. A posteriori estimates

The standard *a priori* estimates (or energy estimates) that are derived to prove existence (and uniqueness) for an evolution equation of the form (17) will depend on $\|h\|_{W^{1,\infty}(Q)}$. This is bad for homogenization, because when $h = h_\varepsilon$ the constants will blow up as ε approaches zero. However, knowing that (17) has a solution it is possible to obtain an *a posteriori* energy estimate, better suited for homogenization, by considering a different formulation of (17) (see (20) below) which seems more natural. Then we combine (20) with the information that $\partial h/\partial t = -\operatorname{div} G$ to obtain the desired estimates. To prove equivalence of the two formulations we need the following lemmas.

Lemma 4.1. *For $u, v \in X$, define $f : (0,T) \to \mathbb{R}$ by*

$$f(t) = \int_\Omega u(t)v(t)\,dx.$$

Then f is absolutely continuous and

$$f(t)-f(s)=\int_s^t \langle u'(t),v(t)\rangle+\langle u(t),v'(t)\rangle\,dt$$

for all $0\leqslant s\leqslant t\leqslant T$.

Proof. See Zeidler [29, Proposition 23.23(iv), p. 422]. □

Lemma 4.2. *For $u\in X$ and $h\in W^{1,\infty}(Q)$, define f by*

$$\langle f(t),v\rangle=\int_\Omega h(t)u(t)v\,dx$$

for all $v\in H_0^1(\Omega)$. Then $f\in H^1(0,T;H^{-1}(\Omega))$ with

$$\langle f'(t),v\rangle=\langle u'(t),h(t)v\rangle+\int_\Omega \frac{\partial h}{\partial t}(t)u(t)v\,dx \tag{18}$$

for all $v\in H_0^1(\Omega)$. We write $f=hu$.

Proof. Put $g(t)=h(t)v$. Then $g\in L^2(0,T;H_0^1(\Omega))$ and $g'(t)=\partial h/\partial t(t)v$ belongs to $L^2(\Omega)\subset H^{-1}(\Omega)$ for a.e. t. Using Lemma 4.1 (with $v=g$)

$$\begin{aligned}\int_0^T\Big(\int_\Omega h(t)u(t)v\,dx\Big)\phi'(t)\,dt&=\int_0^T\Big(\int_\Omega u(t)g(t)\,dx\Big)\phi'(t)\,dt\\&=-\int_0^T\big(\langle u'(t),g(t)\rangle+\langle u(t),g'(t)\rangle\big)\phi(t)\,dt\\&=-\int_0^T\Big(\langle u'(t),h(t)v\rangle+\int_\Omega u(t)\frac{\partial h}{\partial t}(t)v\,dx\Big)\phi(t)\,dt\end{aligned} \tag{19}$$

for all $\phi\in C_c^\infty(0,T)$. This proves that f is weakly differentiable and that (18) holds. From (18) we deduce that $f'\in L^2(0,T;H^{-1}(\Omega))$. □

In view of Lemma 4.2 and the identity (14) the following result holds.

Theorem 4.3. *The weak formulation* (17) *is equivalent to*

$$\langle (hu)'(t),v\rangle+R\big(u(t),v;t\big)=\langle f(t),v\rangle\quad \text{a.e. } t\in(0,T] \tag{20}$$

for all $v\in H_0^1(\Omega)$, where

$$R(u,v;t)=\int_\Omega \lambda h^3\nabla u\cdot\nabla v-hu(\Lambda\cdot\nabla v)\,dx$$

and $f\in L^2(0,T;H^{-1}(\Omega))$ is defined by

$$\langle f(t),v\rangle=\int_\Omega \varphi v+F\cdot\nabla v\,dx.$$

Theorem 4.4. *Let $h\in W^{1,\infty}(Q)$ and let $u\in X$ be a solution of* (20)*. Suppose there exists a vector field $G\in L^\infty(Q)$ such that*

$$\frac{\partial h}{\partial t}=-\operatorname{div}G\quad \text{in } Q. \tag{21}$$

Then there exists a constant C such that

$$\|u\|_{C([0,T];L^2(\Omega))}+\|u\|_{L^2(0,T;H_0^1(\Omega))}+\|(hu)'\|_{L^2(0,T;H^{-1}(\Omega))}\leqslant C\big(\|u(0)\|_{L^2(\Omega)}+\|f\|_{L^2(0,T;H^{-1}(\Omega))}\big). \tag{22}$$

C depends only on α, β, λ, Λ, $\|G\|_{L^\infty(Q)}$, T and Ω.

Proof. Define

$$\eta(t)=\int_\Omega h(t)u(t)^2\,dx.$$

In view of Lemmas 4.1 and 4.2, η is absolutely continuous and

$$\eta'=\langle (hu)',u\rangle+\langle u',hu\rangle=2\langle (hu)',u\rangle-\int_\Omega \frac{\partial h}{\partial t}u^2\,dx.$$

Combining this and (20) we obtain

$$\eta'+\int_\Omega \frac{\partial h}{\partial t}u^2\,dx+2R(u,u;t)=2\langle f,u\rangle \quad \text{a.e. in } (0,T]. \tag{23}$$

By standard estimates there exist positive constants γ and θ such that

$$R(u,u;t)\geqslant \theta\|\nabla u\|^2_{L^2(\Omega)}-\gamma\|u\|^2_{L^2(\Omega)}.$$

Owing to (21),

$$\begin{aligned}\left|\int_\Omega \frac{\partial h}{\partial t}u^2\,dx\right|&=\left|\int_\Omega G\cdot\nabla\big(u^2\big)\,dx\right|\leqslant 2\|G\|_{L^\infty(Q)}\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\\&\leqslant \frac{2}{\theta}\|G\|^2_{L^\infty(Q)}\|u\|^2_{L^2(\Omega)}+\frac{\theta}{2}\|\nabla u\|_{L^2(\Omega)}.\end{aligned}$$

The free term can be estimated as

$$\big|2\langle f,u\rangle\big|\leqslant 2C\|f\|_{H^{-1}(\Omega)}\|\nabla u\|_{L^2(\Omega)}\leqslant \frac{2C^2}{\theta}\|f\|^2_{H^{-1}(\Omega)}+\frac{\theta}{2}\|\nabla u\|^2_{L^2(\Omega)},$$

where the constant C depends on Ω. Using these estimates in (23), we obtain

$$\eta'+\theta\|\nabla u\|^2_{L^2(\Omega)}\leqslant \frac{2C^2}{\theta}\|f\|^2_{H^{-1}(\Omega)}+\frac{2}{\theta}\|G\|^2_{L^\infty(Q)}\|u\|^2_{L^2(\Omega)}+\gamma\|u\|^2_{L^2(\Omega)}. \tag{24}$$

Since $\|u(t)\|^2_{L^2(\Omega)}\leqslant \beta^{-2}\eta(t)$ we obtain

$$\eta'(t)\leqslant \lambda_1\big\|f(t)\big\|^2_{H^{-1}(\Omega)}+\lambda_2\eta(t),$$

for positive constants λ_1 and λ_2. From Grönwall's inequality we deduce

$$\eta(t)\leqslant e^{\lambda_2 t}\left(\eta(0)+\lambda_1\int_0^t\big\|f(\tau)\big\|^2_{H^{-1}(\Omega)}\,d\tau\right).$$

Since $\|u(t)\|^2_{L^2(\Omega)}\leqslant \alpha^{-2}\eta(t)$ there exists a positive constant C such that

$$\max_{0\leqslant t\leqslant T}\big\|u(t)\big\|^2_{L^2(\Omega)}\leqslant C\big(\big\|u(0)\big\|^2_{L^2(\Omega)}+\|f\|^2_{L^2(0,T;H^{-1}(\Omega))}\big).$$

Next, integrating (24) from $t=0$ to T and using the above estimates yields

$$\|\nabla u\|^2_{L^2(0,T;L^2(\Omega))}\leqslant C\big(\big\|u(0)\big\|^2_{L^2(\Omega)}+\|f\|^2_{L^2(0,T;H^{-1}(\Omega))}\big),$$

where the constant C depends on α, β, λ, Λ, $\|G\|_{L^\infty(Q)}$, T and Ω.

From (20), with $\|v\|_{H^1_0(\Omega)}\leqslant 1$ we obtain

$$\begin{aligned}\big|\langle (hu)'(t),v\rangle\big|&=\big|\langle f(t),v\rangle-R\big(u(t),v;t\big)\big|\\&\leqslant \big\|f(t)\big\|_{H^{-1}(\Omega)}+\gamma_1\big\|\nabla u(t)\big\|_{L^2(\Omega)}+\gamma_2\big\|u(t)\big\|_{L^2(\Omega)}.\end{aligned}$$

Hence

$$\big\|(hu)'(t)\big\|_{H^{-1}(\Omega)}\leqslant \big\|f(t)\big\|_{H^{-1}(\Omega)}+\gamma_3\big\|u(t)\big\|_{H^1_0(\Omega)}.$$

By squaring this inequality and integrating from $t=0$ to T and using the above estimates we obtain

$$\big\|(hu)'\big\|^2_{L^2(0,T;H^{-1}(\Omega))}\leqslant C\big(\big\|u(0)\big\|^2_{L^2(\Omega)}+\|f\|^2_{L^2(0,T;H^{-1}(\Omega))}\big)$$

for some constant C with dependence as above. □

5. Homogenization

In this section we consider the homogenization of the generalized equation corresponding to (7), i.e. Eq. (20). For each $\varepsilon > 0$, let $u_\varepsilon \in X$ be the unique solution of the evolution equation

$$\begin{cases} \langle (h_\varepsilon u_\varepsilon)'(t), v\rangle + \displaystyle\int_\Omega \big(\lambda h_\varepsilon(t)^3 \nabla u_\varepsilon(t) - h_\varepsilon(t) u_\varepsilon(t)\Lambda\big)\cdot \nabla v\,dx = \langle f_\varepsilon(t), v\rangle \quad \text{a.e. in } (0,T], \\ u_\varepsilon(0) = \rho_0 - \rho_a := u_0 \end{cases} \tag{25}$$

for all $v \in H_0^1(\Omega)$, where $f_\varepsilon(t) = -\rho_a \partial h_\varepsilon/\partial t(t) - \rho_a \operatorname{div}(h_\varepsilon(t)\Lambda)$. We assume that $h_\varepsilon(x,t) = h(x,t,x/\varepsilon,t/\varepsilon)$ according to Eq. (4) with $h^\pm, r^\pm \in W^{1,\infty}(Q \times \square \times \mathbb{T})$ and that $\alpha \leqslant h \leqslant \beta$ for positive constants α and β. Recall also the relation (6). Under these assumptions we conclude that there exists a constant C such that

$$\|G_\varepsilon\|_{L^\infty(Q)}, \quad \|u_\varepsilon(0)\|_{L^2(\Omega)}, \quad \|f_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C$$

for all $\varepsilon > 0$. Owing to the estimates in Theorem 4.4, there exists a constant C (independent of ε) such that

$$\|u_\varepsilon\|_{C([0,T];L^2(\Omega))} + \|u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} + \|(h_\varepsilon u_\varepsilon)'\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C. \tag{26}$$

Since the above bound is independent of ε, we can apply the compactness theorem of two-scale convergence in the parabolic setting (see e.g. Holmbom, Svanstedt and Wellander [20, Proposition 1] and Lukkassen, Meidell and Wall [25, Theorem 3]). Thus we can extract a subsequence (still denoted by ε) such that u_ε two-scale converges to $u(x,t,\tau) \in L^2((0,T)\times\mathbb{T}; H_0^1(\Omega))$ and ∇u_ε two-scale converges to $\nabla_x u(x,t,\tau) + \nabla_\xi u_1(x,t,\xi,\tau)$, where $u_1 \in L^2(Q\times\mathbb{T}; H^1_{\mathrm{per}}(\square))$. Here it can be noted that the energy estimate does not involve the time derivative of u_ε so we can not draw the standard conclusion that the limit function u does not depend on τ. However, by taking into account that h is given by (4), one can conclude as demonstrated below that $u = u(x,t)$, i.e. u does not depend on the variable τ.

Lemma 5.1. *Suppose that*

- *h has the special form* (4),
- *u_ε two-scale converges to $u = u(x,t,\tau) \in L^2((0,T)\times\mathbb{T}; H_0^1(\Omega))$,*
- *$q_\varepsilon = (h_\varepsilon u_\varepsilon)'$ converges weakly to q in $L^2(0,T;H^{-1}(\Omega))$.*

Then

(i) *$u = u(x,t)$ belongs to $L^2(0,T;H_0^1(\Omega))$,*
(ii) *$q = (\bar h u)'$,*

where $\bar h = \bar h(x,t) \in W^{1,\infty}(Q)$ denotes the average of h over $\square$.

Proof. On the one hand, since $q_\varepsilon = (h_\varepsilon u_\varepsilon)'$ is bounded in $L^2(0,T;H^{-1}(\Omega))$ it follows that

$$\lim_{\varepsilon\to 0} \int_0^T \langle q_\varepsilon(t), v\rangle \varepsilon \phi_1(t)\phi_2(t/\varepsilon)\,dt = 0$$

for all $v \in H_0^1(\Omega)$, $\phi_1 \in C_c^1(0,T)$ and $\phi_2 \in C^1(\mathbb{T})$. On the other hand

$$\int_0^T \langle q_\varepsilon, v\rangle \varepsilon\phi(t,t/\varepsilon)\,dt = -\int_0^T \left(\int_\Omega h_\varepsilon u_\varepsilon v\,dx\right)\big(\varepsilon\phi_1'(t)\phi_2(t/\varepsilon) + \phi_1(t)\phi_2'(t/\varepsilon)\big)\,dt,$$

by integration by parts. Passing to the limit in this equality, and using the definition of two-scale convergence for the right-hand side yields

$$0 = \int_\Omega \left(\int_0^T \left(\int_\mathbb{T} \bar h(x,t,\tau) u(x,t,\tau)\phi_2'(\tau)\,d\tau\right)\phi_1(t)\,dt\right) v(x)\,dx,$$

where $\bar h(x,t,\tau) = \int_\square h(x,t,\xi,\tau)\,d\xi$. Since v, ϕ_1 and ϕ_2 are arbitrary we conclude that the product $\bar h(x,t,\tau)u(x,t,\tau)$ does not depend on τ. But using the translation invariance of the periodic cell $\square$ and the special form (4) of h, we see that

$$\begin{aligned}\bar{h}(x,t,\tau) &= h^0(x,t) + \int_{\Box} r^+\big(\xi - \tau V^+\big) - r^-\big(\xi - \tau V^-\big)\,d\xi \\ &= h^0(x,t) + \int_{\Box} r^0(\xi)\,d\xi\end{aligned}$$

is in fact independent of τ. Hence u does not depend on τ.

If $q_\varepsilon = (h_\varepsilon u_\varepsilon)'$, an integration by parts shows that

$$\int_0^T \langle q_\varepsilon, v\rangle \phi\,dt = -\int_0^T \bigg(\int_\Omega h_\varepsilon u_\varepsilon v\,dx\bigg)\phi'\,dt$$

for all $v \in H_0^1(\Omega)$ and $\phi \in C_c^1(0,T)$. By weak and two-scale convergence we obtain in the limit as $\varepsilon \to 0$

$$\int_0^T \langle q, v\rangle \phi\,dt = -\int_0^T \bigg(\int_\Omega \bar{h} u v\,dx\bigg)\phi'\,dt.$$

This proves $q = (\bar{h}u)'$. $\Box$

Theorem 5.2 *(Parabolic two-scale compactness). Let u_ε be a sequence in X satisfying the bound* (26)*. Then, on passing to a subsequence, there exist $u \in X$ and $u_1 \in L^2(Q \times \mathbb{T}; H^1_{\mathrm{per}}(\Box))$ such that*

(i) *u_ε two-scale converges to $u(x,t)$ in $L^2(Q \times \Box \times \mathbb{T})$,*
(ii) *∇u_ε two-scale converges to $\nabla u(x,t) + \nabla_\xi u_1(x,t,\xi,\tau)$ in $L^2(Q \times \Box \times \mathbb{T}; \mathbb{R}^2)$,*
(iii) *$h_\varepsilon u_\varepsilon$ converges to $\bar{h}u$ weakly in $H^1(0,T; H^{-1}(\Omega))$.*

Proof. (i) and (ii) with $u \in L^2(0,T; H_0^1(\Omega))$ follow from the bound (26), standard compactness results of two-scale convergence (see Holmbom, Svanstedt and Wellander [20] and the references therein for the details) and Lemma 5.1.

(iii) follows from Lemma 5.1. To prove that $u \in X$, set $\bar{u} = \bar{h}u$. By (iii), $\bar{u}' \in L^2(0,T; H^{-1}(\Omega))$ and since $\bar{h} \in W^{1,\infty}(Q)$ we have $\bar{u} \in X$. By our assumptions on h it follows that $1/\bar{h} \in W^{1,\infty}(Q)$. Applying Lemma 4.2 (with $h = 1/\bar{h}$ and $u = \bar{u}$) we conclude that $u = \bar{u}/\bar{h} \in X$. $\Box$

Theorem 5.3 *(Homogenization). The whole sequence u_ε of solutions to* (25) *in X converges weakly in $L^2(0,T; H_0^1(\Omega))$ to the unique solution $u \in X$ of an evolution equation of the form*

$$\begin{cases} \big\langle (\bar{h}u)'(t), v\big\rangle + \displaystyle\int_\Omega \big(A\nabla u(t) - ub(t)\big)\cdot\nabla v\,dx = \int_\Omega -\rho_a \frac{\partial \bar{h}}{\partial t}(t) v + \rho_a b(t)\cdot\nabla v\,dx \quad \text{a.e. in } (0,T], \\ u(0) = \rho_0 - \rho_a, \end{cases} \tag{27}$$

where $\bar{h} = h^0 + \int_\Box r^0(\xi)\,d\xi$, the matrix function A is defined by (38) *and the vector field b is defined by* (39) *in terms of solutions to the local problems* (35)–(37)*. In other words, $\rho = u + \rho_a$ is a generalized solution of the initial–boundary value problem*

$$\begin{cases} \dfrac{\partial}{\partial t}(\bar{h}\rho) + \operatorname{div}(-A\nabla\rho + \rho b) = 0 & \text{in } \Omega \times (0,T], \\ \rho = \rho_a & \text{on } \partial\Omega \times [0,T], \\ u = \rho_0 & \text{on } \Omega \times \{0\}. \end{cases} \tag{28}$$

Moreover the whole sequence ∇u_ε two-scale converges to $\nabla u + \nabla_\xi u_1$, with $u_1 \in L^2(Q \times \mathbb{T}; H^1_{\mathrm{per}}(\Box))$. The function u_1 can be written as

$$u_1(x,t,\xi,\tau) = \frac{\partial u}{\partial x_1}(x,t)\chi_1(x,t,\xi,\tau) + \frac{\partial u}{\partial x_2}(x,t)\chi_2(x,t,\xi,\tau) + \big(u(x,t) + \rho_a\big)\chi_3(x,t,\xi,\tau), \tag{29}$$

where χ_1, χ_2 and χ_3 are unique ξ-periodic solutions of

$$\begin{aligned} &\operatorname{div}_\xi\big(\lambda h^3(\nabla_\xi \chi_1 + e_1)\big) = 0, \\ &\operatorname{div}_\xi\big(\lambda h^3(\nabla_\xi \chi_2 + e_2)\big) = 0, \\ &\frac{\partial h}{\partial \tau} + \operatorname{div}_\xi\big(-\lambda h^3 \nabla_\xi \chi_3 + h\Lambda\big) = 0 \end{aligned}$$

in $Q \times \Box \times \mathbb{T}$, where $e_1 = (1,0)$, $e_2 = (0,1)$ are vectors in $\mathbb{R}^2$.

Proof. Extract from u_ε a subsequence (also denoted by u_ε) with the properties of Theorem 5.2 converging to $u \in X$.

Multiplying (25) with $\phi \in C_c^1(0,T)$ and integrating by parts we obtain

$$\int_Q -h_\varepsilon(u_\varepsilon+\rho_a)v\phi' + \big(\lambda h_\varepsilon^3 \nabla u_\varepsilon - h_\varepsilon(u_\varepsilon+\rho_a)\Lambda\big)\cdot\nabla v\phi\,dx\,dt = 0. \tag{30}$$

Passing to the limit as $\varepsilon \to 0$ and using the definition of two-scale convergence we obtain

$$\int_Q \int_{\square\times\mathbb{T}} -h(u+\rho_a)v\phi' + \big(\lambda h^3(\nabla u + \nabla_\xi u_1) - h(u+\rho_a)\Lambda\big)\cdot\nabla v\phi\,d\xi\,d\tau\,dx\,dt = 0, \tag{31}$$

or on integrating by parts with respect to t

$$\int_Q \big\langle(\bar{h}u)', v\big\rangle\phi + \rho_a\frac{\partial\bar{h}}{\partial t}v\phi + \left(\int_{\square\times\mathbb{T}} \big(\lambda h^3(\nabla u+\nabla_\xi u_1) - h(u+\rho_a)\Lambda\big)\,d\xi\,d\tau\right)\cdot\nabla v\phi\,dx\,dt = 0 \tag{32}$$

for all $v \in H_0^1(\Omega)$ and all $\phi \in C_c^1(0,T)$.

Next replace $v(x)$ and $\phi(t)$ in (30) with $v_\varepsilon(x) = \varepsilon v_1(x, x/\varepsilon)$ and $\phi_\varepsilon(t) = \phi_1(t, t/\varepsilon)$, where $v_1 = v_1(x,\xi)$ belongs to a set of smooth ξ-periodic functions that is dense in $L^2(\Omega; H^1_{\mathrm{per}}(\square))$; and $\phi_1 \in C_c^1(0,T; C^1_{\mathrm{per}}(\mathbb{T}))$. On passing to the limit we obtain

$$\int_Q \int_{\square\times\mathbb{T}} -h(u+\rho_a)v_1\frac{\partial\phi_1}{\partial\tau} + \big(\lambda h^3(\nabla u+\nabla_\xi u_1) - h(u+\rho_a)\Lambda\big)\cdot\nabla_\xi v_1\phi_1\,d\xi\,d\tau\,dx\,dt = 0.$$

Integrating by parts and using that u does not depend on τ, we obtain

$$\int_Q \int_{\square\times\mathbb{T}} (u+\rho_a)\frac{\partial h}{\partial\tau}v_1\phi_1 + \big(\lambda h^3(\nabla u+\nabla_\xi u_1) - h(u+\rho_a)\Lambda\big)\cdot\nabla_\xi v_1\phi_1\,d\xi\,d\tau\,dx\,dt = 0.$$

As linear combinations of functions of the form $v_1(x,\xi)\phi_1(t,\tau)$ are dense in $L^2(Q\times\mathbb{T}; H^1_{\mathrm{per}}(\square))$ we conclude that

$$\int_Q \int_{\square\times\mathbb{T}} (u+\rho_a)\frac{\partial h}{\partial\tau}v_1 + \big(\lambda h^3(\nabla u+\nabla_\xi u_1) - h(u+\rho_a)\Lambda\big)\cdot\nabla_\xi v_1\,d\xi\,d\tau\,dx\,dt = 0 \tag{33}$$

for all $v_1 \in L^2(Q\times\mathbb{T}; H^1_{\mathrm{per}}(\square))$. The integral identities (32) and (33) define a coupled system of equations for u and u_1. One can prove energy estimates for this system, similarly to what was done in Section 4. From these energy estimates one infers that u and u_1 are uniquely determined by (32) and (33). Since the limit functions u and u_1 are unique we must then have convergence for the whole sequence u_ε.

Moreover, the linearity of Eq. (33) implies that u_1 can be written in the form

$$u_1 = \frac{\partial u}{\partial x_1}\chi_1 + \frac{\partial u}{\partial x_2}\chi_2 + (u-\rho_a)\chi_3, \tag{34}$$

where $\chi_i \in L^2(Q\times\mathbb{T}; H^1_{\mathrm{per}}(\square))$, $i = 1,2,3$, solve the local problems

$$\int_\square \lambda h^3(\nabla_\xi\chi_1 + e_1)\cdot\nabla\varphi\,d\xi = 0, \tag{35}$$

$$\int_\square \lambda h^3(\nabla_\xi\chi_2 + e_2)\cdot\nabla\varphi\,d\xi = 0, \tag{36}$$

$$\int_\square \big(\lambda h^3\nabla_\xi\chi_3 - h\Lambda\big)\cdot\nabla\varphi - \frac{\partial h}{\partial\tau}\varphi\,d\xi = 0 \tag{37}$$

for all $\varphi \in H^1_{\mathrm{per}}(\square)$. Define

$$A(x,t) = \int_{\square\times\mathbb{T}} \lambda h^3 \begin{pmatrix} 1+\frac{\partial\chi_1}{\partial\xi_1} & \frac{\partial\chi_2}{\partial\xi_1} \\ \frac{\partial\chi_1}{\partial\xi_2} & 1+\frac{\partial\chi_2}{\partial\xi_2} \end{pmatrix} d\xi\,d\tau, \tag{38}$$

$$b(x,t) = \int_{\square\times\mathbb{T}} h\Lambda - \lambda h^3 \nabla_\xi \chi_3 \, d\xi \, d\tau$$
$$= \bar{h}\Lambda - \int_{\square\times\mathbb{T}} \lambda h^3 \nabla_\xi \chi_3 \, d\xi \, d\tau. \tag{39}$$

Then we see that (32) becomes

$$\int_Q \langle (\bar{h}u)', v\rangle \phi + \rho_a \frac{\partial \bar{h}}{\partial t} v\phi + \big(A\nabla u - (u+\rho_a)b\big) \cdot \nabla v \phi \, dx \, dt = 0. \tag{40}$$

This in turn implies that $u \in X$ is a solution of the evolution equation (27) or equivalently $\rho = u + \rho_a$ is a generalized solution of the initial–boundary value problem (28).

It remains to show that the limit function u satisfies the initial condition $u(0) = \rho_0 - \rho_a$. To this end, multiply (25) with $\phi \in C^1(0,T)$ such that $\varphi(0) = 1$, $\varphi(T) = 0$ and integrate by parts. Thus we obtain

$$\int_\Omega h_\varepsilon u_\varepsilon(0) v \, dx + \int_Q -h_\varepsilon(u_\varepsilon + \rho_a) v\phi' + \big(\lambda h_\varepsilon^3 \nabla u_\varepsilon - h_\varepsilon(u_\varepsilon + \rho_a)\Lambda\big) \cdot \nabla v \phi \, dx \, dt = 0. \tag{41}$$

Letting $\varepsilon \to 0$ yields

$$\int_\Omega \bar{h}(0)(\rho_0 - \rho_a) v \, dx + \int_{Q\times\square\times\mathbb{T}} -h(u+\rho_a)v\phi' + \big(\lambda h^3(\nabla u + \nabla_\xi u_1) - h(u+\rho_a)\Lambda\big) \cdot \nabla v \phi \, d\xi \, d\tau \, dx \, dt = 0. \tag{42}$$

From (32) it follows that

$$\int_\Omega \bar{h}u(0) v \, dx + \int_{Q\times\square\times\mathbb{T}} -h(u+\rho_a)v\phi' + \big(\lambda h^3(\nabla u + \nabla_\xi u_1) - h(u+\rho_a)\Lambda\big) \cdot \nabla v \phi \, d\xi \, d\tau \, dx \, dt = 0 \tag{43}$$

for all $\phi \in C^1(0,T)$ such that $\phi(0) = 1$ and $\phi(T) = 1$. Subtracting (42) from (43) gives

$$\int_\Omega \bar{h}\big(u(0) - \rho_0 + \rho_a\big) v \, dx = 0$$

for all $v \in H_0^1(\Omega)$. Since $\bar{h} > 0$ this implies $u(0) = \rho_0 - \rho_a$ a.e. in Ω. $\square$

6. A corrector result

According to Theorem 5.3 $u_\varepsilon \to u$ weakly in $L^2(0,T;H_0^1(\Omega))$. Here it is proved that the convergence is actually strong provided that a small corrector term is added to u_ε. The key result is

Lemma 6.1. *Let u_ε be the sequence of solutions of* (25)*. Then*

$$\lim_{\varepsilon\to 0} \int_Q \varphi_\varepsilon u_\varepsilon + F_\varepsilon \cdot \nabla u_\varepsilon \, dx \, dt = 0$$

for all $\varphi_\varepsilon \in L^2(Q)$ and all $F_\varepsilon \in L^2(Q;\mathbb{R}^2)$ two-scale converging to zero.

To prove this lemma we introduce a new function q_ε that solves a “dual problem” (cf. (46)). By homogenizing the dual problem, we analyze

$$\frac{d}{dt} \int_\Omega h_\varepsilon u_\varepsilon q_\varepsilon \, dx$$

as $\varepsilon \to 0$. Then Lemma 6.1 follows, if q_ε is chosen in the right way.

6.1. A dual problem

Let $u_\varepsilon \in X$ be the generalized solution of the Reynolds equation, i.e.

$$\begin{cases} \big\langle (h_\varepsilon u_\varepsilon)'(t), v\big\rangle + R_\varepsilon\big(u_\varepsilon(t), v; t\big) = \displaystyle\int_\Omega \rho_a \frac{\partial h_\varepsilon}{\partial t}(t)v + \rho_a h_\varepsilon(t)\Lambda \cdot \nabla v\, dx \quad \text{a.e. in } (0,T], \\ u_\varepsilon(0) = u_0 \end{cases} \tag{44}$$

for all $v \in H_0^1(\Omega)$, where R_ε is the bilinear form defined by

$$R_\varepsilon(u, v; t) = \int_\Omega \big(\lambda h_\varepsilon(t)^3 \nabla u - h_\varepsilon(t)\Lambda\big) \cdot \nabla v\, dx$$

for all $u, v \in H_0^1(\Omega)$.

Consider now the following problem

$$\begin{cases} \big\langle w_\varepsilon'(t), h_\varepsilon(T-t)v\big\rangle + R_\varepsilon\big(v, w_\varepsilon(t), v; T-t\big) = \big\langle f_\varepsilon(T-t), v\big\rangle \quad \text{a.e. in } (0,T], \\ w_\varepsilon(0) = 0, \end{cases} \tag{45}$$

where f_ε is chosen later on. For the time being we shall assume that f_ε converges weakly in $L^2(0,T;H^{-1}(\Omega))$. It can be shown that under the same assumptions that existence for (44) was proved, (45) has a unique solution $w_\varepsilon \in X$. Set $q_\varepsilon(t) = -w_\varepsilon(T-t)$. Then $q_\varepsilon'(t) = w_\varepsilon'(T-t)$ and therefore q_ε solves

$$\begin{cases} \big\langle q_\varepsilon'(t), h_\varepsilon(t)v\big\rangle - R_\varepsilon\big(v, q_\varepsilon(t), v; t\big) = \big\langle f_\varepsilon(t), v\big\rangle \quad \text{a.e. in } (0,T], \\ q_\varepsilon(T) = 0. \end{cases} \tag{46}$$

6.2. Homogenization of the dual problem

Choose f_ε to be of the form

$$\big\langle f_\varepsilon(t), v\big\rangle = \int_Q \varphi_\varepsilon(t)v + F_\varepsilon(t) \cdot \nabla v\, dx, \tag{47}$$

where $\varphi_\varepsilon \in L^2(Q)$ and $F_\varepsilon \in L^2(Q;\mathbb{R}^2)$ both two-scale converge to zero.

With this definition of f_ε we find the homogenized equations for (46). An energy estimate of the type

$$\|q_\varepsilon\|_{C([0,T];L^2(\Omega))} + \|q_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} + \big\|(h_\varepsilon q_\varepsilon)'\big\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C \tag{48}$$

can be derived in a similar fashion to that for u_ε. Thus we can apply Theorem 5.2 and homogenize Eq. (46). In the limit we obtain a system of equations for $q \in X$ with $q(T) = 0$ and $q_1 \in L^2(Q \times \mathbb{T}; H^1_{\text{per}}(\Box))$.

Note that in view of Lemma 4.2 and relation (6) we can write

$$\begin{aligned} \big\langle q_\varepsilon'(t), h_\varepsilon(t)v\big\rangle &= \big\langle (h_\varepsilon q_\varepsilon)'(t), v\big\rangle - \int_\Omega \frac{\partial h_\varepsilon}{\partial t} q_\varepsilon(t) v\, dx \\ &= \big\langle (h_\varepsilon q_\varepsilon)'(t), v\big\rangle - \int_\Omega G_\varepsilon \cdot \nabla q_\varepsilon(t) v + q_\varepsilon(t) G_\varepsilon \cdot \nabla v\, dx. \end{aligned} \tag{49}$$

We rewrite (46) using (49), multiply with $\phi \in C_c^1(0,T)$ and integrate by parts, thus obtaining

$$\int_Q -h_\varepsilon q_\varepsilon v \phi' + \big(-\big(\lambda h_\varepsilon^3 \nabla q_\varepsilon + q_\varepsilon G_\varepsilon\big) \cdot \nabla v + (h_\varepsilon \Lambda - G_\varepsilon) \cdot \nabla q_\varepsilon v\big)\phi\, dx\, dt = \int_Q (\varphi_\varepsilon v + F_\varepsilon \cdot \nabla v)\phi\, dx\, dt. \tag{50}$$

Sending $\varepsilon \to 0$ yields

$$\int_Q \bar{h} q v \phi' + \left(\int_{\Box\times\mathbb{T}} -\big(\lambda h^3(\nabla q + \nabla_\xi q_1) + qG\big) \cdot \nabla v + (h\Lambda - G) \cdot (\nabla q + \nabla_\xi q_1) v\, d\xi\, d\tau\right)\phi\, dx\, dt = 0 \tag{51}$$

for all $v \in H_0^1(\Omega)$ and $\phi \in C_c^1(0,T)$. By using test functions of the second form we obtain

$$\int_Q \int_{\Box\times\mathbb{T}} -hqv_1 \frac{\partial \phi_1}{\partial \tau} - \big(\lambda h^3(\nabla q + \nabla_\xi q_1) + qG\big) \cdot \nabla_\xi v_1 \phi_1\, d\xi\, d\tau\, dx\, dt = 0 \tag{52}$$

or equivalently

$$\int_{Q}\int_{\square\times\mathbb{T}} q\frac{\partial h}{\partial \tau}v_1 - \big(\lambda h^3(\nabla q + \nabla_\xi q_1) + qG\big)\cdot\nabla_\xi v_1\,d\xi\,d\tau\,dx\,dt = 0 \tag{53}$$

for all $v_1 \in L^2(Q\times\mathbb{T}; H^1_{\text{per}}(\square))$. One checks that the unique solution of the system (51) and (53) is $q = q_1 = 0$.

6.3. Proof of Lemma 6.1

By the product rule

$$\begin{aligned}\frac{d}{dt}\int_\Omega h_\varepsilon u_\varepsilon q_\varepsilon\,dx &= \big\langle (h_\varepsilon u_\varepsilon)', q_\varepsilon\big\rangle + \big\langle q_\varepsilon', h_\varepsilon u_\varepsilon\big\rangle \\ &= -R_\varepsilon(u_\varepsilon, q_\varepsilon; t) + \int_\Omega \rho_a\frac{\partial h_\varepsilon}{\partial t}q_\varepsilon + \rho_a h_\varepsilon\Lambda\cdot\nabla q_\varepsilon\,dx + R_\varepsilon(u_\varepsilon, q_\varepsilon; t) - \langle f_\varepsilon, u_\varepsilon\rangle \\ &= \int_\Omega \rho_a\frac{\partial h_\varepsilon}{\partial t}q_\varepsilon + \rho_a h_\varepsilon\Lambda\cdot\nabla q_\varepsilon\,dx - \langle f_\varepsilon, u_\varepsilon\rangle.\end{aligned} \tag{54}$$

Integrating this equality from $t = 0$ to T yields

$$\int_\Omega h_\varepsilon u_\varepsilon q_\varepsilon\,dx\bigg|_{t=0}^{T} = \int_Q \rho_a\frac{\partial h_\varepsilon}{\partial t}q_\varepsilon + \rho_a h_\varepsilon\Lambda\cdot\nabla q_\varepsilon\,dx\,dt - \int_0^T\langle f_\varepsilon, u_\varepsilon\rangle\,dt. \tag{55}$$

Taking into account that $u_\varepsilon(0) = u_0$, $q_\varepsilon(T) = 0$ and relation (6) we obtain

$$\int_0^T\langle f_\varepsilon, u_\varepsilon\rangle\,dt = \int_\Omega h_\varepsilon(0)u_0q_\varepsilon(0)\,dx + \int_Q \rho_a(G_\varepsilon + h_\varepsilon\Lambda)\cdot\nabla q_\varepsilon\,dx\,dt. \tag{56}$$

Passing to the limit in (56) proves Lemma 6.1.

Theorem 6.2. *Let u_ε be the sequence of solutions to* (25) *and let u and u_1 be the solutions of the homogenized system* (32) *and* (33)*. Suppose that the vector field $\nabla_\xi u_1(x,t,x/\varepsilon,t/\varepsilon)$ is measurable on Q and two-scale converges to $\nabla_\xi u_1(x,t,\xi,\tau)$, this is for example the case if $\nabla_\xi u_1 \in L^2(Q; C_{\text{per}}(\square\times\mathbb{T}))$. Then*

(i) $\lim_{\varepsilon\to 0}\|u - u_\varepsilon\|_{L^2(Q)} = 0$,
(ii) $\lim_{\varepsilon\to 0}\|\nabla u + \nabla_\xi u_1(x,t,x/\varepsilon,t/\varepsilon) - \nabla u_\varepsilon\|_{L^2(Q)} = 0$,
(iii) $\lim_{\varepsilon\to 0}\|u + \varepsilon u_1(x,t,x/\varepsilon,t/\varepsilon) - u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} = 0$.

Proof. One verifies that $\varphi_\varepsilon(x,t) = u(x,t) - u_\varepsilon(x,t)$ two-scale converges to 0 in $L^2(Q\times\square\times\mathbb{T})$ and that $F_\varepsilon(x,t) = \nabla u(x,t) + \nabla_\xi u_1(x,t,x/\varepsilon,t/\varepsilon) - \nabla u_\varepsilon(x,t)$ two-scale converges to 0 in $L^2(Q\times\square\times\mathbb{T};\mathbb{R}^2)$. Thus (i) and (ii) follow from Lemma 6.1 and the identity

$$\begin{aligned}&\|u - u_\varepsilon\|^2_{L^2(Q)} + \big\|\nabla u + \nabla_\xi u_1(x,t,x/\varepsilon,t/\varepsilon) - \nabla u_\varepsilon\big\|^2_{L^2(Q)} \\ &\quad = \int_Q \varphi_\varepsilon(u - u_\varepsilon) + F_\varepsilon\cdot\big(\nabla u + \nabla_\xi u_1(x,t,x/\varepsilon,t/\varepsilon) - \nabla u_\varepsilon\big)\,dx\,dt.\end{aligned}$$

(iii) is a direct consequence of (i) and (ii). □

7. Remarks on convergence of pressure, friction and load

Recall that the pressure is defined as

$$p_\varepsilon = \frac{1}{\beta}\log\bigg(\frac{\rho_\varepsilon}{\rho_a}\bigg), \qquad \rho_\varepsilon = u_\varepsilon + \rho_a,$$

which makes sense only if $\rho_\varepsilon > 0$. By Corollary 2.4 it is clear that this is the case provided that Lemma 2.1 holds. This maximum principle was proved for $\rho_\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ and it is not known to the authors if it remains true under weakened regularity assumptions. Taking into account the special dependence of h_ε on ε leads to

$$\rho_\varepsilon(x,t) \geqslant K_1 e^{-K_2 t/\varepsilon} \quad (x,t) \in Q.$$

This information, however, is not sufficient to conclude that p_ε is bounded, let alone converges, in $L^2(Q)$, as shown by the following example:

For $\varepsilon > 0$, define

$$\rho_\varepsilon(t) = \begin{cases} e^{-1/\varepsilon} & 0 \leqslant t < \varepsilon, \\ 1 & \varepsilon \leqslant t \leqslant 1. \end{cases}$$

Then $\lim \|\rho_\varepsilon - 1\|_{L^2(0,1)} = 0$, although $\|\log \rho_\varepsilon\|_{L^2(0,1)} = \varepsilon^{-1/2}$.

Since the maximum principle for the homogenized equation (Corollary 2.5) asserts that $\rho \geqslant \alpha$ in Q for some positive constant, $\rho = u + \rho_a$ being the strong limit in $L^2(Q)$ of ρ_ε, the problem seems to be that the maximum principle does not take into account the averaging effect of the oscillations upon ρ_ε. It should be noted that

$$\lim_{\varepsilon \to 0} \|p_\varepsilon - p\|_{L^2(Q')} = 0, \quad p = \frac{1}{\beta} \log\left(\frac{\rho}{\rho_a}\right),$$

for all $Q' \subset Q$ where the sequence ρ_ε is uniformly bounded from below by a positive constant, say $\rho_\varepsilon \geqslant \alpha$ a.e. in Q'. Thus the present analysis is inconclusive, inasmuch as the question of existence of such a set Q' is not resolved.

Not being able to prove the convergence of p_ε, the present analysis is consequently also inconclusive in regard to the convergence of the load carrying force L_ε defined by (9). Nevertheless, one would expect, as confirmed by the numerical experiments in [5], that at least

$$L_\varepsilon(t) \to L(t) = \int_\Omega \frac{1}{\beta} \log\left(\frac{u + \rho_a}{\rho_a}\right)$$

weakly in $L^2(0,T)$. As to the convergence of friction force F_ε, defined by (8), one would expect that

$$F_\varepsilon(t) \to F(t,\tau) = \int_\Omega \int_\square -\frac{h}{2\beta(u+\rho_a)}(\nabla u + \nabla_\xi u_1) + \frac{\mu}{h}\left(V^+ - V^-\right) d\xi \, dx$$

in the two-scale sense, but due to the above stated reasons it is not even clear that

$$\nabla p_\varepsilon = \frac{1}{\beta \rho_\varepsilon} \nabla \rho_\varepsilon$$

is bounded in $L^2(Q)$.

References

[1] A. Almqvist, J. Dasht, The homogenization process of the Reynolds equation describing compressible liquid flow, Tribol. Int. 39 (9) (2006) 994–1002.

[2] A. Almqvist, E.K. Essel, J. Fabricius, P. Wall, Reiterated homogenization applied in hydrodynamic lubrication, in: Proceedings of the Institution of Mechanical Engineers. Part J, J. Engrg. Tribology 222 (7) (2008) 827–841.

[3] A. Almqvist, E.K. Essel, J. Fabricius, P. Wall, Variational bounds applied to unstationary hydrodynamic lubrication, Internat. J. Engrg. Sci. 46 (9) (2008) 891–906.

[4] A. Almqvist, E.K. Essel, L.-E. Persson, P. Wall, Homogenization of the unstationary incompressible Reynolds equation, Tribol. Int. 40 (9) (2007) 1344–1350.

[5] A. Almqvist, R. Larsson, P. Wall, The homogenization process of the time dependent Reynolds equation describing liquid flow, in: Tribologia, Finnish J. Tribology 26 (4) (2007) 30–44.

[6] A. Almqvist, D. Lukkassen, A. Meidell, P. Wall, New concepts of homogenization applied in rough surface hydrodynamic lubrication, Internat. J. Engrg. Sci. 45 (1) (2007) 139–154.

[7] G. Bayada, M. Chambat, Homogenization of the Stokes system in a thin film flow with rapidly varying thickness, RAIRO Modél. Math. Anal. Numér. 23 (2) (1989) 205–234.

[8] G. Bayada, I. Ciuperca, M. Jai, Homogenized elliptic equations and variational inequalities with oscillating parameters. Application to the study of thin flow behavior with rough surfaces, Nonlinear Anal. Real World Appl. 7 (5) (2006) 950–966.

[9] G. Bayada, J.B. Faure, Double scale analysis approach of the Reynolds roughness comments and application to the journal bearing, J. Tribology 111 (2) (1989) 323–330.

[10] G. Bayada, S. Martin, C. Vázquez, An average flow model of the Reynolds roughness including a mass-flow preserving cavitation model, J. Tribology 127 (4) (2005) 793–802.

[11] G. Bayada, S. Martin, C. Vázquez, Two-scale homogenization of a hydrodynamic Elrod–Adams model, Asymptot. Anal. 44 (1–2) (2005) 75–110.

[12] G. Bayada, S. Martin, C. Vázquez, Micro-roughness effects in (elasto)hydrodynamic lubrication including a mass-flow preserving cavitation model, Tribol. Int. 39 (12) (2006) 1707–1718.

[13] N. Benhaboucha, M. Chambat, I. Ciuperca, Asymptotic behaviour of pressure and stresses in a thin film flow with a rough boundary, Quart. Appl. Math. 63 (2) (2005) 369–400.
[14] G. Buscaglia, I. Ciuperca, M. Jai, Homogenization of the transient Reynolds equation, Asymptot. Anal. 32 (2) (2002) 131–152.
[15] G.C. Buscaglia, M. Jai, Homogenization of the generalized Reynolds equation for ultra-thin gas films and its resolution by FEM, J. Tribology 126 (3) (2004) 547–552.
[16] M. Chambat, G. Bayada, J.B. Faure, Some effects of the boundary roughness in a thin film, in: Boundary Control and Boundary Variations, Nice, 1986, in: Lecture Notes in Comput. Sci., vol. 100, Springer, Berlin, 1988, pp. 96–115.
[17] D. Cioranescu, P. Donato, An Introduction to Homogenization, Oxford Lecture Ser. Math. Appl., vol. 17, The Clarendon Press/Oxford University Press, New York, 1999.
[18] H.G. Elrod, M.L. Adams, A computer program for cavitation, in: Cavitation and Related Phenomena in Lubrication, Mechanical Engineering Publications, New York, NY, USA, 1975, pp. 37–42.
[19] J. Fabricius, Homogenization of some problems in hydrodynamic lubrication involving rough boundaries, PhD thesis, Luleå University of Technology, 2011.
[20] A. Holmbom, N. Svanstedt, N. Wellander, Multiscale convergence and reiterated homogenization of parabolic problems, Appl. Math. 50 (2) (2005) 131–151.
[21] M. Jai, Homogenization and two-scale convergence of the compressible Reynolds lubrication equation modelling the flying characteristics of a rough magnetic head over a rough rigid-disk surface, RAIRO Modél. Math. Anal. Numér. 29 (2) (1995) 199–233.
[22] M. Kane, B. Bou-Said, Comparison of homogenization and direct techniques for the treatment of roughness in incompressible lubrication, J. Tribology 126 (4) (2004) 733–737.
[23] G.M. Lieberman, Second Order Parabolic Differential Equations, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.
[24] D. Lukkassen, A. Meidell, P. Wall, Bounds on the effective behavior of a homogenized generalized Reynolds equation, J. Funct. Spaces Appl. 5 (2) (2007) 133–150.
[25] D. Lukkassen, A. Meidell, P. Wall, Homogenization of some variational problems connected to the theory of lubrication, Internat. J. Engrg. Sci. 47 (1) (2009) 153–162.
[26] S. Martin, Influence of multiscale roughness patterns in cavitated flows: applications to journal bearings, Math. Probl. Eng. 26 (2008), Art. ID 439319.
[27] O. Reynolds, On the theory of lubrication and its application to Mr. Beauchamp Tower's experiments, including an experimental determination of the viscosity of olive oil, Philos. Trans. R. Soc. Lond. 177 (1886) 157–234.
[28] P. Wall, Homogenization of Reynolds equation by two-scale convergence, Chinese Ann. Math. Ser. B 28 (3) (2007) 363–374.
[29] E. Zeidler, Nonlinear Functional Analysis and Its Applications. II/A, Springer-Verlag, New York, 1990.