



Note

On Liouvillian integrability of the first-order polynomial ordinary differential equations

Jaume Giné^{a,*}, Jaume Llibre^b^a Departament de Matemàtica, Universitat de Lleida, Avenida Jaume II 69, 25001 Lleida, Catalonia, Spain^b Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

ARTICLE INFO

Article history:

Received 26 December 2011

Available online 9 June 2012

Submitted by Roman O. Popovych

Keywords:

Liouvillian integrability

Invariant algebraic curve

Riccati differential equation

Abel differential equation

ABSTRACT

Recently, the authors provided an example of an integrable Liouvillian planar polynomial differential system that has no finite invariant algebraic curves; see Giné and Llibre (2012) [8]. In this note, we prove that, if a complex differential equation of the form $y' = a_0(x) + a_1(x)y + \dots + a_n(x)y^n$, with $a_i(x)$ polynomials for $i = 0, 1, \dots, n$, $a_n(x) \neq 0$, and $n \geq 2$, has a Liouvillian first integral, then it has a finite invariant algebraic curve. So, this result applies to Riccati and Abel polynomial differential equations. We shall prove that in general this result is not true when $n = 1$, i.e., for linear polynomial differential equations.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction and statement of the main results

By definition, a *complex planar polynomial differential system*, or simply a *polynomial system*, is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \quad \frac{dy}{dt} = \dot{y} = Q(x, y), \quad (1)$$

where the dependent variables x and y are complex, and the independent one (the *time*) t can be real or complex, and $P, Q \in \mathbb{C}[x, y]$, where $\mathbb{C}[x, y]$ is the ring of all polynomials in the variables x and y with coefficients in \mathbb{C} . We denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of the polynomial system.

Let $f = f(x, y) = 0$ be an algebraic curve in \mathbb{C}^2 . We say that it is *invariant* or that it is a *finite invariant algebraic curve* by the polynomial system (1) if $P \partial f / \partial x + Q \partial f / \partial y = kf$, for some polynomial $k = k(x, y) \in \mathbb{C}[x, y]$, called the *cofactor* of the algebraic curve $f = 0$. Note that the degree of the polynomial k is at most $m - 1$.

Let $h, g \in \mathbb{C}[x, y]$, and assume that h and g are relatively prime in the ring $\mathbb{C}[x, y]$. Then the function $\exp(g/h)$ is called an *exponential factor* of the polynomial system (1) if, for some polynomial $k \in \mathbb{C}[x, y]$ of degree at most $m - 1$, it satisfies equation $P \partial \exp(g/h) / \partial x + Q \partial \exp(g/h) / \partial y = k \exp(g/h)$. If $\exp(g/h)$ is an exponential factor, it is easy to show that $h = 0$ is an invariant algebraic curve.

Let U be an open subset of \mathbb{C}^2 . We say that a non-constant function $H : U \rightarrow \mathbb{C}$ is a *first integral* of the polynomial system (1) in U if H is constant on the trajectories of the polynomial system (1) contained in U .

We say that a non-constant function $R : U \rightarrow \mathbb{C}$ is an *integrating factor* of the polynomial system (1) in U if R satisfies that $\partial(RP) / \partial x + \partial(RQ) / \partial y = 0$, in the points $(x, y) \in U$.

* Corresponding author.

E-mail addresses: gine@matematica.udl.cat (J. Giné), llibre@mat.uab.cat (J. Llibre).

In 1992, Singer [13] showed that, if a polynomial system has a Liouvillian first integral, then the system has an integrating factor of the form

$$R(x, y) = \exp \left(\int_{(x_0, y_0)}^{(x, y)} U(x, y) dx + V(x, y) dy \right), \quad (2)$$

where U and V are rational functions which satisfy $\partial U / \partial y = \partial V / \partial x$. In 1999, Christopher [3] showed that the integrating factor (2) can be written in the form

$$R = \exp(g/h) \prod f_i^{\lambda_i}, \quad (3)$$

where g, h , and f_i are polynomials, and $\lambda_i \in \mathbb{C}$. This condition guarantees the existence of a first integral that can be expressed by quadratures of elementary functions (Liouvillian functions). This type of integrability is known since then as *Liouvillian integrability theory*. For more details on all these notions mentioned until here see the paper [8] and the references quoted there.

Non-algebraic invariant curves with polynomial cofactor can also be used in order to find a first integral for a system. This observation permits the generalization of the Liouvillian integrability theory given in [5–7,10], where a new kind of first integrals, not only the Liouvillian ones, is described.

There was the belief that a Liouvillian integrable system always has an invariant algebraic curve in \mathbb{C}^2 . Moreover, this claim was proved under certain hypotheses; see [14]. However, this claim was refuted in [8], where it is proved that there exist Liouvillian integrable polynomial systems without any finite invariant algebraic curve. For proving that result in [8], a Liouvillian integrable planar polynomial system of degree 2 in \mathbb{C}^2 without finite invariant algebraic curves is provided.

The main result of this note is the following one.

Theorem 1. *If a complex differential equation of the form*

$$y' = \frac{dy}{dx} = a_0(x) + a_1(x)y + \cdots + a_n(x)y^n, \quad (4)$$

with $a_i(x)$ polynomials in the variable x , $a_n(x) \neq 0$, and $n \geq 2$, has a Liouvillian first integral, then it has a finite invariant algebraic curve.

Theorem 1 is proved in Section 2.

This kind of differential equation has been studied by several authors; see [11] and the references cited therein.

Corollary 2. *A Riccati polynomial differential equation (Eq. (4) with $n = 2$) is Liouvillian integrable if and only if has a finite invariant algebraic curve.*

The “if” part follows directly from **Theorem 1**. The “only if” part is proved in Section 2. In any case **Corollary 2** was obtained by first time by Ritt in [12], and later on by Singer in [13].

Corollary 2 has a close relationship with the classical result of Kolchin (see [1]) that a Riccati equation has an algebraic curve solution over a certain differential field \mathcal{K} if and only if is integrable in the Picard–Vessiot sense.

Corollary 3. *If an Abel polynomial differential equation (Eq. (4) with $n = 3$) is Liouvillian integrable, then it has a finite invariant algebraic curve.*

Corollary 3 follows directly from **Theorem 1**.

Proposition 4. *Theorem 1 does not hold for $n = 1$.*

Proposition 4 is proved in Section 2.

2. Proof of the main results

In this section, we shall prove the theorem and the rest of the results given above.

Proof of Theorem 1. For proving **Theorem 1** we shall work with the autonomous planar polynomial differential system equation (1) associated to the non-autonomous differential equation (4); i.e.

$$P(x, y) = 1, \quad Q(x, y) = a_0(x) + a_1(x)y + \cdots + a_n(x)y^n. \quad (5)$$

The proof of **Theorem 1** is by contradiction. We assume that the differential equation (4) is Liouvillian integrable, i.e., has a Liouvillian first integral, and does not have finite invariant algebraic curves. By the results of Singer in [13] (see also [3]) we know that if equation (5) is Liouvillian integrable then it has an integrating factor of the form (3). We recall that $f_i = 0$ and $h = 0$ in (3) are invariant algebraic curves and that $\exp(g/h)$ is an exponential factor for system (5); for more details see [2]. Therefore if system (5) is a planar Liouvillian integrable polynomial differential system without finite invariant algebraic

curves, then it must have an integrating factor of the form $R = \exp(g(x, y))$, where g is a polynomial. Note that $g = 0$ does not need to be an invariant algebraic curve of system (5). From [4] $\exp(g(x, y))$ is an exponential factor coming from the fact that the invariant straight line at infinity has multiplicity larger than one, but this fact is not relevant here.

Now we assume that the degree of g with respect to the variable y is m . Then we write g as a polynomial in the variable y with coefficients polynomials in the variable x , and we impose that $R = \exp(\sum_{j=0}^m g_j(x)y^j)$ is an integrating factor of system with $g_m(x) \neq 0$, i.e.,

$$\frac{\partial R}{\partial x}P + \frac{\partial R}{\partial y}Q + \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)R = 0. \quad (6)$$

After dividing R in the previous equality, the highest power is y^{m+n-1} and its coefficient is $ma_n(x)g_m(x)$. The vanishing of this coefficient is a contradiction with the hypothesis of the theorem unless $m = 0$. Consequently the integrating factor is of the form $R = \exp(g_0(x))$. Now substituting this integrating factor in (6) we get

$$g'_0(x) + a_1(x) + 2a_2(x)y + \cdots + na_n(x)y^{n-1} = 0,$$

which also is contradiction with the hypothesis of the theorem because implies that $a_n(x) = 0$. Therefore, if equation (4) is Liouvillian integrable then it has a finite invariant algebraic curve. \square

Proof of the “only if” part of Corollary 2. If a Riccati polynomial differential equation has a finite invariant algebraic curve, then, by the classical method of resolution of the Riccati equations (see, for instance, [9]), we can transform it into a Bernoulli differential equation, and finally into a first-order linear differential equation. Moreover, any first-order linear differential equation is integrable by quadratures, and consequently the Riccati equation is Liouvillian integrable. \square

For the case $n = 1$, there is no contradiction, because the highest power is y^m , and its coefficient is given by $ma_1(x)g_m(x) + g'_m(x)$. The vanishing of this coefficient does not imply that $g_m(x) = 0$, and allows us to compute $g_m(x)$ as a function of $a_1(x)$, i.e.,

$$g_m(x) = \exp\left(-\int^x ma_1(s)ds\right).$$

Hence, when the differential equation (4) is a linear differential equation, we cannot apply the arguments used when $n \geq 2$.

Proof of Proposition 4. In [8], it is proved that the quadratic polynomial differential system

$$\dot{x} = -1 - x(2x + y), \quad \dot{y} = 2x(2x + y), \quad (7)$$

is Liouvillian integrable and has no finite invariant algebraic curves. In fact, system (7) is Liouvillian integrable, because it has the integrating factor $R = e^{-(2x+y)^2/4}$, and the Liouvillian first integral

$$H = 2e^{-\frac{1}{4}(2x+y)^2}x - \sqrt{\pi} \operatorname{erf}\left(\frac{1}{2}(2x+y)\right), \quad (8)$$

where $\operatorname{erf}(z)$ is the error function, i.e., the integral of the Gaussian distribution, given by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

Moreover, system (7) can be transformed, by making the change of variables $(x, y) \rightarrow (x, z)$, where $z = 2x + y$, into the linear polynomial differential equation

$$\frac{dx}{dz} = \frac{1}{2}(1 + xz).$$

Hence the proposition is proved. \square

Acknowledgments

The first author is partially supported by an MICINN/FEDER grant, number MTM2011-22877, and by a Generalitat de Catalunya grant, number 2009SGR 381. The second author is partially supported by an MICINN/FEDER grant, number MTM2008-03437, by a Generalitat de Catalunya grant, number 2009SGR 410, and by ICREA Academia.

References

- [1] P.D. Acosta-Humánez, J.T. Lázaro, J.J. Morales-Ruiz, C.H. Pantazi, On the integrability of polynomials fields in the plane by means of Picard–Vessiot theory, 2012. Preprint.
- [2] J. Chavarriga, H. Giacomini, J. Giné, J. Llibre, Darboux integrability and the inverse integrating factor, *J. Differential Equations* 194 (2003) 116–139.
- [3] C.J. Christopher, Liouvillian first integrals of second order polynomials differential equations, *Electron. J. Differ. Equ.* 1999 (1999) 7 (electronic).
- [4] C. Christopher, J. Llibre, J.V. Pereira, Multiplicity of invariant algebraic curves in polynomial vector fields, *Pacific J. Math.* 229 (2007) 63–117.
- [5] I.A. García, J. Giné, Generalized cofactors and nonlinear superposition principles, *Appl. Math. Lett.* 16 (2003) 1137–1141.
- [6] I.A. García, J. Giné, Non-algebraic invariant curves for polynomial planar vector fields, *Discrete Contin. Dyn. Syst.* 10 (2004) 755–768.
- [7] J. Giné, M. Grau, Weierstrass integrability of differential equations, *Appl. Math. Lett.* 23 (2010) 523–526.
- [8] J. Giné, J. Llibre, A note on Liouvillian integrability, *J. Math. Anal. Appl.* 387 (2012) 1044–1049.
- [9] E.L. Ince, *Ordinary Differential Equations*, Dover Publications, New York, 1956.
- [10] J. Llibre, S. Walcher, X. Zhang, Local Darboux first integrals of analytic differential systems, 2011. Preprint.
- [11] N.G. Lloyd, The number of periodic solutions of the equation $z' = z^N + p_1(t)z^{N-1} + \dots + p_N(t)$, *Proc. Lond. Math. Soc.* (3) 27 (1973) 667–700.
- [12] J.F. Ritt, On the integration in finite terms of linear differential equations of the second order, *Bull. Amer. Math. Soc.* 33 (1927) 51–57.
- [13] M.F. Singer, Liouvillian first integrals of differential equations, *Trans. Amer. Math. Soc.* 333 (1992) 673–688.
- [14] A.E. Zernov, B.A. Scárdua, Integration of polynomial ordinary differential equations in the real plane, *Aequationes Math.* 61 (2001) 190–200.