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Stability and eigenvalue estimates of linear Weingarten hypersurfaces in a sphere

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A B S T R A C T

Let M be an n-dimensional compact hypersurface without boundary in a unit sphere $\mathbb{S}^{n+1}(1)$. M is called a linear Weingarten hypersurface if $cR + dH + e = 0$, where c, d and e are constants with $c^2+d^2 > 0$, R and H denote the scalar curvature and the mean curvature of M, respectively. By the Gauss equation, we can rewrite the condition $cR + dH + e = 0$ as $(n-1)\tilde{e}H_2 + aH = b$, where H_2 is the 2nd mean curvature, a, b and $\tilde{e}$ are constants such that $a^2 + \tilde{e}^2 > 0$, when $\tilde{e} = 0$, it reduces to the constant mean curvature case.

In this paper, we obtain some stability results about linear Weingarten hypersurfaces, which generalize the stability results about the hypersurfaces with constant mean curvature or with constant scalar curvature. We show that linear Weingarten hypersurfaces satisfying $(n-1)H_2 + aH = b$, where a and b are constants, can be characterized as critical points of the functional $\int_M (a + nH)\, dv$ for volume-preserving variations. We prove that such a linear Weingarten hypersurface is stable if and only if it is totally umbilical and non-totally geodesic. We also obtain optimal upper bounds for the first and second eigenvalues of the Jacobi operator of linear Weingarten hypersurfaces.

1. Introduction

It is well known that hypersurfaces with constant mean curvature in real space forms ($\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}(1)$ or $\mathbb{H}^{n+1}(-1)$) are characterized as critical points of the area functional for volume-preserving variations, and hypersurfaces with constant scalar curvature in real space forms are critical points of the functional $\int_M H\, dv$ for volume-preserving variations, where H is the mean curvature. There exist many results about hypersurfaces with constant mean curvature or constant scalar curvature in a unit sphere $\mathbb{S}^{n+1}(1)$ (see [1–12] and the refs therein). Among these results, Barbosa et al. (see [5]) proved that geodesic sphere is the only stable compact hypersurface with constant mean curvature in a sphere. In [1], Alencar et al. showed that, if a hypersurface M has constant scalar curvature $n(n-1)r$ and is contained in an open hemisphere of $\mathbb{S}^{n+1}(1)$ (which implies that $r > 1$), then M is stable if and only if it is a geodesic sphere.

In this paper, we prove some stability results about linear Weingarten hypersurfaces, which generalize the stability results about the hypersurfaces with constant mean curvature or with constant scalar curvature. We also obtain optimal estimates for the first and second eigenvalues of the Jacobi operator of linear Weingarten hypersurfaces.

The notion of linear Weingarten hypersurfaces in a unit sphere was introduced by Li et al. (see [13]), where they showed some rigidity results about linear Weingarten hypersurfaces.

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Definition 1.1 (*[13]*)**.** Let M be a hypersurface in a unit sphere $\mathbb{S}^{n+1}(1)$. We call M a linear Weingarten hypersurface if $cR + dH + e = 0$, where c, d and e are constants such that $c^2 + d^2 > 0$, R and H denote the scalar curvature and the mean curvature of M, respectively.

Remark 1.2. In Definition 1.1, when $c = 0$, M has constant mean curvature; when $d = 0$, M has constant scalar curvature. By the Gauss equation, we can rewrite the condition $cR + dH + e = 0$ as $(n-1)\tilde{e}H_2 + aH = b$, where H_2 is the 2nd mean curvature, a, b and $\tilde{e}$ are constants such that $a^2 + \tilde{e}^2 > 0$, when $\tilde{e} = 0$, it reduces to the constant mean curvature case, when $\tilde{e} \neq 0$, without loss of generality, we can assume $\tilde{e} = 1$.

In Section 3, we show that linear Weingarten hypersurfaces satisfying $(n-1)H_2 + aH = b$, can be characterized as critical points of the functional $\int_M (a + nH)\, dv$ for volume-preserving variations, where a is a constant and H is the mean curvature. We obtain the following stability result (see Section 4).

Theorem 1.3. *Let M be a compact orientable hypersurface in $\mathbb{S}^{n+1}(1)$, M satisfies that $H_2 > 0$, since $H^2 \geq H_2 > 0$, we choose the orientation such that $H > 0$, and assume that $(n-1)H_2 + aH = b$, where $a \geq 0$ and b are constants. Then M is stable if and only if M is totally umbilical and non-totally geodesic.*

From the point of view of spectral theory, the spectral behavior of the Jacobi operator associated to the corresponding variational problem is directly related to the instability of the critical points (submanifolds). There have been many results (see [2,8,11,14–17,12,18–20] and the refs therein) on the eigenvalue estimates of both the Jacobi operator $J_m = -\Delta - S - n$ and the Jacobi operator $J_s = -\Box - \{n(n-1)H + nHS - f_3\}$, where the spectral behavior of J_m is related to the instability of minimal hypersurfaces and constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}(1)$, and the spectral behavior of J_s is related to the instability of constant scalar curvature hypersurfaces in $\mathbb{S}^{n+1}(1)$. Here Δ is the Laplacian operator and $\Box$ is a differential operator defined in (2.1).

With the same hypothesis as in Theorem 1.3, we prove that the Jacobi operator J_w (see (3.3)) associated to the variational problem for linear Weingarten hypersurfaces is elliptic (see Lemma 4.1). In the last two sections, we get optimal estimates for the first and second eigenvalues of the Jacobi operator J_w. More precisely, under the same assumptions of Theorem 1.3, we have the following theorem.

Theorem 1.4. *The first eigenvalue $\lambda_1^{J_w}$ of the Jacobi operator J_w satisfies:*

$$\lambda_1^{J_w} \leq an^2H_{\min}^2 - [2bn^2 + 4n(n-1) - a^2n]H_{\min} + \frac{2bn(b+1)}{H_{\min}} - 3an(b+1),$$

and the equality holds if and only if M is totally umbilical and non-totally geodesic, or M is a Riemannian product $\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$ with $H_2 > 0$.

We have the following estimate for the second eigenvalue of $J_w = -(2\Box + a\Delta) - 2[S_1S_2 - 3S_3 + (n-1)S_1] - a[(S_1^2 - 2S_2) + n]$ (see (3.3)).

Theorem 1.5. *Let M be a compact orientable hypersurface in $\mathbb{S}^{n+1}(1)$, M satisfies that $H_2 > 0$, we choose the orientation such that $H > 0$. Assume $n \geq 5$, $a \geq 0$, then J_w is elliptic and the second eigenvalue $\lambda_2^{J_w}$ of J_w satisfies:*

$$\lambda_2^{J_w} \leq 0,$$

and $\lambda_2^{J_w} = 0$ if and only if M is totally umbilical and non-totally geodesic.

Remark 1.6. We note that in Theorem 1.5, we have weaker assumptions than in Theorem 1.3 and in Theorem 1.4. Here $(n-1)H_2 + aH = b$ is not assumed.

2. Preliminaries

Let $x : M \to \mathbb{S}^{n+1}(1)$ be an n-dimensional hypersurface in a unit sphere $\mathbb{S}^{n+1}(1)$. We assume that all manifolds are smooth and connected without boundary. We make the following convention on the range of indices:

$$1 \leq i, j, k, \ldots \leq n.$$

We denote the principal curvatures of M by $k_1, \ldots, k_n$. Let H, H_2 and H_3 denote the mean curvature, the 2nd mean curvature and the 3rd mean curvature of M respectively, namely,

$$H = \frac{1}{n}S_1 = \frac{1}{n}\sum_{i=1}^{n} k_i, \qquad H_2 = \frac{2}{n(n-1)}S_2 = \frac{2}{n(n-1)}\sum_{1\leq i_1<i_2\leq n} k_{i_1}k_{i_2},$$
$$H_3 = \frac{6}{n(n-1)(n-2)}S_3 = \frac{6}{n(n-1)(n-2)}\sum_{1\leq i_1<i_2<i_3\leq n} k_{i_1}k_{i_2}k_{i_3}.$$

We choose a local orthonormal frame $\{e_1, \dots, e_n, e_{n+1}\}$ and the dual coframe $\{\omega_1, \dots, \omega_n, \omega_{n+1}\}$ such that when restricted on M, $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M. Hence we have $\omega_{n+1} = 0$ on M and we have the following structure equations (see [21,22,10,18]):

$$dx = \sum_i \omega_i e_i,$$
$$de_i = \sum_j \omega_{ij} e_j + \sum_j h_{ij}\omega_j e_{n+1} - \omega_i x,$$
$$de_{n+1} = -\sum_{i,j} h_{ij}\omega_j e_i,$$

where h_{ij} denote the components of the second fundamental form of M.

The Gauss equations are (see [22,10])

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + h_{ik}h_{jl} - h_{il}h_{jk},$$
$$R_{ik} = (n-1)\delta_{ik} + nHh_{ik} - \sum_j h_{ij}h_{jk},$$
$$R = n(n-1)r = n(n-1) + n^2H^2 - S,$$

where R is the scalar curvature of M, r is the normalized scalar curvature of M and $S = \sum_{i,j} h_{ij}^2$ is the norm square of the second fundamental form, $H = \frac{1}{n}\sum_i h_{ii}$ is the mean curvature of M.

The Codazzi equations are given by (see [22,10])

$$h_{ijk} = h_{ikj}.$$

Let f be a smooth function on M, we define its gradient and Hessian by (see [22,10])

$$df = \sum_{i=1}^n f_i\omega_i,$$
$$\sum_{j=1}^n f_{ij}\omega_j = df_i + \sum_{j=1}^n f_j\omega_{ji}.$$

Let $\phi = \sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M, where

$$\phi_{ij} = nH\delta_{ij} - h_{ij}.$$

We introduce Cheng–Yau's operator $\square$ associated to ϕ acting on any smooth function f by (see [9])

$$\square f = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}, \tag{2.1}$$

where f_{ij} are the components of the Hessian of f. The Laplacian of f is defined by

$$\Delta f = \operatorname{tr}(\operatorname{Hess}(f)) = \sum_i f_{ii}. \tag{2.2}$$

3. The variation problem for linear Weingarten hypersurfaces

Let $x : M \to \mathbb{S}^{n+1}(1)$ be an immersion of an n-dimensional compact, connected, orientable manifold M without boundary into a unit sphere $\mathbb{S}^{n+1}(1)$.

Definition 3.1 (*Cf. [1,5]*). Let $X : (-\varepsilon, \varepsilon) \times M \to \mathbb{S}^{n+1}(1)$, $\varepsilon > 0$, be a differentiable map. We call X *a variation* of x if

(1) for each $t \in (-\varepsilon, \varepsilon)$, $X_t(p) = X(t, p)$, $p \in M$, is an immersion.
(2) $X_0 = x$.

Let X be a variation of x and $W(p) = \frac{\partial X}{\partial t}|_{t=0}$ be the *variational vector* of X. We denote by N the unit normal vector along the immersion x and define the volume function $V : (-\varepsilon, \varepsilon) \to \mathbb{R}$ of X by

$$V(t) = \int_{[0,t]\times M} X^* dw,$$

where dw is the volume element on $\mathbb{S}^{n+1}(1)$. X is called *a normal variation* of x if W is parallel to N. X is called *a volume-preserving variation* of x if $V(t) = V(0)$ for all $t \in (-\varepsilon, \varepsilon)$.

Set $u = \langle W, N\rangle$, which is the normal projection of the variation vector field. We have the following lemmas.

Lemma 3.2 (*Cf. [4,5,1]*).

$$\left.\frac{dV}{dt}\right|_{t=0} = \int_M u dv.$$

Lemma 3.3 (*Cf. [17,3,21]*). *Let M be an n-dimensional compact hypersurface in $\mathbb{S}^{n+1}(1)$. We have*

$$\frac{dS_1}{dt} = \Delta u + (S_1^2 - 2S_2)u + nu + \sum_i W_i S_{1,i};$$

$$\frac{dS_2}{dt} = \Box u + (S_1S_2 - 3S_3)u + (n-1)S_1 u + \sum_i W_i S_{2,i};$$

$$\frac{d(dv_t)}{dt} = \left(-S_1 u + \operatorname{div}\left(\sum_i W_i e_i\right)\right) dv_t,$$

where $W = \sum_i W_i e_i + uN$ is the variational vector field of $x : M \to \mathbb{S}^{n+1}(1)$, $S_{r,i}$ is the first covariant derivative of S_r in the e_i direction, $r = 1, 2$.

For constant a, we consider the following functional

$$\mathcal{F} = \int_M (a + S_1)\, dv. \tag{3.1}$$

Proposition 3.4 (*First Variation Formula*). *Let M be an n-dimensional compact hypersurface in $\mathbb{S}^{n+1}(1)$. For any variation of $x : M \to \mathbb{S}^{n+1}(1)$, we have*

$$\mathcal{F}'(0) = \int_M [-2S_2 + n - aS_1]u\, dv.$$

Proof. We use the same notations as in Lemma 3.3. From Lemma 3.3, we have

$$\begin{aligned}
\mathcal{F}'(0) &= \int_M (a+S_1)\left(-S_1 u + \operatorname{div}\left(\sum_i W_i e_i\right)\right) dv + \int_M \left[\Delta u + (S_1^2 - 2S_2 + n)u + \sum_i W_i S_{1,i}\right] dv \\
&= \int_M [(-2S_2 + n) - aS_1]u\, dv + \int_M \Delta u\, dv + \int_M \operatorname{div}\left(\sum_i (S_1 + a) W_i e_i\right) dv \\
&= \int_M [(-2S_2 + n) - aS_1]u\, dv. \quad \Box
\end{aligned}$$

For volume-preserving variations, by Lemma 3.2, we have

$$\int_M u\, dv = 0, \tag{3.2}$$

by using Proposition 3.4, we get that the critical points of volume-preserving variational problem are the immersions x for which $S_2 + aS_1$ is constant, namely,

$$(n-1)H_2 + aH = b,$$

where a and b are constants. Hence, the critical points of each volume-preserving variation $X : (-\varepsilon, \varepsilon) \times M \to \mathbb{S}^{n+1}(1)$, are the linear Weingarten hypersurfaces satisfying $(n-1)H_2 + aH = b$, where a and b are constants; vice versa.

As a direct application of Lemma 3.3, Proposition 3.4 and Eq. (3.2), after a long but direct computation, we have the following.

Proposition 3.5 (*Second Variation Formula*). *Let $x : M \to \mathbb{S}^{n+1}(1)$ be an isometric immersion for which $(n-1)H_2 + aH = b$, where a and b are constants. For each volume-preserving variation, the second derivative of $\mathcal{F}$ at $t = 0$ is given by*

$$\mathcal{F}''(0) = -\int_M u\{(2\Box + a\Delta)u + 2[S_1S_2 - 3S_3 + (n-1)S_1]u + a(S_1^2 - 2S_2 + n)u\}\, dv.$$

Definition 3.6. Let $x : M \to \mathbb{S}^{n+1}(1)$ be a linear Weingarten hypersurface satisfying $(n-1)H_2 + aH = b$, where a and b are constants. The immersion x is called *stable* if $\mathcal{F}''(0) \geq 0$ for all volume-preserving variations of x.

We denote the operators L and the Jacobi operator J_w respectively by

$$\begin{aligned} &L = 2\Box + a\Delta, \\ &J_w = -L - 2[S_1S_2 - 3S_3 + (n-1)S_1] - a(S_1^2 - 2S_2 + n). \end{aligned} \tag{3.3}$$

It have been proved (see [9]) that $\Box$ is self-adjoint; hence we have that L and J_w are both self-adjoint.

Let $\mathcal{U}$ be the set of all differentiable functions $u : M \to \mathbb{R}$ satisfying $\int_M u dv = 0$. Then a linear Weingarten hypersurface satisfying $(n-1)H_2 + aH = b$ is stable if and only if $\int_M uJ_w u\, dv \geq 0$ for all $u \in \mathcal{U}$. This can be proved after a similar argument as in [5], we omit the details here.

Proposition 3.7. *Let M be an n-dimensional totally umbilical and non-totally geodesic sphere in $\mathbb{S}^{n+1}(1)$. Since M has constant positive principal curvatures, M can be regarded as a linear Weingarten hypersurface satisfying $(n-1)H_2 + aH = b$. We choose the orientation such that $H > 0$ and $a \geq 0$. Then M is stable.*

Proof. Assume M is totally umbilical, we only have to prove that $\int_M uJ_w u\, dv \geq 0$ for all u satisfying $\int_M u\, dv = 0$.

Since M is totally umbilical, H is constant, we have

$$S_2 = \frac{n(n-1)}{2}H^2, \qquad S_3 = \frac{n(n-1)(n-2)}{6}H^3$$

and

$$\Box = (n-1)H\Delta.$$

For any u such that $\int_M u\, dv = 0$, we have

$$\begin{aligned} \int_M uJ_w u\, dv &= -(a + 2(n-1)H)\int_M [n(1+H^2)u^2 - \|\nabla u\|^2]\, dv \\ &\geq -(a + 2(n-1)H)\int_M [n(1+H^2) - \mu(M)]u^2 dv, \end{aligned}$$

where $\mu(M)$ is the first non-zero eigenvalue of the Laplacian Δ on M. Since M is totally umbilical, M is a sphere, we have $\mu(M) = n(1+H^2)$.

Hence we get $\int_M uJ_w u\, dv \geq 0$ for all u satisfying $\int_M u\, dv = 0$. This completes the proof of Proposition 3.7. $\Box$

4. Proof of Theorem 1.3

Lemma 4.1 (*Cf.* [13]). *Let $x : M^n \to \mathbb{S}^{n+1}(1)$ be an isometric immersion for which $(n-1)H_2 + aH = b$, where a and b are constants. If $a^2 + 4nb > 0$, then*

(1) *the Jacobi operator J_w (see (3.3)) associated to the variational problem defined in Section 3 is elliptic;*
(2) *$|\nabla h|^2 \geq |\nabla(nH)|^2$. When the equality holds, we have M is either totally umbilical or a Riemannian product $\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$.*

Proof. (1) We only need to show that $L = 2\Box + a\Delta$ is elliptic.

From $0 \leq S = (nH)^2 - n(n-1)H_2 = (nH)^2 + naH - nb$, we have that H satisfies

$$H \geq \frac{-a + \sqrt{a^2 + 4nb}}{2n} \quad \text{or} \quad H \leq \frac{-a - \sqrt{a^2 + 4nb}}{2n}.$$

So we get

$$2nH + a \geq \sqrt{a^2 + 4nb} > 0 \quad \text{or} \quad 2nH + a \leq -\sqrt{a^2 + 4nb} < 0.$$

This means that $2nH + a$ has the same sign on M. Without loss of generality, we assume $2nH + a > 0$. Since

$$\begin{aligned} (2nH + a)^2 &= 4(nH)^2 + 4naH + a^2 \\ &> 4[(nH)^2 + naH - nb] \\ &= 4S \geq 4k_i^2, \quad \forall i, \end{aligned}$$

we get $2nH + a > 2k_i$, which implies L is elliptic.

(2) From $S = (nH)^2 - n(n-1)H_2 = (nH)^2 + naH - nb$, by taking the covariant derivative with respect to e_k, we have

$$2\sum_{i,j} h_{ij}h_{ijk} = (2nH + a)nH_{,k}.$$

By the Cauchy–Schwarz inequality, we get

$$(2nH+a)^2(nH_{,k})^2 = 4\sum_{i,j}(h_{ij}h_{ijk})^2 \le 4\left(\sum_{i,j}h_{ij}^2\right)\left(\sum_{i,j}h_{ijk}^2\right).$$

After making summation on k, we have

$$(2nH+a)^2|\nabla(nH)|^2 \le 4S|\nabla h|^2 \le (2nH+a)^2|\nabla h|^2,$$

which implies $|\nabla h|^2 \ge |\nabla(nH)|^2$.

When the equality holds, we must have $|\nabla h|^2 = |\nabla(nH)|^2 = 0$. Hence H is constant and the second fundamental form is parallel, which means M is an isoparametric hypersurface with at most two distinct principal curvatures (cf. [23]). Therefore, we have M is either totally umbilical or a Riemannian product $\mathbb{S}^m(c)\times\mathbb{S}^{n-m}(\sqrt{1-c^2})$, $1\le m\le n-1$. $\square$

In the following, we always assume that $x: M^n \to \mathbb{S}^{n+1}(1)$ is an isometric immersion for which $(n-1)H_2 + aH = b$, where a and b are constants.

Lemma 4.2.

$$HLH = \frac{2H}{n}(|\nabla h|^2 - |\nabla(nH)|^2 + nS - (nH)^2 + nHf_3 - S^2),$$

where H is the mean curvature, $f_3 = \sum_{i=1}^n (k_i)^3 = S_1^3 - 3S_1S_2 + 3S_3$.

Proof. By a standard computation of Simons' type formula (see [18,9,8]), we have

$$\frac{1}{2}\Delta S = |\nabla h|^2 + \sum_i k_i(nH)_{,ii} + nS - (nH)^2 + nHf_3 - S^2, \tag{4.1}$$

$$\square H = nH\Delta H - \sum_{i,j} h_{ij}H_{,ij}, \tag{4.2}$$

$$\frac{1}{2}\Delta(nH)^2 = nH\Delta(nH) + |\nabla(nH)|^2, \tag{4.3}$$

$$S = S_1^2 - n(n-1)H_2. \tag{4.4}$$

Then from $naH + (nH)^2 - S = n[(n-1)H_2 + aH] = nb$ is constant, we get

$$na\Delta H = \Delta S - \Delta(nH)^2. \tag{4.5}$$

Hence, by (4.1)–(4.5), we have

$$\begin{aligned} HLH &= H\left(a\Delta H + 2nH\Delta H - 2\sum_{i,j}h_{ij}H_{,ij}\right)\\ &= H\left(\frac{1}{n}\Delta S - \frac{1}{n}\Delta(nH)^2 + 2nH\Delta H - 2\sum_{i,j}h_{ij}H_{,ij}\right)\\ &= H\left(\frac{1}{n}\Delta S - \frac{2}{n}|\nabla(nH)|^2 - 2\sum_i k_iH_{,ii}\right)\\ &= \frac{2H}{n}(|\nabla h|^2 - |\nabla(nH)|^2 + nS - (nH)^2 + nHf_3 - S^2). \quad \square \end{aligned}$$

For any isometric immersion $x: M \to \mathbb{S}^{n+1}(1)$, let E be a fixed vector of $\mathbb{R}^{n+2}$, we define

$$f = \langle N, E\rangle, \qquad g = \langle x, E\rangle.$$

Lemma 4.3 (*Cf. [3,21]*)**.** *For any isometric immersion* $x: M \to \mathbb{S}^{n+1}(1)$, *we have*

$$\begin{aligned} \square g &= 2S_2 f - (n-1)S_1 g,\\ \square f &= -\sum_k S_{2,k}\langle e_k, E\rangle - (S_1S_2 - 3S_3)f + 2S_2 g,\\ \Delta g &= S_1 f - ng,\\ \Delta f &= -\sum_k S_{1,k}\langle e_k, E\rangle - (S_1^2 - 2S_2)f + S_1 g. \end{aligned}$$

Set $p=(n-1)H+a$, $u=bf-pg$, we have

$$\int_M u\,dv=\frac{1}{n}\int_M(a\Delta g+\Box g)=0.$$

By Lemma 4.3, using that L is self-adjoint, we have

$$\begin{aligned}\int_M uLu\,dv &= \int_M (bf-pg)(bLf-L(pg))\,dv\\ &= \int_M [(bf-pg)bLf-bpgLf+pgL(pg)]\,dv\\ &= \int_M \left\{[b^2f-2bpg]Lf+[2a(n-1)H+a^2]gLg+(n-1)^2HgL(Hg)\right\}dv\\ &= \int_M \left\{[b^2f-2bpg][-2(S_1S_2-3S_3)f+4S_2g-a(S_1^2-2S_2)f+aS_1g]\right.\\ &\quad \left.+[2a(n-1)H+a^2]g[4S_2f-2(n-1)S_1g+aS_1f-ang]+(n-1)^2HgL(Hg)\right\}dv.\end{aligned}$$

We take an orthonormal basis $E_1,\dots,E_{n+2}$ of $\mathbb{R}^{n+2}$ and define

$$f_A=\langle N,E_A\rangle,\qquad g_A=\langle x,E_A\rangle,\qquad u_A=bf_A-pg_A,\quad A=1,\dots,n+2.$$

Note that $\sum_{A=1}^{n+2}(f_A)^2=\sum_{A=1}^{n+2}(g_A)^2=1$, $\sum_{A=1}^{n+2}f_Ag_A=0$, we have

$$\begin{aligned}-\sum_{A=1}^{n+2}\int_M u_AJ_wu_A\,dv &= \sum_{A=1}^{n+2}\int_M u_ALu_A\,dv+\sum_{A=1}^{n+2}\int_M(bf_A-pg_A)^2\{2[(S_1S_2-3S_3)+(n-1)S_1]\\ &\quad +a[(S_1^2-2S_2)+n]\}\,dv\\ &= \int_M b^2[2(n-1)S_1+an]\,dv+\int_M(n-1)^2H^2\{2[(S_1S_2-3S_3)+(n-1)S_1]\\ &\quad +a[(S_1^2-2S_2)+n]\}\,dv+\int_M[a^2+2(n-1)aH][2(S_1S_2-3S_3)+a(S_1^2-2S_2)]\,dv\\ &\quad -2\int_M b[(n-1)H+a](4S_2+aS_1)\,dv+(n-1)^2\sum_{A=1}^{n+2}\int_M(Hg_A)L(Hg_A)\,dv.\end{aligned}$$

To deal with the last term above, we need the following lemma.

Lemma 4.4.

$$\sum_{A=1}^{n+2}(Hg_A)L(Hg_A)=HLH-anH^2-2n(n-1)H^3.$$

Proof. By direct computations, we have

$$\begin{aligned}\sum_{A=1}^{n+2}Hg_A\Delta(Hg_A) &= \sum_{A=1}^{n+2}Hg_A\left[g_A\Delta H+H\Delta g_A+2\sum_i(H)_{,i}(g_A)_i\right]\\ &= H\Delta H+H^2\sum_{A=1}^{n+2}g_A(S_1f_A-ng_A)\\ &= H\Delta H-nH^2,\\ \sum_{A=1}^{n+2}Hg_A\Box(Hg_A) &= \sum_{A=1}^{n+2}Hg_A\left[g_A\Box H+H\Box g_A+2\sum_{i,j}(nH\delta_{ij}-h_{ij})(H)_{,i}(g_A)_j\right]\\ &= H\Box H+\sum_{A=1}^{n+2}H^2g_A(2S_2f_A-(n-1)S_1g_A)\\ &= H\Box H-(n-1)H^2S_1,\end{aligned}$$

where we used the fact that $\sum_{A=1}^{n+2}g_A(g_A)_i=\sum_{A=1}^{n+2}\langle x,E_A\rangle\langle e_i,E_A\rangle=\langle x,e_i\rangle=0$ for any $1\le i\le n$. Hence,

$$\begin{aligned}\sum_{A=1}^{n+2}(Hg_A)L(Hg_A) &= 2\sum_{A=1}^{n+2}Hg_A\Box(Hg_A)+a\sum_{A=1}^{n+2}Hg_A\Delta(Hg_A)\\ &= HLH-anH^2-2n(n-1)H^3.\quad\Box\end{aligned}$$

Proof of Theorem 1.3. By Lemmas 4.2 and 4.4 we get that

$$\begin{aligned}-\sum_{A=1}^{n+2}\int_M u_A J_w u_A\,dv &= \frac{2(n-1)^2}{n}\int_M H(|\nabla h|^2-|\nabla(nH)|^2)\,dv+2n(n-1)^3\int_M H(H^2-H_2)(1+nH_2)\,dv\\&\quad+n(n-1)^2a\int_M[(nH^2+3H_2)(H^2-H_2)+2(n-2)H(HH_2-H_3)]\,dv\\&\quad+n(n-1)a^2\int_M[2nH(H^2-H_2)+(n-2)(HH_2-H_3)]\,dv\\&\quad+n(n-1)a^3\int_M(H^2-H_2)\,dv.\end{aligned}$$

Since $H_2>0$, $a\geq 0$, we get $a^2+4nb>0$, $H>0$, $H^2-H_2\geq 0$, $HH_2-H_3\geq 0$. So we have

$$-\sum_{A=1}^{n+2}\int_M u_A J_w u_A\,dv\geq 0.$$

On the other hand, by the condition of stability, we have

$$\sum_{A=1}^{n+2}\int_M u_A J_w u_A\,dv\geq 0.$$

So we have

$$\sum_{A=1}^{n+2}\int_M u_A J_w u_A\,dv= 0.$$

Then we get $H^2=H_2>0$, which implies M is totally umbilical and non-totally geodesic. This combined with Proposition 3.7 completes the proof of Theorem 1.3. □

5. Estimates for the first and second eigenvalues of the Jacobi operator

In this section, we will give optimal estimates for the upper bounds of both the first and second eigenvalues of the Jacobi operator J_w.

Definition 5.1. We call $\lambda_i^{J_w}$ an eigenvalue of J_w if there exists a non-zero function f on M such that $J_w f=\lambda_i^{J_w}f$, we call $\lambda_i^{\Box}$ an eigenvalue of $\Box$ if there exists a non-zero function f on M such that $\Box f+\lambda_i^{\Box}f=0$, and we call λ_i^{Δ} an eigenvalue of Δ if there exists a non-zero function f on M such that $\Delta f+\lambda_i^{\Delta}f=0$.

First of all, we consider the first and second eigenvalues of the Jacobi operator J_w of the totally umbilical and non-totally geodesic hypersurface in $\mathbb{S}^{n+1}(1)$ with positive mean curvature, and the Riemannian product $\mathbb{S}^m(c)\times\mathbb{S}^{n-m}(\sqrt{1-c^2})$, $1\leq m\leq n-1$ with $H_2>0$.

Example 5.2. Let M be a totally umbilical and non-totally geodesic hypersurface in $\mathbb{S}^{n+1}(1)$. Then the mean curvature H is a non-zero constant. We choose the orientation such that $H>0$ and assume that $(n-1)H_2+aH=b$, where $a\geq 0$. Then we have $H_2=H^2$, $H_3=H^3$. From (3.3) we have

$$J_w=-(2(n-1)H+a)\Delta-2[(S_1S_2-3S_3)+(n-1)S_1]+a[(S_1^2-2S_2)+n].$$

Hence

$$\begin{aligned}\lambda_1^{J_w}&=-2[(S_1S_2-3S_3)+(n-1)S_1]+a[(S_1^2-2S_2)+n]\\&=-2n(n-1)H^3-anH^2-2n(n-1)H-an\\&=an^2H^2-[2bn^2+4n(n-1)-a^2n]H+\frac{2bn(b+1)}{H}-3an(b+1),\end{aligned}$$

where we used the fact $b=(n-1)H^2+aH$.

Since the first non-zero eigenvalue of Δ is $n(1+H^2)$, we have

$$\lambda_2^{J_w}=[2(n-1)H+a]n(1+H^2)-2[(S_1S_2-3S_3)+(n-1)S_1]+a[(S_1^2-2S_2)+n]=0.$$

Example 5.3. Let $M = \mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2}),\ 1 \le m \le n-1$, be a hypersurface with $H_2 > 0$ in $\mathbb{S}^{n+1}(1)$. Following the computations of [8,11], we know that all the principal curvatures are constant, and the positive constant c satisfies

$$0 < c^2 < \frac{m}{n} - \frac{\sqrt{m(n-m)}}{n\sqrt{n-1}} \quad \text{or} \quad \frac{m}{n} + \frac{\sqrt{m(n-m)}}{n\sqrt{n-1}} < c^2 < 1. \tag{5.1}$$

We choose the orientation such that $H > 0$ and assume that $(n-1)H_2 + aH = b$, where $a \ge 0$. Since the operator L is self-adjoint and elliptic,

$$\lambda_1^{J_w} = -2[(S_1S_2 - 3S_3) + (n-1)S_1] - a[(S_1^2 - 2S_2) + n].$$

By a long and direct computation, we have

$$\begin{aligned} -[(S_1S_2 - 3S_3) + (n-1)S_1] &= -[2n(n-1) + n^2(n-1)(r-1)]H \\ &\quad + n(n-1)(r-1)[(n-1)(r-1)+1]\frac{1}{H}, \end{aligned} \tag{5.2}$$

where $n(n-1)(r-1) = S_1^2 - S = 2S_2 = nb - anH$.

Hence

$$\begin{aligned} \lambda_1^{J_w} &= 2[-2n(n-1) - n(nb - anH)]H + 2[(nb-anH)(b - aH + 1)]\frac{1}{H} - a[(n^2H^2 + anH - nb) + n] \\ &= an^2H^2 - [2bn^2 + 4n(n-1) - a^2n]H + \frac{2bn(b+1)}{H} - 3an(b+1). \end{aligned}$$

To compute the 2nd eigenvalue of J_w, we choose the unit normal vector such that

$$k_1 = \cdots = k_m = -\frac{\sqrt{1-c^2}}{c}, \qquad k_{m+1} = \cdots = k_n = \frac{c}{\sqrt{1-c^2}},$$

we have

$$S_1 = \frac{nc^2 - m}{c\sqrt{1-c^2}} > 0, \qquad S = \frac{m(1-c^2)}{c^2} + \frac{(n-m)c^2}{1-c^2},$$

$$2S_2 = S_1^2 - S = \frac{f(c^2)}{c^2(1-c^2)} > 0,$$

where $f(t) = n(n-1)t^2 - 2m(n-1)t + m(m-1)$.

After an analogous argument with Example 3.2 in [11], we have

$$Lf = (2(nH - k_1) + a)\Delta_1 f + (2(nH - k_n) + a)\Delta_2 f,$$

where Δ_1 and Δ_2 are the Laplacian on $\mathbb{S}^m(c)$ and $\mathbb{S}^{n-m}(\sqrt{1-c^2})$ respectively. Then

$$\lambda_2^L = \min\{(2(nH - k_1) + a)\lambda_2^{\Delta_1}, (2(nH - k_n) + a)\lambda_2^{\Delta_2}\},$$

where $\lambda_2^{\Delta_i}$ is the 2nd eigenvalue (i.e. the 1st non-zero eigenvalue) of Δ_i given by

$$\lambda_2^{\Delta_1} = \frac{m}{c^2} < \frac{n-m}{1-c^2} = \lambda_2^{\Delta_2},$$

as $S_1 > 0$. Then we have

$$\begin{aligned} \lambda_2^{J_w} &= \lambda_2^L - 2[(S_1S_2 - 3S_3) + (n-1)S_1] - a[(S_1^2 - 2S_2) + n] \\ &= (2(nH - k_1) + a)\frac{m}{c^2} - 2[(S_1S_2 - 3S_3) + (n-1)S_1] - a[(S_1^2 - 2S_2) + n] \\ &= 2\left[\frac{(n-1)c^2 - (m-1)}{c\sqrt{1-c^2}}\frac{m}{c^2} - \big((S_1S_2 - 3S_3) + (n-1)S_1\big)\right] + a\left(\frac{m}{c^2} - S - n\right). \end{aligned}$$

Using (5.2), we have

$$\frac{(n-1)c^2 - (m-1)}{c\sqrt{1-c^2}}\frac{m}{c^2} - \big((S_1S_2 - 3S_3) + (n-1)S_1\big) = \frac{-2c^2}{c\sqrt{1-c^2}}S_2 - (n-1)S_1 < 0.$$

We also have

$$\frac{m}{c^2} - S - n = -(n-m) - \frac{(n-m)c^2}{1-c^2} < 0.$$

Note that $a \ge 0$, we finally get $\lambda_2^{J_w} < 0$.

Proof of Theorem 1.4. Denote $H_{\min} = \min_{x\in M} H(x) > 0$, note that $H_2 = \frac{b-aH}{(n-1)}$, from Lemmas 4.1 and 4.2, we have

$$\begin{aligned}
-\int_M HJ_wH\,dv &= \frac{2}{n}\int_M H(|\nabla h|^2 - |\nabla(nH)|^2)\,dv + \int_M [anH^2(1+H^2) + an(n-1)H^2(H^2-H_2)]dv \\
&\quad + \int_M [2n(n-1)H^3(2+nH_2) - 2n(n-1)HH_2((n-1)H_2+1)]dv \\
&= \frac{2}{n}\int_M H(|\nabla h|^2 - |\nabla(nH)|^2)\,dv + \int_M H^2\Big\{-an^2H^2 + [a^2n + 4n(n-1) + 2bn^2]H \\
&\quad - \frac{2n}{H}(b-aH)(b-aH+1) + an - abn\Big\}\,dv \\
&\geq \int_M H^2\left\{-an^2H^2 + [a^2n + 4n(n-1) + 2bn^2]H - \frac{2n}{H}(b-aH)(b-aH+1) + an - abn\right\}dv.
\end{aligned}$$

Note that $H_{\min} \leq H \leq \frac{b}{a}$, we have

$$2b \geq a(H + H_{\min}).$$

By multiplying both sides of the above inequality by $(H - H_{\min})$, we get

$$2bH - aH^2 \geq 2bH_{\min} - aH_{\min}^2.$$

Hence

$$\begin{aligned}
-\int_M HJ_wH\,dv &\geq \int_M H^2\Big\{-an^2H_{\min}^2 + [a^2n + 4n(n-1) + 2bn^2]H_{\min} \\
&\quad - \frac{2n}{H_{\min}}(b-aH_{\min})(b-aH_{\min}+1) + an - abn\Big\}\,dv \\
&= \int_M H^2\left\{-an^2H_{\min}^2 + [2bn^2 + 4n(n-1) - a^2n]H_{\min} - \frac{2bn(b+1)}{H_{\min}} + 3an(b+1)\right\}dv.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\lambda_1^{J_w} &\leq \frac{\int_M HJ_wH\,dv}{\int_M H^2\,dv} \\
&\leq an^2H_{\min}^2 - [2bn^2 + 4n(n-1) - a^2n]H_{\min} + \frac{2bn(b+1)}{H_{\min}} - 3an(b+1). \qquad (5.3)
\end{aligned}$$

When the equality holds, we have $|\nabla h|^2 = |\nabla(nH)|^2 = 0$, by Lemma 4.1, we have M is totally umbilical or a Riemannian product $\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$. On the other hand, if M is totally umbilical and non-totally geodesic, or a Riemannian product $\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$ with $H_2 > 0$, from Examples 5.2 and 5.3 in this section, we know that the equality in (5.3) is attained. This completes the proof of Theorem 1.4. □

Remark 5.4. We note that when $a = 0$, we get a same optimal estimate as in [8].

In the following, we will prove Theorem 1.5. First, we have $H^2 \geq H_2 > 0$ and $a \geq 0$; hence J_w is elliptic (cf. [8,11]). In order to get the optimal estimate of the second eigenvalue of J_w, we will use a technique which was introduced by Li and Yau in [14] and was later used by other authors (see [15,16,24,19,11]). Let B^{n+2} be the open unit ball in $\mathbb{R}^{n+2}$. For each point $l \in B^{n+2}$, we consider the map

$$F_l(p) = \frac{p + (\mu\langle p, l\rangle + \lambda)l}{\lambda(\langle p, l\rangle + 1)}, \quad \forall\, p \in \mathbb{S}^{n+1}(1) \subset \mathbb{R}^{n+2},$$

where $\lambda = (1 - \|l\|^2)^{-1/2}$, $\mu = (\lambda - 1)\|l\|^{-2}$ and $\langle , \rangle$ denotes the inner product on $\mathbb{R}^{n+2}$. A direct computation (see [15,19]) shows that F_l is a conformal transformation from $\mathbb{S}^{n+1}(1)$ to $\mathbb{S}^{n+1}(1)$ and the differential map dF_l of F_l is given by

$$dF_l(v) = \lambda^{-2}(\langle p, l\rangle + 1)^{-2}\{\lambda(\langle p, l\rangle + 1)v - \lambda\langle v, l\rangle p + \langle v, l\rangle(1-\lambda)\|l\|^{-2}l\},$$

where v is a tangent vector to $\mathbb{S}^{n+1}$ at the point p. Hence, for two vectors $v, w \in T_p\mathbb{S}^{n+1}$ we have (see [15,16,19])

$$\langle dF_l(v), dF_l(w)\rangle = \frac{1-\|l\|^2}{(\langle p, l\rangle + 1)^2}\langle v, w\rangle.$$

By the use of the technique in [14], we have the following result.

Lemma 5.5 (*See [15,16,24,19]*). *Let M be a compact orientable hypersurface in $\mathbb{S}^{n+1}(1)$, M satisfies that $H_2 > 0$, we choose the orientation such that $H > 0$, assume $a \geq 0$; then J_w is elliptic. Let u be a positive first eigenfunction of J_w on M; then there exists $l \in B^{n+2}$ such that $\int_M u(F_l \circ x)dv = (0, \dots, 0)$.*

For any isometric immersion $x : M \to \mathbb{S}^{n+1}(1)$, let $\{E^A\}_{A=1}^{n+2}$ be a fixed orthonormal basis of $\mathbb{R}^{n+2}$, for a fixed point $l \in B^{n+2}$, we define functions $f^A : M \to \mathbb{R}(1 \leq A \leq n+2)$ by

$$f^A = \langle E^A, F_l \circ x\rangle = \frac{\langle E^A, x\rangle + (\mu\langle x, l\rangle + \lambda)\langle l, E^A\rangle}{\lambda(\langle x, l\rangle + 1)}. \tag{5.4}$$

Lemma 5.6 (*See [11]*). *The gradient of f^A is given by*

$$f_i^A = \frac{\langle E^A, e_i\rangle}{\lambda(\langle x, l\rangle + 1)} + \frac{\langle l, e_i\rangle}{\lambda(\langle x, l\rangle + 1)^2}\cdot\left(-\langle E^A, x\rangle + \frac{1-\lambda}{\lambda\|l\|^2}\langle l, E^A\rangle\right).$$

Lemma 5.7. *For any isometric immersion $x : M \to \mathbb{S}^{n+1}(1)$, we have*

$$\sum_{A=1}^{n+2}\int_M (J_w f^A \cdot f^A)dv = \int_M \frac{(2n(n-1)H + an)(1-\|l\|^2)}{(\langle x, l\rangle + 1)^2}dv$$
$$- \int_M \{n(n-1)(2H - (n-2)H_3 + nHH_2)\}dv - \int_M \{a(n^2H^2 - n(n-1)H_2 + n)\}dv. \tag{5.5}$$

Proof. By the divergence theorem and Lemma 5.6 we have

$$-\sum_{A=1}^{n+2}\int_M (\Delta f^A \cdot f^A)dv = \int_M \frac{n(1-\|l\|^2)}{(\langle x, l\rangle + 1)^2}dv. \tag{5.6}$$

From the proof of Lemma 3.6 in [11], we have

$$-\sum_{A=1}^{n+2}\int_M (\Box f^A \cdot f^A)dv = \int_M \frac{n(n-1)H(1-\|l\|^2)}{(\langle x, l\rangle + 1)^2}dv. \tag{5.7}$$

Then (5.5) follows immediately from (5.6) and (5.7). $\Box$

For a fixed point $l \in B^{n+2}$, let

$$\tilde{f} = \langle x, l\rangle, \qquad \tilde{l} = \langle N, l\rangle, \qquad \rho = -\ln\lambda - \ln(1+\tilde{f}), \tag{5.8}$$

where $\lambda = (1-\|l\|^2)^{-1/2}$, x is the position vector and N is the unit normal vector. We have

$$e^{2\rho} = \frac{1}{\lambda^2(1+\tilde{f})^2} = \frac{1-\|l\|^2}{(\langle x, l\rangle + 1)^2}, \qquad \rho_i = \frac{-\tilde{f}_i}{1+\tilde{f}}, \qquad \rho_{ij} = \frac{-\tilde{f}_{ij}}{1+\tilde{f}} + \frac{\tilde{f}_i\tilde{f}_j}{(1+\tilde{f})^2}. \tag{5.9}$$

Lemma 5.8. *Let M be a compact orientable hypersurface in $\mathbb{S}^{n+1}(1)$, M satisfies that $H_2 > 0$, we choose the orientation such that $H > 0$. Let $\tilde{f}, \tilde{l}, \rho$ be the functions defined by (5.8), we have*

(i)

$$\int_M \frac{(1-\|l\|^2)}{(\langle x, l\rangle + 1)^2}dv \leq \int_M (1+H^2)dv, \tag{5.10}$$

and the equality holds if and only if $H + \frac{\tilde{l}}{1+\tilde{f}} \equiv 0$ and $\langle l, e_i\rangle \equiv 0$ on M.

(ii) *(Cf. Lemma 3.7 in [11])*

$$\int_M \frac{H(1-\|l\|^2)}{(\langle x, l\rangle + 1)^2}dv \leq \int_M \left(H + \frac{H_2^2}{H}\right)dv - \int_M \left[H\|\nabla\rho\|^2 - \frac{2}{n(n-1)}\sum_{i,j}(nH\delta_{ij} - h_{ij})\rho_i\rho_j\right]dv, \tag{5.11}$$

and the equality holds if and only if $H_2 + \frac{\tilde{l}H}{1+\tilde{f}} \equiv 0$ on M.

Proof. We only have to prove (i). We have

$$0=\int_M \Delta\rho dv=\frac{n}{2}\int_M\left\{(1+H^2)-\left(H+\frac{\tilde{l}}{1+\tilde{f}}\right)^2-\frac{n-2}{n}\|\nabla\rho\|^2-\frac{1-\|l\|^2}{(1+\tilde{f})^2}\right\}dv,$$

which immediately leads to (5.10). □

Following the same arguments as in Lemma 3.8 of [11], we have the following.

Lemma 5.9. *Let M be a compact orientable hypersurface in $\mathbb{S}^{n+1}(1)$, M satisfies that $H_2>0$, we choose the orientation such that $H>0$. Assume $n\geq 5$, we have*

$$\int_M\left[H\|\nabla\rho\|^2-\frac{2}{n(n-1)}\sum_{i,j}(nH\delta_{ij}-h_{ij})\rho_i\rho_j\right]dv\geq 0.$$

Proof of Theorem 1.5. Since $H_2>0$, $a\geq 0$, we have $2\Box+a\Delta$ an elliptic operator and $H\neq 0$. Hence, we can assume $H>0$. Let u be a first eigenfunction of J_w, we can assume u is positive on M, by Lemma 5.5 there exists $l\in B^{n+2}$ such that

$$\int_M u(F_l\circ x)dv=(0,\ldots,0),$$

which implies that the functions $\{f^A,\ 1\leq A\leq n+2\}$ given by (5.4) are perpendicular to the function u, i.e., $\int_M u\cdot f^A dv=0,\ \forall\, 1\leq A\leq n+2$. Then by using the min–max characterization of eigenvalues for elliptic operators, we have

$$\lambda_2^{J_w}\cdot\int_M(f^A\cdot f^A)dv\leq\int_M(J_wf^A\cdot f^A)dv,\quad\forall\, 1\leq A\leq n+2.$$

Summing up and using the fact that $\sum_{A=1}^{n+2}f^A\cdot f^A=1$, we obtain

$$\lambda_2^{J_w}\cdot\mathrm{Vol}(M)\leq\sum_{A=1}^{n+2}\int_M(J_wf^A\cdot f^A)dv. \tag{5.12}$$

From Lemma 5.7 and Eq. (5.12) we have

$$\begin{aligned}\lambda_2^{J_w}\cdot\mathrm{Vol}(M)\ \leq\ &\int_M\frac{(2n(n-1)H+an)(1-\|l\|^2)}{(\langle x,l\rangle+1)^2}dv-\int_M\{n(n-1)(2H-(n-2)H_3+nHH_2)\}dv\\&-\int_M\{a(n^2H^2-n(n-1)H_2+n)\}dv.\end{aligned} \tag{5.13}$$

Then by Eq. (5.13), Lemmas 5.8 and 5.9, we get

$$\begin{aligned}\lambda_2^{J_w}\cdot\mathrm{Vol}(M)\ &\leq\ 2n(n-1)\cdot\int_M\left(H+\frac{H_2^2}{H}\right)dv+an\cdot\int_M(1+H^2)dv\\&\quad-\int_M n(n-1)(2H-(n-2)H_3+nHH_2)dv-\int_M\{a(n^2H^2-n(n-1)H_2+n)\}dv\\&=2n(n-1)\cdot\int_M\left(\frac{H_2^2}{H}+\frac{n-2}{2}H_3-\frac{nHH_2}{2}\right)dv+an(n-1)\int_M(H_2-H^2)dv.\end{aligned}$$

Since $H_2>0$, we have $H_3\leq\frac{H_2^2}{H}$ and $H_2\leq H^2$ (see [25, p. 52]) and hence

$$\begin{aligned}\lambda_2^{J_w}\cdot\mathrm{Vol}(M)\ &\leq\ 2n(n-1)\cdot\int_M\left(\frac{H_2^2}{H}+\frac{n-2}{2}H_3-\frac{nHH_2}{2}\right)dv\\&\leq 2n(n-1)\cdot\int_M\left(\frac{H_2^2}{H}+\frac{n-2}{2}\frac{H_2^2}{H}-\frac{nHH_2}{2}\right)dv\\&=2n(n-1)\cdot\int_M\frac{nH_2}{2}\left(\frac{H_2}{H}-H\right)dv\leq 0,\end{aligned} \tag{5.14}$$

therefore we get $\lambda_2^{J_w}\leq 0$. When $\lambda_2^{J_w}=0$, all the inequalities become equalities. When the equality in Lemma 5.8 (or in (5.14)) holds, we have $H_2\equiv H^2$ on M, since H_2 is positive, we get M is a totally umbilical and non-totally geodesic hypersurface. On the other hand, if M is a totally umbilical and non-totally geodesic hypersurface, from Example 5.2 in this section, we know that $\lambda_2^{J_w}=0$. This completes the proof of Theorem 1.5. □

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