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Optimization problems for eigenvalues of p-Laplace equations

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ABSTRACT

We study minimization and maximization problems for the principal eigenvalue of a p-Laplace equation in a bounded domain Ω, with weight χ_D, where $D \subset \Omega$ is a variable subset with a fixed measure α. We investigate monotonicity, continuity and differentiability with respect to α of the optimizing eigenvalues.

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1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. For $D \subset \Omega$ and $1 < p$, we consider the eigenvalue problem

$$-\Delta_p u = \lambda \chi_D u^{p-1}, \quad u > 0 \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega. \tag{1}$$

Here λ is the principal eigenvalue, which depends on Ω, p and D. In what follows, Ω and p will be fixed, whereas, the subset D may change, therefore we shall write $\lambda = \lambda_D$. It is well known that

$$\lambda_D = \inf\left\{ \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega \chi_D |v|^p dx} : \; v \in H_0^{1,p}(\Omega), \; \int_\Omega \chi_D |v|^p dx > 0 \right\} = \frac{\int_\Omega |\nabla u_D|^p dx}{\int_\Omega \chi_D u_D^p dx},$$

where $u_D \in H_0^{1,p}(\Omega)$ is the principal (positive) eigenfunction, which we normalize so that $\int_\Omega u_D^p dx = 1$. The eigenvalues of the p-Laplacian have been investigated in several papers, we refer to [1,2] and references therein. For regularity of solutions of p-Laplace equations we refer to [3–5]. In particular, we recall that the eigenfunctions of problem (1) are continuous.

If $D \subset \Omega$ is a measurable set we denote with $|D|$ its Lebesgue measure. Fix $0 < \alpha < |\Omega|$, and consider the minimization problem

$$\inf_{|D|=\alpha} \lambda_D = \inf_{|D|=\alpha} \frac{\int_\Omega |\nabla u_D|^p dx}{\int_\Omega \chi_D u_D^p dx}.$$

It is well known that this problem has (at least) a solution $\hat{D}$ and that

$$\hat{D} = \{x \in \Omega : u_{\hat{D}}(x) > t\}$$

for some $t > 0$. For a proof of this result we refer to [6,7] in the case $p = 2$, and to [8] for general p.

In this paper, we define the function $f(\alpha) = \inf_{|D|=\alpha} \lambda_D$, and we shall prove that $f(\alpha)$ is strictly decreasing and continuous for $\alpha \in (0, |\Omega|)$. In the case $p = 2$, these facts have been observed in [9].

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Furthermore, we consider the maximization problem

$$\sup_{|D|=\alpha} \lambda_D = \sup_{|D|=\alpha} \frac{\int_\Omega |\nabla u_D|^p dx}{\int_\Omega \chi_D u_D^p dx}.$$

This problem has a unique solution $\check{D}$ and

$$\check{D} = \{x \in \Omega : u_{\check{D}}(x) < \tau\}$$

for some $\tau > 0$. For a proof of this result in the case $p = 2$, we refer to [6,7]. For general p, this maximization problem is discussed in [8] in the case where Ω is a ball. For general domains, see the Appendix of the present paper.

If $g(\alpha) = \sup_{|D|=\alpha} \lambda_D$, we have $f(\alpha) < g(\alpha)$ for $\alpha \in (0, |\Omega|)$, and $f(|\Omega|) = g(|\Omega|)$. We shall prove that $g(\alpha)$ is strictly decreasing and differentiable.

The literature on the shape optimization for eigenvalues of elliptic operators is much rich. We quote the books/articles [10–13] and references therein.

2. Main results

Theorem 2.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, let D be a measurable subset of Ω, and let λ_D be the principal eigenvalue of problem (1). The functions*

$$f(\alpha) = \inf_{|D|=\alpha} \lambda_D$$

and

$$g(\alpha) = \sup_{|D|=\alpha} \lambda_D$$

are decreasing for $\alpha \in (0, |\Omega|)$. Furthermore, the function $f(\alpha)$ is continuous, and the function $g(\alpha)$ is differentiable for $0 < \alpha < |\Omega|$.

Proof. Let us show that $f(\alpha)$ is decreasing. We know that there is $\hat{D}$ such that $|\hat{D}| = \alpha$ and

$$f(\alpha) = \lambda_{\hat{D}} = \frac{\int_\Omega |\nabla u_{\hat{D}}|^p dx}{\int_\Omega \chi_{\hat{D}} u_{\hat{D}}^p dx},$$

where $u_{\hat{D}}$ is the normalized eigenfunction corresponding to $\lambda_{\hat{D}}$. If $\alpha < \beta \le |\Omega|$, take $\tilde{D}$ so that $\hat{D} \subset \tilde{D} \subset \Omega$ with $|\tilde{D}| = \beta$. Since $u_{\hat{D}}(x) > 0$ in Ω, we have

$$f(\alpha) = \frac{\int_\Omega |\nabla u_{\hat{D}}|^p dx}{\int_\Omega \chi_{\hat{D}} u_{\hat{D}}^p dx} > \frac{\int_\Omega |\nabla u_{\hat{D}}|^p dx}{\int_\Omega \chi_{\tilde{D}} u_{\hat{D}}^p dx} \ge \lambda_{\tilde{D}} \ge f(\beta).$$

The monotonicity of $f(\alpha)$ is proved.

Let us show that $f(\alpha)$ is continuous from the left. Let D with $|D| = \alpha$ and $u = u_D > 0$ such that

$$f(\alpha) = \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_D u^p dx}. \tag{2}$$

Take $h < 0$ such that $\alpha + h > 0$. Let D_h with $|D_h| = \alpha + h$ and $u_h = u_{D_h} > 0$ such that

$$f(\alpha + h) = \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{D_h} u_h^p dx}.$$

Although $|D_h| < |D|$, we do not have, in general, $D_h \subset D$. Take $\tilde{D}_h \subset D$ with $|\tilde{D}_h| = \alpha + h$. Recall that

$$f(\alpha) = \inf_{|D|=\alpha} \inf_{v \in H_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega \chi_D v^p dx}. \tag{3}$$

Therefore, since $|D_h| = |\tilde{D}_h|$, we have

$$f(\alpha + h) = \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{D_h} u_h^p dx} \le \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_{\tilde{D}_h} u^p dx},$$

where u is the same function as in (2). Hence (recall that $h < 0$),

$$0 < f(\alpha + h) - f(\alpha) \le \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_{\tilde{D}_h} u^p dx} - \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_D u^p dx} = Q_h \int_{D \setminus \tilde{D}_h} u^p dx,$$

where

$$Q_h = \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_D u^p dx \int_\Omega \chi_{\tilde{D}_h} u^p dx} = f(\alpha)\frac{1}{\int_\Omega \chi_{\tilde{D}_h} u^p dx}.$$

Since $u = u(x) > 0$, and $|\tilde{D}_h| = \alpha + h > 0$, Q_h is bounded as $h \to 0^-$. Hence, since $|D \setminus \tilde{D}_h| = |h|$, we have

$$0 \le \lim_{h\to 0^-} [f(\alpha + h) - f(\alpha)] \le \lim_{h\to 0^-} Q_h \int_{D\setminus\tilde{D}_h} u^p dx = 0.$$

The continuity of $f(\alpha)$ from the left is proved.

Let us show that $f(\alpha)$ is continuous from the right. As in the previous case, let D with $|D| = \alpha$ and $u = u_D > 0$ as in (2). If $h > 0$, let D_h with $|D_h| = \alpha + h$ and $u_h = u_{D_h} > 0$ such that

$$f(\alpha + h) = \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{D_h} u_h^p dx}. \tag{4}$$

Although $|D| < |D_h|$, we do not have, in general, $D \subset D_h$. Take $\tilde{D}_h \subset D_h$ with $|\tilde{D}_h| = \alpha$. By (2) and (3) we have

$$f(\alpha) = \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_D u^p dx} \le \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx}.$$

Therefore,

$$0 < f(\alpha) - f(\alpha + h) \le \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx} - \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{D_h} u_h^p dx} = \tilde{Q}_h \int_{D_h\setminus\tilde{D}_h} u_h^p dx,$$

where

$$\tilde{Q}_h = \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx \int_\Omega \chi_{D_h} u_h^p dx} = f(\alpha + h)\frac{1}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx} < f(\alpha)\frac{1}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx}.$$

We claim that $\tilde{Q}_h$ is bounded as $h \to 0^+$. By contradiction, suppose $\tilde{Q}_{h_i} \to \infty$ as $i \to \infty$, where $h_i \to 0$. Since $\int_\Omega u_{h_i}^p dx = 1$, by (4) we find

$$\int_\Omega |\nabla u_{h_i}|^p dx = f(\alpha + h_i) \int_\Omega \chi_{D_{h_i}} u_{h_i}^p dx < f(\alpha) \int_\Omega u_{h_i}^p dx = f(\alpha).$$

As a consequence, a subsequence (still denoted as) u_{h_i}, converges to some $v \in H_0^{1,p}(\Omega)$ in the weak topology of $H^{1,p}$ and in the norm of $L^p(\Omega)$. We have

$$\int_\Omega |\nabla u_{h_i}|^{p-2}\nabla u_{h_i} \cdot \nabla\psi \, dx = \lambda_i \int_\Omega \chi_{D_{h_i}} u_{h_i}^{p-1} \psi \, dx, \quad \forall\psi \in H_0^{1,p}(\Omega),$$

where $\lambda_i = f(\alpha + h_i)$. We may suppose that the sequence λ_i converges to some $\lambda > 0$ and that $\chi_{D_{h_i}}$ (or some of its subsequences) converge in the weak* topology of $L^\infty(\Omega)$ to a function η such that $0 \le \eta \le 1$ and $\int_\Omega \eta \, dx = \alpha$. Hence, v satisfies

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\psi \, dx = \lambda \int_\Omega \eta \, v^{p-1} \psi \, dx, \quad \forall\psi \in H_0^{1,p}(\Omega).$$

This means that v is the first eigenfunction corresponding to η, therefore, $v(x) > 0$. Finally, since $\chi_{\tilde{D}_{h_i}}$ (or some of its subsequences) converge in the weak* topology of $L^\infty(\Omega)$ to a function ξ such that $0 \le \xi \le 1$ and $\int_\Omega \xi \, dx = \alpha$, we have

$$\lim_{i\to\infty} \int_\Omega \chi_{\tilde{D}_{h_i}} u_{h_i}^p dx = \int_\Omega \xi \, v^p dx > 0.$$

It follows that we cannot have $\tilde{Q}_{h_i} \to \infty$, and $\tilde{Q}_h$ is bounded as $h \to 0^+$.

Hence, since $|D_h \setminus \tilde{D}_h| = h$, we have

$$0 \le \lim_{h\to 0^+} [f(\alpha) - f(\alpha + h)] \le \lim_{h\to 0^+} \tilde{Q}_h \int_{D_h\setminus\tilde{D}_h} u_h^p dx = 0.$$

The continuity of $f(\alpha)$ follows.

Let us prove that $g(\alpha)$ is decreasing. Let $\alpha < \beta$. We know that there is $\check{D}$ such that $|\check{D}| = \beta$ and

$$g(\beta) = \lambda_{\check{D}} = \frac{\int_\Omega |\nabla u_{\check{D}}|^p dx}{\int_\Omega \chi_{\check{D}} u_{\check{D}}^p dx},$$

where $u_{\check{D}}$ is the normalized eigenfunction corresponding to $\lambda_{\check{D}}$. Take $\bar{D}$ so that $\bar{D} \subset \check{D}$ and $|\bar{D}| = \alpha$. If $u_{\bar{D}}$ is the eigenfunction corresponding to $\bar{D}$, we have $u_{\bar{D}}(x) > 0$ in Ω, and

$$g(\beta) = \frac{\int_\Omega |\nabla u_{\check{D}}|^p dx}{\int_\Omega \chi_{\check{D}} u_{\check{D}}^p dx} \le \frac{\int_\Omega |\nabla u_{\bar{D}}|^p dx}{\int_\Omega \chi_{\check{D}} u_{\bar{D}}^p dx} < \frac{\int_\Omega |\nabla u_{\bar{D}}|^p dx}{\int_\Omega \chi_{\bar{D}} u_{\bar{D}}^p dx} = \lambda_{\bar{D}} \le g(\alpha).$$

The monotonicity of $g(\alpha)$ is proved.

By using arguments similar to those used to prove the continuity of $f(\alpha)$, one proves that also $g(\alpha)$ is continuous. Let D with $|D| = \alpha$ be the set relative to $g(\alpha)$ (recall that D is unique), and let D_h with $|D_h| = \alpha + h$ be the set relative to $g(\alpha + h)$. If $u = u_D$ and $u_h = u_{D_h}$ are the corresponding eigenfunctions, we claim that, when $h \to 0$, $\chi_{D_h} \to \chi_D$ in the weak* topology of $L^\infty(\Omega)$, and $u_h \to u$ in the norm of $L^p(\Omega)$. Indeed, one finds easily that $\int_\Omega |\nabla u_h|^p dx$ is bounded as $h \to 0$. Hence, there are a sequence h_i (with $h_i \to 0$ as $i \to \infty$) and a function $z = z(x) \ge 0$, $z \in H_0^{1,p}(\Omega)$, such that $u_{h_i} \to z$ in the weak topology of $H^{1,p}(\Omega)$ and in the norm of $L^p(\Omega)$. Furthermore, one finds η with $0 \le \eta(x) \le 1$ and $\int_\Omega \eta(x)dx = \alpha$ such that $\chi_{D_{h_i}} \to \eta$ in the weak* topology of $L^\infty(\Omega)$. Therefore, by

$$\int_\Omega |\nabla u_{h_i}|^{p-2} \nabla u_{h_i} \cdot \nabla \psi \, dx = g(\alpha + h_i) \int_\Omega \chi_{D_{h_i}} u_{h_i}^{p-1} \psi \, dx, \quad \forall \psi \in H_0^{1,p}(\Omega),$$

it follows that

$$\int_\Omega |\nabla z|^{p-2} \nabla z \cdot \nabla \psi \, dx = g(\alpha) \int_\Omega \eta z^{p-1} \psi \, dx, \quad \forall \psi \in H_0^{1,p}(\Omega).$$

This implies that $g(\alpha)$ is the principal eigenvalue corresponding to η. Observe that, if $\mathcal{G}$ is the class of all functions of the kind χ_D with $|D| = \alpha$, then $\eta \in \overline{\mathcal{G}}$, the closure of $\mathcal{G}$ with respect to the weak* topology of $L^\infty(\Omega)$. But (see the Appendix below) we know that there is a unique $\eta \in \overline{\mathcal{G}}$ which corresponds to $g(\alpha)$, and that $\eta \in \mathcal{G}$, that is, $\eta = \chi_{\check{D}}$ for some $\check{D}$ with $|\check{D}| = \alpha$. Hence, with $\psi = z$ we find

$$g(\alpha) = \frac{\int_\Omega |\nabla z|^p dx}{\int_\Omega \chi_{\check{D}} z^p dx} = \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_D u^p dx}.$$

By uniqueness, we must have $D = \check{D}$ and $z = u$. The claim follows.

Now we prove that $g(\alpha)$ is differentiable. We have

$$g(\alpha) = \sup_{|D|=\alpha} \inf_{v \in H_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega \chi_D |v|^p dx} \le \inf_{v \in H_0^{1,p}(\Omega)} \sup_{|D|=\alpha} \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega \chi_D |v|^p dx}.$$

Note that, if $v \in H_0^{1,p}(\Omega)$, $v > 0$, we have (see, for example, [9] page 321)

$$\inf_{|D|=\alpha} \int_\Omega \chi_D v^p dx = \int_\Omega \chi_{\check{D}} v^p dx,$$

for some $\check{D}$ with $|\check{D}| = \alpha$ and

$$\{v(x) < t\} \subset \check{D} \subset \{v(x) \le t\}, \quad t = \sup\{s : |\{v(x) < s\}| < \alpha\}.$$

This implies that, for $v > 0$,

$$g(\alpha) \le \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega \chi_D v^p dx}, \qquad \{v(x) < t\} \subset D \subset \{v(x) \le t\}, \quad |D| = \alpha. \tag{5}$$

If $h > 0$, let D_h with $|D_h| = \alpha + h$ and $u_h = u_{D_h}$ such that

$$g(\alpha + h) = \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{D_h} u_h^p dx}. \tag{6}$$

We know that (see Remark 1 of the Appendix) there is $t > 0$ such that

$$D_h = \{x \in \Omega : u_h(x) < t\} \quad |D_h| = \alpha + h.$$

By using Eq. (1) with $D = D_h$, one finds that u_h has not flat zones on D_h. Therefore, there is $\tau < t$ so that

$$\tilde{D}_h = \{x \in \Omega : u_h(x) < \tau\}, \quad |\tilde{D}_h| = \alpha.$$

Note that, in this situation, the sets $\tilde{D}_h$ and $\{x \in \Omega : u_h(x) \le \tau\}$ have the same measure. Hence, by using (5) we find

$$g(\alpha) = \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_D u^p dx} \le \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx},$$

where, as usual, D is the set corresponding to $g(\alpha)$ and $u = u_D > 0$ is a corresponding eigenfunction. The latter inequality and (6) yield

$$\begin{aligned} -\frac{g(\alpha+h)-g(\alpha)}{h} &\le \frac{1}{h}\left[\frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx} - \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{D_h} u_h^p dx}\right] \\ &= \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx \int_\Omega \chi_{D_h} u_h^p dx} \frac{\int_{D_h \setminus \tilde{D}_h} u_h^p dx}{h} = \frac{g(\alpha+h)}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx} \frac{\int_{D_h \setminus \tilde{D}_h} u_h^p dx}{h}. \end{aligned}$$

Since $h > 0$, we have $g(\alpha + h) < g(\alpha)$. Moreover, as $h \to 0$, u_h converges to the eigenfunction $u = u_D$ in the $L^p(\Omega)$ norm, and χ_{D_h} converges to χ_D in the weak* topology of $L^\infty(\Omega)$. Since

$$\int_\Omega \chi_{\tilde{D}_h} u_h^p dx - \int_\Omega \chi_D u^p dx = -\int_\Omega (\chi_{D_h} - \chi_{\tilde{D}_h}) u_h^p dx + \int_\Omega \chi_{D_h}(u_h^p - u^p) dx + \int_\Omega (\chi_{D_h} - \chi_D) u^p dx,$$

it follows that

$$\lim_{h\to 0^+} \int_\Omega \chi_{\tilde{D}_h} u_h^p dx = \int_\Omega \chi_D u^p dx.$$

Furthermore, since $\tilde{D}_h \subset D_h$ and $|D_h \setminus \tilde{D}_h| = h$, we have

$$\frac{\int_{D_h \setminus \tilde{D}_h} u_h^p dx}{h} \le \sup_{D_h \setminus \tilde{D}_h} u_h^p \le \sup_\Omega u_h^p.$$

Observe that, by regularity (see [4,5]), all eigenfunctions are Hölder continuous in Ω. Since $\alpha/2 < |D_h| < |\Omega|$, we may also assume that the Hölder constants are independent of h. Therefore, by Ascoli–Arzelá Theorem, a sequence u_{h_i} converges to u uniformly. It follows that

$$\limsup_{h\to 0^+} \frac{\int_{D_h \setminus \tilde{D}_h} u_h^p dx}{h} \le \sup_\Omega u^p.$$

Hence,

$$\limsup_{h\to 0^+}\left(-\frac{g(\alpha+h)-g(\alpha)}{h}\right) \le \frac{g(\alpha)}{\int_\Omega \chi_D u^p dx} \sup_\Omega u^p. \tag{7}$$

To prove the reverse inequality we observe that, by Eq. (1) we have $\Delta_p u = 0$ in $\Omega \setminus D$. This implies that $u(x)$ is a constant on $\Omega \setminus D$. More precisely, on $\Omega \setminus D$ we have

$$u(x) = u(x)|_{\partial(\Omega \setminus D)} = \sup_\Omega u(x).$$

Now, if $\tau = \sup_\Omega u(x)$, take $\bar{D}_h$ such that

$$\{u(x) < \tau\} \subset \bar{D}_h \subset \{u(x) \le \tau\}, \quad |\bar{D}_h| = \alpha + h.$$

By (6) and (5) (with $\alpha + h$ in place of α) we have

$$g(\alpha+h) = \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{D_h} u_h^p dx} \le \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_{\bar{D}_h} u^p dx}.$$

Therefore,

$$\begin{aligned} g(\alpha) - g(\alpha+h) &\ge \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_D u^p dx} - \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_{\bar{D}_h} u^p dx} \\ &= \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_D u^p dx \int_\Omega \chi_{\bar{D}_h} u^p dx} \frac{\int_{\bar{D}_h \setminus D} u^p dx}{h} = \frac{g(\alpha)}{\int_\Omega \chi_{\bar{D}_h} u^p dx} \frac{\int_{\bar{D}_h \setminus D} u^p dx}{h}. \end{aligned}$$

Since u is a constant on $\bar{D}_h \setminus D$ and $|\bar{D}_h \setminus D| = h$, we have

$$\frac{\int_{\bar{D}_h \setminus D} u^p dx}{h} = \sup_{\Omega} u^p.$$

Hence

$$\liminf_{h\to 0^+}\Big(-\frac{g(\alpha+h)-g(\alpha)}{h}\Big) \geq \frac{g(\alpha)}{\int_\Omega \chi_D u^p dx} \sup_\Omega u^p. \tag{8}$$

From (7) and (8) we find

$$\lim_{h\to 0^+}\Big(-\frac{g(\alpha+h)-g(\alpha)}{h}\Big) = \frac{g(\alpha)}{\int_\Omega \chi_D u^p dx} \sup_\Omega u^p. \tag{9}$$

Now let $h < 0$ such that $\alpha + h > 0$. We know that there are D_h with $|D_h| = \alpha + h$ and $u_h = u_{D_h} > 0$ such that

$$g(\alpha+h) = \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{D_h} u_h^p dx},$$

and

$$D_h = \{x \in \Omega : u_h(x) < t\}$$

for $t = \sup_\Omega u_h(x)$. Let us take $\tilde{D}_h \supset D_h$ with $|\tilde{D}_h| = \alpha$. Note that

$$\{u_h(x) < t\} \subset \tilde{D}_h \subset \{u_h(x) \leq t\}.$$

Using (5) (with u_h in place of v), we find

$$g(\alpha) = \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_D u^p dx} \leq \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx}.$$

Hence,

$$\begin{aligned} g(\alpha) - g(\alpha+h) &\leq \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx} - \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{D_h} u_h^p dx} \\ &= -\frac{g(\alpha+h)}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx} \int_{\tilde{D}_h \setminus D_h} u_h^p dx. \end{aligned}$$

Since $|\tilde{D}_h \setminus D_h| = |h|$ and since u_h is a constant outside D_h, it follows that

$$\frac{g(\alpha)-g(\alpha+h)}{h} \geq \frac{g(\alpha+h)}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx} \frac{\int_{\tilde{D}_h \setminus D_h} u_h^p dx}{|h|} = \frac{g(\alpha+h)}{\int_\Omega \chi_{\tilde{D}_h} u_h^p dx} \sup_\Omega u_h^p.$$

Since $h < 0$, we have $g(\alpha+h) > g(\alpha)$, and

$$\liminf_{h\to 0^-}\Big(-\frac{g(\alpha+h)-g(\alpha)}{h}\Big) \geq \frac{g(\alpha)}{\int_\Omega \chi_D u^p dx} \sup_\Omega u^p. \tag{10}$$

On the other hand, since u has not flat zones on D, we can find τ such that

$$\bar{D}_h = \{x \in \Omega : u(x) < \tau\}, \quad |\bar{D}_h| = \alpha + h.$$

In this situation, the sets $\bar{D}_h$ and $\{x \in \Omega : u(x) \leq \tau\}$ have the same measure. By (5) (with $\alpha + h$ in place of α), we have

$$g(\alpha+h) = \frac{\int_\Omega |\nabla u_h|^p dx}{\int_\Omega \chi_{D_h} u_h^p dx} \leq \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_{\bar{D}_h} u^p dx}.$$

Since $\bar{D}_h \subset D$, we find

$$\begin{aligned} g(\alpha) - g(\alpha+h) &\geq \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_D u^p dx} - \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \chi_{\bar{D}_h} u^p dx} \\ &= -\frac{g(\alpha)}{\int_\Omega \chi_{\bar{D}_h} u^p dx} \int_{D \setminus \bar{D}_h} u^p dx. \end{aligned}$$

Finally, since $h < 0$, we have

$$\frac{g(\alpha) - g(\alpha + h)}{h} \leq \frac{g(\alpha)}{\int_\Omega \chi_{\bar{D}_h} u^p dx} \frac{\int_{D \setminus \bar{D}_h} u^p dx}{|h|}.$$

Hence,

$$\limsup_{h \to 0^-} \Big(-\frac{g(\alpha + h) - g(\alpha)}{h} \Big) \leq \frac{g(\alpha)}{\int_\Omega \chi_D u^p dx} \sup_\Omega u^p. \tag{11}$$

From (10) and (11) we find

$$\lim_{h \to 0^-} \Big(-\frac{g(\alpha + h) - g(\alpha)}{h} \Big) = \frac{g(\alpha)}{\int_\Omega \chi_D u^p dx} \sup_\Omega u^p. \tag{12}$$

From (9) and (12) it follows that $g(\alpha)$ is differentiable and

$$-g'(\alpha) = \frac{g(\alpha)}{\int_\Omega \chi_D u^p dx} \sup_\Omega u^p.$$

The proof of the theorem is completed. □

Remarks. (1) A similar proof for the differentiability of $f(\alpha)$ fails to hold in general. For example, in the case where Ω is a dumbbell (see [9]), for a particular value of α we have two different sets D_1 and D_2 such that $|D_1| = |D_2| = \alpha$ and $\lambda_{D_1} = \lambda_{D_2} = f(\alpha)$. In this situation, a sequence of domains D_{h_i} corresponding to $\lambda_{D_{h_i}}$, as $i \to \infty$ may converge to D_1, whereas, another sequence D_{h_j} corresponding to $\lambda_{D_{h_j}}$, as $j \to \infty$ may converge to D_2. In this case our method does not guarantee the differentiability of $f(\alpha)$. We think that differentiability occurs for domains Ω where uniqueness of a minimizer holds for any $\alpha \in (0, |\Omega|)$.

(2) The results for maximization and minimization are different. Indeed, for maximization we always have uniqueness, whereas, for minimization we may have different solutions (as in the case of a dumbbell). We think that uniqueness of a minimizer occurs for convex domains, but we do not have a proof. In the case $p = 2$ and where Ω is Steiner symmetric, in [9] it is shown that any minimizer $\hat{D}$ is symmetric. This implies uniqueness of the minimizer when $\Omega = B$ is a ball. Therefore, in the latter case, the minimizer $\hat{D}$ is a ball concentric with B, whereas, the maximizer $\check{D}$ is the annulus whose exterior boundary coincides with the boundary of B.

(3) From the definition of principal eigenvalue, it follows that $\lambda_D \to \infty$ as $|D| \to 0$. Therefore, $f(\alpha)$ and $g(\alpha)$ diverge to ∞ as $\alpha \to 0$.

Appendix

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. We say that two measurable functions $f(x)$ and $g(x)$ defined in Ω have the same rearrangement if

$$|\{x \in \Omega : f(x) \geq \beta\}| = |\{x \in \Omega : g(x) \geq \beta\}| \quad \forall \beta \in \mathbb{R}.$$

If $g_0(x) \geq 0$ is a bounded function defined in Ω, we denote by $\mathcal{G} = \mathcal{G}(g_0)$ the class of its rearrangements. We assume $g_0(x) > 0$ in a subset of positive measure, and suppose $g_0 \neq$ *constant*. Let $\overline{\mathcal{G}}$ be the closure of $\mathcal{G}$ in the weak* topology of $L^\infty(\Omega)$. We make use of the following result on minimization of a linear functional.

Lemma A.1. *Let $\check{g} \in \overline{\mathcal{G}}$, and let u be a measurable function that has not flat zones in the set $F = \{x \in \Omega : \check{g}(x) > 0\}$. If*

$$\int_\Omega g\, u\, dx \geq \int_\Omega \check{g}\, u\, dx \quad \forall g \in \overline{\mathcal{G}},$$

then, $\check{g} \in \mathcal{G}$, and there is a decreasing function $\psi(t)$ such that $\check{g} = \psi(u)$.

Proof. We refer to [14], proof of Theorem 2.2, part II. The proof uses ideas from [15–17]. □

For $1 < p < \infty$ and $g \in \overline{\mathcal{G}}$, we consider the eigenvalue problem

$$-\Delta_p u = \lambda g(x) u^{p-1}, \quad u > 0 \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega. \tag{13}$$

Here λ is the principal eigenvalue, which depends on Ω, p and g. In what follows, Ω and p will be fixed, whereas, the function g may change, therefore we shall write $\lambda = \lambda_g$. It is well known that

$$\lambda_g = \inf \left\{ \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega g |v|^p dx} : \; v \in H_0^{1,p}(\Omega), \; \int_\Omega g |v|^p dx > 0 \right\} = \frac{\int_\Omega |\nabla u_g|^p dx}{\int_\Omega g u_g^p dx},$$

where $u_g \in H_0^{1,p}(\Omega)$ is the principal eigenfunction and $u_g(x) > 0$ (see [1]). Fix g_0 and define the corresponding class $\mathcal{G} = \mathcal{G}(g_0)$. Consider the problem

$$\sup_{g\in\mathcal{G}} \lambda_g = \sup_{g\in\mathcal{G}} \frac{\int_\Omega |\nabla u_g|^p dx}{\int_\Omega g u_g^p dx}. \tag{14}$$

Note that

$$\sup_{g\in\mathcal{G}} \lambda_g = \sup_{g\in\mathcal{G}} \inf_{v\in H_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega g|v|^p dx},$$

and that we cannot change, in general, the order of $\sup_{g\in\mathcal{G}}$ with $\inf_{v\in H_0^{1,p}(\Omega)}$. In the case $p = 2$ problem (14) is discussed in [7], but the method used there does not seem to work for general p. We use here a different method inspired by that used in [14] in a different situation.

For $g \in \overline{\mathcal{G}}$, we define

$$J(g) = \frac{1}{\lambda_g}.$$

We have

$$J(g) = \sup_{v\in H_0^{1,p}(\Omega)} \frac{\int_\Omega g|v|^p dx}{\int_\Omega |\nabla v|^p dx} = \frac{\int_\Omega g u_g^p dx}{\int_\Omega |\nabla u_g|^p dx}. \tag{15}$$

Note that problem (14) is equivalent to problem

$$\inf_{g\in\mathcal{G}} J(g). \tag{16}$$

Theorem A.2. *Let g_0 be a non negative bounded function. Let $\mathcal{G}$ be the class of rearrangements generated by g_0, and let $J(g)$ be defined as in (15). Then, problem (16) has a unique solution $\check{g}$; furthermore, if $u_{\check{g}} > 0$ is a corresponding eigenfunction then $\check{g} = \psi\left(u_{\check{g}}\right)$ for some decreasing function $\psi(t)$.*

Proof. By Lemma 4.2 of [8], the map $g \to \lambda_g$ is continuous. Equivalently, the map $g \to J(g)$ is continuous. Let us show that this map in Gateaux differentiable. If $g,\ g_i \in \overline{\mathcal{G}}$ we have

$$\begin{aligned} J(g) + \frac{\int_\Omega (g_i - g) u_g^p\, dx}{\int_\Omega |\nabla u_g|^p dx} &= \frac{\int_\Omega g_i u_g^p\, dx}{\int_\Omega |\nabla u_g|^p dx} \le J(g_i) = \frac{\int_\Omega g_i u_{g_i}^p\, dx}{\int_\Omega |\nabla u_{g_i}|^p dx} \\ &= \frac{\int_\Omega g u_{g_i}^p\, dx}{\int_\Omega |\nabla u_{g_i}|^p dx} + \frac{\int_\Omega (g_i - g) u_{g_i}^p\, dx}{\int_\Omega |\nabla u_{g_i}|^p dx} \le J(g) + \frac{\int_\Omega (g_i - g) u_{g_i}^p\, dx}{\int_\Omega |\nabla u_{g_i}|^p dx}. \end{aligned} \tag{17}$$

Let $t_i > 0$ be a sequence such that $t_i \to 0$ as $i \to \infty$. Let $g, h \in \overline{\mathcal{G}}$ and let $g_i = g + t_i(h - g)$. Then, by (17) we find

$$\begin{aligned} J(g) + t_i \frac{\int_\Omega (h - g) u_g^p\, dx}{\int_\Omega |\nabla u_g|^p dx} &\le J(g_i) \\ &\le J(g) + t_i \frac{\int_\Omega (h - g) u_{g_i}^p\, dx}{\int_\Omega |\nabla u_{g_i}|^p dx}. \end{aligned} \tag{18}$$

Recall that we are using the normalization $\int_\Omega u_{g_i}^p\, dx = 1$. Since $t_i \to 0$ as $i \to \infty$, we have $g_i \to g$ in the norm of $L^\infty(\Omega)$. As a consequence, the sequence u_{g_i} is bounded in the norm of $H^{1,p}(\Omega)$, and a subsequence (denoted again by u_{g_i}) converges in the norm of $L^p(\Omega)$ to some function $z \in H_0^{1,p}(\Omega)$. We find

$$\liminf_{i\to\infty} \lambda_{g_i} = \liminf_{i\to\infty} \frac{\int_\Omega |\nabla u_{g_i}|^p dx}{\int_\Omega g_i u_{g_i}^p\, dx} \ge \frac{\int_\Omega |\nabla z|^p dx}{\int_\Omega g\, z^p\, dx} \ge \lambda_g.$$

Since, by continuity,

$$\lim_{i\to\infty} \lambda_{g_i} = \lambda_g,$$

we must have $z = u_g$ and $u_{g_i} \to u_g$ in the norm of $H^{1,p}(\Omega)$. Therefore, from (18) we get

$$\lim_{t\to 0^+} \frac{J(g + t(h - g)) - J(g)}{t} = \frac{\int_\Omega (h - g) u_g^p\, dx}{\int_\Omega |\nabla u_g|^p dx}. \tag{19}$$

To discuss problem (16), we first consider the problem

$$\inf_{g\in\overline{\mathcal{G}}} J(g),$$

where $\overline{\mathcal{G}}$ is the closure of $\mathcal{G}$ with respect to the weak* topology of $L^\infty(\Omega)$. By continuity of $J(g)$ (see Lemma 4.2 of [8]) and compactness of $\overline{\mathcal{G}}$, the latter problem has a solution $\check{g} \in \overline{\mathcal{G}}$. By strict convexity of $J(g)$ (see Lemma 4.1 of [8]), this solution is unique. Let us prove that $\check{g} \in \mathcal{G}$.

If $0 < t < 1$ and if $g_t = \check{g} + t(g - \check{g})$, by the minimality of $\check{g}$ and by (19) with $g = \check{g}$ and $h = g$, we have

$$J(\check{g}) \le J(g_t) = J(\check{g}) + t\frac{\int_\Omega (g-\check{g})u_{\check{g}}^p\,dx}{\int_\Omega |\nabla u_{\check{g}}|^p dx} + o(t) \quad \text{as } t \to 0^+.$$

Hence,

$$\int_\Omega (g-\check{g})u_{\check{g}}^p dx \ge 0.$$

Equivalently,

$$\int_\Omega g\, u_{\check{g}}^p dx \ge \int_\Omega \check{g}\, u_{\check{g}}^p dx \quad \forall g \in \overline{\mathcal{G}}.$$

On the other hand, by the equation

$$-\Delta_p u_{\check{g}} = \lambda \check{g}\, u_{\check{g}}^{p-1},$$

it follows that the function $u_{\check{g}}$ cannot have flat zones in the set

$$F = \{x \in \Omega : \check{g}(x) > 0\}.$$

The theorem follows now from Lemma A.1. $\square$

Remarks. (1) In the case $g_0 = \chi_D$, the function ψ of Theorem A.2 is $\psi(s) = H(\tau - s)$, where H is the Heaviside function and τ is the superior of $u_{\check{D}}(x)$ in Ω. It follows that $\check{g} = \chi_{\check{D}}$ with

$$\check{D} = \{x \in \Omega : u_{\check{D}}(x) < \tau\}.$$

(2) A different approach to prove Theorem A.2 could be the use of the following functional

$$A(g, v) = p\int_\Omega g|v|^p dx - \left(\int_\Omega |\nabla v|^p dx\right)^2.$$

This functional is inspired by Auchmuty [18]. For $t > 0$ we have

$$A(g, tv) = t^p p\int_\Omega g|v|^p dx - t^{2p}\left(\int_\Omega |\nabla v|^p dx\right)^2 \le A(g, t_0 v),$$

with

$$t_0^p = \frac{p}{2}\frac{\int_\Omega g|v|^p dx}{\left(\int_\Omega |\nabla v|^p dx\right)^2}. \tag{20}$$

It follows that

$$A(g, tv) \le \frac{p^2}{4}\left(\frac{\int_\Omega g|v|^p dx}{\int_\Omega |\nabla v|^p dx}\right)^2,$$

and

$$\sup_{v\in H_0^{1,p}(\Omega)} A(g, v) = \frac{p^2}{4}\frac{1}{\lambda_g^2}.$$

Note that a maximizer $u > 0$ of $v \mapsto A(g, v)$ is normalized so that $t_0 = 1$ in (20), that is,

$$\frac{p}{2}\int_\Omega g u^p dx = \left(\int_\Omega |\nabla u|^p dx\right)^2.$$

References

[1] B. Kawohl, M. Lucia, S. Prashanth, Simplicity of the first eigenvalue for indefinite quasilinear problems, Adv. Differential Equations 12 (2007) 407–434.
[2] P. Lindqvist, On the equation div($|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc. 109 (1990) 157–164;
P. Lindqvist, On the equation div($|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc. 116 (1992) 583–584 (addendum).
[3] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Clarendon Press, Oxford, New York, Tokyo, 1993.
[4] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations 51 (1984) 126–150.
[5] N.S. Trudinger, On Harnack type inequalities and their applications to quasilinear elliptic equations, Comm. Pure Appl. Math. XX (1967) 721–747.
[6] S.J. Cox, J.R. McLaughlin, Extremal eigenvalue problems for composite membranes, I, Appl. Math. Optim. 22 (1990) 153–167.
[7] S.J. Cox, J.R. McLaughlin, Extremal eigenvalue problems for composite membranes, II, Appl. Math. Optim. 22 (1990) 169–187.
[8] F. Cuccu, B. Emamizadeh, G. Porru, Optimization of the first eigenvalue in problems involving the p-Laplacian, Proc. Amer. Math. Soc. 137 (2009) 1677–1687.
[9] S. Chanillo, D. Grieser, M. Imai, K. Kurata, I. Ohnishi, Symmetry breaking and other phenomena in the optimization of eigenvalues for composite membranes, Comm. Math. Phys. 214 (2000) 315–337.
[10] D. Bucur, G. Buttazzo, Variational Methods in Shape Optimization Problems, in: Progress in Nonlinear Differential Equations, vol. 65, Birkhäuser Verlag, Basel, 2005.
[11] G. Buttazzo, Spectral optimization problems, Rev. Mat. Complut. 24 (2) (2011) 277–322.
[12] A. Henrot, Extremum Problems for Eigenvalues of Elliptic Operators, in: Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006.
[13] A. Henrot, M. Pierre, Variation et optimisation de formes, in: Une Analyse Géométrique, in: Mathématiques & Applications, vol. 48, Springer-Verlag, Berlin, 2005.
[14] F. Cuccu, G. Porru, S. Sakaguchi, Optimization problems on general classes of rearrangements, Nonlinear Anal. 74 (2011) 5554–5565.
[15] G.R. Burton, Rearrangements of functions, maximization of convex functionals and vortex rings, Math. Ann. 276 (1987) 225–253.
[16] G.R. Burton, Variational problems on classes of rearrangements and multiple configurations for steady vortices, Ann. Inst. Henri Poincarè 6 (1989) 295–319.
[17] G.R. Burton, J.B. McLeod, Maximisation and minimisation on classes of rearrangements, Proc. Roy. Soc. Edinburgh Sect. A 119 (1991) 287–300.
[18] G. Auchmuty, Dual principles for eigenvalue problems, in: F. Browder (Ed.), Nonlinear Functional Analysis and its Applications, A.M.S, Providence, RI, 1986, pp. 55–71.