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Strong duality for generalized monotropic programming in infinite dimensions

Regina S. Burachik, Saba N. Majeed *

School of Mathematics and Statistics, University of South Australia, Mawson Lakes, SA 5095, Australia

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ABSTRACT

We establish duality results for the generalized monotropic programming problem in separated locally convex spaces. We formulate the generalized monotropic programming (GMP) as the minimization of a (possibly infinite) sum of separable proper convex functions, restricted to a closed and convex cone. We obtain strong duality under a constraint qualification based on the closedness of the sum of the epigraphs of the conjugates of the convex functions. When the objective function is the sum of finitely many proper closed convex functions, we consider two types of constraint qualifications, both of which extend those introduced in the literature. The first constraint qualification ensures strong duality, and is equivalent to the one introduced by Boţ and Wanka. The second constraint qualification is an extension of Bertsekas' constraint qualification and we use it to prove zero duality gap.

1. Introduction

The Monotropic Programming (MP) problem was introduced by Rockafellar in [1] and has been widely studied (cf. [2–4]). In its classical form, MP is defined in a finite dimensional setting and it involves minimizing a finite sum of proper and convex separable functions restricted to a closed subspace. We focus on a generalization of this problem, the generalized monotropic programming (GMP), which consists of minimizing a (finite or infinite) sum of proper and convex functions defined on (possibly different) locally convex spaces. To define the problem, denote $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and let I be an arbitrary index set. Consider a family of real separated locally convex spaces $\{X_i\}_{i\in I}$, and a family $\{f_i\}_{i\in I}$ of proper and convex functions such that $f_i : X_i \to \overline{\mathbb{R}}$ for all $i \in I$. Take $X := \prod_{i\in I} X_i$ and consider the sum of $\{f_i\}_{i\in I}$, defined as $f : X \to \overline{\mathbb{R}}$ such that $f(x) := \sum_{i\in I} f_i(x_i)$. The meaning of the right-hand side of the last expression in the case in which I is infinite is recalled later on in Definition 2.2. The GMP problem we study is as follows:

$$\begin{aligned} &\min \sum_{i\in I} f_i(x_i) \\ &\text{subject to } x \in K, \end{aligned} \tag{P}$$

where $x_i \in X_i$ for all $i \in I$, and the constraint set K is a closed and convex cone contained in $X = \prod_{i\in I} X_i$. As far as we know, the only work dealing with an infinite sum of convex functions is [5]. The functions in [5] are defined in Banach spaces, however, the definition and the properties of the infinite sums mentioned in [5] can be stated for functions defined on separated locally convex spaces.

Following [6,3], we say that strong duality holds when the optimal primal and dual values coincide, and the dual value is attained. If we only have equality of the primal and dual optimal values, we say that we have zero duality gap. Rockafellar

* Corresponding author.
E-mail addresses: regina.burachik@unisa.edu.au (R.S. Burachik), saba.majeed@mymail.unisa.edu.au, majsy001@mymail.unisa.edu.au (S.N. Majeed).

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[1,4] was the first to use a variant of the ϵ-descent method to prove zero duality gap for the MP problem. More recently, Bertsekas [7] has modified Rockafellar's method and applied it for solving the extended MP problem. The latter problem has for the objective function a finite sum of extended real-valued functions which can have domains in different finite dimensional spaces [7], and use a subspace S as a constraint set. To obtain zero duality gap in this context, Bertsekas used projections on an outer approximation of the ϵ-subdifferential and used a constraint qualification involving the closedness of the Minkowski sum of ϵ-subdifferentials. Bertsekas' constraint qualification (cf. [7, Proposition 4.1]) requires that the set

$$A_\epsilon(x) = \partial_\epsilon \delta_S(x) + \partial_\epsilon \bar{f}_1(x) + \cdots + \partial_\epsilon \bar{f}_m(x) \tag{1}$$

is closed for all feasible solutions $x = (x_1, \ldots, x_m)$ and every $\epsilon > 0$, where $\bar{f}_i(x) := f_i(x_i)$ for each $i = 1, \ldots, m$ (for the definition of $\bar{f}_i$, see Remark 3.1(1)).

In locally convex spaces, Boţ and Csetnek [3] proved zero duality gap for the extended MP problem under alternative assumptions. Boţ and Csetnek used in [3] an extension to separated locally convex spaces of Bertsekas' constraint qualification (1). Our purpose is to study strong duality for our general version GMP of the MP. We obtain strong duality under new constraint qualifications (see Theorems 3.4 and 3.5). In Theorem 3.4 we prove that, when I is finite and the constraint set K is nonempty closed and convex, strong duality for the Problem (P) holds if the set

$$epi\delta_K^* + epi\bar{f}_1^* + \cdots + epi\bar{f}_m^* \tag{2}$$

is $weak^*$ closed. The $epi\bar{f}_i^*$ is the epigraph of the conjugate function of $\bar{f}_i$ defined above. We also show, in Theorem 3.4, that the above constraint qualification is equivalent to the ones used in [8, p. 2798], [6, Theorem 3.2.6] for the case of locally convex spaces and [9, Corollary 3] for the case of Banach spaces, to obtain generalized Fenchel's duality. Namely, it is equivalent to the $weak^*$ closedness of the set $epif^* + epig^*$ in case $f(x)$ is defined as in Problem (P) and $g(x) = \delta_K(x)$. Still for the finite sum, we use in Theorem 3.6, an extension of Bertsekas' constraint qualification (1) to obtain zero duality gap. This constraint qualification requires that the set

$$A_\epsilon(x) = \partial_\epsilon \delta_K(x) + \partial_\epsilon \bar{f}_1(x) + \cdots + \partial_\epsilon \bar{f}_m(x) \tag{3}$$

is $weak^*$ closed for all feasible solutions $x = (x_1, \ldots, x_m)$ and every $\epsilon > 0$, where $\partial_\epsilon \bar{f}_i(x)$ is the ϵ-subdifferential of $\bar{f}_i$ at x. Note that the constraint subspace S used in (1) has been replaced in (3) by any closed convex cone K.

Theorem 3.5 considers the case of I infinite and an arbitrary closed and convex constraint set K. For this case we prove strong duality when

$$epi\delta_K^* + \sum_{i\in I}^{*} epi\bar{f}_i^* \tag{4a}$$

is $weak^*$ closed, and

$$epif^* = \sum_{i\in I}^{*} epi\bar{f}_i^*. \tag{4b}$$

This constraint qualification, which is new in the literature, involves the $weak^*$ closedness of the sum of the epigraphs of the conjugate functions and an additional condition on the summability of the epigraph of the conjugate of the infinite sum. Corollary 3.1 describes conditions under which the constraint qualification (4a) is enough to ensure strong duality.

The outline of the paper is as follows. In Section 2, we review the necessary definitions and preliminary results. In Section 3, we introduce the GMP problem and its dual in a separated locally convex space and introduce our new constraint qualifications (2) and (4a)–(4b) to obtain strong duality for the GMP problem. Still in Section 3, we introduce the constraint qualification (3) to show zero duality gap which generalizes the one used in [7]. We end Section 3 with an example illustrating the fact that our new constraint qualification (2) is not weaker than (3). Section 4 contains our conclusions. Some needed technical facts from real analysis are proved in the Appendix.

2. Preliminaries

We collect in this section some definitions and properties from convex analysis which can be found e.g., in [6,10,11]. Let X denote a locally convex space, and X^* its topological dual space endowed with the $weak^*$ topology $w^*(X^*, X)$. If D is a subset of X^*, the *$weak^*$ closure* of D will be denoted by $\overline{D}^{w^*}$. Let C be a non empty subset of X. The *indicator function* associated with the set C, $\delta_C : X \to \overline{\mathbb{R}}$ is defined by

$$\delta_C(x) := \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

The *support function* $\sigma_C : X^* \to \overline{\mathbb{R}}$ is defined by $\sigma_C(v) = \sup\{\langle v, x\rangle : x \in C\}$, where $\langle \cdot, \cdot\rangle$ is the *duality product* in $X^* \times X$. Recall that, given $\epsilon \geq 0$, the *ϵ-normal set* of C, which is denoted by $N_C^\epsilon(x)$, is defined as

$$N_C^\epsilon(x) := \begin{cases} \{x^* \in X^* : \langle x^*, y - x\rangle \leq \epsilon, \forall y \in C\} & \text{if } x \in C, \\ \phi & \text{otherwise.} \end{cases} \tag{4}$$

A subset K of X is called a closed *convex cone* if K is a closed and convex set such that for every $x \in K$ and $\lambda \geq 0$ it holds that $\lambda x \in K$. The *polar* convex cone of K is defined as $K^\circ = \{x^* \in X^* \mid \langle x, x^* \rangle \leq 0 \, \forall x \in K\}$, and the *dual* cone will be denoted by $K^* := -K^\circ = \{x^* \in X^* \mid \langle x, x^* \rangle \geq 0 \, \forall x \in K\}$. A convex function $f : X \to \overline{\mathbb{R}}$ is said to be *proper* if $f(x) > -\infty$ for every $x \in X$, and the set $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\} \neq \emptyset$. The set $\operatorname{dom} f$ is called the *domain* of f. The *epigraph* of f is defined as $epi f := \{(x, \alpha) \in X \times \mathbb{R} \mid f(x) \leq \alpha\}$. The convex function f is called *lower semi continuous* (lsc) at x if and only if

$$f(x) = \liminf_{y \to x} f(y) = \lim_{\epsilon \to 0} \inf\{f(y) \mid |y - x| \leq \epsilon\}.$$

The function $clf : X \to \overline{\mathbb{R}}$ is called the *lower semicontinuous hull of* f, and is defined as $(clf)(x) := \inf\{t : (x, t) \in cl(epif)\}$. If f is lsc at every point in the space then f is called *closed*, this fact is denoted as $clf = f$. The function $f^* : X^* \to \overline{\mathbb{R}}$, defined as $f^*(v) := \sup\{\langle v, x \rangle - f(x) \mid x \in X\}$ is called the *conjugate function* of f. In addition, if a convex function f is proper then $f^*(v) > -\infty \quad \forall v \in X^*$. The *subdifferential* of f at $x \in X$ is the point-to-set mapping $\partial f : X \rightrightarrows X^*$ defined as

$$\partial f(x) := \{v \in X^* \mid f(y) \geq f(x) + \langle v, y - x \rangle \text{ for all } y \in X\},$$

and the ϵ-*subdifferential* of f is defined as

$$\partial_\epsilon f(x) := \{v \in X^* \mid f(y) \geq f(x) + \langle v, y - x \rangle - \epsilon \text{ for all } y \in X\},$$

for all $x \in X$ and all $\epsilon \geq 0$. The definition of $\partial_\epsilon f(x)$ entails that, for every $x \in \operatorname{dom} f$ and every $\epsilon \geq 0$, $v \in \partial_\epsilon f(x)$, if and only if $f^*(v) + f(x) - \langle v, x \rangle \leq \epsilon$. If $x \in \operatorname{dom} f$ and $\epsilon \geq 0$, $\partial_\epsilon f(x)$ is *weak** closed and convex (see [10]). If f is a proper and lsc convex function, $\partial_\epsilon f(x) \neq \phi$ for all $\epsilon > 0$ and $x \in \operatorname{dom} f$ (see [10]). If f is a proper and lsc convex function, $x \in \operatorname{dom} f$ and $\epsilon > 0$, the *support function* of $\partial_\epsilon f(x)$ is given by the formula

$$\sigma[y|\partial_\epsilon f(x)] := \sup\{\langle v, y \rangle \mid v \in \partial_\epsilon f(x)\} = \inf_{\alpha > 0} \frac{f(x + \alpha y) - f(x) + \epsilon}{\alpha}, \tag{5}$$

(see [10]). Recall that for a non-empty convex subset C of X we have

$$\partial_\epsilon \delta_C(x) = N_C^\epsilon(x), \tag{6}$$

where $N_C^\epsilon(x)$ is defined in (4). When C is a subspace it is easy to show that

$$N_C(x) = \partial \delta_C(x) = \partial_\epsilon \delta_C(x) = C^\perp, \quad \forall x \in C, \forall \epsilon \geq 0$$

where $C^\perp := \{y \in X : \langle x, y \rangle = 0 \, \forall x \in C\}$ is the orthogonal subspace of C. The infimal convolution of two proper convex functions $f, g : X \to \overline{\mathbb{R}}$ is denoted by $f \oplus g$ and defined as $(f \oplus g)(x) = \inf_{x = x_1 + x_2}\{f(x_1) + f(x_2)\}$. The infimal convolution is called *exact* if the infimum is achieved for every $x \in X$ [11]. For finite proper convex functions $f_1, \ldots, f_m$, the infimal convolution is defined as $(f_1 \oplus \cdots \oplus f_m)(x) = \inf\{f_1(x_1) + \cdots + f_m(x_m) \mid x = x_1 + \cdots + x_m\}$. We recall now the following useful result.

Lemma 2.1 ([6, *Theorem 2.3.10, Proposition 2.3.9*]). *Let $f_i : X \to \overline{\mathbb{R}}$, $\forall i = 1, \ldots, m$ be proper and closed convex functions such that $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \phi$. Then*

(1)

$$epi\left(\sum_{i=1}^m f_i\right)^* = cl\left(\sum_{i=1}^m epif_i^*\right). \tag{7}$$

(2) *The following statements are equivalent:*
 (i) $epi(\sum_{i=1}^m f_i)^* = \sum_{i=1}^m epif_i^*$
 (ii) $(\sum_{i=1}^m f_i)^* = f_1^* \oplus \cdots \oplus f_m^*$ *and the infimal convolution is exact.*

The ϵ-descent direction and the ϵ-descent method are main tools for obtaining zero duality gap for the extended MP problem in [7]. We will use them in our analysis. To the best of our knowledge, the ϵ-descent method is defined in a finite dimensional setting. For references on the ϵ-descent method, see [12,2].

Definition 2.1. Let $f : X \to \overline{\mathbb{R}}$ be a function. A vector y is called an ϵ-*descent direction* at $x \in \operatorname{dom} f$ if and only if

$$\inf_{\alpha > 0} f(x + \alpha y) < f(x) - \epsilon. \tag{8}$$

Combining the expression above with (5), we conclude that y is an ϵ-descent direction at x if and only if $\sup_{v \in \partial_\epsilon f(x)} \langle v, y \rangle < 0$.

The ϵ-descent method is defined as follows. Select a vector $x_0 \in \operatorname{dom} f$ and generate a sequence $\{x_k\} \subset \operatorname{dom} f$ such that $x_{k+1} = x_k + \alpha_k y_k$, where y_k is an ϵ-descent direction (if one can be found) at x_k and α_k is a positive stepsize such that

$$f(x_k + \alpha_k y_k) < f(x_k) - \epsilon. \tag{9}$$

From Definition 2.1, α_k as above exists if an ϵ-descent direction y_k can be found. The ϵ-descent method stops if and only if $0 \in \partial_\epsilon f(x)$. Indeed, $0 \in \partial_\epsilon f(x)$ if and only if $f(x + \alpha y) \geq f(x) - \epsilon\ \forall y \in X, \forall \alpha \geq 0$ and the latter inequality yields $\inf_{\alpha>0}(x + \alpha y) \geq f(x) - \epsilon\ \forall y \in X$. By (8), no ϵ-descent direction can be obtained and the method has to stop. We will use this method in Theorem 3.6.

We mentioned above that the index set I may be infinite. For analyzing this situation we recall the relevant definitions, taken from [5].

Definition 2.2. Let I be an index set, and let $\mathcal{F}(I) = \{J \subseteq I \mid J \text{ is finite}\}$. Note that $\mathcal{F}(I)$ is a directed set ordered by the inclusion relation.

(1) Let $\{w_i \mid i \in I\} \subseteq \mathbb{R} \cup \{+\infty\}$. We define *limit inferior* (liminf) and *limit superior* (limsup) of $\sum_{i\in I} w_i$, respectively, as follows.

$$\liminf_{i\to+\infty} \sum_{i\in I} w_i = \sup_{n\in N} \left(\inf_{\substack{|A|\leq n\\ A\in\mathcal{F}(I)}} \sum_{i\in A} w_i \right),$$

and

$$\limsup_{i\to+\infty} \sum_{i\in I} w_i = \inf_{n\in N} \left(\sup_{\substack{|A|\leq n\\ A\in\mathcal{F}(I)}} \sum_{i\in A} w_i \right).$$

If the limits above exist and coincide, we denote the common limit by

$$\lim_{\substack{|A|\to+\infty\\ A\in\mathcal{F}(I)}} \sum_{i\in A} w_i = \sum_{i\in I} w_i, \tag{10}$$

where $\sum_{i\in I} w_i$ belongs to $\mathbb{R} \cup \{+\infty\}$. In this case, we say that the sum of $\{w_i\}_{i\in I}$ is convergent.

(2) Let $\{f_i \mid i \in I\}$ be a family of extended real valued functions defined on a separable locally convex space X_i, for all $i \in I$. Define, for $x = (x_i)_{i\in I} \in \prod_{i\in I} X_i$, the infinite sum of $\{f_i \mid i \in I\}$ as $f(x) := \sum_{i\in I} f_i(x_i)$, where $f : \prod_{i\in I} X_i \to \overline{\mathbb{R}}$. The sum on the right-hand side is understood as in (10) such that $w_i = f_i(x_i) \in \mathbb{R}$, and $\operatorname{dom} f := \{x \in \prod_{i\in I} X_i \mid \sum_{i\in I} f_i(x_i) < +\infty\}$.

(3) Let $\{Z_i \mid i \in I\}$ be a family of locally convex spaces. Let $P_j : \prod_{i\in I} Z_i \to Z_j$ be the projection of $\prod_{i\in I} Z_i$ onto Z_j for each $j \in I$, *i.e.*, $P_j((z_i)_{i\in I}) = z_j$.

(4) Let $v \in X^*$ and $\{v_i\}_{i\in I} \subseteq X^*$, we say that $v := \sum_{i\in I}^{*} v_i$ if and only if $\langle v, x\rangle = \sum_{i\in I} \langle v_i, x\rangle\ \forall x \in X$. In this case, the sum of $\{v_i\}_{i\in I}$ is "weakly convergent". This happens if and only if $\sum_{i\in I} \langle v_i, x\rangle$ converges in $\mathbb{R}$, $\forall x \in X$, and its limit coincides with $\langle v, x\rangle$.

(5) We now define the arbitrary sum of subsets of X^* as follows. The sum of $\{A_i\}_{i\in I} \subset X^*$ is defined as

$$\sum_{i\in I}^{*} A_i := \left\{ v \in X^* \mid \forall i \in I \exists v_i \in A_i \text{ such that } v = \sum_{i\in I}^{*} v_i \right\}.$$

Remark 2.1. Let $X = \prod_{i\in I} X_i$. Then

(i) Denote by $A \simeq B$ the fact that two vector spaces are algebraically isomorphic. Then, following [13, Theorem 4.3], we know that there exists an algebraic isomorphism $\gamma : X^* \to \prod_{i\in I} X_i^*$. In other words, for all $v \in X^*$ there exist a unique $(v_i)_{i\in I}$ such that $\gamma(v) = (v_i)_{i\in I}$. Moreover, from [13, Theorem 4.3 and Section 4.1] and [14, p. 3], it holds that

$$\langle v, x\rangle = \sum_{i\in I} \langle v_i, x_i\rangle, \tag{11}$$

for each $x = (x_i)_{i\in I} \in X$ where the sum above has only a finite number of nonzero terms.

(ii) Let $f : X \to \overline{\mathbb{R}}$ be a proper and convex separable function such that $f(x) := \sum_{i\in I} f_i(x_i)$ for all $x \in X$. Then $f^*(v) := \sum_{i\in I} f_i^*(v_i)$ for all $v \in \operatorname{dom} f^*$ where $f^* : \prod_{i\in I} X_i^* \to \overline{\mathbb{R}}$. Indeed, if $v \in \operatorname{dom} f^*$ then $f^*(v) = \sup_{z\in X}\{\langle v, z\rangle - f(z)\} < +\infty$. By using (i) and the definition of f, the last expression becomes $f^*(v) = \sup_{z_i\in X_i}\{\sum_{i\in I}\langle v_i, z_i\rangle - \sum_{i\in I} f_i(z_i)\} < +\infty$. From Lemma A.1(iv), we get $f^*(v) = \sup_{z_i\in X_i}\{\sum_{i\in I}(\langle v_i, z_i\rangle - f_i(z_i))\} < +\infty$. Using now Lemma A.2(ii–iii)

$$f^*(v) = \sum_{i\in I} \sup_{z_i\in X_i}\{\langle v_i, z_i\rangle - f_i(z_i)\} = \sum_{i\in I} f_i^*(v_i), \tag{12}$$

as claimed.

3. Duality results for the GMP problem

We start by describing the primal and dual problems for GMP.

3.1. Primal and dual problems

To derive the dual problem of GMP problem (P), we recall from the introduction that the GMP problem can be formulated as

$$\begin{aligned} &\min \sum_{i\in I} f_i(x_i) \\ &\text{subject to } x \in K. \end{aligned} \tag{13}$$

The above problem is the unconstrained minimization of the function

$$\tilde{f}(x) = \delta_K(x) + \sum_{i\in I} f_i(x_i) = \delta_K(x) + f(x) \quad \forall x \in D := K \cap \left(\prod_{i\in I} \operatorname{dom} f_i \right). \tag{14}$$

We construct the dual in a canonical way using Fenchel's duality [15, p. 454]. As in [2], we introduce an auxiliary vector $z \in X$ and re-write GMP problem (13) in the following equivalent form

$$\begin{aligned} &\min f(z) \\ &\text{subject to } z = x, \quad x \in K. \end{aligned} \tag{15}$$

We then append the constraint to the objective function through a Lagrange multiplier vector $v \in X^*$. Namely, we define the Lagrangian function $L : X \times X \times X^* \to \overline{\mathbb{R}}$ such that $L(x, z, v) = f(z) + \delta_K(x) + \langle v, x - z\rangle$. By taking the infimum of $L(\cdot, \cdot, v)$ over $K \times X$ we obtain the dual function $q(v)$, *i.e.*,

$$\begin{aligned} q(v) &= \inf_{x\in K, z\in X} L(x, z, v) \\ &= \inf_{x\in K} \langle v, x\rangle + \inf_{z\in X} \{f(z) - \langle v, z\rangle\} \\ &= \inf_{x\in K} \langle v, x\rangle - \sup_{z\in X} \{\langle v, z\rangle - f(z)\} \\ &= \begin{cases} -f^*(v) & \text{if } v \in K^*, v \in \operatorname{dom} f^* \\ -\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{16}$$

By applying Remark 2.1 (Eq. (12)), the last expression yields

$$= \begin{cases} -\sum_{i\in I} f_i^*(v_i) & \text{if } v \in \operatorname{dom} f^* \cap K^* \\ -\infty & \text{otherwise.} \end{cases}$$

Thus, the conjugate dual problem can be written as follows

$$\begin{aligned} &\max - \sum_{i\in I} f_i^*(v_i) \\ &\text{subject to } v \in K^*, \end{aligned} \tag{17}$$

where $v_i^* \in X_i^*, f_i^* : X_i^* \to \overline{\mathbb{R}}$ is the conjugate convex function of f_i for each $i \in I$, and K^* is the dual cone, $K^* \subseteq \prod_{i\in I} X_i^*$.

Remark 3.1. (1) The GMP problem involves the sum of separable proper and convex functions f_i defined on X_i. However, we can define the GMP problem as a sum of functions $\bar{f}_i$ defined on $X = \prod_{i\in I} X_i$. Namely, given $f_i : X_i \to \overline{\mathbb{R}}$, define $\bar{f}_i : \prod_{i\in I} X_i \to \overline{\mathbb{R}}$ such that $\bar{f}_i(x) := f_i(x_i)$ for all $i \in I$. With this definition, we can write $f(x) = \sum_{i\in I} f_i(x_i) = \sum_{i\in I} \bar{f}_i(x)$. Thus, we can now re-write (13) as

$$\begin{aligned} &\min \sum_{i\in I} \bar{f}_i(x) \\ &\text{subject to } x \in K. \end{aligned} \tag{18}$$

Similarly, the dual problem (17) will be equivalent to

$$\begin{aligned} &\max - \sum_{i\in I} \overline{f_i^*}(v) \\ &\text{subject to } v \in K^*, \end{aligned} \tag{19}$$

where, $\overline{f_i^*} : \prod_{i\in I} X_i^* \to \overline{\mathbb{R}}$ such that $\overline{f_i^*}(v) := f_i^*(v_i)$ for all $i \in I$.

(2) At this point, it is important to distinguish the difference between the functions $\bar{f_i}^*$ and $\overline{f_i^*}$ for all $i \in I$. The dom $\overline{f_i^*} = \{v = (v_j)_{j\in I} \in \prod_{j\in I} X_j^* : \overline{f_i^*}(v) < +\infty\} = \{v \mid P_i(v) = v_i \in \operatorname{dom} f_i^*\}$, where P_i is as in Definition 2.2(3) for $Z_i = X_i^*$. Hence, the dom $\overline{f_i^*} = (\prod_{j\neq i} X_j^*) \times \operatorname{dom} f_i^*$. On the other hand, the function $\bar{f_i}^* : \prod_{i\in I} X_i^* \to \overline{\mathbb{R}}$ is the conjugate of $\bar{f_i}$ and is defined as

$$\begin{aligned}\bar{f_i}^*(v) &= \sup_{x\in X}\{\langle v, x\rangle - \bar{f_i}(x)\}\\ &= \sup_{x\in X}\left\{\sum_{i\in I}\langle v_i, x_i\rangle - f_i(x_i)\right\}.\end{aligned}$$

If $\bar{f_i}^*(v) < +\infty$, then by using Lemma A.1(iv), the last equality can be written as follows

$$\begin{aligned}&= \sup_{x\in X}\left\{\sum_{j\neq i}\langle v_j, x_j\rangle + (\langle v_i, x_i\rangle - f_i(x_i))\right\}\\ &= \sup_{\substack{x_j\in X_j\\ \forall j\neq i}}\sum_{j\neq i}\langle v_j, x_j\rangle + \sup_{x_i\in X_i}\{\langle v_i, x_i\rangle - f_i(x_i)\}.\end{aligned}$$

Since $\bar{f_i}^*(v) < +\infty$ and $\sup_{\substack{x_j\in X_j\\ j\neq i}} \sum_{j\neq i}\langle v_j, x_j\rangle \geq 0$ then $\sup_{x_i\in X_i}\{\langle v_i, x_i\rangle - f_i(x_i)\} < +\infty$, which yields $v_i \in \operatorname{dom} f_i^*$. Thus, using Lemma A.2(i)–(iii), we have $\sup_{x_j\in X_j}\langle v_j, x_j\rangle < +\infty$ for each $j \neq i$ and this forces $v_j = 0$ for all $j \neq i$. Thus, we get

$$\begin{aligned}\bar{f_i}^*(v) &= \begin{cases}\sup_{x_i\in X_i}\{\langle v_i, x_i\rangle - f_i(x_i)\} & \text{if } v_j = 0\ \forall j\neq i,\ v_i \in \operatorname{dom} f_i^*\\ +\infty & \text{otherwise,}\end{cases}\\ &= \begin{cases} f_i^*(v_i) & \text{if } v_j = 0\ \forall j\neq i,\ v_i \in \operatorname{dom} f_i^*\\ +\infty & \text{otherwise.}\end{cases}\end{aligned} \tag{20}$$

In other words, $v = (v_j)_{j\in I} \in \operatorname{dom} \bar{f_i}^*$ if and only if the ith component $v_i \in \operatorname{dom} f_i^*$ and $v_j = 0$ for all $i \neq j$, *i.e.*,

$$\operatorname{dom} \bar{f_i}^* = \left(\prod_{j\neq i}\{0\}\right) \times \operatorname{dom} f_i^*. \tag{21}$$

The aim of the rest of this section is to obtain strong duality for the primal–dual problems (13) and (17) mentioned above. Namely, we develop a constraint qualification which ensures

$$\inf\left\{\delta_K(x) + \sum_{i\in I} f_i(x_i) \mid x \in X\right\} = \max\left\{-\delta_K^*(-v) - \sum_{i\in I} f_i^*(v_i) \mid v \in X^*\right\},$$

which is, from (18) and (19), equivalent to

$$\inf\left\{\sum_{i\in I}\bar{f_i}(x) \mid x \in K\right\} = \max\left\{-\sum_{i\in I}\overline{f_i^*}(v) \mid v \in K^*\right\}.$$

The equality above can be seen as a Fenchel's duality result for GMP problems with infinite I.

3.2. Strong duality

In this subsection we deal with GMP problem (13) and (18) and its dual (17) and (19) respectively such that each function f_i in the primal problem is closed, *i.e.* each function is lower semi continuous on the whole space X_i. Our aim is to show strong duality for this problem under the constraint qualifications (2) and (4a)–(4b), respectively. When the objective function in the primal problem is the sum of finitely many proper closed convex functions, our constraint qualification (2) turns out to be equivalent to the geometric condition introduced by Boţ and Wanka in [8] to obtain Fenchel's duality in locally convex spaces (see also [6, Theorem 3.2.6]). We need the following properties.

Lemma 3.1. *Let f and $\{f_i : i \in I\}$ be defined as in Remark 3.1(1). Then*

(1) *The relation between the ϵ-subdifferentials of $\bar{f_i}$ and f_i is given as*

$$\partial_\epsilon \bar{f_i}(x) = \{v = (v_i)_{i\in I} \mid P_i(v) = v_i \in \partial_\epsilon f_i(x_i) \text{ and } P_j(v) = v_j = 0\ \forall j \neq i\}, \tag{22}$$

for all $i \in I$.

(2) *The epigraph of* $\bar{f}_i^*$ *can be expressed in terms of the epigraph of* f_i^* *as follows*

$$epi\bar{f}_i^* = \left\{(v,\alpha) \in \prod_{i\in I} X_i^* \times \mathbb{R} \mid P_i(v) = v_i \text{ with } f_i^*(v_i) \le \alpha \text{ and } P_j(v) = v_j = 0\ \forall j \ne i\right\}, \quad \text{for all } i \in I. \tag{23}$$

Proof. Statement (22) is straightforward from the definition of ϵ-subdifferential because $\bar{f}_i$ depends only on the coordinate x_i, and is constant with respect to other coordinates. Statement (23) follows directly from (20). □

It was proved in [5] that the equality in (7) is no longer true when we have the sum of infinite proper and closed convex functions. We quote below [5, Lemma 3.1] which was given for Banach spaces. However, its proof is also valid in separated locally convex spaces.

Lemma 3.2. *Let* $\{f, f_i : i \in I\}$ *be a family of proper closed convex functions such that* $f(x) = \sum_{i\in I} f_i(x)$ *for all* $x \in X$*. Then*

$$\overline{\sum_{i\in I}{}^{*} epif_i^*}^{w^*} \subseteq epif^*. \tag{24}$$

In [5], the authors present conditions under which (24) holds as an equality. Their results [5, Theorems 4.1 and 4.3] are stated for Banach spaces. We quote them below, and we point out that one of them (Theorem 4.3) is also valid in our framework of separated locally convex spaces. We note that assumption (1) in Theorem 3.1 below, together with the closedness of the functions, implies that the functions are continuous (cf. [16, Theorem 2.2.20]).

Theorem 3.1. *Let* $\{f, f_i : i \in I\}$ *be a family of proper closed convex functions such that* $f(x) = \sum_{i\in I} f_i(x)$ *for all* $x \in X$*. Then* $epif^* = \overline{\sum_{i\in I}{}^{*} epif_i^*}^{w^*}$*, if one of the following conditions hold.*

(1) *Assume that X is a Banach space and that each function in the family* $\{f, f_i : i \in I\}$ *is real valued.*
(2) *Each function in the family* $\{f, f_i : i \in I\}$ *is nonnegative on X.*

It is shown in [5, Theorem 4.2] that, by using condition (1) in Theorem 3.1 with additional assumptions, the sum of (possibly infinite) closed epigraphs of the conjugate of convex functions is $weak^*$ closed. We quote this result next.

Theorem 3.2. *Let X be a Banach space, I be a countable set, and* $\{f, f_i : i \in I\}$ *be a family of continuous convex real valued functions on X such that* $f(x) = \sum_{i\in I} f_i(x)$ *for all* $x \in X$*. Assume that* $\operatorname{dom} f^* = \operatorname{Im}\partial f$ *where* $\operatorname{Im}\partial f = \{v \in X^* :$ *there exists* $x \in X$ *such that* $v \in \partial f(x)\}$*. Then* $epif^* = \sum_{i\in I}{}^{*} epif_i^*$*.*

It is clear from the preceding Theorems 3.1–3.2 that the sum of (possibly infinite) closed epigraphs of the conjugate of convex functions is not necessarily closed. However, for the convex functions $\bar{f}_i$ in GMP problem (18), this sum is always closed as we show in the next lemma.

Lemma 3.3. *Assume that* $\bar{f}_i$ *is defined as in Remark 3.1(1). Then* $\sum_{i\in I}^{*} epi\bar{f}_i^*$ *is* $weak^*$ *closed.*

Proof. It is enough to prove that $\overline{\sum_{i\in I}^{*} epi\bar{f}_i^*}^{w^*} \subseteq \sum_{i\in I}^{*} epi\bar{f}_i^*$, so take $(v,\alpha) \in \overline{\sum_{i\in I}^{*} epi\bar{f}_i^*}^{w^*}$, so $v = (v_i)_{i\in I}$. We need to show that $(v,\alpha) \in \sum_{i\in I}^{*} epi\bar{f}_i^*$, that is, for each $i \in I$, there exists $(\bar{v}_i, \alpha_i) \in \prod_{i\in I} X_i^* \times \mathbb{R}$, with $\bar{f}_i^*(\bar{v}_i) \le \alpha_i$ such that $\sum_{i\in I} \alpha_i = \alpha$ and $\langle v, x\rangle = \sum_{i\in I}\langle \bar{v}_i, x\rangle$ for each $x \in \prod_{i\in I} X_i$. From Lemma 3.2, $(v,\alpha) \in epi(\sum_{i\in I}\bar{f}_i)^*$, *i.e.*, $(\sum_{i\in I}\bar{f}_i)^*(v) \le \alpha$. Using also (11) and definition of the conjugate function

$$\sup_{x\in \operatorname{dom} f}\left\{\sum_{i\in I}\langle v_i, x_i\rangle - \sum_{i\in I}\bar{f}_i(x)\right\} \le \sup_{x\in X}\left\{\sum_{i\in I}\langle v_i, x_i\rangle - \sum_{i\in I}\bar{f}_i(x)\right\} \le \alpha. \tag{25}$$

Applying Lemma A.1(iv), Lemma A.2(ii)–(iii) and the definition of $\bar{f}_i$, we obtain

$$\sum_{i\in I}\sup_{x_i\in \operatorname{dom} f_i}\{\langle v_i, x_i\rangle - f_i(x_i)\} \le \alpha.$$

Thus, $\sum_{i\in I} f_i^*(v_i) \le \alpha$. We claim that this implies $f_i^*(v_i) < +\infty$ for all $i \in I$. Indeed, if there exist $j \in I$ such that $f_j^*(v_j) = +\infty$, then because $f_i^*(v_i) > -\infty\ \forall i \ne j$, we will have $\sup_{\substack{|A|\le n\\ A\in\mathcal{F}(I)}} \sum_{i\in A} f_i^*(v_i) = +\infty$ for each $n \ge 1$, that is, $\sum_{i\in I} f_i^*(v_i) = +\infty$ which contradicts the fact that $\sum_{i\in I} f_i^*(v_i) \le \alpha$. Hence, $f_i^*(v_i) \in \mathbb{R}\ \forall i \in I$. From Lemma A.1(v), there exist some $\alpha_i \in \mathbb{R}$ and $i \in I$ such that

$$\sum_{i\in I}\alpha_i = \alpha, \tag{26}$$

and $f_i^*(v_i) \leq \alpha_i$. Hence, $(v_i, \alpha_i) \in epif_i^*$ for each $i \in I$. Define $\bar{v}_i \in \prod_{i\in I} X_i^*$ such that $P_j(\bar{v}_i) = 0\ \forall j \neq i$ and $P_i(\bar{v}_i) = v_i$. Thus, from (20),

$$\bar{f}_i^*(\bar{v}_i) = f_i^*(v_i) \leq \alpha_i. \tag{27}$$

Finally, using $\langle v_i, x_i\rangle = \langle \bar{v}_i, x\rangle$ we obtain

$$\begin{aligned} \langle v, x\rangle &= \sum_{i\in I}\langle v_i, x_i\rangle \\ &= \sum_{i\in I}\langle \bar{v}_i, x\rangle \quad \forall x \in X. \end{aligned} \tag{28}$$

By making use of (26)–(28), we can conclude that $(v, \alpha) \in \sum_{i\in I}^* epi\bar{f}_i^*$ as required. $\square$

We next recall a generalized Fenchel's duality result proved in [6, Theorem 3.2.6] (see also the quote below Theorem 4.2 in [8]) for the case of locally convex spaces, and the result proved in [9, Corollary 3] for a Banach space setting.

Theorem 3.3. *Let $f, g : X \to \overline{\mathbb{R}}$ be proper lower semicontinuous (closed) convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \phi$. *If the set $epif^* + epig^*$ is weak* closed then*

$$\inf_{x\in X}\{f(x) + g(x)\} = \max_{v\in X^*}\{-f^*(-v) - g^*(v)\}.$$

The closed epigraph condition used above has been also used to characterize the subdifferential sum formula in the case where the functions involved in the formula are lower semi-continuous and sublinear [17], and as a necessary and sufficient condition for a stable Fenchel–Rockafellar duality theorem [18]. We are now ready to prove strong duality for GMP problem (18), for the case when I is finite, as a direct conclusion of Theorem 3.3.

Theorem 3.4. *Let $\{f, f_i : i \in I\}$ be a family of proper closed convex functions. Consider GMP problem (18) and its dual (19) respectively such that K is a nonempty closed convex set. Assume that $g(x) = \delta_K(x)$, $f(x) = \sum_{i=1}^m f_i(x_i) = \sum_{i=1}^m \bar{f}_i(x)$ and the set*

$$epi\delta_K^* + epi\bar{f}_1^* + \cdots + epi\bar{f}_m^* \tag{29}$$

is weak closed. Then*

$$\inf\{g(x) + f(x) : x \in X\} = \max\{-g^*(-v) - f^*(v) : v \in X^*\},$$

i.e., $\inf\{\delta_K(x) + \sum_{i=1}^m \bar{f}_i(x) : x \in X\} = \max\{-\delta_K^*(-v) - \sum_{i=1}^m \overline{f_i^*}(v) : v \in X^*\}$.

Proof. It is clear, from (19), that $f^*(v) = \sum_{i=1}^m \overline{f_i^*}(v)$. Also, $g^*(-v) = \sup_{x\in X}\{\langle -v, x\rangle - g(x)\} = \sup_{x\in K}\{\langle -v, x\rangle\} = \delta_K^*(-v)$. Thus, in view of Theorem 3.3, we need to show that $epig^* + epif^*$ is *weak** closed. Note that from Lemmas 2.1(1) and 3.3 we have $epig^* + epif^* = epi\delta_K^* + epi(\sum_{i=1}^m \bar{f}_i)^* = epi\delta_K^* + cl(\sum_{i=1}^m epi\bar{f}_i^*) = epi\delta_K^* + \sum_{i=1}^m epi\bar{f}_i^*$. Since the right most expression is *weak** closed by assumption then $epig^* + epif^*$ is *weak** closed as required. $\square$

Now we consider GMP problem (13) when I is infinite. Lemma 3.2 shows that the constraint qualification (29) is not enough to obtain strong duality.

Theorem 3.5. *Let $\{f, \bar{f}_i : i \in I\}$ be a family of proper closed convex functions. Consider GMP problem (18) and its dual (19) respectively such that K is a nonempty closed convex set. Assume that $g(x) = \delta_K(x)$, $f(x) = \sum_{i\in I}\bar{f}_i(x)$. Assume that the set*

(i) *$epi\delta_K^* + \sum_{i\in I}^* epi\bar{f}_i^*$ is weak* closed, and*
(ii) *$epif^* = \sum_{i\in I}^* epi\bar{f}_i^*$.*

Then $\inf\{g(x) + f(x) : x \in X\} = \max\{-g^*(-v) - f^*(v) : v \in X^*\}$,

$$i.e., \ \inf\left\{\delta_K(x) + \sum_{i\in I}\bar{f}_i(x) : x \in X\right\} = \max\left\{-\delta_K^*(-v) - \sum_{i\in I}\overline{f_i^*}(v) : v \in X^*\right\}. \tag{30}$$

Proof. As in the preceding theorem we must show that $epif^* + epig^*$ is *weak** closed. Note first that $g^*(-v) = \delta_K^*(-v)$ and, from (19), $f^*(v) = \sum_{i\in I}\overline{f_i^*}(v)$.

$$epig^* + epif^* = epi\delta_K^* + epi\left(\sum_{i\in I}\bar{f}_i\right)^* = epi\delta_K^* + \sum_{i\in I}^* epi\bar{f}_i^*,$$

which is *weak** closed by (i). The second equality in the expression above follows from (ii). Thus, $epif^* + epig^*$ is *weak** closed and from Theorem 3.3 we obtain (30). $\square$

The constraint qualification (i) in the preceding theorem is enough to obtain strong duality whenever the objective functions in problem (18) satisfy the assumptions of either condition (1) or (2) of Theorem 3.1, or the assumptions of Theorem 3.2.

Corollary 3.1. *Let $\{f, \bar{f}_i : i \in I\}$ be a family of proper closed convex functions. Consider GMP problem (18) and its dual (19) respectively such that K is a nonempty closed convex set and $g(x) = \delta_K(x)$. Assume that the set*

$$epi\delta_K^* + \sum_{i\in I}{}^{*} epi\bar{f}_i^{*} \tag{31}$$

is weak closed, and one of the following conditions hold.*

(1) *X is a Banach space, and each function in the family $\{f, \bar{f}_i : i \in I\}$ is real valued on X.*
(2) *Each function in the family $\{f, \bar{f}_i : i \in I\}$ is nonnegative on X.*
(3) *If X is a Banach space, I is a countable set, $\operatorname{dom} f^* = \operatorname{Im}\partial f$, and each function in the family $\{f, \bar{f}_i : i \in I\}$ is real valued on X.*

Then $\inf_{x\in X}\{g(x) + f(x)\} = \max_{v\in X^*}\{-g^*(-v) - f^*(v)\}$.

Proof. Assume that condition (1) or (2) holds, then from Theorem 3.1 and Lemma 3.3, we have that $epif^* = \overline{\sum_{i\in I}{}^{*} epi\bar{f}_i^{*}}^{w^*} = \sum_{i\in I}{}^{*} epi\bar{f}_i^{*}$. Hence, condition (ii) of Theorem 3.5 holds. Using now Theorem 3.5, we obtain the conclusion. If assumption (3) holds, then by using Theorem 3.2, we obtain that $epif^* = \sum_{i\in I}{}^{*} epi\bar{f}_i^{*}$. Thus, condition (ii) of Theorem 3.5 is satisfied. Hence the conclusion. □

3.3. Bertsekas's constraint qualification

Bertsekas [7, Proposition 4.1] has recently proved zero duality gap for the extended MP problem in the case of extended real valued functions in which $X_i = R^{n_i}$, $n_i \geq 0$, $i \in I$. For I finite and K is a closed subspace, he proved that zero duality gap holds if the set

$$A_\epsilon(x) = \partial_\epsilon\delta_K(x) + \partial_\epsilon\bar{f}_1(x) + \cdots + \partial_\epsilon\bar{f}_m(x), \tag{32}$$

is closed for all feasible solutions $x = (x_1, \ldots, x_m)$ and for all $\epsilon > 0$. The proof in [7] uses the ϵ-descent algorithm to approximate $\partial_\epsilon f(x)$ by the set $A_\epsilon(x)$ such that $\partial_\epsilon f(x) \subset A_\epsilon(x) \subset \partial_{m\epsilon} f(x)$, where $f(x) = \delta_K(x) + \sum_{i=1}^m f_i(x_i) = \delta_K(x) + \sum_{i=1}^m \bar{f}_i(x)$.

We will prove that the constraint qualification (32) above ensures zero duality gap for infinite dimensional GMP problem (13) such that the index set I is finite and K is a closed convex cone. We start with the following result which has been cited in [7] for finite dimensional case. We restate the result in [7, Proposition 3.1] in separated locally convex spaces. We omit the proof for the infinite dimensional setting because it is the same as the one used for finite dimension.

Proposition 3.1. *Let $f(x) = \sum_{i=1}^m f_i(x)$ and $f_i : X \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$ be proper closed convex functions. Choose a vector $x \in \operatorname{dom} f$ and a positive scalar ϵ. Then*

$$\partial_\epsilon f(x) \subset \overline{A_\epsilon(x)}^{w^*} \subset \partial_{m\epsilon} f(x),$$

where $A_\epsilon(x) = \partial_\epsilon f_1(x) + \cdots + \partial_\epsilon f_m(x)$.

Corollary 3.2. *Let $\{f, \bar{f}_i : i = 1, \ldots, m\}$ be a family of proper closed convex functions defined as in Remark 3.1. Let $\tilde{f}(x) = \delta_K(x) + \sum_{i=1}^m f_i(x_i) = \delta_K(x) + \sum_{i=1}^m \bar{f}_i(x)$ such that $x \in D = \operatorname{dom}\delta_K \cap (\bigcap_{i=1}^m \operatorname{dom}\bar{f}_i)$ where K is a closed convex cone subset of $X = \prod_{i=1}^m X_i$. From Proposition 3.1,*

$$\partial_\epsilon\tilde{f}(x) \subset \overline{A_\epsilon(x)}^{w^*} \subset \partial_{(m+1)\epsilon}\tilde{f}(x), \tag{33}$$

where $A_\epsilon(x) = \partial_\epsilon\delta_K(x) + \partial_\epsilon\bar{f}_1(x) + \cdots + \partial_\epsilon\bar{f}_m(x) = N_K^\epsilon(x) + \partial_\epsilon\bar{f}_1(x) + \cdots + \partial_\epsilon\bar{f}_m(x)$.

Inclusion (33) has been used by Bertsekas to prove zero duality gap in the extended MP problem when K is a closed subspace S. Bertsekas obtained zero duality gap when the set $A_\epsilon(x) = S^\perp + \partial_\epsilon\bar{f}_1(x) + \cdots + \partial_\epsilon\bar{f}_m(x)$ is closed (see [7, Proposition 4.1]).

Under the *weak** closedness of the set $A_\epsilon(x)$ in Corollary 3.2, we obtain zero duality gap for our primal–dual GMP problem as we show next.

Theorem 3.6. *Let $\{f, \bar{f}_i : i = 1, \ldots, m\}$ be a family of proper closed convex functions. Consider problem (13), for $I = \{1, \ldots, m\}$, and its dual (17) respectively, and define the functions $\{\bar{f}_i : i = 1, \ldots, m\}$ as in Remark 3.1(1). Assume also problem (13) has feasible solutions x such that $x \in D = K \cap (\bigcap_{i=1}^m \operatorname{dom}\bar{f}_i)$ and the set*

$$A_\epsilon(x) = \partial_\epsilon\delta_K(x) + \partial_\epsilon\bar{f}_1(x) + \cdots + \partial_\epsilon\bar{f}_m(x) \tag{34}$$

is weak closed for all* $\epsilon > 0$, $i = 1, \ldots, m$. *Then*

$$\inf\left\{\sum_{i=1}^{m} f_i(x_i) : x \in K\right\} = \sup\left\{-\sum_{i=1}^{m} f_i^*(v_i) : v \in K^*\right\}. \tag{35}$$

Proof. From weak duality, we have $\inf\{\sum_{i=1}^{m} f_i(x_i) : x \in K\} \geq \sup\{-\sum_{i=1}^{m} f_i^*(v) : v \in K^*\}$. Thus, it remains to show that

$$\inf\left\{\sum_{i=1}^{m} f_i(x_i) : x \in K\right\} \leq \sup\left\{-\sum_{i=1}^{m} f_i^*(v_i) : v \in K^*\right\}. \tag{36}$$

If $\inf\{\sum_{i=1}^{m} f_i(x_i) : x \in K\} = -\infty$, then from weak duality, (35) is automatically fulfilled, so let us assume that $p^* := \inf\{\sum_{i=1}^{m} f_i(x_i) : x \in K\} > -\infty$. As in [7, Proposition 4.1], we apply the ϵ-descent method to obtain (36). We choose an initial vector $x_0 \in D$ and generate a sequence $\{x_k\} \subset D$ such that the $(k+1)$th iteration is

$$x_{k+1} = x_k + \alpha_k y_k. \tag{37}$$

The iteration can be implemented as follows. We find the projection λ_k of the origin on the closed set $A_\epsilon(x_k)$, *i.e.*, $\lambda_k = \operatorname{argmin}_{\lambda \in A_\epsilon(x_k)} \|\lambda\|$. If $\lambda_k = 0$, which means that $0 \in A_\epsilon(x_k)$, then the method stops because there is no ϵ-descent direction, *i.e.*, $x_{k+1} = x_k$. In this case, x_k is within $(m+1)\epsilon$ of being optimal. If $\lambda_k \neq 0$ $(0 \notin A_\epsilon(x_k))$, hence $0 \notin \partial_\epsilon \tilde{f}(x_k)$ (from Corollary 3.2), we generate $x_{k+1} \in D$ as in (37) as follows. By using a separation theorem (see e.g., [19, Corollary 4.22] and [20, Corollary 5.80]), there exists a hyperplane strongly separating 0 from $\partial_\epsilon \tilde{f}(x_k)$. This means there exist a continuous linear functional γ_k defined in X^* such that

$$\sup_{\lambda \in \partial_\epsilon \tilde{f}(x_k)} \gamma_k(\lambda) < 0. \tag{38}$$

Since γ_k is linear and continuous, there exists $y_k \in X$ such that $\gamma_k(\lambda) = \langle \lambda, y_k \rangle$, $\quad \forall \lambda \in X^*$ (cf. [21, p. 112, Theorem 1]). Using this in (38) gives $\sup_{\lambda \in \partial_\epsilon \tilde{f}(x_k)} \langle \lambda, y_k \rangle < 0$. By Definition 2.1, y_k is an ϵ-descent direction which yields $\inf_{\alpha > 0} \tilde{f}(x_k + \alpha y_k) < \tilde{f}(x_k) - \epsilon$. Therefore, there exists $\alpha_k > 0$ such that $\tilde{f}(x_k + \alpha_k y_k) < \tilde{f}(x_k) - \epsilon < \tilde{f}(x_k)$. The last expression ensures that the current iteration $x_{k+1} = x_k + \alpha_k y_k$ reduces the cost function $\tilde{f}$ by more than ϵ. We claim that this method should stop at a finite number of iterations $\hat{k} \in N$. Consequently, for $\hat{k} \in N$, we have $0 \in A(x_{\hat{k}}, \epsilon)$. To prove this claim, let $L \in N$ be such that

$$\tilde{f}(x_0) - p^* < (L-1)\epsilon \tag{39}$$

where x_0 is the initial point. We claim that the method should stop at some $\hat{k} \leq L-1$. Indeed, assume that, on the contrary, we generate $x_0, \ldots, x_L$ such that $\tilde{f}(x_{i+1}) < \tilde{f}(x_i) - \epsilon$, $\forall i = 0, \ldots, L-1$. This yields,

$$\tilde{f}(x_0) - \tilde{f}(x_L) = \sum_{i=0}^{L-1} \tilde{f}(x_i) - \tilde{f}(x_{i+1}) > L\epsilon.$$

By re-arranging the expression above and using the fact that $\tilde{f}$ is bounded below by p^*, it follows that

$$p^* \leq \tilde{f}(x_L) < \tilde{f}(x_0) - L\epsilon = (\tilde{f}(x_0) - p^*) + p^* - L\epsilon < (L-1)\epsilon + p^* - L\epsilon = p^* - \epsilon,$$

where the right most inequality follows from (39). The last expression yields a contradiction. Therefore, the process has to stop at some $\hat{k}$ such that $0 \in A_\epsilon(x_{\hat{k}})$ as claimed. We proceed now to show that this implies zero duality gap. Denote $x_{\hat{k}}$ by x, we have $0 \in A_\epsilon(x)$. Since $A_\epsilon(x) = \partial_\epsilon \delta_K(x) + \partial_\epsilon \bar{f}_1(x) + \cdots + \partial_\epsilon \bar{f}_m(x)$, there exist some vectors $v = (v_1, \ldots, v_m)$ with $v \in \partial_\epsilon \bar{f}_1(x) + \cdots + \partial_\epsilon \bar{f}_m(x)$, and $-v \in \partial_\epsilon \delta_K(x)$. Note that the vector v can be expressed as $v = \sum_{i=1}^{m} \bar{v}_i$ where $\bar{v}_i = (0, \ldots, 0, v_i, 0, \ldots, 0)$ has m components and the only possibly nonzero element in $\bar{v}_i$ is in the ith position. From (6), $-v \in N_K^\epsilon(x)$, and in view of (22), $v_i \in \partial_\epsilon f_i(x_i)$ for $i = 1, \ldots, m$. Using the definition of the ϵ-subdifferential we obtain $\forall x_i \in X_i$

$$f_i(x_i) \leq -f_i^*(v_i) + \langle v_i, x_i \rangle + \epsilon, \quad i = 1, \ldots, m. \tag{40}$$

Also, from the definition of ϵ-normal set (see (4)) we have

$$\langle -v, y \rangle \leq \epsilon + \langle -v, x \rangle \quad \forall y \in K. \tag{41}$$

Since $\langle -v, x \rangle \leq 0$, from (41) we get $\sup_{y \in K} \langle -v, y \rangle \leq \epsilon$. This forces $-v$ to be in the polar cone of K. Namely, $-v \in K^\circ$ and hence $v \in K^*$. By summing up for all i in (40), we obtain

$$\sum_{i=1}^{m} f_i(x_i) \leq -\sum_{i=1}^{m} f_i^*(v_i) + \langle v, x \rangle + m\epsilon.$$

In (41), take $y = 0$. This yields $\langle v, x\rangle \le \epsilon$

$$\sum_{i=1}^{m} f_i(x_i) \le -\sum_{i=1}^{m} f_i^*(v_i) + (m+1)\epsilon.$$

Since x is primal feasible and v is dual feasible, then

$$\inf\left\{\sum_{i=1}^{m} f_i(x_i) : x \in K\right\} \le \sup\left\{-\sum_{i=1}^{m} f_i^*(v_i) : v \in K^*\right\} + (m+1)\epsilon.$$

By taking $\epsilon \to 0$, we get $\inf\{\sum_{i=1}^{m} f_i(x_i) : x \in K\} \le \sup\{-\sum_{i=1}^{m} f_i^*(v_i) : v \in K^*\}$. Thus, $\inf\{\sum_{i=1}^{m} f_i(x_i) : x \in K\} = \sup\{-\sum_{i=1}^{m} f_i^*(v_i) : v \in K^*\}$ as required. □

Remark 3.2. Theorems 3.4 and 3.6 use two types of constraint qualifications that ensure zero duality gap for the GMP problem. The following example shows that the constraint qualification (29) used in Theorem 3.4 is not weaker than the constraint qualification (34) used in Theorem 3.6. We also show below how the primal and the dual GMP problems (13) and (17) can be formulated for this specific example. The example is inspired by the one introduced in [8, p. 2798].

Example 3.1. Let $X = \mathbb{R}^2$ and $K \subseteq X$ be a non empty closed convex cone such that $K = \{(x_1, x_2) : x_1 \le 0\}$. Consider also two convex sets C and D, respectively where $C = \{(x_1, x_2) : 2x_1 + x_2^2 \le 0\}$, and $D = \{(x_1, x_2) : x_1 \ge 0\}$. Let $g = \delta_K$ be a proper lsc and convex function, and $f_1, f_2 : X \to \overline{\mathbb{R}}$ be two separable proper lsc and convex functions such that $X_1 = X_2 = \mathbb{R}^2$, $f_1 = \delta_C$ and $f_2 = \delta_D$. Clearly, $K \cap \operatorname{dom} f_1 \cap \operatorname{dom} f_2 = \{(0, 0)\}$. First we show that for all $\epsilon > 0$, the sum

$$A_\epsilon(0, 0) := \partial_\epsilon \delta_K(0, 0) + \partial_\epsilon f_1(0, 0) + \partial_\epsilon f_2(0, 0),$$

is closed. One can check that the conjugate functions g^*, f_1^* and f_2^* are

$$g^*(w_1, w_2) = \delta_K^*(w_1, w_2) = \begin{cases} 0 & \text{if } w_1 \ge 0,\ w_2 = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

$$f_1^*(u_1, u_2) = \delta_C^*(u_1, u_2) = \begin{cases} \dfrac{u_2^2}{2u_1} & \text{if } u_1 > 0 \\ 0 & \text{if } u_1 = u_2 = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$f_2^*(v_1, v_2) = \delta_D^*(v_1, v_2) = \begin{cases} 0 & \text{if } v_1 \le 0,\ v_2 = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

By direct calculation we have $\partial_\epsilon \delta_K(0, 0) = [0, +\infty) \times \{0\}$, $\partial_\epsilon f_1(0, 0) = (0, 0) \cup \left(\cup_{u_1>0}(u_1 \times [-\sqrt{2\epsilon u_1}, \sqrt{2\epsilon u_1}])\right)$ and $\partial_\epsilon f_2(0, 0) = (-\infty, 0] \times \{0\}$. Therefore the set

$$A_\epsilon(0, 0) = \mathbb{R} \times \{0\} + (0, 0) \cup \left(\cup_{u_1>0}(u_1 \times [-\sqrt{2\epsilon u_1}, \sqrt{2\epsilon u_1}])\right) = \mathbb{R}^2$$

is closed. However, the set $epig^* + epif_1^* + epif_2^*$ is not closed. To show this, we will find the infimal convolution of g^*, f_1^*, f_2^*, and prove that there exist a point in $\mathbb{R}^2$ at which the infimal convolution is not exact.

$$\begin{aligned}
(g^* \oplus f_1^* \oplus f_2^*)(x_1^*, x_2^*) &= \inf_{\substack{w_1+u_1+v_1=x_1^* \\ w_2+u_2+v_2=x_2^*}} \{\delta_K^*(w_1, w_2) + \delta_C^*(u_1, u_2) + \delta_D^*(v_1, v_2)\} \\
&= \inf_{\substack{w_1+u_1+v_1=x_1^* \\ w_2+u_2+v_2=x_2^*}} \begin{cases} \dfrac{u_2^2}{2u_1} & \text{if } w_1 \ge 0,\ w_2 = 0,\ u_1 > 0,\ v_1 \le 0,\ v_2 = 0 \\ 0 & \text{if } w_1 \ge 0,\ w_2 = 0,\ u_1 = u_2 = 0,\ v_1 \le 0,\ v_2 = 0 \end{cases} \\
&= \inf_{\substack{w_1+u_1 \ge x_1^* \\ u_2=x_2^*}} \begin{cases} \dfrac{u_2^2}{2u_1} & \text{if } w_1 \ge 0,\ u_1 > 0 \\ 0 & \text{if } u_1 = u_2 = 0 \end{cases} \\
&= 0.
\end{aligned}$$

It is clear that $g^* \oplus f_1^* \oplus f_2^*$ is lsc on $\mathbb{R}^2$. However, it is not exact at $(x_1^*, x_2^*) = (1/2, 1/4)$. In fact, the infimal convolution $(g^* \oplus f_1^* \oplus f_2^*)(1/2, 1/4)$ is not attained. Thus, in the view of Lemma 2.1(2) $epig^* + epif_1^* + epif_2^* \neq epi(g + f_1 + f_2)^*$, and hence by Lemma 2.1(1) $epig^* + epif_1^* + epif_2^*$ is not closed.

Let us see now how the primal and dual problems (13) and (17) can be formulated for this example. The primal problem can be written as

$$\min f_1(w_1) + f_2(w_2)$$
$$\text{subject to } (w_1, w_2) \in K \times K \subset \mathbb{R}^2 \times \mathbb{R}^2.$$

In other words, we duplicate the space $\mathbb{R}^2$ in order to minimize the sum of two separable functions over the closed convex cone $K \times K$.

To write the dual problem as in (17), we first define $\bar{f}_1, \bar{f}_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \overline{\mathbb{R}}$ such that $\bar{f}_i(w_1, w_2) = f_i(w_i)$, $i = 1, 2$. Note that each $\bar{f}_i$ is defined as in Remark 3.1(1). Then we calculate the conjugate of $\bar{f}_i$, to obtain

$$\bar{f}_1^*(v_1, v_2) = \sigma_C(v_1) + \delta_{\{(0,0)\}}(v_2) = f_1^*(v_1) + \delta_{\{(0,0)\}}(v_2). \tag{42}$$

We also have that

$$\bar{f}_2^*(v_1, v_2) = \sigma_D(v_2) + \delta_{\{(0,0)\}}(v_1) = f_2^*(v_2) + \delta_{\{(0,0)\}}(v_1), \tag{43}$$

where $(v_1, v_2) \in \mathbb{R}^2 \times \mathbb{R}^2$. Note that from (42) and (43), we have that

$$\operatorname{dom} \bar{f}_1^* = \operatorname{dom} f_1^* \times \{0\} \quad \text{and} \quad \operatorname{dom} \bar{f}_2^* = \{0\} \times \operatorname{dom} f_2^*.$$

Thus,

$$\operatorname{dom} \bar{f}_1^* \cap \operatorname{dom} \bar{f}_2^* = \{(0, 0)\}. \tag{44}$$

Hence, by using (42)–(43), the dual problem (in the form of problem (17)) trivializes to

$$\max -(\sigma_C(v_1) + \delta_{\{(0,0)\}}(v_2)) - (\sigma_D(v_2) + \delta_{\{(0,0)\}}(v_1))$$
$$\text{subject to } (v_1, v_2) \in (K \times K)^* = K^* \times K^*.$$

It is clear, from (44), that only $(v_1, v_2) = (0, 0)$ belongs to the domain of the dual objective function. Hence, both primal and dual have 0 as the optimal value, and the optimal dual value is attained at (0, 0).

4. Conclusion

In this paper we define the generalized monotropic programming (GMP) problem in locally convex spaces and we obtain strong duality for the primal–dual problem in this setting. The GMP problem is the minimization of a possibly infinite sum of separable proper convex functions subject to a closed convex cone. Two new constraint qualifications are studied. Namely, when we have a finite sum on the objective, we show that the constraint qualification (29) implies strong duality. Moreover, the constraint qualification in [7, Proposition 4.1] is used in Theorem 3.6 to obtain zero duality gap, for the case in which the constraint set for the (GMP) problem is a closed and convex cone. For the case of infinite sum of the separable functions, we use the constraint qualification (4a)–(4b). Still for the infinite sum, under additional assumptions on the primal objective functions, the constraint qualification (31) is enough to obtain strong duality for the GMP problem.

A natural question, for the case of the sum of finite objective functions, is what is the connection between the constraint qualifications (29) and (34)?

Example 3.1 confirms that the constraint qualification (29) is not weaker than (34), but we have not been able to find an example to show that (34) is not weaker than (29). It is also a question of future research to investigate new constraint qualifications which can be equivalent to condition (34).

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Appendix

Lemma A.1. *Let I be an indexed set. Then*

(i) *If I is a countable set then there exists a real sequence $a = (a_i)_{i\in I}$ with $a_i > 0$ for all $i \in I$ such that $\sum_{i\in I} a_i = 1$.*

(ii) *If I is uncountable set then there exists a net $a = (a_i)_{i\in I}$ with $a_i \geq 0$ for all $i \in I$ such that $\sum_{i\in I} a_i = 1$.*

(iii) *Let $\{a_i : i \in I\} \subseteq \mathbb{R}$ such that $\sum_{i\in I} a_i = a$ then $\sum_{i\in I} ka_i = ka$ for all $k \in \mathbb{R}$.*

(iv) *Let* $\{b_i : i \in I\} \subseteq \mathbb{R}$ *and* $\{c_i : i \in I\} \subseteq \mathbb{R}$ *such that* $\sum_{i\in I} b_i = b$ *and* $\sum_{i\in I} c_i = c$*, where* $b, c \in \mathbb{R}$*. Then* $\sum_{i\in I}(b_i + c_i) = \sum_{i\in I} b_i + \sum_{i\in I} c_i$.

(v) *If* $\delta, \alpha \in \mathbb{R}$ *and the net* $\{\delta_i : i \in I\} \subset \mathbb{R}$ *such that* $\sum_{i\in I} \delta_i = \delta \le \alpha$*. Then there exist some* $\alpha_i \in \mathbb{R}$ *such that* $\sum_{i\in I} \alpha_i = \alpha$ *and* $\delta_i \le \alpha_i$ *for all* $i \in I$.

Proof. To prove (i), if $|I| = m$ then there exists a positive sequence $(a_i) = \{\frac{1}{m} : i \in \{1, \ldots, m\} \subset N\}$. Thus, $\sum_{i\in I} a_i = \sum_{i=1}^{m} \frac{1}{m} = 1$. If the cardinal of I is infinite, *i.e.*, $|I| = N$, the set of natural numbers, then take $(a_i)_{i\in I} = \{\frac{1}{2^i} : i \in N, i \ge 1\}$ where $a_i > 0$ for all $i \in I$, and $\sum_{i\in I} a_i = 1$. To show (ii), let $J \subset I$ be a countable set. From (i), there exists a real sequence $\tilde{a} = (\tilde{a}_i)_{i\in J}$ with $\tilde{a}_i > 0$ for all $i \in J$ such that $\sum_{i\in J} \tilde{a}_i = 1$. Let us define a net $a = (a_i)_{i\in I}$ such that

$$a_i := \begin{cases} 0 & \text{if } i \notin J, \\ \tilde{a}_i & \text{if } i \in J. \end{cases}$$

From (10), $\sum_{i\in I} a_i = \lim_{\substack{|A|\to+\infty \\ A\in\mathcal{F}(I)}} \sum_{i\in A} a_i$. Note that $\sum_{i\in A} a_i = \sum_{i\in A\cap J} a_i + \sum_{i\in A\cap J^c} a_i = \sum_{i\in A\cap J} a_i$. Thus, from Definition 2.2(1)

$$\begin{aligned} \sum_{i\in I} a_i &= \sup_{n\in N} \left(\inf_{|A|\le n} \sum_{i\in A} a_i \right) = \sup_{n\in N} \left(\inf_{\substack{|A|\le n \\ A\cap J\neq\phi}} \sum_{i\in A\cap J} a_i \right) \\ &\ge \sup_{n\in N} \left(\inf_{\substack{|A\cap J|\le n \\ A\cap J\neq\phi}} \sum_{i\in A\cap J} a_i \right) \\ &= \liminf_{i\to\infty} \sum_{i\in J} a_i = \liminf_{i\to\infty} \sum_{i\in J} \tilde{a}_i = 1. \end{aligned} \tag{A.1}$$

The last equality above follows from the definition of $a = (a_i)_{i\in I}$ and part (i). In a similar way and using Definition 2.2(1) $\sum_{i\in I} a_i = \inf_{n\in N} (\sup_{|A|\le n} \sum_{i\in A} a_i)$, we obtain

$$\sum_{i\in I} a_i \le 1. \tag{A.2}$$

By combining (A.1) and (A.2), we obtain $\sum_{i\in I} a_i = 1$. To prove (iii), for I countable, it has been shown in [22, Theorem 3.47], that $\sum_{i\in I} ka_i = ka$ for all $k \in \mathbb{R}$. Assume now I is uncountable and $k \ge 0$, from Definition 2.2(1) we have that

$$\sum_{i\in I} ka_i = \liminf_{i\to\infty} \sum_{i\in I} ka_i = k \liminf_{i\to\infty} \sum_{i\in I} a_i = ka.$$

If $k < 0$, then $\sum_{i\in I} ka_i = \sup_{n\in N} (-\sup_{\substack{|A|\le n \\ A\in\mathcal{F}(I)}} \sum_{i\in A}(-k)a_i) = -\inf_{n\in N} (\sup_{\substack{|A|\le n \\ A\in\mathcal{F}(I)}} \sum_{i\in A}(-k)a_i) = k \limsup_{i\to\infty} \sum_{i\in I} a_i = ka$.

To obtain (iv), if I is countable, see [22, Theorem 3.47]. Assume I is uncountable, from Definition 2.2(1) and the assumption, we have that

$$b = \sum_{i\in I} b_i = \sup_{n\in N} \left(\inf_{\substack{|A|\le n \\ A\in\mathcal{F}(I)}} \sum_{i\in A} b_i \right),$$

$$c = \sum_{i\in I} c_i = \sup_{n\in N} \left(\inf_{\substack{|A|\le n \\ A\in\mathcal{F}(I)}} \sum_{i\in A} c_i \right).$$

From the definition of supremum, the above expressions imply that, for each $\epsilon > 0$ there exist $n_0, \tilde{n}_0 \in N$ such that

$$\inf_{\substack{|A|\le n_0 \\ A\in\mathcal{F}(I)}} \sum_{i\in A} b_i > b - \frac{\epsilon}{2} \quad \text{and} \quad \inf_{\substack{|A|\le \tilde{n}_0 \\ A\in\mathcal{F}(I)}} \sum_{i\in A} c_i > c - \frac{\epsilon}{2}.$$

Take $\bar{n}_0 = \min\{n_0, \tilde{n}_0\}$. Note that if $|A| \le \bar{n}_0$ then $|A| \le n_0$ and $|A| \le \tilde{n}_0$. Hence

$$\inf_{\substack{|A|\le \bar{n}_0 \\ A\in\mathcal{F}(I)}} \sum_{i\in A} b_i \ge \inf_{\substack{|A|\le n_0 \\ A\in\mathcal{F}(I)}} \sum_{i\in A} b_i > b - \frac{\epsilon}{2} \tag{A.3}$$

similarly,

$$\inf_{\substack{|A|\le \bar{n}_0 \\ A\in\mathcal{F}(I)}} \sum_{i\in A} c_i > c - \frac{\epsilon}{2}. \tag{A.4}$$

From (A.3) and (A.4), we conclude $\inf_{\substack{|A|\le \bar{n}_0\\ A\in\mathcal{F}(I)}} \sum_{i\in A} b_i + \inf_{\substack{|A|\le \bar{n}_0\\ A\in\mathcal{F}(I)}} \sum_{i\in A} c_i > b + c - \epsilon$. This yields,

$$\begin{aligned}\liminf_{i\to\infty}\sum_{i\in I}(b_i+c_i) &= \sup_{n\in N}\left(\inf_{\substack{|A|\le n\\ A\in\mathcal{F}(I)}}\sum_{i\in A}(b_i+c_i)\right)\\ &\ge \inf_{\substack{|A|\le \bar{n}_0\\ A\in\mathcal{F}(I)}}\sum_{i\in A}(b_i+c_i)\\ &\ge \inf_{\substack{|A|\le \bar{n}_0\\ A\in\mathcal{F}(I)}}\sum_{i\in A} b_i + \inf_{\substack{|A|\le \bar{n}_0\\ A\in\mathcal{F}(I)}}\sum_{i\in A} c_i\\ &> b+c-\epsilon. \end{aligned} \tag{A.5}$$

Using now

$$b = \sum_{i\in I} b_i = \inf_{n\in N}\left(\sup_{\substack{|A|\le n\\ A\in\mathcal{F}(I)}}\sum_{i\in A} b_i\right),$$

$$c = \sum_{i\in I} c_i = \inf_{n\in N}\left(\sup_{\substack{|A|\le n\\ A\in\mathcal{F}(I)}}\sum_{i\in A} c_i\right),$$

we can show, in a similar way described above, that

$$\limsup_{i\to\infty}\sum_{i\in I}(b_i+c_i) < b+c+\epsilon \tag{A.6}$$

for all $\epsilon > 0$. This means, $\sum_{i\in I}(b_i+c_i) = b + c$. Finally we prove (v), from the assumption, we know that $\delta \le \alpha$. If $\delta = \alpha$ then the required result is obtained. Assume now $\delta < \alpha$, so $\alpha - \delta > 0$. From (iii), for $k = (\alpha - \delta)$, we have that $(\alpha - \delta) = \sum_{i\in I} a_i(\alpha - \delta)$, where $\{a_i : a_i \ge 0\}$ are as in (ii). Take

$$\alpha = \delta + (\alpha-\delta) = \sum_{i\in I}\delta_i + \sum_{i\in I} a_i(\alpha-\delta) = \sum_{i\in I}(\delta_i + a_i(\alpha-\delta)).$$

The last equality follows from (iv). Call $\alpha_i := \delta_i + a_i(\alpha - \delta) \ge \delta_i$. Hence, we constructed $\alpha_i \ge \delta_i$ for each $i \in I$ with $\alpha = \sum_{i\in I}\alpha_i$. □

Lemma A.2. *Let I be an indexed set, X, $\{X_i\}_{i\in I}$ be locally convex spaces such that $X = \prod_{i\in I} X_i$ and let $\{\varphi_i : i \in I\}$ be a family of separable functions such that $\varphi_i : X_i \to \overline{\mathbb{R}}$ for all $i \in I$. Define $\varphi(x) := \sum_{i\in I}\varphi_i(x_i)$ where $\varphi : \prod_{i\in I} X_i \to \overline{\mathbb{R}}$. Assume also that, for $x = (x_i)_{i\in I}$, $\sup_{x\in X}\sum_{i\in I}\varphi_i(x_i) < +\infty$. Then*

(i) *$S_i := \sup_{x_i\in X_i}\varphi_i(x_i) < +\infty\ \forall i \in I$,*
(ii) *$\sum_{i\in I} S_i = \sum_{i\in I}\sup_{x_i\in X_i}\varphi_i(x_i)$ exist and finite,*
(iii) *$\sup_{x\in X}\sum_{i\in I}\varphi_i(x_i) = \sum_{i\in I}\sup_{x_i\in X_i}\varphi_i(x_i) = \sum_{i\in I} S_i$.*

Proof. To show (i), we assume that there exist $j \in I$ such that $\sup_{x_j\in X_j}\varphi_j(x_j) = +\infty$. This means, for each $M > 0$ there exist $x_j(M) \in X_j$ such that $\varphi_j(x_j(M)) > M$. Choose $\tilde{x}(M) \in X$ such that $(\tilde{x}(M))_j = x_j(M)$. The coordinates $i \ne j$ are irrelevant. Since $\varphi(x) := \sum_{i\in I}\varphi_i(x_i)$ for all $x \in X$, then in particular $\varphi(\tilde{x}(M)) := \sum_{i\in I}\varphi_i((\tilde{x}(M))_i)$. From Definition 2.2(1), the latter expression is defined as

$$\varphi(\tilde{x}(M)) = \inf_{n\in N}\left(\sup_{\substack{|A|\le n\\ A\in\mathcal{F}(I)}}\sum_{i\in A}\varphi_i(\tilde{x}(M))_i\right).$$

Because $\sup_{|A|\le n}\sum_{i\in A}\varphi_i((\tilde{x}(M))_i) \ge \sup_{|A|=1}\sum_{i\in A}\varphi_i((\tilde{x}(M))_i) = \sup_{i\in I}\varphi_i((\tilde{x}(M))_i)$, we have that $\varphi(\tilde{x}(M)) \ge \inf_{n\in N}\ (\sup_{|A|=1}\sum_{i\in A}\varphi_i(\tilde{x}(M))_i) = \sup_{i\in I}\varphi_i(\tilde{x}_i(M)) \ge \varphi_j(\tilde{x}_j(M)) > M$. Therefore, for each $M > 0$ there exist $\tilde{x}(M) \in X$ such that $\sup_{x\in X}\varphi(x) \ge \varphi(\tilde{x}(M)) > M$ which means that $\sup_{x\in X}\varphi(x) = +\infty$. This contradicts the assumption. Hence, S_i must be finite for each $i \in I$, which shows (i). To obtain (ii), we first show that $\sum_{i\in I} S_i = \limsup_{i\to\infty}\sum_{i\in I}\sup_{x_i\in X_i}\varphi_i(x_i)$ is finite. From the assumption, we have $\sup_{x\in X}\sum_{i\in I}\varphi_i(x_i) < +\infty$, which yields that there exists $\gamma > 0$ such that

$$\sup_{x\in X}\sum_{i\in I}\varphi_i(x_i) < \gamma. \tag{A.7}$$

Assume that $\limsup_{i\to\infty}\sum_{i\in I}\sup_{x_i\in X_i}\varphi_i(x_i) = +\infty$. By definition of lim sup, this implies that $\inf_{n\in N}\ (\sup_{\substack{|A|\le n\\ A\in\mathcal{F}(I)}}\sum_{i\in A}\sup_{x_i\in X_i}\varphi_i(x_i)) > 3\gamma$. Thus, for every fixed $n\in N$ there exist $A(n)\in\mathcal{F}(I)$ with $|A(n)|\le n$ such that

$$\sum_{i\in A(n)}\sup_{x_i\in X_i}\varphi_i(x_i) > 3\gamma. \tag{A.8}$$

From (i), we know that $\sup_{x_i\in X_i}\varphi_i(x_i) < +\infty$ for all $i\in I$. Thus, using the definition of supremum, for each $i\in A(n)$ there exist $x_i'\in X_i$ such that

$$\sup_{x_i\in X_i}\varphi_i(x_i) - \frac{\gamma}{n} < \varphi_i(x_i') \le \sup_{x_i\in X_i}\varphi_i(x_i),$$

by taking the finite sum over all $i\in A(n)$ and using $\sum_{i\in A(n)}\frac{\gamma}{n}\le\gamma$ we obtain

$$\sum_{i\in A(n)}\sup_{x_i\in X_i}\varphi_i(x_i) - \gamma < \sum_{i\in A(n)}\varphi_i(x_i') \le \sum_{i\in A(n)}\sup_{x_i\in X_i}\varphi_i(x_i). \tag{A.9}$$

By re-arranging (A.9) and using (A.8) we get $2\gamma < \sum_{i\in A(n)}\varphi_i(x_i')$, which yields $2\gamma < \sum_{i\in A(n)}\varphi_i(x_i') \le \sup_{|A|\le n}\sum_{i\in A}\varphi_i(\tilde{x}_i)$, where $\tilde{x}_i = x_i'$ if $i\in A(n)$. Thus, we have $2\gamma < \inf_{n\in N}(\sup_{|A|\le n}\sum_{i\in A}\varphi_i(\tilde{x}_i)) = \sum_{i\in I}\varphi_i(\tilde{x}_i)$. Using (A.7) and the fact that $\sum_{i\in I}\varphi_i(\tilde{x}_i)\le\sup_{x\in X}\sum_{i\in I}\varphi_i(x_i)$ we get $2\gamma < \sup_{x\in X}\sum_{i\in I}\varphi_i(x_i) < \gamma$, which contradicts (A.7). Thus,

$$\limsup_{i\to\infty}\sum_{i\in I}\sup_{x_i\in X_i}\varphi_i(x_i) < +\infty. \tag{A.10}$$

Now we must show that $\limsup_{i\to\infty}\sum_{i\in I}\sup_{x_i\in X_i}\varphi_i(x_i) = \liminf_{i\to\infty}\sum_{i\in I}\sup_{x_i\in X_i}\varphi_i(x_i)$, which consequently yields that $\sum_{i\in I}\sup_{x_i\in X_i}\varphi_i(x_i)$ exist. For simplicity, denote

$$L := \liminf_{i\to\infty}\sum_{i\in I}\sup_{x_i\in X_i}\varphi_i(x_i), \quad \text{and} \quad R := \limsup_{i\to\infty}\sum_{i\in I}\sup_{x_i\in X_i}\varphi_i(x_i).$$

Since $L\le R$, by using (A.10), we conclude $L < +\infty$. Thus, it is enough to prove that $R\le L$. Assume that $L < R$. Choose an arbitrary positive number δ. Since R is finite, we can say that

$$R - \frac{\delta}{2} < R := \inf_{n\in N}\left(\sup_{\substack{|A|\le n\\ A\in\mathcal{F}(I)}}\sum_{i\in A}\sup_{x_i\in X_i}\varphi_i(x_i)\right).$$

So for each $n\in N$ there exists a finite set $A(n)\in\mathcal{F}(I)$ with $|A(n)|\le n$ such that

$$R - \frac{\delta}{2} < \sum_{i\in A(n)}\sup_{x_i\in X_i}\varphi_i(x_i). \tag{A.11}$$

On the other hand, from the definition of the supremum, for each $i\in A(n)$ there exist $\tilde{x}_i^n\in X_i$ such that $\varphi_i(\tilde{x}_i^n) > \sup_{x_i\in X_i}\varphi_i(x_i) - \frac{\delta}{2n}$. By taking the sum over all $i\in A(n)$ for the last inequality and then combine it with (A.11) we get

$$R - \delta < \sum_{i\in A(n)}\varphi_i(\tilde{x}_i^n) \quad \text{for each } n\in N. \tag{A.12}$$

Because $L+\frac{\delta}{2} > L := \sup_{n\in N}(\inf_{|B|\le n}\sum_{i\in B}\sup_{x_i\in X_i}\varphi_i(x_i))$, for each $n\in N$ there exist a finite set $B(n)\in\mathcal{F}(I)$ with $|B(n)|\le n$ such that

$$L + \frac{\delta}{2} > \sum_{i\in B(n)}\sup_{x_i\in X_i}\varphi_i(x_i) \ge \sum_{i\in B(n)}\varphi_i(x_i') \quad \forall x_i'\in X_i.$$

Since the last inequality holds for each $n\in N$ and for each $x_i\in X_i$, then

$$L+\delta > L + \frac{\delta}{2} \ge \sup_{n\in N}\left(\inf_{|B|\le n}\sum_{i\in B}\varphi_i(x_i')\right) = \inf_{n\in N}\left(\sup_{|B|\le n}\sum_{i\in B}\varphi_i(x_i')\right) = \varphi(x') \quad \forall x'\in X.$$

This yields, there exists $n_0\in N$ such that

$$L+\delta > \sup_{|B|\le n_0}\sum_{i\in B}\varphi_i(x_i) \ge \sum_{i\in A(n_0)}\varphi_i(\tilde{x}_i^{n_0}).$$

Using (A.12) (for $n = n_0$) with the inequalities above yields

$$L + \delta > \sum_{i \in A(n_0)} \varphi_i(\tilde{x}_i^{n_0}) > R - \delta.$$

We proved that for every $\delta > 0$ we have $L+\delta > R-\delta$. Hence $L \geq R$ as required. Thus, we have showed that $\sum_{i\in I} \sup_{x_i \in X_i} \varphi_i(x_i)$ exists. Finally we prove (iii), It is clear that

$$\sup_{(x_i)_{i\in I} \in \prod_{i \in I} X_i} \sum_{i\in I} \varphi_i(x_i) \leq \sum_{i \in I} \sup_{x_i \in X_i} \varphi_i(x_i).$$

So, we only need to show the opposite inequality. Assume that there exist $\gamma \in \mathbb{R}$ such that

$$\sup_{(x_i)_{i\in I} \in \prod_{i \in I} X_i} \sum_{i\in I} \varphi_i(x_i) < \gamma < \sum_{i \in I} \sup_{x_i \in X_i} \varphi_i(x_i) = \sup_{n \in N} \left(\inf_{|A| \leq n} \sum_{i \in A} \sup_{x_i \in X_i} \varphi_i(x_i) \right). \tag{A.13}$$

From (A.13), there exists $\delta > 0$ and $n_0 \in N$ such that

$$\gamma + \delta < \inf_{|A| \leq n_0} \sum_{i \in A} \sup_{x_i \in X_i} \varphi_i(x_i). \tag{A.14}$$

From the inequality on the left hand side of (A.13), we have that for each $x \in X$

$$\sum_{i \in I} \varphi_i(x_i) = \sup_{n \in N} \left(\inf_{|A| \leq n} \sum_{i \in A} \varphi_i(x_i) \right) < \gamma - \delta.$$

Thus, $\inf_{|A|\leq n} \sum_{i\in A} \varphi_i(x_i) < \gamma - \delta$ for each $n \in N$ and for each $x \in X$. Fix $n_0 \in N$, from the definition of infimum, there exist $\tilde{A}(n_0) \in \mathcal{F}(I)$ with $|\tilde{A}(n_0)| \leq n_0$ such that

$$\sum_{i \in \tilde{A}(n_0)} \varphi_i(x_i) < \gamma - \delta \quad \text{for each } x_i \in X_i. \tag{A.15}$$

From (i) and the definition of supremum, there exist $\tilde{x}_i \in X_i$ such that

$$\varphi_i(\tilde{x}_i) > \sup_{x_i \in X_i} \varphi_i(x_i) - \frac{\delta}{2n_0}, \tag{A.16}$$

where $n_0 \in N$ and $i \in \tilde{A}(n_0)$. By taking the sum over all $i \in \tilde{A}(n_0)$ and using the fact that $\sum_{i \in \tilde{A}(n_0)} \frac{\delta}{2n_0} \leq \frac{\delta}{2}$, the inequality (A.16) yields

$$\frac{\delta}{2} + \sum_{i \in \tilde{A}(n_0)} \varphi_i(\tilde{x}_i) > \sum_{i \in \tilde{A}(n_0)} \sup_{x_i \in X_i} \varphi_i(x_i) \geq \inf_{|A| \leq n_0} \sum_{i \in A} \sup_{x_i \in X_i} \varphi_i(x_i) > \gamma + \delta,$$

where we used (A.14) in the right most expression above. Thus, the last expression yields $\sum_{i \in \tilde{A}(n_0)} \varphi_i(\tilde{x}_i) > \gamma + \frac{\delta}{2}$, and by combining the last inequality with (A.15), we get $\gamma - \delta > \sum_{i \in \tilde{A}(n_0)} \varphi_i(\tilde{x}_i) > \gamma + \frac{\delta}{2}$. This entails a contradiction. Thus,

$$\sup_{(x_i)_{i\in I} \in \prod_{i \in I} X_i} \sum_{i\in I} \varphi_i(x_i) \geq \sum_{i \in I} \sup_{x_i \in X_i} \varphi_i(x_i),$$

as required. $\square$

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