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On the dimensions of Cantor Julia sets of rational maps

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ABSTRACT

In this paper, we study the dimensions associated with the Cantor Julia set of a rational map whose Fatou set is an attracting domain. We prove that if the Julia set of such a map contains no persistently recurrent critical points, then the conformal dimension and the Hausdorff dimension of the Julia set are equal.

1. Introduction and main result

Let f be a rational map from the Riemann sphere $\widehat{\mathbb{C}}$ onto itself of degree $d \geqslant 2$. The Fatou set $F(f)$ and the Julia set $J(f)$ of f represent the stable part and the chaotic part of the complex dynamical systems generated by the iteration of f respectively. For the background of the complex dynamical systems, readers can refer to the book [10]. The purpose of this paper is to study the geometrical properties of the Julia sets, such as the measures and dimensions relative to $J(f)$. The most important dimension associated with the Julia sets is the Hausdorff dimension. It is a fractal dimension which can describe the complexity of the Julia sets. The definitions of Hausdorff measure and Hausdorff dimension can be found in [5]. In this paper, we denote by $\dim_H$ the Hausdorff dimension and H_t the t-dimensional Hausdorff measure.

Conformal measure is an important geometrical measure associated with the Julia set, which is first introduced to complex dynamical systems by Sullivan in [17]. It is defined as follows.

Definition 1. A probability measure μ supported on $J(f)$ is called a conformal measure with exponent α or an α-conformal measure $(0 < \alpha \leqslant 2)$ for f if the equation

$$\mu(f(A)) = \int_A |f'(z)|^\alpha d\mu$$

holds for every Borel set $A \subset J(f)$ such that $f|_A$ is injective.

The existence of the conformal measure for any rational map was shown in [17] by Sullivan, see also [3]. Conformal measure is one of the tools to study the Hausdorff dimension of Julia sets because of the following principle, which is very useful to estimate the Hausdorff dimension.

Theorem 1 (*See Theorem 1.1 of [18]*)**.** *Assume that X is a compact subset of the plane and μ is a Borel probability measure on X. Then for every $t \geqslant 0$, there exist constants $h_1(t)$ and $h_2(t)$ with the following properties: If A is a Borel subset of X and C is a*

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constant such that

(1) *for all (but countable many)* $x \in A$

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^t} \geqslant C^{-1},$$

then for every Borel subset $E \subset A$ *we have* $H_t(E) \leqslant h_1(t)C\mu(E)$. *In particular,* $H_t(A) < \infty$, $t \geqslant \dim_H(A)$; *or*

(2) *for all* $x \in A$

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^t} \leqslant C^{-1},$$

then for every Borel subset $E \subset A$ *we have* $H_t(E) \geqslant h_2(t)C\mu(E)$.

According to the construction of the conformal measure, all the conformal measures for a rational map f form a closed set in the sense of weak convergence, so there exists a conformal measure whose exponent is the infimum of all the exponents of conformal measures for f. Denote by

$$\alpha_*(f) = \inf\{\alpha \in (0, 2] : \exists \text{ an } \alpha\text{-conformal measure supported on } J(f)\}.$$

This minimal exponent $\alpha_*(f)$ is called the conformal dimension of $J(f)$.

In [17], Sullivan also proved that for any hyperbolic rational map f, there is only one conformal measure μ supported on $J(f)$, whose exponent δ equals the Hausdorff dimension of $J(f)$. It implies that the conformal dimension of $J(f)$ equals the Hausdorff dimension of $J(f)$ in the expanding case.

Naturally, we want to know that for a general rational map, whether or not the above two dimensions of its Julia set are equal. This problem has been studied in several cases before and almost all the answers are affirmative:

(1) Parabolic rational maps, [19].
(2) Rational maps with no recurrent critical points in their Julia sets, [20].
(3) Collet–Eckmann rational maps and non-renormalizable quadratic polynomial $z \mapsto z^2+c$ with c not in the main cardioid in the Mandelbrot set, [11].
(4) Quadratic Feigenbaum map whose Julia set has zero 2-dimensional Lebesgue measure, [1]. However, if a quadratic Feigenbaum map has a Julia set with positive area, then the above conclusion fails.
(5) Rational maps satisfying the backward contraction property, [8].

In this paper, we will concentrate on a special class of rational maps with Cantor Julia sets and obtain a positive result about the problem mentioned above. Notice that if the Julia set $J(f)$ of a rational map f is a Cantor set, then the Fatou set $F(f)$ is connected and $F(f)$ is either an attracting domain or a parabolic domain.

Let $\mathcal{F}$ be the set of rational maps f with Cantor Julia sets satisfying the following hypotheses.

(1) $F(f)$ is an attracting domain. In other words, there is an attracting fixed point in $F(f)$.
(2) $J(f)$ contains no persistently recurrent critical points.

The definition of the persistently recurrent critical point will be given in the next section. The main result in this paper is the following

Main Theorem. *If* $f \in \mathcal{F}$, *then* $\alpha_*(f) = \dim_H(J(f))$.

Notations. (1) $\mathbb{C}$ denotes the complex plane; $\widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ denotes the Riemann sphere; $\mathbb{R}$ denotes the set of real numbers.
(2) $B(z, r)$ denotes the ball of radius r centered at z. Specially, $\mathbb{D}_r = \{z : |z| < r\}$ and $\mathbb{D} = \mathbb{D}_1$ denotes the unit disk.
(3) diam denotes the diameter and dist denotes the distance.

2. Branner–Hubbard puzzle and KSS nest

The combinatorial tool we used in this paper is the Branner–Hubbard puzzle. The Fatou set of a rational map with Cantor Julia set is either an attracting domain or a parabolic domain. In other words, there is either an attracting fixed point in $F(f)$ or a parabolic fixed point in $J(f)$. In this paper, we focus on the attracting case.

Now, we construct the Branner–Hubbard puzzle for $f \in \mathcal{F}$. We always assume that ∞ is the attracting fixed point of f.

Take a simply connected neighborhood $U_0 \subset F(f)$ of ∞ such that $\overline{U_0} \subset f^{-1}(U_0)$. In particular, we can take the disk $\{z : |z| > R\}$ as U_0, where R is a sufficiently large number and $\{z : |z| = R\}$ is disjoint from the critical orbits of f. Let U_n be the component of $f^{-n}(U_0)$ containing ∞. Then $\overline{U_n} \subset U_{n+1}$ and $F(f) = \bigcup_{n=0}^{\infty} U_n$. For an integer N_0 large enough, $f^{-n}(U_{N_0})$ has only one component for any $n \geqslant 0$. The set $f^{-n}(\widehat{\mathbb{C}} - \overline{U_{N_0}})$ is a disjoint union of finitely many topological disks. For each $n \geqslant 0$, let $\mathbf{P_n}$ be the collection of all the components of $f^{-n}(\widehat{\mathbb{C}} - \overline{U_{N_0}})$, which are called puzzle pieces of depth n. For arbitrary two different puzzle pieces P_1 and P_2, there are three possibilities: $\overline{P_1} \cap \overline{P_2} = \emptyset$, $\overline{P_1} \subset P_2$ or $P_1 \supset \overline{P_2}$.

For any point $x \in J(f)$ and any $n \geqslant 0$, there is only one puzzle piece $P_n(x) \in \mathbf{P_n}$ containing x. Thus each point $x \in J(f)$ determines a nested sequence $P_0(x) \supset P_1(x) \supset \cdots$ and $\bigcap_{n\geqslant 0} P_n(x) = \{x\}$, since each component of $J(f)$ is a singleton.

Take N_0 large enough such that U_{N_0} contains all the critical points in $F(f)$ and each puzzle piece contains at most one critical point.

We say that a critical point is recurrent if $c \in \omega(c)$, where $\omega(c)$ is the set of limit points of forward orbit of c. Otherwise, c is called non-recurrent. Let

$$\mathrm{Crit} = \{c \in J(f) : c \text{ is a critical point of } f\}.$$

Definition 2. (1) We say x is combinatorially convergent to y, write as $x \to y$, if for any $n \geqslant 0$, there exists $j > 0$ such that $f^j(x) \in P_n(y)$. It is clear that $x \to y$ if and only if $y \in \bigcup_{n\geqslant 1} f^{-n}(x)$ or $y \in \omega(x)$, the limit set of the forward orbit of x. If $x \to y$ and $y \to z$, then $x \to z$. For each recurrent critical point $c \in \mathrm{Crit}$, let

$$[c] = \{c' \in \mathrm{Crit} : c \to c' \text{ and } c' \to c\}.$$

$[c]$ is called the combinatorially equivalent class of c.

(2) We say x is non-critical if $x \nrightarrow c$ for any $c \in \mathrm{Crit}$.

(3) We say $P_{n+k}(c')$ is a child of $P_n(c)$ if $c' \in [c]$, $f^k(P_{n+k}(c')) = P_n(c)$, and $f^{k-1} : P_{n+k-1}(f(c')) \to P_n(c)$ is conformal.

(4) Suppose $c \to c$, i.e. $[c] \neq \emptyset$. We say c is persistently recurrent if $P_n(c_1)$ has only finitely many children for all $n \geqslant 0$ and all $c_1 \in [c]$. Otherwise, c is said to be reluctantly recurrent.

Remark 1. We say that a critical point c is combinatorially recurrent if $c \to c$. This terminology is equivalent to recurrent in the usual sense that $c \in \omega(c)$ when $J(f)$ is a Cantor set.

Remark 2. We consider the preperiodic critical points as non-recurrent ones since their orbits are finite. All the statements and proofs about non-recurrent critical points are available for preperiodic critical points.

While we construct the Branner–Hubbard puzzle piece, take N_0 large enough if necessary to make sure that for each $c \in \mathrm{Crit}$, $f^n(c) \notin P_0(c')$ holds for any $n \geqslant 0$ and any $c' \in \mathrm{Crit}$ if $c \nrightarrow c'$.

Let

$$\begin{aligned}
&\mathrm{Crit}_n = \{c \in \mathrm{Crit} : c \text{ is non-critical}\},\\
&\mathrm{Crit}_r = \{c \in \mathrm{Crit} : c \text{ is reluctantly recurrent}\},\\
&\mathrm{Crit}_{en} = \{c' \in \mathrm{Crit} : c' \nrightarrow c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit}_n\},\\
&\mathrm{Crit}_{er} = \{c' \in \mathrm{Crit} : c' \nrightarrow c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit}_r\}.
\end{aligned}$$

Then

$$\mathrm{Crit} = \mathrm{Crit}_n \cup \mathrm{Crit}_r \cup \mathrm{Crit}_{en} \cup \mathrm{Crit}_{er}.$$

This is not a classification because these sets might intersect.

Let A be an open set and $x \in A$. The connected component of A containing x will be denoted by $\mathrm{Comp}_x A$. Given a puzzle piece I, let

$$D(I) = \{z \in \mathbb{C} : \text{ There exists } k \geqslant 1 \text{ such that } f^k(z) \in I\} = \bigcup_{k\geqslant 1} f^{-k}(I).$$

For any $z \in D(I)$, let $k \geqslant 1$ be the smallest integer such that $f^k(z) \in I$ and let n_0 be the depth of I. Then there is at most one piece in

$$\{P_{n_0+k}(z) = \mathrm{Comp}_z f^{-k}(I), f(P_{n_0+k}(z)), \ldots, f^{k-1}(P_{n_0+k}(z))\}$$

which contains a critical point c for any $c \in \mathrm{Crit}$. Hence

$$\deg(f^k : P_{n_0+k}(z) \to I) \leqslant D$$

for a constant $D < \infty$ depending only on Crit.

3. Distortion lemmas

This section is devoted to proving several distortion results about the holomorphic maps which are concluded by the famous Koebe distortion theorem.

Theorem 2 (*Koebe Distortion Theorem*). *Suppose that $\varphi : \mathbb{D} \to \varphi(\mathbb{D}) \subset \mathbb{C}$ is a conformal map, then for any $z \in \mathbb{D}$, we have*

$$\begin{aligned}
&|\varphi'(0)|\frac{|z|}{(1+|z|)^2} \leqslant |\varphi(z) - \varphi(0)| \leqslant |\varphi'(0)|\frac{|z|}{(1-|z|)^2},\\
&|\varphi'(0)|\frac{1-|z|}{(1+|z|)^3} \leqslant |\varphi'(z)| \leqslant |\varphi'(0)|\frac{1+|z|}{(1-|z|)^3}.
\end{aligned}$$

Corollary. *If $\varphi : \mathbb{D} \to \varphi(\mathbb{D})$ is a conformal map, then we have*

$$\frac{1}{4}(1-|z|^2)|\varphi'(z)| \leqslant \operatorname{dist}(\varphi(z), \partial\varphi(\mathbb{D})) \leqslant (1-|z|^2)|\varphi'(z)|, \quad \forall z \in \mathbb{D}.$$

In this paper, we also need the following theorem which is the generalized version of Koebe distortion theorem.

Theorem 3. *Let U, V be two simply-connected domains in $\mathbb{C}$ satisfying $\overline{U} \subset V$ and g be a conformal map in V. Suppose the conformal modulus of $V - \overline{U}$ equals $\nu > 0$. Then there is a constant $C \geqslant 1$ depending only on ν such that the inequalities*

$$\frac{1}{C} \leqslant \frac{|g'(x)|}{|g'(y)|} \leqslant C$$

hold for all $x, y \in U$.

Let U be a subset of $\mathbb{C}$ and $x_0 \in U$, the shape of U about the point x_0 is defined as

$$\operatorname{shape}(U, x_0) = \frac{\max_{z\in\partial U} \operatorname{dist}(x_0, z)}{\min_{z\in\partial U} \operatorname{dist}(x_0, z)} = \frac{\max_{z\in\partial U} \operatorname{dist}(x_0, z)}{\operatorname{dist}(x_0, \partial U)}.$$

Lemma 1. *Assume that $U \subset\subset P$ and $\widetilde{U} \subset\subset \widetilde{P}$ are two pairs of simply connected domains, $x_0 \in U$, and $\varphi : (P, U) \to (\widetilde{P}, \widetilde{U})$ is a conformal map satisfying $\widetilde{x}_0 = \varphi(x_0) \in \widetilde{U}$. If $\operatorname{mod}(P - \overline{U}) \geqslant \nu > 0$, then there exists a constant $C_0 = C_0(\nu)$ such that*

$$\frac{1}{C_0} \cdot \operatorname{shape}(U, x_0) \leqslant \operatorname{shape}(\widetilde{U}, \widetilde{x}_0) \leqslant C_0 \cdot \operatorname{shape}(U, x_0).$$

Proof. Let $h_1 : (P, U) \to (\mathbb{D}, V)$ and $h_2 : (\widetilde{P}, \widetilde{U}) \to (\mathbb{D}, \widetilde{V})$ be the Riemann maps with $h_1(x_0) = 0$ and $h_2(\widetilde{x}_0) = 0$ respectively, so we have

$$\operatorname{mod}(P - \overline{U}) = \operatorname{mod}(\widetilde{P} - \overline{\widetilde{U}}) = \operatorname{mod}(\mathbb{D} - \overline{V}) = \operatorname{mod}(\mathbb{D} - \overline{\widetilde{V}}) \geqslant \nu.$$

By Grötzsch Theorem, there is a constant $r_0 = r_0(\nu) < 1$ such that $V, \widetilde{V} \subset \mathbb{D}_{r_0}$ (in two different unit disks). Clearly, $g = h_1^{-1} \circ \varphi \circ h_2$ is a conformal map from $\mathbb{D}$ onto itself with $g(0) = 0$, so $g(z)$ must be a rigid rotation $e^{i\theta}z$ for some $\theta \in \mathbb{R}$. Moreover, we have $g(V) = \widetilde{V}$.

Suppose that the points $x_R \in \partial U$ and $x_r \in \partial U$, $\widetilde{x}_R \in \partial\widetilde{U}$ and $\widetilde{x}_r \in \partial\widetilde{U}$ satisfy

$$|x_R - x_0| = \max_{x\in\partial U} |x - x_0|, \qquad |x_r - x_0| = \min_{x\in\partial U} |x - x_0|$$

and

$$|\widetilde{x}_R - \widetilde{x}_0| = \max_{\widetilde{x}\in\partial\widetilde{U}} |\widetilde{x} - \widetilde{x}_0|, \qquad |\widetilde{x}_r - \widetilde{x}_0| = \min_{\widetilde{x}\in\partial\widetilde{U}} |\widetilde{x} - \widetilde{x}_0|$$

respectively. Correspondingly, there are points $z_R, z_r \in \partial V$ and $\widetilde{z}_R, \widetilde{z}_r \in \partial\widetilde{V}$ satisfying

$$|z_R| = \max_{z\in\partial V} |z|, \qquad |z_r| = \min_{z\in\partial V} |z|;$$
$$|\widetilde{z}_R| = \max_{\widetilde{z}\in\partial\widetilde{V}} |\widetilde{z}|, \qquad |\widetilde{z}_r| = \min_{\widetilde{z}\in\partial\widetilde{V}} |\widetilde{z}|.$$

Consider the conformal map h_1^{-1} first, by Koebe distortion theorem, we have

$$|x_R - x_0| \geqslant |h_1^{-1}(z_R) - h_1^{-1}(0)| \geqslant |(h_1^{-1})'(0)| \frac{|z_R|}{(1+|z_R|)^2},$$
$$|x_r - x_0| \leqslant |h_1^{-1}(z_r) - h_1^{-1}(0)| \leqslant |(h_1^{-1})'(0)| \frac{|z_r|}{(1-|z_r|)^2}.$$

Therefore we can conclude that

$$\operatorname{shape}(U, x_0) = \frac{|x_R - x_0|}{|x_r - x_0|} \geqslant \frac{|z_R|}{|z_r|} \frac{(1-|z_r|)^2}{(1+|z_R|)^2} \geqslant \operatorname{shape}(V, 0) \frac{(1-r_0)^2}{(1+r_0)^2}.$$

Now we consider the map h_2. We have the following inequalities,

$$|\widetilde{z}_R| \geqslant |h_2(\widetilde{x}_R)| = |h_2(\widetilde{x}_R) - h_2(\widetilde{x}_0)| \geqslant \frac{1}{C} \cdot |h_2'(\widetilde{x}_0)\| \widetilde{x}_R - \widetilde{x}_0|,$$
$$|\widetilde{z}_r| \leqslant |h_2(\widetilde{x}_r)| = |h_2(\widetilde{x}_r) - h_2(\widetilde{x}_0)| \leqslant C \cdot |h_2'(\widetilde{x}_0)\| \widetilde{x}_r - \widetilde{x}_0|,$$

where C is the constant given in Theorem 3. It follows that

$$\mathrm{shape}(\widetilde{V}, 0) = \frac{|\widetilde{z}_R|}{|\widetilde{z}_r|} \geqslant \frac{|\widetilde{x}_R - \widetilde{x}_0|}{|\widetilde{x}_r - \widetilde{x}_0|}\frac{1}{C^2} = \frac{1}{C^2} \cdot \mathrm{shape}(\widetilde{U}, \widetilde{x}_0).$$

Since g is a rigid rotation, it follows immediately that

$$\mathrm{shape}(V, 0) = \mathrm{shape}(g(V), 0) = \mathrm{shape}(\widetilde{V}, 0).$$

Combining the above facts together, finally we deduce that

$$\mathrm{shape}(\widetilde{U}, \widetilde{x}_0) \leqslant C^2 \cdot \frac{(1 + r_0)^2}{(1 - r_0)^2}\mathrm{shape}(U, x_0) = C_0 \cdot \mathrm{shape}(U, x_0),$$

where C_0 is a constant depending only on ν.

Another inequality can be proved by the same method if we deal with the map $g^{-1} = h_2^{-1} \circ \varphi^{-1} \circ h_1$ instead of g. $\square$

Lemma 2. *Suppose $g : \mathbb{D} \to \mathbb{D}$ is holomorphic proper map of degree $d \geqslant 1$ and $g(0) = 0$. Let $V \subset \mathbb{D}$ be a simply connected domain containing 0. Then there exists a constant $C_1 = C_1(d)$ such that*

$$\mathrm{shape}(U, 0) \leqslant C_1 \cdot \mathrm{shape}(V, 0),$$

where $U = \mathrm{Comp}_0 g^{-1}(V)$.

Proof. Let

$$L_1 = \max_{w\in\partial V} |w|, \qquad l_1 = \min_{w\in\partial V} |w|;$$
$$L_2 = \max_{z\in\partial U} |z|, \qquad l_2 = \min_{z\in\partial U} |z|.$$

By Schwarz's Lemma, we know that $l_2 \geqslant l_1$.

If $L_1 \geqslant \frac{1}{2}$, then

$$\mathrm{shape}(U, 0) = \frac{L_2}{l_2} \leqslant \frac{1}{l_1} \leqslant \frac{2L_1}{l_1} = 2\mathrm{shape}(V, 0).$$

Now we consider the case of $L_1 < \frac{1}{2}$. Denote by D the component of $g^{-1}(B(0, 2L_1))$ containing 0, so D is simply connected. Let $\varphi : \mathbb{D} \to D$ be the Riemann map with $\varphi(0) = 0$. Denote by W the set $\varphi^{-1}(U)$.

The composed map $G(x) = \frac{1}{2L_1} g \circ \varphi(x) : \mathbb{D} \to \mathbb{D}$ is a holomorphic map of degree d. It is easy to see that $\max_{w\in\partial G(W)} |w| = \frac{1}{2}$ and $\mathrm{shape}(V, 0) = \mathrm{shape}(G(W), 0)$. Obviously,

$$\mathrm{mod}(\mathbb{D}_{2L_1} - \overline{V}) \geqslant \frac{1}{2\pi} \ln 2,$$

so, $\mathrm{mod}(D - \overline{U}) = \mathrm{mod}(\mathbb{D} - \overline{W}) \geqslant \frac{\ln 2}{2\pi d}$. According to the former case, we have

$$\mathrm{shape}(W, 0) \leqslant 2\mathrm{shape}(G(W), 0) = 2\mathrm{shape}(V, 0).$$

By Lemma 1, there exists a constant $C_0 = C_0(\frac{\ln 2}{2\pi d})$ such that $\mathrm{shape}(U, 0) \leqslant C_0 \cdot \mathrm{shape}(W, 0)$. Hence,

$$\mathrm{shape}(U, 0) \leqslant 2C_0 \cdot \mathrm{shape}(V, 0) = C_1 \cdot \mathrm{shape}(V, 0),$$

where C_1 is a constant depending only on d. $\square$

Lemma 2 is a version of distortion lemma of the d-to-1 holomorphic maps, one can refer to the Refs. [2,6,14,16,21] to find out the previous work about this topic.

4. Proof of Main Theorem

In this section, we start with the following lemma that gives a control of the degrees of iterations from arbitrarily deep critical puzzle pieces to a fixed puzzle piece.

Lemma 3 (*See Lemma 7 of [22]*)**.** *There is a constant $D < \infty$ such that for any $c \in \mathrm{Crit}$ and all $n > 0$, there exist a puzzle piece P_0 of depth 0 and infinitely many i_n satisfying*

$$\deg(f^{i_n} : P_{i_n}(c) \to P_0) \leqslant D.$$

Proof. If $c \in \mathrm{Crit}_n$, then there must be an integer $n_0 \geqslant 0$ such that for any $n \geqslant 1, f^n(c) \notin \bigcup_{c_1\in\mathrm{Crit}} P_{n_0}(c_1)$, so

$$\deg(f^n : P_{n_0+n}(c) \to P_{n_0}(f^n(c))) = \deg(f|_{P_{n_0+n}(c)}).$$

Thus, there exists an integer D_1 such that

$$\deg(f^{n_0+n} : P_{n_0+n}(c) \to P_0(f^{n_0+n}(c))) \leqslant D_1.$$

Take a subsequence i_n of $n_0 + n$ such that $P_0(f^{i_n}(c)) = P_0$ for some fixed puzzle piece P_0.

If $c \in \mathrm{Crit}_r$, according to the definition, there exist an integer $n_0 \geqslant 0$, $c' \in [c]$, $c_1 \in [c]$ and infinitely many integers $k_n \geqslant 1$ such that $\{P_{n_0+k_n}(c')\}_{n\geqslant 1}$ are children of $P_{n_0}(c_1)$. Since $c' \in [c]$, we have $c \to c'$. For each n, let m_n be the first moment that $f^{m_n}(c) \in P_{n_0+k_n}(c')$. Let $P_{i_n}(c) = \mathrm{Comp}_c f^{-m_n}(P_{n_0+k_n}(c'))$, then

$$\deg(f^{m_n+k_n} : P_{i_n}(c) \to P_{n_0}(c_1))$$

are uniformly bounded from above. So there must be an integer D_2 such that

$$\deg(f^{m_n+k_n+n_0} : P_{i_n}(c) \to P_0(f^{n_0}(c_1))) \leqslant D_2.$$

For a critical point c of other kind, it must combinatorially converge to a critical point $c_0 \in \mathrm{Crit}_n \bigcup \mathrm{Crit}_r$. For each n, suppose l_n is the smallest integer such that $f^{l_n}(c) \in P_{i_n}(c_0)$. Denote by $P_{i_n}(c)$ the component $\mathrm{Comp}_c f^{-l_n}(P_{i_n}(c_0))$. So $\deg(f^{i_n} : P_{i_n}(c) \to P_0)$ are uniformly bounded from above. Pay attention to the symbols, for different critical points, the indices i_n are also different. $\square$

Lemma 4 (*See Lemma 5 of [8]*). *Suppose that μ is an α-conformal measure for a rational map f. Then there is a constant K depending on f satisfying*

$$\frac{\mu(U)}{(\mathrm{diam}(U))^\alpha} \geqslant K \frac{\mu(V)}{(\mathrm{diam}(V))^\alpha},$$

where V is a simply connected open set and U is a simply connected component of $f^{-1}(V)$.

Proposition 1. *There is a constant $M \geqslant 1$ such that for each critical point $c \in \mathrm{Crit}$, we have*

$$\mathrm{shape}(P_{i_n+1}(c), c) \leqslant M,$$

where the puzzle pieces $P_{i_n}(c)$ are given in Lemma 3.

Proof. This proposition is an easy conclusion of Lemma 2 by normalizing the puzzle pieces to the unit disk by the Riemann maps, see also the proof of Proposition 2 of [22]. $\square$

The following proposition is the main result of this section.

Proposition 2. *Assume that μ is an α-conformal measure for $f \in \mathcal{F}$, then there is a constant $L > 0$ such that for all $c \in \mathrm{Crit}$ and $n \geqslant 0$, we have*

$$\frac{\mu(P_{i_n+1}(c))}{(\mathrm{diam} P_{i_n+1}(c))^\alpha} \geqslant L,$$

where the pieces $P_{i_n}(c)$ are given in Lemma 3 and the constant L is independent of n.

Proof. By Lemma 3, we know that there is a constant D depending on Crit such that for each $c \in \mathrm{Crit}$, there exist a sequence of nested puzzle pieces $P_{i_n}(c)$ and a fixed puzzle piece P_0 of depth 0 satisfying

$$\deg(f^{i_n} : P_{i_n+1}(c) \to P_1(f^{i_n}(c))) \leqslant \deg(f^{i_n} : P_{i_n}(c) \to P_0) \leqslant D.$$

By passing to a subsequence, we may also assume that $P_1(f^{i_n}(c)) = P_1$ is a fixed puzzle piece of depth 1. Let j_k, $k = 1, \ldots, m$, $0 = j_1 < j_2 < \cdots < j_m < i_n$, be the moments that $f^{j_k}(P_{i_n+1}(c))$ contains a critical point. Obviously, for each n, the iteration $f^{i_n}|_{P_{i_n+1}(c)}$ can be decomposed into several steps as follows:

$$\begin{aligned}
&f : P_{i_n+1}(c) \to f(P_{i_n+1}(c)),\\
&f^{j_2-1} : f(P_{i_n+1}(c)) \to f^{j_2}(P_{i_n+1}(c)),\\
&f : f^{j_2}(P_{i_n+1}(c)) \to f^{j_2+1}(P_{i_n+1}(c)),\\
&f^{j_3-j_2-1} : f^{j_2+1}(P_{i_n+1}(c)) \to f^{j_3}(P_{i_n+1}(c)),\\
&\ldots,\\
&f : f^{j_m}(P_{i_n+1}(c)) \to f^{j_m+1}(P_{i_n+1}(c)),\\
&f^{i_n-j_m-1} : f^{j_m+1}(P_{i_n+1}(c)) \to P_1.
\end{aligned}$$

It is easy to see that the following m iterations

$$\begin{aligned}
&f^{j_k-j_{k-1}-1} : f^{j_{k-1}+1}(P_{i_n+1}(c)) \to f^{j_k}(P_{i_n+1}(c)), \quad k = 2, \ldots, m,\\
&f^{i_n-j_m-1} : f^{j_m+1}(P_{i_n+1}(c)) \to P_1
\end{aligned}$$

are all conformal maps.

Note that we can find a neighborhood $\widetilde{P_1}$ of P_1 satisfying $\overline{P_1} \subset \widetilde{P_1}$ and the annulus $\widetilde{P_1} - \overline{P_1}$ is disjoint from the critical orbits, since $J(f)$ is contained in the union of puzzle piece of depth 1 and all the critical orbits in $F(f)$ are attracted to the fixed point ∞. Moreover, because there are only finitely many puzzle pieces of depth 1, so there must exist a constant $\nu > 0$ such that

$$\min_{P_1 \in \mathbf{P}_1} \{ \operatorname{mod}(\widetilde{P_1} - \overline{P_1}) \} \geqslant \nu,$$

which means that all the distortion theorems stated in Section 3 are applicable in P_1. Therefore, there exists a universal constant $K_1 = K_1(\nu)$ (see Subproposition below) such that

$$\frac{\mu(f^{j_{k-1}+1}(P_{i_n+1}(c)))}{(\mathrm{diam} f^{j_{k-1}+1}(P_{i_n+1}(c)))^\alpha} \geqslant K_1 \frac{\mu(f^{j_k}(P_{i_n+1}(c)))}{(\mathrm{diam} f^{j_k}(P_{i_n+1}(c)))^\alpha}, \quad k = 2, \ldots, m, \tag{1}$$

$$\frac{\mu(f^{j_m+1}(P_{i_n+1}(c)))}{(\mathrm{diam} P_{i_n+1}(c))^\alpha} \geqslant K_1 \frac{\mu(P_1)}{(\mathrm{diam} P_1)^\alpha}. \tag{2}$$

Subproposition. *Let f and μ be as in Proposition 2, $\widetilde{U} \supset\supset U$ be two topological disks intersecting $J(f)$ with $\operatorname{mod}(\widetilde{U} - \overline{U}) \geqslant \nu > 0$. Suppose that $z_0 \in U$ and $\operatorname{shape}(U, z_0) \leqslant M$, $\operatorname{shape}(f^m(U), f^m(z_0)) \leqslant M$. If f^m is injective in $\widetilde{U}$ for some positive integer m, then we have*

$$K_1 \cdot \frac{\mu(f^m(U))}{(\mathrm{diam} f^m(U))^\alpha} \leqslant \frac{\mu(U)}{(\mathrm{diam} U)^\alpha} \leqslant K_1' \cdot \frac{\mu(f^m(U))}{(\mathrm{diam} f^m(U))^\alpha},$$

where K_1 and K_1' are constants depending on ν and M.

Proof. Let $\varphi : \mathbb{D} \to U$ and $\psi : \mathbb{D} \to f^m(U)$ be the Riemann maps with $\varphi(0) = z_0$ and $\psi(0) = f^m(z_0)$ respectively. Then the composed map $F(w) = \psi^{-1} \circ f^m \circ \varphi(w)$ is a conformal map from $\mathbb{D}$ onto $\mathbb{D}$ satisfying $F(0) = 0$. Thus F must be a rigid rotation, that is, $F(w) = e^{i\beta} w$ for some $\beta \in \mathbb{R}$. In particular,

$$|F'(0)| = 1 = |(\psi^{-1} \circ f^m \circ \varphi)'(0)| = \frac{|(f^m)'(z_0) \cdot \varphi'(0)|}{|\psi'(0)|}.$$

So taking advantage of the corollary in Section 3, we have

$$|(f^m)'(z_0)| = \frac{|\psi'(0)|}{|\varphi'(0)|} \leqslant \frac{4\mathrm{diam} f^m(U)}{\mathrm{dist}(z_0, \partial U)} \leqslant \frac{4\mathrm{diam} f^m(U)}{\mathrm{diam} U/2M} = 8M \cdot \frac{\mathrm{diam} f^m(U)}{\mathrm{diam} U}.$$

On the other hand, we have

$$|(f^m)'(z_0)| = \frac{|\psi'(0)|}{|\varphi'(0)|} \geqslant \frac{\mathrm{dist}(f^m(z_0), \partial f^m(U))}{4\mathrm{diam} U} \geqslant \frac{\mathrm{diam} f^m(U)/2M}{4\mathrm{diam} U} = \frac{1}{8M} \cdot \frac{\mathrm{diam} f^m(U)}{\mathrm{diam} U}.$$

According to the definition of conformal measure, we conclude that

$$\mu(f^m(U)) = \int_U |(f^m)'(z)|^\alpha d\mu \leqslant C^\alpha |(f^m)'(z_0)|^\alpha \mu(U) \leqslant (8MC)^\alpha \cdot \frac{(\mathrm{diam} f^m(U))^\alpha}{(\mathrm{diam} U)^\alpha} \mu(U),$$

where $C \geqslant 1$ is given in Theorem 3. In other words,

$$\frac{\mu(U)}{(\mathrm{diam} U)^\alpha} \geqslant K_1 \cdot \frac{\mu(f^m(U))}{(\mathrm{diam} f^m(U))^\alpha}$$

is true for some constant K_1 depending only on ν and M.

The inequality in another direction can be proved if we use another estimate of $|(f^m)'(z_0)|$. □

The rest of the iterations $f : f^{j_k}(P_{i_n+1}(c)) \to f^{j_k+1}(P_{i_n+1}(c))$, $k = 1, \ldots, m$, are all branched coverings of degree bigger than one. Therefore, by Lemma 4, there exists a constant K_2 depending on f such that

$$\frac{\mu(f^{j_k}(P_{i_n+1}(c)))}{(\mathrm{diam} f^{j_k}(P_{i_n+1}(c)))^\alpha} \geqslant K_2 \frac{\mu(f^{j_k+1}(P_{i_n+1}(c)))}{(\mathrm{diam} f^{j_k+1}(P_{i_n+1}(c)))^\alpha}, \quad k = 1, \ldots, m. \tag{3}$$

Multiplying the above inequalities (1)–(3), we obtain that

$$\frac{\mu(P_{i_n+1}(c))}{(\mathrm{diam} P_{i_n+1}(c))^\alpha} \geqslant K_1^m K_2^m \frac{\mu(P_1)}{(\mathrm{diam} P_1)^\alpha}.$$

Notice that $2^m \leqslant D \leqslant d_{\max}^m$, which means that

$$\frac{\log D}{\log d_{\max}} \leqslant m \leqslant \frac{\log D}{\log 2},$$

where $d_{\max}$ is the maximal local degree of the critical points in Crit. Moreover, since there are only finitely many puzzle pieces of depth 1, we have

$$\min_{P_1 \in \mathbf{P_1}} \left\{ \frac{\mu(P_1)}{(\mathrm{diam} P_1)^\alpha} \right\} > 0.$$

Finally, we can claim that there is a constant L such that for all $c \in \mathrm{Crit}$, we have

$$\frac{\mu(P_{i_n+1}(c))}{(\mathrm{diam} P_{i_n+1}(c))^\alpha} \geqslant L,$$

where L is not dependant on n. □

Another important dimension associated with the Julia set is the hyperbolic dimension which is introduced by Shishikura in [15].

Definition 3. We call a compact forward invariant subset $X \subset J(f)$ hyperbolic if there exists $n \geqslant 1$ such that $|(f^n)'(x)| > 1$ for every $x \in X$.

The hyperbolic dimension of the Julia set $J(f)$, denoted by $\mathrm{hypdim}_H(J(f))$, is defined as the supremum of the dimensions of all the hyperbolic subsets of $J(f)$.

The relation between the conformal dimension and the hyperbolic dimension of the Julia set is as follows.

Theorem 4 (*See [4,12]*). *For any rational map f, $\alpha_*(f) = \mathrm{hypdim}_H(J(f))$.*

Proof of Main Theorem. For every non-negative integer n, define the set

$$Y_n = \left\{ z \in J(f) : f^k(z) \notin \bigcup_{c \in \mathrm{Crit}} P_{i_n+1}(c) \text{ for every } k \in \mathbb{N} \right\} \subset J(f),$$

where the puzzle pieces $P_{i_n}(c)$, $c \in \mathrm{Crit}$, are given in Lemma 3. So we have $Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ and the set

$$Y = \bigcup_n Y_n$$

is the collection of points which do not combinatorially converge to any critical point. For each n, Y_n is a hyperbolic set (see the proof of Fact 5.1 of [7]). Then $\dim_H(Y_n) \leqslant \mathrm{hypdim}_H(J(f))$ holds for each $n \geqslant 0$, so

$$\dim_H(Y) = \sup_n\{\dim_H(Y_n)\} \leqslant \mathrm{hypdim}_H(J(f)) = \alpha_*(f).$$

For each point $z \in J(f) - Y$, here, we may assume that z is not critical, or we can use Propositions 1 and 2 directly. The orbit of z enters the union of critical puzzle pieces $\bigcup_{c \in \mathrm{Crit}} P_{i_n+1}(c)$ infinitely many times for each n. Let p_n be the smallest positive integer such that $f^{p_n}(z) \in \bigcup_{c \in \mathrm{Crit}} P_{i_n+1}(c)$. Assume that $f^{p_n}(z) \in P_{i_n+1}(c_0)$ for some $c_0 \in \mathrm{Crit}$, denote by $U_n(z)$ the component $\mathrm{Comp}_z f^{-p_n}(P_{i_n+1}(c_0))$. Then for n large enough, $f^{p_n} : U_n(z) \to P_{i_n+1}(c_0)$ are conformal maps, since $J(f)$ is a Cantor set, the pieces $U_n(z)$ with sufficiently large depth contains no critical points if z is not critical. Using the similar proof of Propositions 1 and 2, we can conclude that there are constants L_0 and M_0 such that for all n, we have

$$\frac{\mu(U_n(z))}{(\mathrm{diam} U_n(z))^\alpha} \geqslant L_0 \quad \text{and} \quad \mathrm{shape}(U_n(z), z) \leqslant M_0.$$

Moreover, $\mathrm{diam} U_n(z) \to 0$ since $J(f)$ is a Cantor set. Combining the above several facts, we can easily conclude that

$$\limsup_{r \to 0} \frac{\mu(B(z,r))}{r^\alpha} \geqslant C$$

for some constant C. It follows that

$$\dim_H(J(f) - Y) \leqslant \alpha$$

by Theorem 1. Actually, the above proof is also true for the conformal measures with arbitrary exponent, so

$$\dim_H(J(f) - Y) \leqslant \alpha_*(f).$$

Finally, we get the relation about the dimensions:

$$\begin{aligned}\alpha_*(f) &= \text{hypdim}_H(J(f)) \leqslant \dim_H(J(f)) \\ &= \max\{\dim_H(J(f)-Y), \dim_H(Y)\} \leqslant \alpha_*(f).\end{aligned}$$

The proof is completed. □

Improvement of the result

Of course, we hope to prove Main Theorem without the assumption that $J(f)$ contains no persistently recurrent critical points. In this case, the difficulty is how to find a good estimate of the moduli of critical nest of puzzle pieces around the persistently recurrent critical points. Precisely speaking, in [7], Kozlovski, Shen and van Strien introduced a critical nest containing the persistently recurrent critical points, which is usually called KSS nest. Qiu and Yin proved that the moduli of the annuli between arbitrary two adjacent pieces in the KSS nest have a positive lower bound if $J(f)$ is a Cantor set, see [13]. But this estimate is not good enough. To improve Main Theorem, we have to show that the moduli of the annuli increase to infinity as the depths tend to infinity. Lyubich obtained a similar result about quadratic polynomials. He proved the linear growth of the moduli about the principal nest around the critical point of a special class of quadratic polynomials, for details, see [9].

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