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On H^∞ on the complement of $C^{1+\alpha}$ curves

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A R T I C L E I N F O

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A B S T R A C T

Let ρ be a quasiconformal mapping on the plane with complex dilatation μ. We show that if μ satisfies a certain Carleson measure condition, then one can transfer H^∞ on the upper half plane onto the corresponding space in the complement of the quasicircle $\Gamma = \rho(\mathbb{R})$, and that this condition on μ characterizes $C^{1+\alpha}$ curves.

0. Introduction

Let f be a quasiconformal self-mapping of the plane with complex dilatation μ. Thus f is a homeomorphism with locally integrable distributional derivatives verifying that $\bar{\partial} f - \mu \partial f = 0$, $\mu \in L^\infty(\mathbb{C})$ and $\|\mu\|_\infty < 1$.

The images of the real line (or the unit circle $\partial\mathbb{D}$) under quasiconformal mappings of the plane are called quasicircles. In general, they are not rectifiable and they do not satisfy any regularity conditions such as local absolute continuity or differentiability a.e., even when $\|\mu\|_\infty$ is small. Then, understanding the properties of the geometry of quasicircles in terms of the complex dilatation becomes one of the main objectives of the quasiconformal analysis and, also, of this article.

One can get some regularity by imposing some stronger smallness condition on the dilatation μ. If μ were zero in a neighborhood of $\partial\mathbb{D}$, then the map would be smooth on $\partial\mathbb{D}$. So, if μ decays to zero in some sense as it approaches the unit circle then we should be able to get some good behavior of the mapping on $\partial\mathbb{D}$.

One of the first results along these lines is due to Carleson [6]. He showed that if f is a quasiconformal self mapping of $\mathbb{C}\backslash\mathbb{D}$ such that

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$$\int_0^1 \frac{M(t)^2}{t}\,dt < \infty,$$

where $M(t) = \sup\{|\mu(z)|\colon 1 < |z| < 1+t\}$, then f is absolutely continuous on the circle and $f' \in L^2_{loc}$. Becker [3] extended this result to the case where f represents a conformal mapping of $\mathbb{D}$ that extends quasiconformally to the whole plane.

In the same context, Dyn'kin [7] proved the following stronger result. Let

$$Q(\mu)(z) = \left(\int_{1<|\zeta|<2} \frac{|\mu(\zeta)|^2}{|\zeta - z|^2}\,d\xi\,d\eta \right)^{1/2}.$$

If there exists a constant $a > 0$ so that $e^{aQ(\mu)^2} \in L^1(\partial\mathbb{D})$, then the curve $\Gamma = f(\partial\mathbb{D})$ is rectifiable.

A particular type of rectifiable quasicircle are the chord-arc or Laurentiev curves. A Jordan curve, Γ, is chord-arc if it satisfies $\Lambda(\Gamma(z_1, z_2)) \leqslant C|z_1 - z_2|$ for some constant $C > 0$, where $\Lambda(\Gamma(z_1, z_2))$ denotes the length of the shortest arc of the curve Γ between $z_1, z_2 \in \Gamma$.

A condition for chord-arc curves with small constant was given by Astala and Zinsmeister [2, Theorem 3], and requires the measure

$$d\tau(z) = \frac{|\mu(z)|^2}{|z| - 1}\,dx\,dy$$

to be a Carleson measure in $\mathbb{C}\backslash\mathbb{D}$ with small norm, i.e., $\|\tau\|_C \leqslant \gamma_0$ for some $\gamma_0 > 0$. The proof relies on estimates for the Schwarzian derivative of f in $\mathbb{D}$, showing that $\log f' \in \mathrm{BMOA}$ with small BMO constant. On the other hand Semmes [14, Theorem 0.1] and MacManus [11, Corollary 6.5] showed that the same result holds with no assumption on f being conformal on $\mathbb{D}$. Semmes' proof is based on estimates for a certain perturbed Cauchy integral operator while the strategy in [11] is to estimate Haar type coefficients for $\log f'$.

For arbitrary constants, the result is no longer true [5]. In fact, if no restriction on the Carleson norm of $|\mu(z)|^2/(|z| - 1)$ is imposed, the quasicircle Γ might not be even rectifiable.

Related to these results is the following one due to Pommerenke [12]. A curve Γ is an asymptotically smooth curve, that is

$$\frac{\Lambda(\Gamma(z_1, z_2))}{|z_1 - z_1|} \to 1, \quad \text{as } |z_1 - z_2| \to 0,$$

if and only if $\log f' \in \mathrm{VMOA}$.

This paper presents a new characterization of smooth curves in terms of μ. In particular, we are interested in $C^{1+\alpha}$ curves. We prove the following result:

Theorem 1. *Let f denote a conformal map of $\mathbb{D}$ onto the inner domain of a Jordan curve Γ. Then Γ is a $C^{1+\alpha}$ curve if and only if f extends to a global quasiconformal map whose dilatation μ satisfies that $|\mu(z)|^2/(|z| - 1)^{1+\varepsilon}$ is a Carleson measure relative to $\mathbb{T}$, where $\varepsilon = \varepsilon(\alpha)$ and $\alpha = \alpha(\varepsilon, \|\mu\|_\infty)$.*

The equivalent result holds if we consider unbounded $C^{1+\alpha}$ curves. In this case μ satisfies that $|\mu(z)|^2/|y|^{1+\varepsilon}$ is a Carleson measure relative to $\mathbb{R}$, where $y = \mathrm{Im}(z)$.

In the second part of the paper we consider quasiconformal mappings of the plane $\rho : \mathbb{C} \to \mathbb{C}$ whose complex dilatation μ satisfies that for some $\varepsilon > 0$, $|\mu|^2/|y|^{1+\varepsilon}$ is a Carleson measure relative to $\mathbb{R}$. We will show that under this condition on μ we can transfer H^∞ on the half plane $\mathbb{R}^2_+$ onto the corresponding space in the complement of the quasicircle $\Gamma = \rho(\mathbb{R})$. More precisely, denoting by $C_\Gamma(g)$ the Cauchy integral of a function $g \in L^\infty(\Gamma)$, that is

$$C_\Gamma(g)(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{g(\omega)}{\omega - z} \, d\omega, \quad z \notin \Gamma,$$

we prove the following result:

Theorem 2. *Let ρ be a quasiconformal map of the plane onto itself whose complex dilatation μ has compact support and satisfies that for some $\varepsilon > 0$, $|\mu|^2/|y|^{1+\varepsilon}$ is a Carleson measure relative to $\mathbb{R}$. Let Ω_+ and Ω_- denote the two regions bounded by the quasicircle $\Gamma = \rho(\mathbb{R})$ and let $g \in L^\infty(\Gamma)$. Then $C_\Gamma(g) \in H^\infty(\Omega_\pm)$ if and only if $C_\mathbb{R}(f) \in H^\infty(\mathbb{R}^2_\pm)$ respectively, where $f = g \circ \rho$.*

The proof is based on an argument by Semmes [13], where the idea is to transform a $\bar\partial$ problem relative to Γ into a $\bar\partial - \mu\partial$ problem relative to $\mathbb{R}$ via a change of variables. As an immediate consequence of Theorem 1 and Theorem 2 we obtain the following corollary.

Corollary. *Let Γ be an unbounded $C^{1+\alpha}$ curve analytic at ∞, and let ρ denote a conformal map of $\mathbb{R}^2_+$ onto any of the regions bounded by Γ. Then, given a function $g \in L^\infty(\Gamma)$, the Cauchy integral $C_\Gamma(g) \in L^\infty(\mathbb{C})$ if and only if $C_\mathbb{R}(f) \in L^\infty(\mathbb{C})$, where f denotes the pullback of g under the conformal mapping ρ.*

The paper is structured as follows: In Section 1, we review some definitions and basic facts, in particular the analytic characterization of $C^{1+\alpha}$ curves. The proof of Theorem 1 is presented in Section 2 whereas Section 3 is devoted to Theorem 2.

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1. Preliminaries

Let us denote complex variables by $z = x + iy$ and $\zeta = \xi + i\eta$. We shall use the following notation throughout this article: $\mathrm{Im}(z) = y$, $\mathbb{D} = \{z\colon\ |z| < 1\}$, $\mathbb{T} = \partial\mathbb{D}$, $B_x(r)$ denotes the ball centered at x and radius r, $|I|$ represents the length of any arc $I \subset \partial\mathbb{D}$, $\Gamma(z_1, z_2)$ is the shortest arc of the curve Γ between $z_1, z_2 \in \Gamma$, and $\Lambda(\Gamma)$ the length of the curve Γ. Also, we shall write $\bar\partial = \partial/\partial\bar z = 1/2(\partial_x + i\partial_y)$ and $\partial = \partial/\partial z = 1/2(\partial_x - i\partial_y)$.

A positive measure λ on $\mathbb{C}$ is called a Carleson measure relative to a given chord-arc curve Γ if there exists a constant $C > 0$ such that $\lambda(B_z(R)) \leqslant CR$ for all $z \in \Gamma$ and $R > 0$. The smallest such C is the norm of λ, $\|\lambda\|_C$. If

$$\lim_{r\to 0} \sup_{R<r} \frac{\lambda(B_z(R))}{R} = 0,$$

the measure is said to be a vanishing Carleson measure or that it satisfies the o(1)-Carleson condition.

We will denote by $H^p(\mathbb{D})$, $0 < p < \infty$, the Hardy space of analytic functions on $\mathbb{D}$ such that

$$\sup_r \frac{1}{2\pi} \int \left|f\left(re^{i\theta}\right)\right|^p d\theta = \|f\|^p_{H_p} < +\infty.$$

If $p = \infty$, $f \in H^\infty(\mathbb{D})$ if $f(z)$ is a bounded analytic function on $\mathbb{D}$, $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$.

A function $f \in L^1(\mathbb{T})$ belongs to the space $\mathrm{BMO}(\mathbb{T})$ if there exists $A > 0$ so that

$$\sup_I \frac{1}{|I|} \int_I \left|f(\zeta) - a_I\right| |d\zeta| \leqslant A, \quad \text{with } a_I = \frac{1}{|I|} \int_I f(\zeta) |d\zeta|$$

and where the supremum is taken over all arcs $I \subseteq \mathbb{T}$. The least possible A in this inequality is called the BMO norm of f, $\|f\|_*$.

The space VMO($\mathbb{T}$) is defined as follows:

$$\mathrm{VMO}(\mathbb{T}) = \left\{ f \in \mathrm{BMO}(\mathbb{T}):\ \lim_{|I|\to 0} \frac{1}{|I|} \int_I \left|f(\zeta) - a_I\right| |d\zeta| = 0 \right\}.$$

We can extend the definitions of BMO and VMO to any locally rectifiable curve by replacing intervals with arcs.

We say that $f \in \mathrm{BMOA}$ or $f \in \mathrm{VMOA}$ if $f \in H^1(\mathbb{D})$ and if the boundary values of f on $\mathbb{T}$ belong to BMO($\mathbb{T}$) or VMO($\mathbb{T}$) respectively. Recall that BMOA is contained in the Bloch space

$$\mathcal{B} = \left\{ f \text{ analytic in } \mathbb{D}:\ \|f\|_{\mathcal{B}} = \sup\left(1 - |z|^2\right)\left|f'(z)\right| < \infty \right\}$$

and VMOA is contained in $\mathcal{B}_0 = \{f \in \mathcal{B}:\ (1 - |z|^2)|f'(z)| \to 0 \text{ as } |z| \to 1 - 0\}$.

The notion of Carleson measures is closely related to BMO functions (see for example [8, Chapter 6, Section 3]). Let F be an analytic function on $\mathbb{R}^2_+$ and $f = F|_{\mathbb{R}}$. Then $f \in \mathrm{BMO}(\mathbb{R})$ if and only if $|F'(z)|^2|y|\,dx\,dy$ is a Carleson measure with respect to $\mathbb{R}$ [8, p. 262].

Let ω be a nonnegative locally integrable function on $\mathbb{R}$. Set $\omega(E) = \int_E \omega(x)\,dx$, and let $|E|$ denote the Lebesgue measure of E. We say that ω is an A_∞ weight if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if I is any interval and $E \subseteq I$, then $|E|/|I| < \delta$ implies $\omega(E)/\omega(I) < \varepsilon$. Note that if h is a bilipschitz map, clearly $|h'| \in A_\infty$.

An important fact is that $\log \omega \in \mathrm{BMO}$ if $\omega \in A_\infty$, and $\{\log \omega:\ \omega \in A_\infty\}$ spans an open subset of real-valued BMO, inducing a natural topology on A_∞. In particular, there is a $\gamma > 0$ so that $e^b \in A_\infty$ if b is real valued and $\|b\|_* \leqslant \gamma$ [12, p. 171].

We now introduce the curves which are the main object of study in this paper. A Jordan curve Γ is said to be of class C^n $(n = 1, 2, \ldots)$ if it has a parametrization $\varphi(\tau) = f(e^{i\tau})$, $0 \leqslant \tau \leqslant 2\pi$, that is n times continuously differentiable and satisfies that $\varphi'(\tau) \neq 0$, $\forall \tau$. Furthermore, it is of class $C^{n+\alpha}$, for $0 < \alpha < 1$, if

$$\left|\varphi^{(n)}(\tau_1) - \varphi^{(n)}(\tau_2)\right| \leqslant C|\tau_1 - \tau_2|^\alpha. \tag{1}$$

It is well known that for $0 < \alpha < 1$ we can consider the parametrization of the curve given by the conformal mapping f that sends $\mathbb{D}$ onto the inner domain bounded by Γ (Kellogg–Warschawski Theorem). In this case, by the Hardy–Littlewood criterion [15, Section V.4], (1) is equivalent to

$$\left|f^{n+1}(z)\right| \leqslant C\left(1 - |z|\right)^{\alpha - 1}, \quad \text{for all } z \in \mathbb{D}. \tag{2}$$

2. Proof of Theorem 1

The main idea is based on an estimate of the logarithmic derivative developed by Dyn'kin [7]. Accordingly, if f is a conformal mapping in the unit disc with a k-quasiconformal extension to the whole plane, that is $\|\mu\|_\infty \leqslant k < 1$, then for all $z \in \mathbb{D}$

$$\left(1 - |z|\right)\left|\frac{f''(z)}{f'(z)}\right| \leqslant C\left(1 - |z|\right)^{1-k}\left[1 + \int_{1-|z|}^{1} \frac{\omega(z,t)}{t^{2-k}}\,dt\right], \tag{3}$$

where

$$\omega(z,t) = \left(\frac{1}{\pi t^2} \int\limits_{|\zeta - z| < t} \left|\mu(\zeta)\right|^2 d\xi\, d\eta\right)^{1/2}, \tag{4}$$

and where the constant C depends on k only.

Using this estimate, he also proved the following result [7, Theorem 2]:

Theorem 3. *If the integral*

$$\int\limits_0^1 \frac{\omega(z,t)}{t}\, dt < \infty$$

converges uniformly in $z \in \mathbb{T}$, then $\log f'$ (and thereby f' and $1/f'$) is continuous in the closed disc.

Theorem 1. *Let f denote a conformal map of $\mathbb{D}$ onto the inner domain of a Jordan curve Γ. Then Γ is a $C^{1+\alpha}$ curve if and only if f extends to a global quasiconformal map whose dilatation μ satisfies that $|\mu(z)|^2/(|z|-1)^{1+\varepsilon}$ is a Carleson measure relative to $\mathbb{T}$, where $\varepsilon = \varepsilon(\alpha)$ and $\alpha = \alpha(\varepsilon, \|\mu\|_\infty)$.*

Proof. Let us assume first that there exists a quasiconformal extension of f so that for some $\varepsilon > 0$, $\nu(z) = |\mu(z)|^2/(|z|-1)^{1+\varepsilon}$ is a Carleson measure relative to $\mathbb{T}$, $\|\mu\|_\infty \leqslant k < 1$. Let $z \in \bar{\mathbb{D}}$, then for any $t > 1 - |z|$, we obtain from (4) that:

$$\begin{aligned}\omega^2(z,t) &\leqslant \frac{1}{\pi t^2}\left(\int\limits_{B_z(t)} \frac{|\mu(\zeta)|^2}{(|\zeta|-1)^{1+\varepsilon}}\, d\xi\, d\eta\right)^{1/2}\left(\int\limits_{B_z(t)} \left|\mu(\zeta)\right|^2\left(|\zeta|-1\right)^{1+\varepsilon} d\xi\, d\eta\right)^{1/2} \\ &\leqslant Ct^{\varepsilon/2},\end{aligned} \tag{5}$$

where $C = C(\|\mu\|_\infty, \|\nu\|_C)$. To prove the last inequality note that the first integral can be always approximated by integrals on balls centered at the boundary. Then, by the Carleson condition on the measure ν, we get that the first integral is bounded by $C(\|\nu\|_C)t^{1/2}$, while the second one is clearly bounded by $C(\|\mu\|_\infty)t^{(3+\varepsilon)/2}$. Therefore, if $z \in \mathbb{D}$ and $\alpha \leqslant \min(\varepsilon/4; 1-k)$, by (3)

$$\left|\frac{f''(z)}{f'(z)}\right| \leqslant C\big(1-|z|\big)^{-k}\left[1 + C\int\limits_{1-|z|}^1 t^{\varepsilon/4+k-2}\, dt\right] \leqslant C\big(1-|z|\big)^{\alpha-1}$$

and $|f''(z)| \leqslant C|f'(z)|(1-|z|)^{\alpha-1}$, where now $C = C(\varepsilon, \|\mu\|_\infty, \|\nu\|_C)$.

We need to prove that $|f'(z)|$ is bounded on $\mathbb{T}$. Let $z \in \mathbb{T}$, then by (5)

$$\int\limits_0^1 \frac{\omega(z,t)}{t}\, dt \leqslant C\int\limits_0^1 \frac{t^{\varepsilon/4}}{t}\, dt < \infty,$$

so we get by Theorem 3 that $|f'|$ is bounded in the closed disc. Therefore, $|f''(z)| \leqslant C(1-|z|)^{\alpha-1}$ which implies by (2) that Γ is $C^{1+\alpha}$.

To prove the second part of the theorem, let us consider the following quasiconformal extension of the Riemann mapping f:

$$f(z) = f\left(\frac{1}{\bar{z}}\right) + f'\left(\frac{1}{\bar{z}}\right)\left(z - \frac{1}{\bar{z}}\right), \quad \text{for } |z| > 1. \tag{6}$$

Note that if $\beta(z)$ denotes the logarithmic derivative, that is $\beta(z) = (1-|z|)|f''(z)/f'(z)|$, $z \in \mathbb{D}$, and μ denotes the complex dilatation of the quasiconformal extension, then for $|z| > 1$

$$\left|\mu(z)\right| \asymp \frac{1}{|z|}\beta\left(\frac{1}{\bar{z}}\right). \tag{7}$$

In general, the mapping (6) is not homeomorphic, but Becker and Pommerenke [4, Theorem 4] proved that (6) is indeed a quasiconformal extension of f to a neighborhood of $\mathbb{T}$ if $f(\mathbb{D})$ is a Jordan domain and $\overline{\lim}_{|z|\to 1-0}\beta(z) < 1$.

If Γ is $C^{1+\alpha}$, it is asymptotically smooth, so by Pommerenke's result $\log f' \in \mathrm{VMOA}$ (see [12, p. 172]). Since $\mathrm{VMOA} \subset \mathcal{B}_0$, we get that $\beta(z) \to 0$ as $|z| \to 1-0$ and the aforementioned extension of f gives a well defined quasiconformal extension to a neighborhood of $\mathbb{T}$, $G = \{z\colon |z| < R_1 \text{ for some } R_1 > 1\}$.

To obtain a global quasiconformal mapping we apply the following theorem [10, Theorem 8.1]: If $f: G \to G'$ is a k-qc map and E is a compact set of the domain G, then there exists a k'-qc map of the whole plane that coincides with f in E and with k' depending only on k, G and E.

Setting $E = \{z\colon |z| \leqslant R_0, 1 < R_0 < R_1\}$, the above result provides a global quasiconformal extension of the conformal mapping f whose complex dilatation μ satisfies (7) for $1 < |z| \leqslant R_0$.

Since Γ is a $C^{1+\alpha}$ curve, $\log f'$ is continuous in $\bar{\mathbb{D}}$ [12, Theorem 3.5] and therefore $|f'|$ is bounded below in $\mathbb{D}$. So, we get by (2) and (7) that

$$\left|\mu(z)\right| \leqslant C\beta\left(\frac{1}{\bar{z}}\right) \leqslant C\left(|z|-1\right)^{\alpha}, \quad \text{for } 1 < |z| \leqslant R_0. \tag{8}$$

To finish the proof of the theorem, it only remains to show the Carleson condition. For that, and without any loss of generality, consider a ball $B_z(R)$ centered at a point $z \in \mathbb{T}$ and radius $R < R_0 - 1$. By using (8) and a change of variables to polar coordinates, we obtain that

$$\int\limits_{B_z(R)} \frac{|\mu(\zeta)|^2}{(|\zeta|-1)^{1+\varepsilon}}\, d\xi\, d\eta \leqslant C \int\limits_{B_z(R)} \frac{(|\zeta|-1)^{2\alpha}}{(|\zeta|-1)^{1+\varepsilon}}\, d\xi\, d\eta \leqslant CR,$$

for $\varepsilon < 2\alpha$. $\square$

3. Proof of Theorem 2

Before proceeding to the proof we need to mention a result on quasiconformal mappings preserving Carleson measures. We say that $F: \mathbb{R}^2_+ \to \mathbb{R}^2_+$ preserves Carleson measures if given any Carleson measure μ in $\mathbb{R}^2_+$, the measure ν defined in $\mathbb{R}^2_+$ as

$$\nu(E) = \int\limits_{F^{-1}(E)} a_F(z)\, d\mu(z)$$

is a Carleson measure, where

$$a_F(z) = \frac{1}{|B_z|} \int\limits_{B_z} J_F(\zeta)^{1/2} \, d\xi \, d\eta,$$

$B_z = B_z(1/2\,y)$ and J_F is the Jacobian of F. The function $a_F(z)$ is somehow the quasiconformal substitute of $|F'|$ in Koebe distortion theorem. In fact $\operatorname{Im} F(z) \simeq a_F(z) y$ [1].

If we consider a quasiconformal map ρ from $\mathbb{R}^2_+$ onto itself, it is well known that ρ preserves Carleson measures if and only if $\rho|_{\mathbb{R}}$ is strongly quasisymmetric, i.e., it is locally absolutely continuous and $|\rho'|_{\mathbb{R}}| \in A_\infty$ [2,9].

We will need an analogous result for a quasiconformal map ρ from $\mathbb{R}^2_+$ onto a domain bounded by a chord-arc curve.

Lemma 1. *Let $\rho : \mathbb{R}^2_+ \to \Omega$ a quasiconformal map and let $\Gamma = \rho(\mathbb{R})$ be a chord-arc curve. If $\rho|_{\mathbb{R}}$ is locally absolutely continuous and $|\rho'|_{\mathbb{R}}| \in A_\infty$, then ρ preserves Carleson measures.*

Proof. Note that since Γ is chord-arc, there exists a global bilipschitz map h that sends $\mathbb{R}^2_+$ onto Ω [12, Theorem 7.9]. As h is bilipschitz, so is h^{-1}. Then $h^{-1} \circ \rho$ is absolutely continuous and $|(h^{-1} \circ \rho)'|_{\mathbb{R}}| \in A_\infty$. Since $h^{-1} \circ \rho$ sends $\mathbb{R}^2_+$ onto itself, the lemma follows from the previous case, that is $\Omega = \mathbb{R}^2_+$. □

Let us now estate more precisely the result mentioned at the Introduction due to Semmes [14, Theorem 0.1] and MacManus [11, Theorem 6.3]: Let $\rho : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping with dilatation μ. Set $\tau = |\mu|^2/|y|$. If there exists γ_0 such that if $\|\tau\|_C \leqslant \gamma_0$ then $\rho(\mathbb{R})$ is a chord-arc curve, $\rho|_{\mathbb{R}}$ is absolutely continuous and $\log \rho' \in \mathrm{BMO}(\mathbb{R})$, with $\|\log \rho'\|_* \leqslant c\|\tau\|_C^{1/2}$ for some constant $c > 0$.

We will apply this result when the measure $|\mu|^2/|y|$ is a vanishing Carleson measure. In fact, in this case not only $\log \rho' \in \mathrm{BMO}(\mathbb{R})$ but also $|\rho'|_{\mathbb{R}}| \in A_\infty$ as the following lemma shows.

Lemma 2. *Let $\rho : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping with complex dilatation μ. If μ has compact support and $|\mu|^2/|y|$ is a vanishing Carleson measure relative to $\mathbb{R}$, then $\rho(\mathbb{R})$ is a chord-arc curve, $\rho|_{\mathbb{R}}$ is absolutely continuous and $|\rho'|_{\mathbb{R}}| \in A_\infty$.*

Proof. To prove this, we may assume that $\mu = 0$ outside a band $|y| < \varepsilon$ for some $\varepsilon > 0$. Indeed, let $\tilde{\rho}$ be the solution of the Beltrami equation $\tilde{\rho}_{\bar{z}}/\tilde{\rho}_z = \mu(z)$ for $|y| < \varepsilon$ and $\tilde{\rho}_{\bar{z}}/\tilde{\rho}_z = 0$ otherwise. Then $\rho = F \circ \tilde{\rho}$ where F is conformal in the quasidisc $\tilde{\rho}(\{z\colon\ |y| < \varepsilon\})$. We can then replace ρ by $\tilde{\rho}$ in the whole plane without any loss of generality. Next, note that by choosing $\varepsilon > 0$ small enough, we can make the Carleson norm of the measures as small as we want and, by the properties of A_∞ weights mentioned in the preliminaries, we can conclude that $|\rho'|_{\mathbb{R}}| \in A_\infty$. □

To prove Theorem 2 we will follow Semmes approach in [13]. Let Γ be locally rectifiable quasicircle in the plane. Given a function g defined on Γ, consider its Cauchy integral

$$C_\Gamma(g)(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{g(\omega)}{\omega - z} \, d\omega, \quad z \notin \Gamma.$$

We define the jump of $G = C_\Gamma(g)$ across Γ at the point z as $g_+ - g_-$, where g_+ and g_- denote the boundary values of G. As the classical Plemelj formula states,

$$g_\pm(z) = \pm\frac{1}{2} g(z) + \frac{1}{2\pi i} P.V. \int\limits_\Gamma \frac{g(\omega)}{\omega - z} \, d\omega, \quad z \in \Gamma.$$

Hence $g_+(z) - g_-(z) = g(z)$. Also, G is holomorphic off Γ, so that $\bar{\partial} G = 0$ on $\mathbb{C} \backslash \Gamma$.

Applying Green's theorem, we can reexpress these two conditions by saying that, in the distributional sense, $\bar{\partial} G = g\, dz_\Gamma$ on $\mathbb{C}$. This means that for any $\varphi \in C^\infty(\mathbb{C})$ with compact support

$$\int_{\mathbb{C}} G(z)\bar{\partial}\varphi(z)\, dz \wedge d\bar{z} = -\int_{\Gamma} \varphi(z)\big(g^+(z) - g^-(z)\big)\, dz_\Gamma,$$

where dz_Γ denotes the usual measure on Γ and $dz \wedge d\bar{z}$ represents the wedge product which is equal to $2\mathrm{i}\, dx\, dy$.

We can also say that G is determined by the equation $\bar{\partial} G = 0$ on $\mathbb{C}\backslash\Gamma$ and the condition $\mathrm{jump}(G) = g$ on Γ, as if $\tilde{G}$ were another function with the same properties, then $\bar{\partial}(G - \tilde{G}) = 0$ in the sense of distributions and therefore, by Weyl's lemma, $G - \tilde{G}$ would be entire and a mild condition at ∞ would force it to be 0.

Let $\tilde{G} = G \circ \rho$ on $\mathbb{C}\backslash\mathbb{R}$ and $f = g \circ \rho$ on $\mathbb{R}$, where $\rho : \mathbb{C} \to \mathbb{C}$ is a quasiconformal map that takes $\mathbb{R}$ into Γ. Then, $\bar{\partial} G = 0$ off Γ transforms into $(\bar{\partial} - \mu\partial)\tilde{G} = 0$ off $\mathbb{R}$ with $\mathrm{jump}(\tilde{G}) = f$ across $\mathbb{R}$, where $\mu = \mu_\rho$. Again, in the distributional sense, we can say that $(\bar{\partial} - \mu\partial)\tilde{G} = f\, dx$.

In order to prove the following result it is convenient to change the problem a bit more. Let $F = C_{\mathbb{R}}(f)$ the Cauchy integral on $\mathbb{R}$ of f. Thus, F is holomorphic off $\mathbb{R}$ and its jump across $\mathbb{R}$ is given by f, i.e., $\bar{\partial} F = f\, dx$.

Let us now define $H = \tilde{G} - F$ on $\mathbb{C}\backslash\mathbb{R}$. Then, H has no jump across $\mathbb{R}$ and $\bar{\partial} H = \mu\partial\tilde{G}$. As H has no jump, we can consider that the previous equation holds on all of $\mathbb{C}$ in the sense of distributions (that there is no boundary piece).

Theorem 2. *Let ρ be a quasiconformal map of the plane onto itself whose complex dilatation μ has compact support and satisfies that for some $\varepsilon > 0$, $|\mu|^2/|y|^{1+\varepsilon}$ is a Carleson measure relative to $\mathbb{R}$. Let Ω_+ and Ω_- denote the two regions bounded by the quasicircle $\Gamma = \rho(\mathbb{R})$ and let $g \in L^\infty(\Gamma)$. Then $C_\Gamma(g) \in H^\infty(\Omega_\pm)$ if and only if $C_{\mathbb{R}}(f) \in H^\infty(\mathbb{R}^2_\pm)$ respectively, where $f = g \circ \rho$.*

Proof. Set $G = C_\Gamma(g)$ where $g \in L^\infty(\Gamma)$ and $\tilde{G} = G \circ \rho \in L^\infty(\mathbb{R}^2_+)$. To show that $F = C_{\mathbb{R}}(f) \in H^\infty(\mathbb{R}^2_+)$ we need to prove that $F|_{\mathbb{R}} \in L^\infty(\mathbb{R})$. Using the notation above, and since $H = \tilde{G} - F$, this is equivalent to prove that $H|_{\mathbb{R}} \in L^\infty(\mathbb{R})$. Since $\bar{\partial} H = \mu\partial\tilde{G}$, μ with compact support,

$$H(a) = \frac{1}{\pi i}\int_{\mathbb{C}} \frac{\bar{\partial} H(z)}{z - a}\, dx\, dy = \frac{1}{\pi i}\int_{\mathbb{C}} \frac{\mu(z)\partial\tilde{G}(z)}{z - a}\, dx\, dy \quad \text{for } a \in \mathbb{R}. \tag{9}$$

For $a \in \mathbb{R}$ and k an integer, let us denote $B_k = B_a(2^{-k})$. Then

$$\begin{aligned} \big|H(a)\big| &\lesssim \sum_k 2^{k+1} \int_{B_k\backslash B_{k+1}} \big|\mu(z)\big|\big|\partial\tilde{G}(z)\big|\, dx\, dy \\ &\lesssim \sum_k 2^{k+1} \bigg(\int_{B_k} \frac{|\mu(z)|^2}{|y|}\, dx\, dy\bigg)^{1/2} \bigg(\int_{B_k} \big|\partial\tilde{G}(z)\big|^2 |y|\, dx\, dy\bigg)^{1/2}. \end{aligned} \tag{10}$$

Since $\nu(z) = |\mu(z)|^2/|y|^{1+\varepsilon}$ is a Carleson measure,

$$\int_{B_k} \frac{|\mu(z)|^2}{|y|}\, dx\, dy = \int_{B_k} \frac{|\mu(z)|^2}{|y|^{1+\varepsilon}} |y|^\varepsilon\, dx\, dy \leqslant \|\nu\|_C 2^{-k(1+\varepsilon)}. \tag{11}$$

If $\tau = |\mu|^2/|y|$, the above inequality shows that $\|\tau\|_C \leqslant \|\nu\|_C 2^{-k\varepsilon}$ and, therefore, that it is a vanishing Carleson measure. By Lemma 2, the quasicircle $\Gamma = \rho(\mathbb{R})$ is actually a chord-arc curve with small constant, $\rho|_{\mathbb{R}}$ is absolute continuous and $|\rho'|_{\mathbb{R}}| \in A_\infty$.

On the other hand, since $G \in H^\infty(\Omega_+)$, then $|G'(\zeta)|^2 \delta_\Gamma(\zeta)\, d\xi\, d\eta$ is a Carleson measure relative to Γ with norm $\leqslant C\|g\|_*^2$ [13, Theorem 5.1].

By Lemma 1,

$$\sigma(B_k) = \int\limits_{\rho(B_k)} a_{\rho^{-1}}(\zeta)\left|G'(\zeta)\right|^2 \delta_\Gamma(\zeta)\, d\xi\, d\eta \lesssim 2^{-k}.$$

So, the measure σ defined in $\mathbb{R}_+^2$ is also a Carleson measure. By the Koebe distortion theorem for quasiconformal mappings [1], $a_{\rho^{-1}}(\zeta)\delta_\Gamma(\zeta) \simeq |\mathrm{Im}(\rho^{-1}(\zeta))| = |y|$. Then

$$\begin{aligned}
\int\limits_{B_k} \left|\partial \tilde{G}(z)\right|^2 |y|\, dx\, dy &= \int\limits_{B_k} \left|G'\big(\rho(z)\big)\right|^2 \left|\partial\rho(z)\right|^2 |y|\, dx\, dy \\
&\simeq \int\limits_{B_k} \left|G'\big(\rho(z)\big)\right|^2 J_\rho(z)|y|\, dx\, dy \\
&\simeq \int\limits_{\rho(B_k)} \left|G'(\zeta)\right|^2 a_{\rho^{-1}}(\zeta)\delta_\Gamma(\zeta)\, d\xi\, d\eta \lesssim 2^{-k}.
\end{aligned} \tag{12}$$

This shows that $\lambda = |\partial\tilde{G}|^2|y|\, dx\, dy$ is a Carleson measure relative to $\mathbb{R}$ and

$$\int\limits_{B_k} \left|\partial \tilde{G}(z)\right|^2 |y|\, dx\, dy \leqslant \|\lambda\|_C 2^{-k}. \tag{13}$$

By (10), (11) and (13) $|H(a)| \leqslant C(\|\nu\|_C, \|\lambda\|_C) \sum_k (2^{-\varepsilon/2})^k < \infty$ as we wanted to prove.

Conversely, if F were bounded, the same argument would show that $\tilde{G}$ is bounded on $\mathbb{R}$ and then, that $G \in H^\infty(\Omega_+)$. $\square$

To conclude this paper we would like to propose the following problem:

Problem. *Find conditions on μ so that Theorem 2 holds.*

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