J. Math. Anal. Appl. 416 (2014) 647–658

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Well-posedness and ill-posedness of KdV equation with higher dispersion

Yin Li [a,c,*], Wei Yan [b]

[a] School of Mathematics and Information Science, Shaoguan University, Shaoguan, 512005, PR China
[b] College of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan 453007, PR China
[c] Department of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou, 510275, PR China

ARTICLE INFO

Article history:
Received 6 November 2012
Available online 20 January 2014
Submitted by J. Mawhin

Keywords:
KdV equation with $2n+1$ order dispersion
Well-posedness and ill-posedness
Fourier restriction norm method

ABSTRACT

First, the Cauchy problem for KdV equation with $2n+1$ order dispersion is studied, and the local well-posedness result for the initial data in Sobolev spaces $H^s(\mathbf{R})$ with $s > -n+\frac{1}{4}$ is established via the Fourier restriction norm method. Second, we prove that the KdV equation with $2n+1$ order dispersion is ill-posed for the initial data in $H^s(\mathbf{R})$ with $s < -n+\frac{1}{4}$, $n \geqslant 2$, $n \in \mathbf{N}^+$ if the flow map is C^2 differentiable at zero form $\dot{H}^s(\mathbf{R})$ to $C([0,T];\dot{H}^s(\mathbf{R}))$. Finally, we obtain the sharp regularity requirement for the KdV equation with $2n+1$ order dispersion $s > -n+\frac{1}{4}$.

1. Introduction

This paper is devoted to the Cauchy problem for the following KdV equation with $2n+1$ order dispersion

$$\partial_t u + \partial_x^{2n+1} u + \frac{1}{2}\partial_x\left(u^2\right) = 0, \quad x,t \in \mathbf{R},\ n \in \mathbf{N}^+,\ n \geqslant 2, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

which arises in the study of propagation of unidirectional nonlinear dispersive waves. Note that (1.1) at least possesses the following three invariant functionals

$$I_1(u) = \int_{\mathbf{R}} u\,dx,$$

$$I_2(u) = \frac{1}{2}\int_{\mathbf{R}} u^2\,dx,$$

* Corresponding author.
E-mail addresses: liyin2009521@163.com (Y. Li), yanwei.scut@yahoo.com.cn (W. Yan).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2014.01.035

$$I_3(u) = \frac{(-1)^{n+1}}{2} \int_{\mathbf{R}} \left(\partial_x^n u\right)^2 dx - \frac{1}{6} \int_{\mathbf{R}} u^3\, dx. \tag{1.3}$$

Consequently, (1.1) at least possesses three conservation laws

$$I_j(u) = I_j(u_0) \quad (j = 1, 2, 3). \tag{1.4}$$

We define Poisson bracket as follows:

$$\{F, G\} = \int_{\mathbf{R}} \frac{\delta F}{\delta u} \frac{\partial}{\partial x} \frac{\delta G}{\delta u}\, dx. \tag{1.5}$$

It is easily checked that the bracket (1.5) is anti-symmetric and satisfies Jacobi identity. Thus ∂_x is a Hamiltonian operator, see [26]. Obviously, (1.1) can be rewritten in the following form

$$\partial_t u = \frac{\partial}{\partial x} \frac{\delta I_3(u)}{\delta u}.$$

Thus (1.1) possesses Hamiltonian structure. When $n = 1$ in (1.1), we have the KdV equation which has been extensively studied by lots of authors, for instance, see [18,27,29,19,2,21–23,11,6,4,12,14,13]. The KdV equation possesses bi-Hamiltonian structure, Lax pairs and infinite conservation laws. It is known that the KdV equation is associated with the Virasoro algebra, see [7]. The KdV equation can be viewed as the geodesic equation on some diffeomorphism group with respect to the invariant L^2 metric, thus the KdV equation is also viewed as the generalized Euler equation. In [28], the authors studied the long time behavior of solutions to a class of Korteweg–de Vries type equations

$$\partial_t u + \partial_x \left(\frac{u^\lambda}{\lambda}\right) + \partial_x \left(-\partial_x^2\right)^\alpha u = 0, \tag{1.6}$$

where $\lambda \in \mathbf{Z}^+$, $\lambda \geqslant 2$ and $\alpha \in \mathbf{R}$, $\alpha \geqslant \frac{1}{2}$. They showed that for $\alpha \geqslant \frac{1}{2}$ and $\lambda > \alpha + \frac{3}{2} + (\alpha^2 + 3\alpha + \frac{5}{4})^{1/2}$, solutions of the nonlinear equation with small initial conditions are smooth in the large and asymptotic when $t \longrightarrow \pm\infty$ to solutions of the linear problem. In [20], the authors also considered the long time behavior of solutions to (1.6), (1.2) and improved the result of [28]. In [2], the authors proposed the Fourier restriction norm method. In [21,23], the authors developed the Fourier restriction norm method. In [23], by using the Cauchy–Schwartz's inequality and the Fourier restriction norm method, the authors established the local well-posedness of the KdV equation for the initial data in $H^s(\mathbf{R})$ with $-\frac{3}{4} < s < 0$. In [3], the author proved that the KdV equation is locally ill-posed for the initial data in $H^s(\mathbf{R})$ with $s < -\frac{3}{4}$ if the flow map is C^3-differentiable at zero from $H^s(\mathbf{R})$ to $C([0,T]; H^s(\mathbf{R}))$. In [30], the author proved that the KdV equation is locally ill-posed for the initial data in $\dot{H}^s(\mathbf{R})$ with $s < -\frac{3}{4}$ if the flow map is C^2-differentiable at zero from $\dot{H}^s(\mathbf{R})$ to $C([0,T]; \dot{H}^s(\mathbf{R}))$. In [5], by using the I-method, the authors established the global well-posedness of the KdV equation for the initial data in $H^s(\mathbf{R})$ with $-\frac{3}{8} < s < 0$. In [6], by using the I-method, the authors proved that the KdV equation is globally well-posed for the initial data in $H^s(\mathbf{R})$ with $-\frac{3}{4} < s < 0$. In [4], the authors proved that the KdV equation is locally well-posed for the initial data in $H^s(\mathbf{R})$ with $s = -\frac{3}{4}$ and that the real KdV equation is ill-posed for the initial data in $H^s(\mathbf{R})$ with $-1 \leqslant s < -\frac{3}{4}$ if the flow map is uniformly continuous. In [13], by using the I-method which can be seen in [6] and the dyadic bilinear estimates and resolution spaces which can be seen in [16,17], the author proved that the KdV equation is globally well-posed for the initial data in $H^s(\mathbf{R})$ with $s = -\frac{3}{4}$. In [24], the authors proved that the complex KdV is not uniformly continuous for the initial data in $H^s(\mathbf{R})$ with $s < -\frac{3}{4}$. In [1], the authors introduced a general well-posedness principle. Recently, in [9], the authors considered the periodic case of (1.1), (1.2) with $n \geqslant 2$ and $n \in \mathbf{N}^+$, they proved that (1.1), (1.2) are locally well-posed for the initial data in $H^s(\mathrm{T})$ with $s \geqslant -\frac{1}{2}$.

In this paper, inspired by [9,15,10], we consider the nonperiodic initial value problem of (1.1), (1.2). By using the Fourier restriction norm method which can be used to establish bilinear estimate and the fixed point argument, we derive the local well-posedness of (1.1), (1.2) for the initial data in $H^s(\mathbf{R})$. We prove that (1.1), (1.2) are ill-posed for the initial data in $H^s(\mathbf{R})$ with $s < -n + \frac{1}{4}$, $n \geqslant 2$, $n \in \mathbf{N}^+$ if we require that the flow map

$$u_0 \longrightarrow u(t), \quad t \in [0, T]$$

is C^2-differentiable at zero from $\dot{H}^s(\mathbf{R})$ to $C([0,T]; \dot{H}^s(\mathbf{R}))$.

We introduce some definitions and notations before giving the main results. Throughout this paper, we denote $\langle \xi \rangle^s = (1 + \xi^2)^{\frac{s}{2}}$ for any $\xi \in \mathbf{R}$. $\mathscr{F}u$ is the Fourier transform of u with respect to its all variables. $\mathscr{F}^{-1}u$ is the Fourier inverse transform of u with respect to its all variables. $\mathscr{F}_x u$ is the Fourier transform of u with respect to its space variable. $\mathscr{F}_x^{-1}u$ is the Fourier inverse transform of u with respect to its space variable. $\mathscr{S}(\mathbf{R}^n)$ is the Schwartz space and $\mathscr{S}'(\mathbf{R}^n)$ is its dual space. $H^s(\mathbf{R})$ is the usual Sobolev space with norm $\|f\|_{H^s(\mathbf{R})} = \|\langle \xi \rangle^s \mathscr{F}_x f(\xi)\|_{L^2_\xi(\mathbf{R})}$. For any $s, \alpha \in \mathbf{R}$, $X_{s,\alpha}(\mathbf{R}^2)$ is the Bourgain space with phase function $\phi(\xi) = (-1)^n \xi^{2n+1}$. That is, a functions $u(x,t)$ in $\mathscr{S}'(\mathbf{R}^2)$ belongs to $X_{s,\alpha}(\mathbf{R}^2)$ if

$$\|u\|_{X_{s,\alpha}(\mathbf{R}^2)} = \left\| \langle \xi \rangle^s \left\langle \tau + (-1)^n \xi^{2n+1} \right\rangle^\alpha \mathscr{F}u(\xi, \tau) \right\|_{L^2_\tau(\mathbf{R}) L^2_\xi(\mathbf{R})} < \infty.$$

For any given interval L, $X_{s,\alpha}(\mathbf{R} \times L)$ is the space of the restriction of all functions in $X_{s,\alpha}(\mathbf{R}^2)$ on $\mathbf{R} \times L$, and for $u \in X_{s,\alpha}(\mathbf{R} \times L)$ its norm is

$$\|u\|_{X_{s,\alpha}(\mathbf{R} \times L)} = \inf\left\{ \|U\|_{X_{s,\alpha}(\mathbf{R}^2)};\ U|_{\mathbf{R} \times L} = u \right\}.$$

When $L = [0,T]$, $X_{s,\alpha}(\mathbf{R} \times L)$ is abbreviated as $X^T_{s,\alpha}$. We always assume that ψ is a smooth function, $\psi_\delta(t) = \psi(\frac{t}{\delta})$, satisfying $0 \leqslant \psi \leqslant 1$, $\psi = 1$ when $t \in [-1,1]$, $\operatorname{supp} \psi \subset [-2,2]$. We define $\sigma = \tau + (-1)^n \xi^{2n+1}$ and $\sigma_j = \tau_j + (-1)^n \xi_j^{2n+1}$ $(j = 1, 2)$. We define

$$W(t)u_0 = C \int_{\mathbf{R}} e^{i(x\xi + t(-1)^n \xi^{2n+1})} \mathscr{F}_x u_0(\xi)\, d\xi$$

and

$$\|f\|_{L^q_t L^p_x} = \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}} \left| f(x,t) \right|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}, \qquad \|f\|_{L^p_t L^p_x} = \|f\|_{L^p_{xt}}.$$

We use $|X| \leqslant C_0|Y|$ to denote $X \preceq Y$, where C_0 is a generic positive constant. We denote $X \sim Y$ by $A_1|X| \leqslant |Y| \leqslant A_2|X|$, where $A_j > 0$ $(j = 1, 2)$, which may depend on C. C is a generic constant which may depend on n and may vary from line to line.

Obviously, (1.1), (1.2) are equivalent to the following integral equation

$$u(t) = W(t)u_0(x) - \frac{1}{2} \int_0^t W(t - \tau) \partial_x \left(u^2(\tau) \right) d\tau. \tag{1.7}$$

The main results of this paper are as follows.

Theorem 1.1. *Let $s > -n+\frac{1}{4}$, $n \in \mathbf{N}^+$. Then (1.1), (1.2) are locally well-posed for the initial data in $H^s(\mathbf{R})$.*

Theorem 1.2. *Let $s < -n + \frac{1}{4}$, $n \geqslant 2$, $n \in \mathbf{N}^+$. Then there does not exist any $T > 0$ such that (1.7) admits a unique local solution defined on the interval $[0, T]$ and such that the flow map*

$$u_0 \longrightarrow u, \quad t \in [0, T]$$

is C^2-differentiable at zero from $\dot{H}^s(\mathbf{R})$ to $C([0, T]; \dot{H}^s(\mathbf{R}))$.

Remark 1. When $n = 1$, (1.1), (1.2) are ill-posed if the flow map is uniformly continuous, see [4]. Thus we only consider the case $n \geqslant 2$ of (1.1), (1.2).

Remark 2. We believe that the local well-posedness for the case $s = -n + \frac{1}{4}$ can be obtained with the aid of the idea of [13].

Remark 3. In Theorem 1.2, inspired by [1], we choose $t = N^{-2n+\frac{2}{4n-3}}$.

Remark 4. In Theorem 1.2, for an arbitrary fixed $T > 0$, we can choose sufficiently N such that $t = N^{-2n+\frac{2}{4n-3}} < T$ for $n \geqslant 2$ and $n \in Z$.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, by using the Fourier restriction norm method, we establish a crucial bilinear estimate. In Section 4, we give the proof of Theorem 1.1. In Section 5, we give the proof of Theorem 1.2.

2. Preliminaries

Lemma 2.1. *Let $n \in \mathbf{N}^+$ and $\xi = \xi_1 + \xi_2$. Then*

$$\left|\xi^{2n+1} - \xi_1^{2n+1} - \xi_2^{2n+1}\right| \sim |\xi_{\min}||\xi_{\max}|^{2n}, \tag{2.1}$$

where $|\xi_{\min}| := \min\{|\xi|, |\xi_1|, |\xi_2|\}$ and $|\xi_{\max}| := \max\{|\xi|, |\xi_1|, |\xi_2|\}$.

Lemma 2.1 can be found in Lemma 2.5 in [31].

Lemma 2.2. *For $0 < \delta < 1$, $s \in \mathbf{R}$ and $\frac{1}{2} < b \leqslant 1$, we have*

$$\left\|\psi_\delta(t)W(t)u_0\right\|_{X_{s,b}} \leqslant C\delta^{\frac{1}{2}-b}\|u_0\|_{H^s}, \tag{2.2}$$

and for $-\frac{1}{2} < b' \leqslant 0 \leqslant b \leqslant b' + 1$, we have

$$\left\|\psi_\delta(t)\int_0^t W(t-\tau)f(\tau)\,d\tau\right\|_{X_{s,b}} \leqslant C\delta^{1+b'-b}\|f\|_{X_{s,b'}}. \tag{2.3}$$

Lemma 2.2 can be seen in Lemmas 3.1 and 3.2 of [8].

3. Bilinear estimates

In this section, we will prove a crucial bilinear estimate.

Lemma 3.1. *Let* $s \geqslant -n + \frac{1}{4} + (6n+3)\epsilon$, $n \in \mathbf{N}^+$, $b = \frac{1}{2} + 2\epsilon$, $b' = -\frac{1}{2} + 3\epsilon$, $0 < \epsilon \ll 1$. *Then*

$$\left\| \partial_x \left(\prod_{j=1}^{2} u_j \right) \right\|_{X_{s,b'}} \leqslant C \prod_{j=1}^{2} \|u_j\|_{X_{s,b}}. \tag{3.1}$$

Proof. Let

$$F_j(\xi_j, \tau_j) = \langle \xi_j \rangle^s \langle \sigma_j \rangle^b \mathscr{F} u_j(\xi_j, \tau_j) \quad (j = 1, 2),$$
$$F(\xi, \tau) = \langle \xi \rangle^{-s} \langle \sigma \rangle^{-b'} \mathscr{F} u(\xi, \tau).$$

By duality and the Plancherel's identity, to derive (3.1), it suffices to prove

$$\int_{\mathbf{R}^2} \int_{\substack{\xi=\xi_1+\xi_2 \\ \tau=\tau_1+\tau_2}} \frac{|\xi| \langle \xi \rangle^s |F| \prod_{j=1}^{2} |F_j|}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \sigma_j \rangle^b \langle \xi_j \rangle^s} \, d\xi_1 \, d\tau_1 \, d\xi \, d\tau \leqslant C \|F\|_{L^2_{\xi\tau}} \prod_{j=1}^{2} \|F_j\|_{L^2_{\xi\tau}}. \tag{3.2}$$

Without loss of generality, we can assume that $F_j(\xi_j, \tau_j)$ $(j = 1, 2) \geqslant 0$ and $F(\xi, \tau) \geqslant 0$ and $|\xi_1| \geqslant |\xi_2|$ since $|\xi_1|$ and $|\xi_2|$ is symmetrical. It is easily checked that $\{|\xi_2| \leqslant |\xi_1|\} \subset \bigcup_{j=1}^{7} \Omega_j$, where,

$$\begin{aligned}
&\Omega_1 = \{|\xi_2| \leqslant |\xi_1| \leqslant 1\}, \\
&\Omega_2 = \{4|\xi_2| < |\xi_1|, |\xi_2| \leqslant 1, |\xi_1| \geqslant 1\}, \\
&\Omega_3 = \{4|\xi_2| < |\xi_1|, |\xi_2| \geqslant 1, |\xi_1| \geqslant 1\}, \\
&\Omega_4 = \{|\xi_2| \leqslant 1 \leqslant |\xi_1| \leqslant 4|\xi_2|\}, \\
&\Omega_5 = \{1 \leqslant |\xi_2| \leqslant |\xi_1| \leqslant 4|\xi_2|, \xi_1\xi_2 \leqslant 0, 2|\xi| \leqslant |\xi_2|\}, \\
&\Omega_6 = \{1 \leqslant |\xi_2| \leqslant |\xi_1| \leqslant 4|\xi_2|, \xi_1\xi_2 \leqslant 0, |\xi_2| \leqslant 2|\xi|\}, \\
&\Omega_7 = \{1 \leqslant |\xi_2| \leqslant |\xi_1| \leqslant 4|\xi_2|, \xi_1\xi_2 \geqslant 0\}.
\end{aligned}$$

In this lemma, integrals over the subregion Ω_k's are respectively denoted as J_k $(1 \leqslant k \leqslant 7)$. Let

$$K_1(\xi_1, \tau_1, \xi, \tau) = \frac{|\xi| \langle \xi \rangle^s}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \sigma_j \rangle^b \langle \xi_j \rangle^s},$$

and

$$\mathscr{F} f_j = \frac{F_j}{\langle \sigma_j \rangle^b} \quad (j = 1, 2), \qquad \mathscr{F} f = \frac{F}{\langle \sigma \rangle^{-b'}}.$$

(1) **Subregion** $\{|\xi_2| \leqslant |\xi_1| \leqslant 1\}$. In this subregion,

$$K_1(\xi_1, \tau_1, \xi, \tau) \leqslant \frac{C}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \sigma_j \rangle^b}.$$

This case can be treated similarly to Subregion (1) of Lemma 3.2 of [25].

(2) **Subregion** $\{|\xi_1| > 4|\xi_2|, |\xi_2| \leqslant 1, |\xi_1| \geqslant 1\}$. In this subregion, $|\xi| \sim |\xi_1|$ which yields

$$K_1(\xi_1, \tau_1, \xi, \tau) \leqslant C \frac{|\xi|}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \sigma_j \rangle^b} \leqslant C \frac{|\xi_1^{2n} - \xi_2^{2n}|^{\frac{1}{2}}}{\langle \sigma \rangle^{-b'} \prod_{j=1}^{2} \langle \sigma_j \rangle^b}.$$

This case can be treated similarly to Subregion (2) of Lemma 3.3 of [25].

(3) **Subregion** $\{|\xi_1| > 4|\xi_2|, |\xi_2| \geqslant 1, |\xi_1| \geqslant 1\}$. In this subregion, $|\xi| \sim |\xi_1|$. By using (2.1), since $\xi = \xi_1 + \xi_2$, we have

$$3\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geqslant |\sigma - \sigma_1 - \sigma_2| = \left|\xi^{2n+1} - \xi_1^{2n+1} - \xi_2^{2n+1}\right| \\ \geqslant C|\xi_{\min}||\xi_{\max}|^{2n}. \tag{3.3}$$

(3.3) implies that one of the following three cases always occurs.

$$|\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geqslant C|\xi_{\min}||\xi_{\max}|^{2n}, \tag{3.4}$$

$$|\sigma_1| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geqslant C|\xi_{\min}||\xi_{\max}|^{2n}, \tag{3.5}$$

$$|\sigma_2| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geqslant C|\xi_{\min}||\xi_{\max}|^{2n}. \tag{3.6}$$

In the case $s \geqslant 0$, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leqslant C\frac{|\xi|}{\langle\sigma\rangle^{-b'}\prod_{j=1}^{2}\langle\sigma_j\rangle^b} \leqslant C\frac{|\xi_1^{2n} - \xi_2^{2n}|^{1/2}}{\langle\sigma\rangle^{-b'}\prod_{j=1}^{2}\langle\sigma_j\rangle^b},$$

can be treated similarly to Subregion (2) of Lemma 3.3 of [25].

In the case $s < 0$, we consider the cases (3.4), (3.5), (3.6), respectively.

Case (1). When (3.4) holds.

If $-s + b' \leqslant 0$, since $b' = -\frac{1}{2} + 3\epsilon$, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leqslant C\frac{|\xi|^{1+2nb'}|\xi_2|^{-s+b'}}{\prod_{j=1}^{2}\langle\sigma_j\rangle^b} \leqslant C\frac{|\xi_1|^{\frac{2n-1}{4}}}{\prod_{j=1}^{2}\langle\sigma_j\rangle^b}.$$

If $-s + b' \geqslant 0$, since $s \geqslant -n + \frac{1}{4} + (6n+3)\epsilon$ and $b' = -\frac{1}{2} + 3\epsilon$, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leqslant C\frac{|\xi|^{1+2nb'}|\xi_2|^{-s+b'}}{\prod_{j=1}^{2}\langle\sigma_j\rangle^b} \leqslant C\frac{|\xi|^{1-s+2nb'+b'}}{\prod_{j=1}^{2}\langle\sigma_j\rangle^b} \leqslant C\frac{|\xi_1|^{\frac{2n-1}{4}}}{\prod_{j=1}^{2}\langle\sigma_j\rangle^b}.$$

This case can be treated similarly to case 3.6 of Subregion (5) of Lemma 3.2 of [25].

Case (2). When (3.5) holds.

In this case, since $\langle\sigma\rangle^{b'+b} \leqslant \langle\sigma_1\rangle^{b'+b}$ which yields $\langle\sigma_1\rangle^{-b}\langle\sigma\rangle^{b'} \leqslant \langle\sigma_1\rangle^{b'}\langle\sigma\rangle^{-b}$, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leqslant C\frac{|\xi||\xi_2|^{-s}\langle\sigma_1\rangle^{b'}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b} \leqslant C\frac{|\xi|^{1+2nb'}|\xi_2|^{-s+b'}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b},$$

if $-s + b' \leqslant 0$, since $b' = -\frac{1}{2} + 3\epsilon$, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leqslant C\frac{|\xi|^{1+2nb'}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b} \leqslant C\frac{|\xi_2^{2n} - \xi^{2n}|^{\frac{1}{2}}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b},$$

if $-s + b' \geqslant 0$, since $s \geqslant -n + \frac{1}{4} + (6n+3)\epsilon$ and $b' = -\frac{1}{2} + 3\epsilon$, then

$$K_1(\xi_1, \tau_1, \xi, \tau) \leqslant C\frac{|\xi|^{1-s+b'+2nb'}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b} \leqslant C\frac{|\xi_2^{2n} - \xi^{2n}|^{\frac{1}{2}}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b}.$$

This case can be treated similarly to case 3.6 of $1 \leqslant n \leqslant 2$ of Subregion (4) of Lemma 3.2 of [25].

Case (3). When (3.6) holds.

In this case, since $\langle\sigma\rangle^{b'+b} \leqslant \langle\sigma_2\rangle^{b'+b}$ which yields $\langle\sigma_2\rangle^{-b}\langle\sigma\rangle^{b'} \leqslant \langle\sigma_2\rangle^{b'}\langle\sigma\rangle^{-b}$, we have

$$K_1(\xi_1,\tau_1,\xi,\tau) \leqslant C\frac{|\xi||\xi_2|^{-s}\langle\sigma_2\rangle^{b'}}{\langle\sigma_1\rangle^b\langle\sigma\rangle^b} \leqslant C\frac{|\xi|^{1+2nb'}|\xi_2|^{-s+b'}}{\langle\sigma_1\rangle^b\langle\sigma\rangle^b},$$

if $-s+b' \leqslant 0$, since $b' = -\frac{1}{2}+3\epsilon$, we have

$$K_1(\xi_1,\tau_1,\xi,\tau) \leqslant C\frac{|\xi|^{1+2nb'}}{\langle\sigma_1\rangle^b\langle\sigma\rangle^b} \leqslant C\frac{|\xi_1|^{\frac{2n-1}{4}}}{\langle\sigma_1\rangle^b\langle\sigma\rangle^b},$$

if $-s+b' \geqslant 0$, since $s \geqslant -n+\frac{1}{4}+(6n+3)\epsilon$ and $b' = -\frac{1}{2}+3\epsilon$, we have

$$K_1(\xi_1,\tau_1,\xi,\tau) \leqslant C\frac{|\xi|^{1-s+b'+2nb'}}{\langle\sigma_1\rangle^b\langle\sigma\rangle^b} \leqslant C\frac{|\xi_1|^{\frac{2n-1}{4}}}{\langle\sigma_1\rangle^b\langle\sigma\rangle^b}.$$

This case can be treated similarly to case 3.7 of Subregion (5) of Lemma 3.2 of [25].

(4) **Subregion** $\{|\xi_2| \leqslant 1 \leqslant |\xi_1| \leqslant 4|\xi_2|\}$. In this subregion,

$$K_1(\xi_1,\tau_1,\xi,\tau) \leqslant \frac{C}{\langle\sigma\rangle^{-b'}\prod_{j=1}^2\langle\sigma_j\rangle^b}.$$

This case can be treated similarly to Subregion (1) of Lemma 3.2 of [25].

(5) **Subregion** $\{1 \leqslant |\xi_2| \leqslant |\xi_1| \leqslant 4|\xi_2|, \xi_1\xi_2 \leqslant 0, 2|\xi| \leqslant |\xi_2|\}$. Obviously, in this subregion, since $\xi_1\xi_2 \leqslant 0$, $2|\xi| \leqslant |\xi_2|$ and $|\xi_2| \leqslant |\xi_1| \leqslant 4|\xi_2|$, we have

$$|\xi_1^{2n}-\xi_2^{2n}|^{\frac{1}{2}} \geqslant C|\xi|^{\frac{1}{2}}|\xi_1|^{n-\frac{1}{2}}, \qquad |\xi^{2n}-\xi_2^{2n}|^{\frac{1}{2}} \geqslant C|\xi_1|^n, \qquad |\xi^{2n}-\xi_1^{2n}|^{\frac{1}{2}} \geqslant C|\xi_1|^n$$

which can be seen in [25].

In the case $s \geqslant 0$, we have

$$K_1(\xi_1,\tau_1,\xi,\tau) \leqslant C\frac{|\xi|}{\langle\sigma\rangle^{-b'}\prod_{j=1}^2\langle\sigma_j\rangle^b} \leqslant C\frac{|\xi_1^{2n}-\xi_2^{2n}|^{1/2}}{\langle\sigma\rangle^{-b'}\prod_{j=1}^2\langle\sigma_j\rangle^b}.$$

This case can be treated similarly to Subregion (2) of Lemma 3.3 of [25].

In the case $s<0$, we consider cases (3.4), (3.5), (3.6), respectively.

Case (1). When (3.4) holds.

By using (3.4), since $-n+\frac{1}{4}+(6n+3)\epsilon \leqslant s < 0$ and $b' = -\frac{1}{2}+3\epsilon$, we have

$$\begin{aligned} K_1(\xi_1,\tau_1,\xi,\tau) &\leqslant C\frac{|\xi|^{1+b'}\prod_{j=1}^2|\xi_j|^{-s+nb'}}{\prod_{j=1}^2\langle\sigma_j\rangle^b} \leqslant C\frac{|\xi|^{\frac{1}{2}}|\xi_1|^{\frac{1}{2}-2s+(2n+1)b'}}{\prod_{j=1}^2\langle\sigma_j\rangle^b} \\ &\leqslant C\frac{|\xi|^{\frac{1}{2}}|\xi_1|^{n-\frac{1}{2}}}{\prod_{j=1}^2\langle\sigma_j\rangle^b} \leqslant C\frac{|\xi_1^{2n}-\xi_2^{2n}|^{\frac{1}{2}}}{\prod_{j=1}^2\langle\sigma_j\rangle^b}. \end{aligned}$$

This case can be treated similarly to case 3.6 of $1 \leqslant n \leqslant 2$ of Subregion (4) of Lemma 3.3 of [25].

Case (2). When (3.5) holds.

In this case, since $\langle\sigma\rangle^{b'+b} \leqslant \langle\sigma_1\rangle^{b'+b}$ which yields $\langle\sigma_1\rangle^{-b}\langle\sigma\rangle^{b'} \leqslant \langle\sigma_1\rangle^{b'}\langle\sigma\rangle^{-b}$, since $-n+\frac{1}{4}+(6n+3)\epsilon \leqslant s<0$ and $b'=-\frac{1}{2}+3\epsilon$, we have

$$K_1(\xi_1,\tau_1,\xi,\tau) \leqslant C\frac{|\xi|\langle\sigma_1\rangle^{b'}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b\prod_{j=1}^2\langle\xi_j\rangle^s} \leqslant C\frac{|\xi|^{1+b'}\prod_{j=1}^2|\xi_j|^{-s+nb'}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b} \leqslant C\frac{|\xi_1|^{1-2s+(2n+1)b'}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b}$$
$$\leqslant C\frac{|\xi_1|^n}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b} \leqslant C\frac{|\xi^{2n}-\xi_2^{2n}|^{\frac{1}{2}}}{\langle\sigma\rangle^b\langle\sigma_2\rangle^b}.$$

This case can be treated similarly to case 3.6 of $1\leqslant n\leqslant 2$ of Subregion (4) of Lemma 3.3 of [25].

Case (3). When (3.6) holds.

By using $|\xi^{2n}-\xi_1^{2n}|^{\frac{1}{2}} \geqslant C|\xi_1|^n$ and an idea similar to Case (3.5), in this case we obtain

$$J_5 \leqslant C\|F\|_{L^2_{\xi\tau}}\prod_{j=1}^2\|F_j\|_{L^2_{\xi\tau}}.$$

(6) **Subregion** $\{1\leqslant|\xi_2|\leqslant|\xi_1|\leqslant 4|\xi_2|, \xi_1\xi_2\leqslant 0, |\xi_2|\leqslant 2|\xi|\}$. In this subregion, $|\xi_1|\sim|\xi_2|\sim|\xi|$ which yields

$$K_1(\xi_1,\tau_1,\xi,\tau) \leqslant C\frac{|\xi_1|^{1-s}}{\langle\sigma\rangle^{-b'}\prod_{j=1}^2\langle\sigma_j\rangle^b}.$$

We consider (3.4), (3.5), (3.6), respectively.

Case (1). When (3.4) holds.

Since $-n+\frac{1}{4}+(6n+3)\epsilon \leqslant s<0$ and $b'=-\frac{1}{2}+3\epsilon$, we have

$$K_1(\xi_1,\tau_1,\xi,\tau) \leqslant C\frac{|\xi_1|^{1-s+(2n+1)b'}}{\prod_{j=1}^2\langle\sigma_j\rangle^b} \leqslant C\frac{|\xi_1|^{\frac{2n-1}{4}}}{\prod_{j=1}^2\langle\sigma_j\rangle^b}.$$

This case can be treated similarly to case 3.6 of Subregion (5) of Lemma 3.2 of [25].

Case (2). When (3.5) holds.

Since $\langle\sigma\rangle^{b'+b} \leqslant \langle\sigma_1\rangle^{b'+b}$ which yields $\langle\sigma_1\rangle^{-b}\langle\sigma\rangle^{b'} \leqslant \langle\sigma_1\rangle^{b'}\langle\sigma\rangle^{-b}$, by a calculation similar to Case (3.4), we have

$$K_1(\xi_1,\tau_1,\xi,\tau) \leqslant C\frac{|\xi_2|^{\frac{2n-1}{4}}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b}.$$

This case can be treated similarly to case 3.7 of Subregion (5) of Lemma 3.2 of [25].

Case (3). When (3.6) holds.

This case can be treated similarly to case 3.7 of Subregion (5) of Lemma 3.2 of [25].

(7) **Subregion** $\{1\leqslant|\xi_2|\leqslant|\xi_1|\leqslant 4|\xi_2|, \xi_1\xi_2\geqslant 0\}$.

This case can be treated similarly to subregion of Lemma 3.2 of [25].

Consequently, by putting the estimates of $J_k(1\leqslant k\leqslant 7)$ together, we obtain

$$\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} K_1(\xi_1,\tau_1,\xi,\tau)F\prod_{j=1}^2 F_j\,d\xi_1\,d\tau_1\,d\xi\,d\tau \leqslant C\|F\|_{L^2_{\xi\tau}}\prod_{j=1}^2\|F_j\|_{L^2_{\xi\tau}}. \tag{3.7}$$

Thus we complete the proof of Lemma 3.1.

4. Proof of Theorem 1.1

In order to prove Theorem 1.1, firstly, for $u_0 \in H^s(\mathbf{R})$ and $\delta \in (0,1]$, $v \in X_{s,b}(\mathbf{R}^2)$, we define $G_{u_0}(v)$ by

$$G_{u_0}(v) = \psi(t)W(t)u_0 - \frac{1}{2}\psi_\delta \int_0^t W\left(t-t'\right)\left(\partial_x v^2\right) dt'. \tag{4.1}$$

Applying Lemma 2.2 and Lemma 3.1 to (4.1), we conclude that for a certain constant C,

$$\left\|G_{u_0}(v)\right\|_{X_{s,b}(\mathbf{R}^2)} \leqslant C\|u_0\|_{H^s(\mathbf{R})} + C\delta^{b'+1-b}\|v\|^2_{X_{s,b}(\mathbf{R}^2)}, \tag{4.2}$$

where s, b, b' of (4.2) concords with s, b and b' of Lemma 3.1. Let

$$\delta = \left(\frac{1}{8C^2(\|u_0\|_{H^s(\mathbf{R})}+2)}\right)^{\frac{1}{b'+1-b}}, \quad \text{and} \quad r = 2C\|u_0\|_{H^s(\mathbf{R})}, \tag{4.3}$$

where $0 < \delta < 1$, thus based on (4.2) and (4.3), we obtain that G is a mapping from the closed ball $B(0,r) = \{u \in X_{s,b}(\mathbf{R}^2),\ \|u\|_{X_{s,b}(\mathbf{R}^2)} \leqslant r\}$ into itself. By a similar calculation, we have

$$\begin{aligned}\left\|G_{u_0}(v) - G_{u_0}(u)\right\|_{X_{s,b}(\mathbf{R}^2)} &\leqslant C\delta^{b'+1-b}\|v-u\|_{X_{s,b}(\mathbf{R}^2)}\left(\|u\|_{X_{s,b}(\mathbf{R}^2)} + \|v\|_{X_{s,b}(\mathbf{R}^2)}\right)\\ &\leqslant \frac{1}{2}\|v-u\|_{X_{s,b}(\mathbf{R}^2)},\end{aligned}$$

thus G is a contraction mapping from the closed ball $B(0,r) = \{u \in X_{s,b}(\mathbf{R}^2),\ \|u\|_{X_{s,b}(\mathbf{R}^2)} \leqslant r\}$ into itself, by using Banach fixed point theorem, we have $G_{u_0}(v) = v$. The rest of local well-posedness of Theorem 1.1 follow from a standard proof.Consequently, we complete the proof of Theorem 1.1.

5. Proof of Theorem 1.2

This section is devoted to proving Theorem 1.2. We give Theorem 5.1 before proving Theorem 1.2.

Theorem 5.1. *Let $s < -n + \frac{1}{4}$ and T be a positive real number. Then there does not exist a space Y_T continuously embedded in $C([0,T];\dot{H}^s(\mathbf{R}))$ such that*

$$\left\|W(t)u_0\right\|_{Y_T} \leqslant C\|u_0\|_{\dot{H}^s}, \quad \forall u_0 \in \dot{H}^s(\mathbf{R}), \tag{5.1}$$

$$\left\|\int_0^t W(t-\tau)\partial_x\left(u^2(\tau)\right) d\tau\right\|_{Y_T} \leqslant C\|u\|^2_{Y_T}, \quad \forall u \in Y_T. \tag{5.2}$$

Proof. We assume that $u = W(t)u_0$ in (5.2) and Y_T satisfies (5.1), (5.2), since Y_T is continuously embedded in $C([0,T];\dot{H}^s(\mathbf{R}))$, for any $t \in [0,T]$, we have

$$\left\|\int_0^t W(t-\tau)\partial_x\left(W(t)u_0\right)^2 d\tau\right\|_{\dot{H}^s} \leqslant C\|u_0\|^2_{\dot{H}^s}, \quad \text{for all } u_0 \in Y_T. \tag{5.3}$$

We prove that (5.3) fails by choosing

$$\mathscr{F}_x u_{0N}(x) = \gamma^{-1/2}N^{-s}\left(\chi_{I_N}(\xi) + \chi_{I_N}(-\xi)\right),$$

where $I_N = [N, N + 2\gamma]$ and $N \gg 1$ and γ will be chosen later. Thus

$$\|u_{0N}\|_{\dot H^s} \sim 1.$$

We define

$$u_{2,N} = \int_0^t W(t-\tau)\partial_x \big(W(t)u_{0N}\big)^2\, d\tau.$$

Thus

$$u_{2,N} = C(f-g),$$

where

$$f = \gamma^{-1} N^{-2s} \int_{K_\xi} \frac{(\xi_1+\xi_2)e^{ix(\xi_1+\xi_2)+it(\phi(\xi_1)+\phi(\xi_2))}}{\phi(\xi_1)+\phi(\xi_2)-\phi(\xi_1+\xi_2)}\, d\xi_1,$$

$$g = \gamma^{-1} N^{-2s} \int_{K_\xi} \frac{(\xi_1+\xi_2)e^{ix(\xi_1+\xi_2)+it\phi(\xi_1+\xi_2)}}{\phi(\xi_1)+\phi(\xi_2)-\phi(\xi_1+\xi_2)}\, d\xi_1.$$

Thus we deduce that

$$\begin{aligned}\big\|u_{2,N}(t)\big\|^2_{\dot H^s} &\geqslant \int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}} |\xi|^{2s} \big|\mathscr{F}_x u_{2,N}(t,\xi)\big|^2\, d\xi \\ &= N^{-4s}\gamma^{-2} \int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}} |\xi|^{2s}|\xi|^2 \left| \int_{K_\xi} \frac{e^{it(\phi(\xi_1)+\phi(\xi_2)-\phi(\xi_1+\xi_2))-1}}{\phi(\xi_1)+\phi(\xi_2)-\phi(\xi_1+\xi_2)}\, d\xi_1 \right|^2 d\xi,\end{aligned}$$

where

$$K_\xi = \{\xi_1 \mid \xi-\xi_1 \in I_N,\ \xi_1 \in -I_N\} \cup \{\xi_1 \mid \xi_1 \in I_N,\ \xi-\xi_1 \in -I_N\}.$$

Notice that $mes(K_\xi) \geqslant \gamma$ where mes denotes the Lebesgue measure. Thus we have

$$\big\|u_{2,N}(t)\big\|^2_{\dot H^s} \geqslant N^{-4s}\gamma^{-2}\gamma^{2s}\gamma^2\gamma N^{-4n}\gamma^{-2}\gamma^2 = N^{-4s-4n}\gamma^{1+2s}.$$

Taking $\gamma = N^{-\frac{2}{4n-3}}$ and $t = N^{-2n+\frac{2}{4n-3}}$ which yields

$$t\big|\big(\phi(\xi_1)+\phi(\xi_2)-\phi(\xi_1+\xi_2)\big)\big| \sim t|\xi||\xi_1|^{2n} \sim 1$$

resulting from (2.1), then we have

$$\big\|u_{2,N}(t)\big\|^2_{\dot H^s} \geqslant N^{-4s-4n}N^{-\frac{4s+2}{4n-3}} = N^{-4s\frac{4n-2}{4n-3}-4n-\frac{2}{4n-3}}. \tag{5.4}$$

When $s < -n + \frac{1}{4}$ which yields $-4s\frac{4n-2}{4n-3} - 4n - \frac{2}{4n-3} > 0$, (5.4) contradicts with (5.3) for sufficiently N.

We complete the proof of Theorem 5.1.

Now we prove Theorem 1.2.

Let u be a solution of (1.1), (1.2), then we have

$$u(x,t,u_0) = W(t)u_0 - \frac{1}{2}\int_0^t W(t-\tau)\partial_x\big(u(\cdot,\tau,u_0)^2\big)\,d\tau.$$

Suppose that the flow-map is C^2. By using the fact that $u(x,t,0)=0$, we derive

$$u_1(x,t) = \frac{\partial u}{\partial u_0}(x,t,0)[h] = W(t)h,$$
$$u_2(x,t) = \frac{\partial^2 u}{\partial^2 u_0}(x,t,0)[h,h] = -\int_0^t W(t-\tau)\partial_x\big(W(t)h\big)^2\,d\tau.$$

By using the fact that the flow-map is C^2, we derive

$$\big\|u_2(t)\big\|_{\dot{H}^s} \leqslant C\|h\|_{\dot{H}^s}^2, \quad \forall h \in \dot{H}^s(\mathbf{R}). \tag{5.5}$$

By using Theorem 5.1, we have that (5.5) does hold. Consequently, we have completed the proof of Theorem 1.2.

Acknowledgments

The authors are deeply indebted to Prof. Yulin Zhao, Prof. Zheng-an Yao of Sun Yat-sen University and Prof. Chun-long Zheng of Shaoguan University for giving some helpful and suggest. This work is supported by the NSF of China (No. 11171355), Foundation for Distinguished Young Talents in Higher Education of Guangdong Province of China (No. 2013LYM0081), Guangdong Provincial NSF of China (No. S2012010010069), the Shaoguan Science and Technology Foundation (No. 313140546) and Science Foundation of Shaoguan University. The authors thank the handling editor and the reviewers for their valuable comments and suggestions, which improved the completeness of the paper.

References

[1] I. Bejennaru, T. Tao, Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation, J. Funct. Anal. 233 (2006) 228–259.

[2] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, Part I: Schrodinger equation, Geom. Funct. Anal. 3 (1993) 107–156;
J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, Part II: The KdV equation, Geom. Funct. Anal. 3 (1993) 209–262.

[3] J. Bourgain, Periodic Korteweg de Vries equation with measures as initial data, Selecta Math. (N.S.) 3 (1997) 115–159.

[4] M. Christ, J. Colliander, T. Tao, Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocussing equations, Amer. J. Math. 125 (2003) 1235–1293.

[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Global well-posedness for the KdV in Sobolev spaces of negative indices, Electron. J. Differential Equations 26 (2001) 1–7.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Sharp global well-posedness for KdV and modified KdV on **R** and **T**, J. Amer. Math. Soc. 16 (2003) 705–749.

[7] I.M. Gel'fand, I.Ya. Dorfman, Hamiltonian operators and infinite dimensional Lie algebras, Funktsional. Anal. i Prilozhen. 15 (1981) 23–40.

[8] J. Ginibre, Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace (d'après Bourgain), Asteŕisque 237 (1996) 163–187.

[9] J. Gorsky, A.A. Himonas, Well-posedness of KdV equation with higher dispersion, Math. Comput. Simulation 80 (2009) 173–183.

[10] J. Gorsky, A. Himonas, C. Holliman, G. Petronilho, The Cauchy problem of a periodic higher order KdV equation in analytic Gevrey spaces, J. Math. Anal. Appl. 405 (2013) 349–361.

[11] A. Grünrock, New applications of the Fourier restriction norm method to well-posedness problems for nonlinear evolution equations, Dissertation, University of Wuppertal, 2002.
[12] A. Grünrock, An improved local well-posedness result for the modified KdV equation, Int. Math. Res. Not. IMRN 61 (2004) 1488–1503.
[13] Z.H. Guo, Global well-posedness of the Korteweg–de Vries equation in $H^{-\frac{3}{4}}(\mathbf{R})$, J. Math. Pures Appl. 91 (2009) 583–597.
[14] S. Herr, Well-posedness results for dispersive equations with derivative nonlinearities, Dissertation, Dem Fachbereich Mathematik der Universität Dortmund vorgelegt von, 2006.
[15] H. Hirayama, Local well-posedness for the periodic higher order KdV type equations, NoDEA Nonlinear Differential Equations Appl. 19 (2012) 677–693.
[16] A.D. Ionescu, C.E. Kenig, Global well-posedness of the Benjamin–Ono equation in low-regularity spaces, J. Amer. Math. Soc. 20 (2007) 753–798.
[17] A.D. Ionescu, C.E. Kenig, D. Tataru, Global well-posedness of the KP-I initial-value problem in the energy space, Invent. Math. 173 (2008) 265–304.
[18] T. Kato, On the Cauchy problem for the generalized Korteweg–Vries equation, Adv. Math. Suppl. Stud. 8 (1983) 93–128.
[19] C.E. Kenig, G. Ponce, L. Vega, Well-posedness of the initial value problem for the Korteweg–de Vries equation, J. Amer. Math. Soc. 4 (1991) 323–347.
[20] C.E. Kenig, G. Ponce, L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Mah. J. 40 (1991) 33–69.
[21] C.E. Kenig, G. Ponce, L. Vega, The Cauchy problem for the Korteweg–de Vries equation in Sobolev spaces of negative indices, Duke Math. J. 71 (1993) 1–21.
[22] C.E. Kenig, G. Ponce, L. Vega, Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle, Comm. Pure Appl. Math. 46 (1993) 527–620.
[23] C.E. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc. 9 (1996) 573–603.
[24] C.E. Kenig, G. Ponce, L. Vega, On the ill-posedness of some canonical dispersive equations, Duke Math. J. 106 (2001) 617–633.
[25] Y.S. Li, W. Yan, X.Y. Yang, Well-posedness of a higher modified Camassa–Holm equation in spaces of low regularity, J. Evol. Equ. 10 (2010) 465–486.
[26] P.J. Olver, Applications of Lie Groups to Differential Equations, second edition, Springer-Verlag, 2000.
[27] J. Saut, R. Temam, Remark on the Korteweg–de Vries equation, Israel J. Math. 24 (1976) 78–87.
[28] A. Sidi, C. Sulem, P.-L. Sulem, On the long time behavior of a generalized KdV equation, Acta Appl. Math. 7 (1986) 35–47.
[29] A. Sjöberg, On the Korteweg–de Vries equation: existence and uniqueness, J. Math. Anal. Appl. 29 (1970) 569–579.
[30] N. Tzvetkov, Remark on the local ill-posedness for KdV equation, C. R. Acad. Sci. Paris 329 (1999) 1043–1047.
[31] W. Yan, Y.S. Li, S.M. Li, Sharp well-posedness and ill-posedness of a higher-order modified Camassa–Holm equation, Differential Integral Equations 25 (2012) 1053–1074.