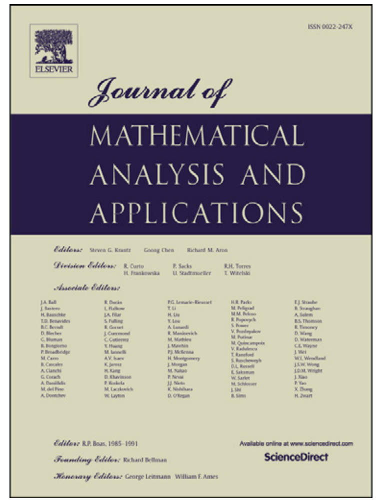
Accepted Manuscript

A criterion for the explicit reconstruction of a holomorphic function from its restrictions on lines

Amadeo Irigoyen



PII: S0022-247X(14)00542-3
DOI: 10.1016/j.jmaa.2014.06.004
Reference: YJMAA 18595

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 7 February 2014

A CRITERION FOR THE EXPLICIT RECONSTRUCTION OF A HOLOMORPHIC FUNCTION FROM ITS RESTRICTIONS ON LINES

AMADEO IRIGOYEN

ABSTRACT. We deal with a problem of the explicit reconstruction of any holomorphic function f on a ball of $\mathbb{C}^2$ from its restrictions on a union of complex lines. The validity of such a reconstruction essentially depends on the mutual repartition of these lines. This criterion can be analytically described and it is also possible to give geometrical sufficient conditions. The motivation of this problem also comes from possible applications in mathematical economics and medical imaging.

CONTENTS

1. INTRODUCTION

1.1. **Presentation of the problem and first results.**

1.1.1. *General formulation of the problem.* In this paper we deal with a problem of the reconstruction of a holomorphic function from its restrictions on analytic submanifolds. f being a holomorphic function on a domain $\Omega \subset \mathbb{C}^n$ and $\{Z_j\}_{j=1}^N$ a family of analytic submanifolds of Ω, we want to find f from the data $f_{|\{Z_j\}_{j=1}^N} := \{f_{|Z_j}\}_{j=1}^N$. One can give interpolating functions $f_N \in \mathcal{O}(\Omega)$ that satisfy $f_{N|\{Z_j\}_{j=1}^N} = f_{|\{Z_j\}_{j=1}^N}$ (for example if Ω is convex, strictly pseudoconvex or $\Omega = \mathbb{C}^n$, see [1]), but generally $f_N \neq f$. Then a natural way is to consider an infinite family of submanifolds $\{Z_j\}_{j=1}^\infty$ and construct the associated interpolating $f_{\{Z_j\}_{j\geq 1}}$ as $\lim_{N\to\infty} f_N$. In this case the uniqueness of the interpolating function will certainly be guaranteed but without any assurance of the convergence of the sequence $(f_N)_{N\geq 1}$. Moreover, this motivates the research of explicit reconstruction formulas.

1.1.2. *An explicit interpolation formula.* Here we deal with the case of $\mathbb{C}^2$, $\Omega = B_2(0, r_0) \subset \mathbb{C}^2$ (where $B_2(0, r_0) = \{z \in \mathbb{C}^2,\ |z_1|^2 + |z_2|^2 < r_0^2\}$), and a family of distinct complex lines that cross the origin. Such a family can be described as

$$\left\{\{z \in \mathbb{C}^2,\ z_1 - \eta_j z_2 = 0\}\right\}_{j\geq 1}, \tag{1.1}$$

with $\eta_j \in \mathbb{C}$ all different, that we will simply denote by $\eta = \{\eta_j\}_{j\geq 1}$ (w.l.o.g. we can forget the line $\{z_2 = 0\}$ that is associated to $\eta_0 = \infty$). On the other hand, $f \in \mathcal{O}(B_2(0, 1))$ being given, a way to give one interpolating function f_N is the one that uses one of the essential ideas from [1], whose computation exploits residues and principal values (see [3] and [7]) and whose motivation is to get a formula that fixes any polynomial function with degree smaller than N. $S_2(0,1)$ being the unit sphere, one has $\forall\, z \in B_2(0,1)$,

$$\begin{aligned} f(z) &= \lim_{\varepsilon\to 0} \frac{1}{(2\pi i)^2} \int_{\zeta\in S_2(0,1), \left|\prod_{j=1}^N(\zeta_1-\eta_j\zeta_2)\right|=\varepsilon} \frac{f(\zeta)\det\left(\overline{\zeta}, P_N(\zeta,z)\right)\omega(\zeta)}{\prod_{j=1}^N(\zeta_1-\eta_j\zeta_2)(1-<\overline{\zeta},z>)} \\ &\quad - \lim_{\varepsilon\to 0} \frac{\prod_{j=1}^N(z_1-\eta_j z_2)}{(2\pi i)^2} \int_{\zeta\in S_2(0,1), \left|\prod_{j=1}^N(\zeta_1-\eta_j\zeta_2)\right|>\varepsilon} \frac{f(\zeta)\,\omega'\left(\overline{\zeta}\right)\wedge\omega(\zeta)}{\prod_{j=1}^N(\zeta_1-\eta_j\zeta_2)(1-<\overline{\zeta},z>)^2}\,, \end{aligned}$$

where $\omega'(\zeta) = \zeta_1 d\zeta_2 - \zeta_2 d\zeta_1$, $\omega(\zeta) = d\zeta_1 \wedge d\zeta_2$, and $P_N(\zeta,z) \in \left(\mathcal{O}\left(\mathbb{C}^2\times\mathbb{C}^2\right)\right)^2$ satisfies $\forall\,(\zeta,z)\in\mathbb{C}^2\times\mathbb{C}^2$,

$$< P_N(\zeta), \zeta - z > = P_{N,1}(\zeta,z)(\zeta_1-z_1) + P_{N,2}(\zeta,z)(\zeta_2-z_2) = \prod_{j=1}^N(\zeta_1-\eta_j\zeta_2) - \prod_{j=1}^N(z_1-\eta_j z_2)\,.$$

Both integrals can be explicited and yield the following relation: let $f \in \mathcal{O}\left(B_2(0,r_0)\right)$ (resp. $f\in\mathcal{O}\left(\mathbb{C}^2\right)$), one has $\forall\, z\in B_2(0,r_0)$ (resp. $z\in\mathbb{C}^2$),

$$f(z) = E_N(f;\eta)(z) - R_N(f;\eta)(z) + \sum_{k+l\geq N} a_{k,l} z_1^k z_2^l\,, \tag{1.2}$$

where $\sum_{k,l\geq 0} a_{k,l} z_1^k z_2^l$ is the Taylor expansion of f,

$$\begin{aligned} E_N(f;\eta)(z) &:= \sum_{p=1}^N \left(\prod_{j=p+1}^N (z_1-\eta_j z_2)\right) \sum_{q=p}^N \frac{1+\eta_p\overline{\eta_q}}{1+|\eta_q|^2} \frac{1}{\prod_{j=p, j\neq q}^N(\eta_q-\eta_j)} \times \\ &\quad \times \sum_{m\geq N-p} \left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{m-N+p} \frac{1}{m!}\frac{\partial^m}{\partial v^m}|_{v=0}[f(\eta_q v, v)] \end{aligned} \tag{1.3}$$

and

$$R_N(f;\eta)(z) := \sum_{p=1}^N \left(\prod_{j=1, j\neq p}^N \frac{z_1-\eta_j z_2}{\eta_p-\eta_j}\right) \sum_{k+l\geq N} a_{k,l}\eta_p^k \left(\frac{z_2+\overline{\eta_p}z_1}{1+|\eta_p|^2}\right)^{k+l-N+1}. \tag{1.4}$$

This relation is an application of the main theorem from [8] that is a more general version for the case of multiple complex lines (i.e. with the restriction of f and its first derivatives on every line). A direct proof of (1.2) is also given in the Appendix (Proposition 3). On the other hand, the formula $E_N(f;\eta)$ is well-defined and has the following properties:

- $E_N(f;\eta) \in \mathcal{O}\left(B_2(0,r_0)\right)$ (resp. $E_N(f;\eta)\in\mathcal{O}\left(\mathbb{C}^2\right)$);
- $E_N(f;\eta)$ is an explicit formula that is constructed with the data $\left\{f_{|\{z_1=\eta_j z_2\}}\right\}_{1\leq j\leq N}$;
- $\forall\, j=1,\ldots,N$, $E_N(f;\eta)_{|\{z_1=\eta_j z_2\}} = f_{|\{z_1=\eta_j z_2\}}$.

In addition, $E_N(f;\eta)$ is essentially the unique formula that fixes any polynomial function with bounded degree: $\forall\, P\in\mathbb{C}[z_1,z_2]$ with $\deg P \leq N-1$, $E_N(P;\eta)\equiv P$.

As $N\to\infty$, the function $f - E_N(f;\eta)$ will be a holomorphic function that will vanish on an increasing number of lines. If $E_N(f;\eta)$ were uniformly bounded on any compact subset (in particular if it converged to some function), then by the Stiltjes-Vitali-Montel Theorem, there would be a subsequence of $f-E_N(f;\eta)$ that would converge to a holomorphic function that would vanish on an infinite number of lines, so this limit would be 0 (in fact, the whole sequence $E_N(f;\eta)$ would converge to f uniformly on any compact subset).

1.1.3. *Applications in Radon transform theory.* Our reconstruction problem is also motivated by possible applications in real Radon transform theory that may have consequences in mathematical economics and medical imaging. Let μ be a measure with compact support $K \subset \mathbb{R}^2$ (w.l.o.g. one can assume that $0 \in K$). We want to reconstruct it from the knowledge of its Radon transforms on a finite number of directions, i.e. from $(\mathcal{R}\mu)\left(\theta^{(j)}, s\right)$ with $\left(\theta^{(j)}, s\right) \in \mathbb{S}^1 \times \mathbb{R}$ and $j = 1, \dots, N$, where $\mathbb{S}^1$ is the unit sphere of $\mathbb{R}^2$ and

$$(\mathcal{R}\mu)\left(\theta^{(j)}, s\right) \quad := \quad \frac{\partial}{\partial s} \int_{\{x \in \mathbb{R}^2,\, \theta_1^{(j)} x_1 + \theta_2^{(j)} x_2 \le s\}} \mu(dx)\,. \tag{1.5}$$

The way is the following: we use some properties of the Fantappie transform of μ (see [13]). We consider the dual space $K^\star \subset \mathbb{C}\mathbf{P}^2$ (the projective complex space) that is defined as the open set of the complex lines ξ of $\mathbb{C}^2 \supset \mathbb{R}^2 \supset K$ that do not cross K, i.e. $K^\star := \left\{\xi = [\xi_0 : \xi_1 : \xi_2] \in \mathbb{C}\mathbf{P}^2,\, < \xi, x > \neq 0,\, \forall\, x \in K\right\}$, where $< \xi, x > := \xi_0 + \xi_1 x_1 + \xi_2 x_2$. The Fantappie transform of μ is defined by

$$\begin{array}{rcl} \Phi_\mu \;:\; K^\star & \longrightarrow & \mathbb{C} \\ \xi & \mapsto & < \mu, \dfrac{\xi_0}{< \xi, x >} > := \displaystyle\int_{x \in K} \frac{\xi_0}{< \xi, x >}\, \mu(dx)\,. \end{array} \tag{1.6}$$

This function is well-defined and holomorphic on $K^\star$. $\mathbb{C}^2 \subset \mathbb{C}\mathbf{P}^2$ being the affine space of $\mathbb{C}\mathbf{P}^2$ defined by the canonical identification $z \in \mathbb{C}^2 \mapsto [1 : z_1 : z_2] \in \left\{\xi \in \mathbb{C}\mathbf{P}^2,\, \xi_0 \neq 0\right\}$, a classical calculation yields $r_K > 0$ such that $B_2(0, r_K) \subset K^\star$ and, for all $\theta \in \mathbb{S}^2$ and all $u \in \mathbb{C}$ with $|u| < r_K$,

$$\Phi_\mu([1 : u\theta_1 : u\theta_2]) \quad = \quad \int_{-\infty}^{+\infty} \frac{(\mathcal{R}\mu)(\theta, s)}{1 + s\, u}\, ds\,.$$

It follows that the knowledge of $(\mathcal{R}\mu)(\theta^{(j)}, s)$, $j = 1, \dots, N$, $s \in \mathbb{R}$, allows to know the restriction of $\Phi_\mu \in \mathcal{O}\left(B_2(0, r_K)\right)$ on every line $L_{\theta^{(j)}} = \{(u\theta_1, u\theta_2),\, u \in \mathbb{C}\} = \{z \in \mathbb{C}^2,\, z_1 = \eta_j z_2\}$ where

$$\eta_j \quad = \quad \theta_1^{(j)} / \theta_2^{(j)} \;\in \mathbb{R},\; j = 1, \dots, N \tag{1.7}$$

(w.l.o.g. one can assume that $\theta_2^{(j)} \neq 0$). Thus, if the interpolation formula $E_N(\Phi_\mu; \eta)$ converges to Φ_μ, then by the Martineau's isomorphism theorem (see [13]), it will be possible to give an explicit family of measures μ_N, $N \ge 1$ (defined from $E_N(\Phi_\mu; \eta)$ under the reciprocal isomorphism Φ^{-1}) that will converge to μ in an appropriate topology.

1.1.4. *A first observation.* The essential problem is that we do not have any control *a priori* of the function $E_N(f; \eta)$ and do not have any idea if $E_N(f; \eta)$ is always uniformly bounded for any given f. Indeed, the following result that will be justified below gives explicit positive and negative examples of sets $\{\eta_j\}_{j \ge 1}$.

Proposition 1. (1) *Let consider the following sequence defined as*

$$\eta_j \quad := \quad \frac{i^j}{j}\,,\; \forall\, j \ge 1\,.$$

Then the associated interpolation formula $E_N(\cdot; \eta)$ does not converge, i.e. there exists $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ such that $E_N(f; \eta)$ does not converge (uniformly in any compact subset $K \subset \mathbb{C}^2$). Similarly, for all $\varepsilon > 0$, there is $f \in \mathcal{O}\left(B_2(0, r_0)\right)$ and a compact subset $K \subset B_2(0, \varepsilon r_0)$ such that $E_N(f; \eta)$ does not converge in K.

(2) *Let consider any set* $\{\eta_j\}_{j\geq 1} \subset \mathbb{R}$. *Then the associated interpolation formula converges, i.e. for all* $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, $E_N(f;\eta)$ *converges to* f *uniformly on any compact subset of* $\mathbb{C}^2$. *Similarly,* r_0 *being given, there is* $\varepsilon_\eta > 0$ *such that, for all* $f \in \mathcal{O}\left(B_2(0,r_0)\right)$, $E_N(f;\eta)$ *converges to* f *uniformly on any compact subset of* $B_2(0,\varepsilon_\eta r_0)$.

The same holds true if we consider any set $\{\eta_j\}_{j\geq 1} \subset \mathbb{C}$ *with* $|\eta_j| = 1$, $\forall\, j \geq 1$.

On one hand, this fact leads to the following questions: first, which are the sets $\{\eta_j\}_{j\geq 1}$ whose interpolation formula $E_N(\cdot;\eta)$ will converge or will not? Next, is there some criterion that allows to know if a given set $\{\eta_j\}_{j\geq 1}$ will make converge (or will not) its associated interpolation formula $E_N(\cdot;\eta)$? These questions will be answered in the following subsection.

On the other hand, the second part of Proposition 1 can be applied in our reconstruction problem in Radon transform theory (the associated η_j's are real by (1.7)). It follows that the measure μ can be reconstructed in an appropriate topology by an explicit family of interpolating measures μ_N (μ_N interpolates μ in the meaning that, for all $N \geq 1$ and $k,\, l \geq 0$ with $k + l \leq N$, $< \mu_N, x_1^k x_2^l > = < \mu, x_1^k x_2^l >$). In addition, an application of some results of Henkin and Shananin from [6] will allow to give some good precision for this reconstruction. These expected estimates may also be compared to the one of Logan and Shepp from [12] where they establish the optimal reconstruction formula for μ in the special case of uniformly distributed lines $\theta^{(j)}$'s. This would allow to give some prognosis for the ability of our reconstruction formula $E_N(\cdot;\eta)$, at least in the case of real η_j's. All these results are handled in [11] that is currently in progress.

1.2. **An equivalent criterion.** Before giving the first essential result, we need to consider the following operator of divided differences Δ_p of any function φ that is defined on the η_j's and that looks like the discrete derivative of φ with order p (see [5]):

$$\begin{aligned} \Delta_{0,\emptyset}(\varphi)(\eta_1) &= \varphi(\eta_1), \\ \forall\, p \geq 1,\ \Delta_{p,(\eta_p \ldots,\eta_1)}(\varphi)(\eta_{p+1}) &= \frac{\Delta_{p-1,(\eta_{p-1},\ldots,\eta_1)}(\varphi)(\eta_{p+1}) - \Delta_{p-1,(\eta_{p-1},\ldots,\eta_1)}(\varphi)(\eta_p)}{\eta_{p+1}-\eta_p}\,. \end{aligned} \tag{1.8}$$

Then we can give the following equivalent criterion for the convergence of $E_N(\cdot;\eta)$ in the case when the subset $\{\eta_j\}_{j\geq 1}$ is bounded.

Theorem 1. *Let* $\{\eta_j\}_{j\geq 1}$ *be bounded and fix any* $r_0 > 0$. *TFAE:*

(1) *there is* $\varepsilon_\eta > 0$ *such that, for all* $f \in \mathcal{O}\left(B_2(0,r_0)\right)$, *the interpolation formula* $E_N(f;\eta)$ *converges to* f, *uniformly on any compact subset of* $B_2(0,\varepsilon_\eta r_0)$;
(2) *for all* $g \in \mathcal{O}\left(\mathbb{C}^2\right)$, *the interpolation formula* $E_N(g;\eta)$ *converges to* g, *uniformly on any compact subset of* $\mathbb{C}^2$;
(3) $\exists\, R_\eta \geq 1,\ \forall\, p, q \geq 0$,

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right| \leq R_\eta^{p+q}\,. \tag{1.9}$$

Furthermore, this result yields the following consequences: first, some precision for the convergence of $E_N(f;\eta)$ to f; next, this convergence is also uniform with respect to f belonging to any given compact subset.

Corollary 1. *When any of the equivalent conditions from the above theorem is satisfied, one has in addition: for all* $\mathcal{K} \subset \mathcal{O}\left(B_2(0,r_0)\right)$ *(resp.* $\mathcal{O}\left(\mathbb{C}^2\right)$*) and* $K \subset B_2(0,\varepsilon_\eta r_0)$ *(resp.* $\mathbb{C}^2$*) compact subsets, there are* $C_{\mathcal{K},K}$ *and* $\varepsilon_K > 0$ *such that, for all* $N \geq 1$,

$$\sup_{f\in\mathcal{K}}\sup_{z\in K}|f(z) - E_N(f;\eta)(z)| \leq C_{\mathcal{K},K}\,(1-\varepsilon_K)^N\,.$$

This analytic criterion gives a condition on the mutual repartition of the points η_j, $j \geq 1$. Since the operator Δ_p looks like the iterated derivative, condition (1.9) can be interpreted as an exponential estimate of the derivatives (as it is always the case for any holomorphic function). Thus Theorem 1 claims that this repartition of the η_j's must be such that under the action of Δ_p, $p \geq 1$, the (non-holomorphic) functions $\overline{\zeta^q}/(1+|\zeta|^2)^q$, $q \geq 1$, should act as if they were holomorphic.

On the other hand, one can notice the equivalence between (1) and (2) in Theorem 1. As we will see in all the following, it will always be the case. That is why we want to simplify some notations and give the following definition.

Definition 1. *Let be any fixed set $\{\eta_j\}_{j\geq1}$.*

We say that the interpolation formula $E_N(\cdot;\eta)$ converges *if statement (1) from Theorem 1 is valid for all $r_0 > 0$ and so is (2).*

Similarly, we say that the interpolation formula $E_N(\cdot;\eta)$ does not converge *if statement (1) is valid for none $r_0 > 0$ and neither is (2).*

Now we want to know what happens when the set $\{\eta_j\}_{j\geq1}$ is not bounded. The way uses the symmetry of the problem under any rotation of the lines. By $\eta^c \notin \{\eta_j\}_{j\geq1}$, we will mean any number that is different from all η_j, $j \geq 1$. Let fix any η^c and set $\theta_j := h_{\eta^c}(\eta_j)$, $\forall\, j \geq 1$, where h_{η^c} is the homographic transformation defined on the Riemann sphere $\overline{\mathbb{C}}$ as

$$h_{\eta^c}(\zeta) \quad := \quad \frac{1+\overline{\eta^c}\,\zeta}{\zeta-\eta^c}, \tag{1.10}$$

Then the set $\{\theta_j\}_{j\geq1}$ is well-defined and one can give an extension of Theorem 1.

Theorem 2. (1) *Let fix any set $\{\eta_j\}_{j\geq1}$. If $E_N(\cdot;\eta)$ converges then for all $\eta^c \notin \{\eta_j\}_{j\geq1}$, there is R_{η^c} such that, $\forall\, p, q \geq 0$,*

$$\left|\Delta_{p,(h_{\eta^c}(\eta_p),\dots,h_{\eta^c}(\eta_1))}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](h_{\eta^c}(\eta_{p+1}))\right| \quad \leq \quad R_{\eta^c}^{p+q}\,. \tag{1.11}$$

(2) *If $\{\eta_j\}_{j\geq1}$ is not dense, then TFAE:*

(a) *$E_N(\cdot;\eta)$ converges;*

(b) *there is R_η such that, for all p, $q \geq 0$,*

$$\sup_{\eta^c\notin\{\eta_j\}_{j\geq1}}\left|\Delta_{p,(h_{\eta^c}(\eta_p),\dots,h_{\eta^c}(\eta_1))}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](h_{\eta^c}(\eta_{p+1}))\right| \;\leq\; R_\eta^{p+q}\,; \tag{1.12}$$

(c) *there is $\eta_\infty \notin \overline{\{\eta_j\}_{j\geq1}}$ (the topological closure of $\{\eta_j\}_{j\geq1}$) such that (1.11) is satisfied with the choice of $\eta^c := \eta_\infty$.*

In addition, when any of these equivalent conditions is satisfied, the conclusion of Corollary 1 holds.

First, in part (2) of this theorem, it suffices to satisfy condition (1.11) for one $\eta_\infty \notin \overline{\{\eta_j\}_{j\geq1}}$ in order to deduce the uniform estimate (1.12). Next, although the assertion is still open, we expect that the equivalence in part (2) will hold for the case where $\{\eta_j\}_{j\geq1}$ is dense (this would complete part (1)). Finally, as it has been commented above as a property of holomorphic functions, the exponential estimates from (1.9), (1.11) and (1.12) motivate us to consider another criterion.

1.3. **A geometric criterion.** First, we begin with giving the following geometric definition for any set $\{\eta_j\}_{j\geq1}$.

Definition 2. *We say that the set* $\{\eta_j\}_{j\geq 1}$ *is* locally interpolable by real-analytic curves *if it can be locally embedded in the zero set of a regular real-analytic function.*

There is an equivalent formulation of this definition that is justified in this paper (Section 5, Lemma 18) and that will be useful in the following: $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves if and only if it can locally holomorphically interpolate the conjugate function, i.e. for all $\zeta \in \overline{\{\eta_j\}_{j\geq 1}}$ (the topological closure of $\{\eta_j\}_{j\geq 1}$), there exist a neighborhood V of ζ and $g \in \mathcal{O}(V)$ such that $\overline{\eta_j} = g(\eta_j)$, $\forall\, \eta_j \in V$. In particular, when $\infty \in \overline{\{\eta_j\}_{j\geq 1}}$, then the associated function g is holomorphic in a neighborhood of ∞, i.e. the function defined by

$$\zeta \neq 0 \;\mapsto\; \frac{1}{g(1/\zeta)}\,,\;\; 0 \;\mapsto\; 0\,, \tag{1.13}$$

is holomorphic in a neighborhood of 0.

First, as it was expected, in such a set the conjugate function (as well as $\overline{\zeta}^q/(1+|\zeta|)^q$, $q \geq 1$) will coincide on the η_j's with a holomorphic one. Next, this geometric condition is easier and more natural to be formulated than (1.9) that also seems difficult to be numerically tested since the computation of Δ_{p+1} does not only require the previous one, but rather the computation of other Δ_p's (i.e. with some other points η_j, see (1.8)). In addition, this criterion can be interpreted as a real-analytic dependence of the family $\left(\{z_1 - \eta_j z_2 = 0\}\right)_{j\geq 1}$, a formulation that can be extended in the case of any family of analytic submanifolds $\{Z_j\}_{j\geq 1}$ of any domain $\Omega \subset \mathbb{C}^n$. Finally, it is a sufficient condition for the convergence of $E_N(\cdot;\eta)$.

Theorem 3. *If* $\{\eta_j\}_{j\geq 1}$ *is locally interpolable by real-analytic curves, then* $E_N(\cdot;\eta)$ *converges. In addition, the conclusion of Corollary 1 holds.*

First, an immediate consequence of this theorem is the proof of the second part of Proposition 1 that we have claimed above since $\mathbb{R} = \{\zeta \in \mathbb{C},\ \overline{\zeta} = \zeta\}$. Similarly, the unit circle can be written as $\{\zeta \in \mathbb{C}\setminus\{0\},\ \overline{\zeta} = 1/\zeta\}$.

Next, we do not know if this condition is also necessary. Our first intuition was negative given the scarcity of the sets $\{\eta_j\}_{j\geq 1}$ that are locally interpolable by real-analytic curves. Later, it has been confirmed by a counterexample of an explicit set $\{\eta_j\}_{j\geq 1}$ whose topological closure has nonempty interior but whose associated interpolation formula $E_N(\cdot;\eta)$ does converge (see [10], Proposition 3). Actually, the essential results from [10] give the following equivalence: $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves if and only if the interpolation formula $E_N(\cdot;\sigma(\eta))$ also converges for all $\sigma \in \mathfrak{S}_{\mathbb{N}}$ (the group of the permutations of $\mathbb{N}$), where $\sigma(\eta) := \left(\eta_{\sigma(j)}\right)_{j\geq 1}$ (see [10], Theorem 3). Nevertheless, we will give in Section 5 the proof of the following result that is a special case of equivalence for Theorem 3.

Proposition 2. *Let* $(\eta_j)_{j\geq 1}$ *be any convergent sequence. If the interpolation formula* $E_N(\cdot;\eta)$ *converges, then* $\{\eta_j\}_{j\geq 1}$ *is locally interpolable by real-analytic curves.*

In addition, this result allows to easily construct examples of families of complex lines whose associated interpolation formula does not converge: any convergent sequence that cannot be embedded in any real-analytic curve. In particular, the set $\{\eta_j\}_{j\geq 1}$ where $\eta_j = i^j/j$, $j \geq 1$, is not locally interpolable by real-analytic curves then the associated interpolation formula $E_N(\cdot;\eta)$ does not converge (Section 5, Corollary 2). This example gives the justification of the first part of Proposition 1 and finally completes its whole proof.

Acknowledgements. I would like to thank G. Henkin for having introduced me this interesting problem and J. Ortega-Cerdà for all the rewarding ideas and discussions about it. I would also like to thank the referee for the useful remarks about the improvement of this paper.

2. Some preliminar results on the divided differences

In this part, $\{\eta_j\}_{j\geq 1}$ will be any set of points all differents, and φ will be any function defined on the points η_j, $j \geq 1$. The following results can be found in [2] and their proofs are given in the references therein. They have also been proved independently by the author in a first version of this paper [9] (except for Lemma 6).

The proof of the first one is an immediate consequence of the definition of Δ_p.

Lemma 1. *For all $p \geq 0$ and $0 \leq q \leq p$,*

$$\begin{aligned} \Delta_{p,(\eta_p,\ldots,\eta_1)}(\varphi)\,(\eta_{p+1}) &= \Delta_{1,\eta_p}\left[\zeta \mapsto \Delta_{p-1,(\eta_{p-1},\ldots,\eta_1)}(\varphi)(\zeta)\right](\eta_{p+1}) \\ &= \Delta_{p-1,(\eta_p,\ldots,\eta_2)}\left[\zeta \mapsto \Delta_{1,\eta_1}(\varphi)(\zeta)\right](\eta_{p+1}) \\ &= \Delta_{p-q,(\eta_p,\ldots,\eta_{q+1})}\left[\zeta \mapsto \Delta_{q,(\eta_q,\ldots,\eta_1)}(\varphi)(\zeta)\right](\eta_{p+1}) \\ &= \Delta_{1,\eta_p}\left[\zeta_p \mapsto \Delta_{1,\eta_{p-1}}\left[\cdots\left[\zeta_1 \mapsto \Delta_{1,\eta_1}(\varphi)\,(\zeta_1)\right]\cdots\right](\zeta_p)\right](\eta_{p+1})\ . \end{aligned}$$

Now φ being any given function that is defined on the η_j's (not necessarily with some regularity condition), the Lagrange polynomial of φ is defined as

$$\mathcal{L}_N[\varphi](X) \;=\; \sum_{p=1}^{N}\left(\prod_{j=1,j\neq p}^{N}\frac{X-\eta_j}{\eta_p-\eta_j}\right)\varphi\,(\eta_p)\ , \tag{2.1}$$

and is the unique polynomial function with degree at most $N-1$ that coincides with φ on the N first points η_j's. The following result is the Newton formula that gives another expression of $\mathcal{L}_N[\varphi](X)$ with the Δ_p's (see [5], Chapter 1, 1)3), or [4], Chapter 4, 7)d)).

Lemma 2. *For all $N \geq 1$, one has*

$$\mathcal{L}_N[\varphi](X) \;=\; \sum_{p=0}^{N-1}\prod_{j=1}^{p}(X-\eta_j)\,\Delta_{p,(\eta_p,\ldots,\eta_1)}(\varphi)\,(\eta_{p+1})\ .$$

The following result is the Leibniz formula for the divided differences (see [15]).

Lemma 3. *For all $p \geq 0$ and φ, ψ functions defined on the η_j's, one has*

$$\Delta_{p,(\eta_p,\ldots,\eta_1)}(\varphi\psi)\,(\eta_{p+1}) \;=\; \sum_{q=0}^{p}\Delta_{p-q,(\eta_p,\ldots,\eta_{q+1})}(\varphi)\,(\eta_{p+1})\ \Delta_{q,(\eta_q,\ldots,\eta_1)}(\psi)\,(\eta_{q+1})\ .$$

This result is an example of the explicit calculation of the Δ_p of a holomorhic function (see [14], Chapter 3, and also [4], Chapter 4, 7)c) and 7)d)).

Lemma 4. *Let be $\varphi \in \mathcal{O}(D(w_0,r))$ and $\varphi(w) = \sum_{n\geq 0}a_n(w-w_0)^n$ its Taylor expansion for all $|w-w_0| < r$. Assume that $\forall\, j \geq 1$, $\eta_j \in D(w_0,r)$. Then for all $p \geq 0$,*

$$\begin{aligned} \Delta_{p,(\eta_p,\ldots,\eta_1)}(\varphi)\,(\eta_{p+1}) &= \sum_{n\geq p}a_n\sum_{l_1=0}^{n-p}(\eta_1-w_0)^{n-p-l_1}\sum_{l_2=0}^{l_1}(\eta_2-w_0)^{l_1-l_2}\ \cdots \\ &\cdots\sum_{l_{p-1}=0}^{l_{p-2}}(\eta_{p-1}-w_0)^{l_{p-2}-l_{p-1}}\sum_{l_p=0}^{l_{p-1}}(\eta_p-w_0)^{l_{p-1}-l_p}(\eta_{p+1}-w_0)^{l_p}\ . \end{aligned} \tag{2.2}$$

In particular,

$$\lim_{\eta_1,\ldots,\eta_p,\eta_{p+1}\to w_0}\Delta_{p,(\eta_p,\ldots,\eta_1)}(\varphi)\,(\eta_{p+1}) \;=\; a_p \;=\; \frac{\varphi^{(p)}(0)}{p!}\ .$$

In addition, if φ is a polynomial function, then for any subset $\{\eta_j\}_{j\geq 1} \subset \mathbb{C}$ and all $p > \deg \varphi$,

$$\Delta_{p,(\eta_p,\ldots,\eta_1)}(\varphi)\,(\eta_{p+1}) \quad = \quad 0\,.$$

Next, we give the following result whose proof is given in [4], Chapter 4, 7)a).

Lemma 5. *For all $p \geq 0$ and $\sigma \in \mathfrak{S}_{p+1}$ (the permutation group of the $p+1$ first integers), one has*

$$\Delta_{p,(\eta_{\sigma(p)},\ldots,\eta_{\sigma(1)})}(\varphi)\left(\eta_{\sigma(p+1)}\right) \quad = \quad \Delta_{p,(\eta_p,\ldots,\eta_1)}(\varphi)\,(\eta_{p+1})\,.$$

This assertion can also be deduced from the following identity whose proof is given in [5], Chapter 1: for all $p \geq 0$,

$$\Delta_{p,(\eta_p,\ldots,\eta_1)}(\varphi)(\eta_{p+1}) \quad = \quad \sum_{q=1}^{p+1} \frac{\varphi(\eta_q)}{\prod_{j=1, j\neq q}^{p+1}(\eta_q - \eta_j)}\,.$$

Now the next result is the Hermite formula of the integral representation for divided differences. Its proof can be found in [5], Chapter 1, 4)3).

Lemma 6. *Let fix $p \geq 0$ and let $D(\zeta_0, r)$ be a disc that contains $\eta_1, \ldots, \eta_{p+1}$. On the other hand, let φ be any holomorphic function defined on a neighborhood of the closed disc $\overline{D(\zeta_0, r)}$. Then one has*

$$\Delta_{p,(\eta_p,\ldots,\eta_1)}(\varphi)(\eta_{p+1}) \quad = \quad \frac{1}{2\pi i}\int_{|\zeta-\zeta_0|=r} \frac{\varphi(\zeta)}{\prod_{j=1}^{p+1}(\zeta - \eta_j)}d\zeta\,.$$

As an application, one has the following estimate:

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}(\varphi)(\eta_{p+1})\right| \quad \leq \quad r \sup_{|\zeta-\zeta_0|=r} \left|\frac{\varphi(\zeta)}{\prod_{j=1}^{p+1}(\zeta - \eta_j)}\right|\,. \tag{2.3}$$

3. On the equivalence with the exponential estimate of the divided differences

In this section, we will prove Theorem 1. Its proof requires some preliminar results that will also be useful in the other sections. In all the following, we will mean the Taylor expansion of any function $f \in \mathcal{O}\left(B_2(0, r_0)\right)$ (resp. $f \in \mathcal{O}\left(\mathbb{C}^2\right)$), that absolutely converges in any compact subset of $B_2(0, r_0)$ (resp. $\mathbb{C}^2$), by

$$f(z) \quad = \quad \sum_{k,l\geq 0} a_{k,l} z_1^k z_2^l\,. \tag{3.1}$$

When it is not specified, the simple expression $\Delta_p(\varphi)$ will mean $\Delta_{p,(\eta_p,\ldots,\eta_1)}(\varphi)\,(\eta_{p+1})$.

We begin with the following preliminar result: it gives the equivalence between the convergence of $E_N(f;\eta)$ to f and the control of $R_N(f;\eta)$.

Lemma 7. *$r_0 > 0$ being fixed, let consider $f \in \mathcal{O}\left(B_2(0, r_0)\right)$ (resp. $f \in \mathcal{O}\left(\mathbb{C}^2\right)$) and K any compact subset of $B_2(0, r_0)$ (resp. $\mathbb{C}^2$). Then for all $N \geq 1$, one has*

$$\sup_{z\in K} |f(z) - E_N(f;\eta)(z)| \quad \leq \quad \sup_{z\in K} |R_N(f;\eta)(z)| + C_K(N+2) \sup_{\|z\|\leq r_K} |f(z)|\,(1-\varepsilon_K)^N\,,$$

where $\|z\| = \sqrt{|z_1|^2 + |z_2|^2}$ is the usual norm on $\mathbb{C}^2$ and C_K, r_K depend only on K.

In particular, $E_N(f;\eta)$ converges to f if and only if so does $R_N(f;\eta)$ to 0.

Proof. First, let fix any $z \in B_2(0, r_0)$. Since $\|z\| < r_0$, there is $\varepsilon > 0$ such that one still has $\|z\|^2 < \|z\|^2 + 2\varepsilon < r_0^2$. Since $\overline{D_2(0, (\sqrt{|z_1|^2+\varepsilon}, \sqrt{|z_2|^2+\varepsilon}))} \subset \overline{B_2(0, \sqrt{\|z\|^2+2\varepsilon})} \subset B_2(0, r_0)$, then one has by Cauchy's estimates for all k, $l \geq 0$,

$$|a_{k,l}| = \left| \frac{1}{(2\pi i)^2} \int_{|\zeta_1|=\sqrt{|z_1|^2+\varepsilon}, |\zeta_2|=\sqrt{|z_2|^2+\varepsilon}} \frac{f(\zeta_1, \zeta_2)\, d\zeta_1 d\zeta_2}{\zeta_1^{k+1}\zeta_2^{l+2}} \right| \leq \frac{\sup_{\zeta \in B_2(0, \sqrt{\|z\|^2+2\varepsilon})} |f(\zeta)|}{(\sqrt{|z_1|^2+\varepsilon})^k (\sqrt{|z_2|^2+\varepsilon})^l}.$$

It follows that, for all $N \geq 1$,

$$\begin{aligned} \sum_{k+l \geq N} |a_{k,l}|\, |z_1|^k |z_2|^l &\leq \sup_{\zeta \in B_2(0, \sqrt{\|z\|^2+2\varepsilon})} |f(\zeta)| \sum_{k+l \geq N} \left(\frac{|z_1|}{\sqrt{|z_1|^2+\varepsilon}} \right)^k \left(\frac{|z_2|}{\sqrt{|z_2|^2+\varepsilon}} \right)^l \\ &\leq \sup_{\zeta \in B_2(0, \sqrt{\|z\|^2+2\varepsilon})} |f(\zeta)| \sum_{m \geq N} (m+1)(1-\varepsilon')^m, \end{aligned}$$

where $\varepsilon' = 1 - \max_{i=1,2} \dfrac{|z_i|}{\sqrt{|z_i|^2+\varepsilon}}$. On the other hand, by a classical calculation (since $\varepsilon' \leq 1$), $\sum_{m \geq N} (m+1)(1-\varepsilon')^m \leq \dfrac{(N+2)(1-\varepsilon')^N}{\varepsilon'^2}$, then

$$\sum_{k+l \geq N} |a_{k,l}|\, |z_1|^k |z_2|^l \leq \frac{N+2}{\varepsilon'^2} \sup_{\zeta \in B_2(0, \sqrt{\|z\|^2+2\varepsilon})} |f(\zeta)|(1-\varepsilon')^N.$$

Now, let $K \subset B_2(0, r_0)$ being fixed. By the maximum principle, there is $z_K \in K$ such that $\sup_{z \in K} \left| \sum_{k+l \geq N} a_{k,l} z_1^k z_2^l \right| = \left| \sum_{k+l \geq N} a_{k,l} (z_K)_1^k (z_K)_2^l \right|$. The lemma follows by (1.2) and the above estimate applied to z_K with the associated choices of ε_K, ε'_K, $C_K = 1/{\varepsilon'_K}^2$ and $r_K = \sqrt{\|z_K\|^2 + 2\varepsilon_K}$.

Now if $f \in \mathcal{O}(\mathbb{C}^2)$, the proof is still valid with the choice of any arbitrary ε_K. Notice that in this case the associated $1 - \varepsilon'_K$ can be made arbitrarily small.

$\surd$

3.1. **A comparison for the convergence of $E_N(f;\eta)$ betwen $\mathcal{O}(B_2(0, r_0))$ and $\mathcal{O}(\mathbb{C}^2)$.** We prove this result that will be useful in all the following.

Lemma 8. *Let $r_0 > 0$ be fixed. If there is $\varepsilon_\eta > 0$ such that, $\forall\, f \in \mathcal{O}(B_2(0, r_0))$, $R_N(f;\eta)$ converges to 0 uniformly on any compact subset of $B_2(0, \varepsilon_\eta r_0)$, then $\forall\, g \in \mathcal{O}(\mathbb{C}^2)$, $R_N(g;\eta)$ converges to 0 uniformly on any compact subset of $\mathbb{C}^2$.*

Proof. Let $g \in \mathcal{O}(\mathbb{C}^2)$ and $K \subset \mathbb{C}^2$ being fixed. There is $R_K > 0$ such that $K \subset B_2(0, R_K)$ then, by setting $\varepsilon_K = \dfrac{\varepsilon_\eta r_0}{R_K}$, one has $\varepsilon_K K \subset B_2(0, \varepsilon_\eta r_0)$.

On the other hand, the function defined as $f(z) := g(z/\varepsilon_K)$, $z \in \mathbb{C}^2$, is still entire (then holomorphic on $B_2(0, r_0)$). Moreover, $g(z) = \sum_{k+l \geq 0} z_1^k z_2^l$ being the Taylor expansion of g, one immediatly has $f(z) = \sum_{k+l \geq 0} \dfrac{a_{k,l}}{\varepsilon_K^{k+l}} z_1^k z_2^l$. This yields for all $z \in \mathbb{C}^2$,

$$\begin{aligned} R_N(f;\eta)(\varepsilon_K z) &= \sum_{p=1}^N \varepsilon_K^{N-1} \prod_{j=1, j \neq p}^N \frac{z_1 - \eta_j z_2}{\eta_p - \eta_j} \sum_{k+l \geq N} \frac{a_{k,l}}{\varepsilon_K^{k+l}} \eta_p^k \varepsilon_K^{k+l-N+1} \left(\frac{z_2 + \overline{\eta_p} z_1}{1 + |\eta_p|^2} \right)^{k+l-N+1} \\ &= \sum_{p=1}^N \prod_{j=1, j \neq p}^N \frac{z_1 - \eta_j z_2}{\eta_p - \eta_j} \sum_{k+l \geq N} a_{k,l} \eta_p^k \left(\frac{z_2 + \overline{\eta_p} z_1}{1 + |\eta_p|^2} \right)^{k+l-N+1}, \end{aligned}$$

that is exactly $R_N(g;\eta)(z)$. It follows that

$$\sup_{z\in K}|R_N(g;\eta)(z)| \;=\; \sup_{z\in\varepsilon_K K}|R_N(f;\eta)(z)| \;\xrightarrow[N\to+\infty]{}\; 0\,, \tag{3.2}$$

the last assertion coming from hypothesis since $\varepsilon_K K$ is a compact subset of $B_2(0,\varepsilon_\eta r_0)$.

$\surd$

3.2. **On the necessity of the estimate (1.9).** In this part, we deal with the necessity of the estimate (1.9) to make converge the associated interpolation formula $E_N(f;\eta)$.

Remark 3.1. In particular, we will see that no one condition is required for the set $\{\eta_j\}_{j\geq 1}$ yet, like boundedness.

We begin with this result.

Lemma 9. *For all $f\in\mathcal{O}\left(\mathbb{C}^2\right)$, $N\geq 1$ and $k_1\geq N$,*

$$\frac{1}{k_1!}\frac{\partial^{k_1}}{\partial z_1^{k_1}}|_{z=0}[R_N(f;\eta)(z)] \;=\; \Delta_{N-1,(\eta_{N-1},\ldots,\eta_1)}\left(\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{k_1-N+1}\sum_{k+l=k_1}a_{k,l}\zeta^k\right)(\eta_N)\,.$$

Proof. First, we claim that

$$R_N(f;\eta)(z)=\sum_{p=0}^{N-1}z_2^{N-1-p}\prod_{j=1}^{p}(z_1-\eta_j z_2)\Delta_{p,(\eta_p,\ldots,\eta_1)}\left(\zeta\mapsto r_N(\zeta,z)\right)(\eta_{p+1})\,, \tag{3.3}$$

with $r_N(\zeta,z):=\sum_{m\geq N}\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{m-N+1}\sum_{k+l=m}a_{k,l}\zeta^k$. Indeed, by Lemma 2,

$$\begin{aligned}
R_N(f;\eta)(z) &= z_2^{N-1}\sum_{p=1}^{N}\prod_{j=1,j\neq p}^{N}\frac{z_1/z_2-\eta_j}{\eta_p-\eta_j}\sum_{k+l\geq N}a_{k,l}\eta_p^k\left(\frac{z_2+\overline{\eta_p}z_1}{1+|\eta_p|^2}\right)^{k+l-N+1}\\
&= z_2^{N-1}\sum_{p=0}^{N-1}\prod_{j=1}^{p}(z_1/z_2-\eta_j)\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\sum_{k+l\geq N}a_{k,l}\zeta^k\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{k+l-N+1}\right](\eta_{p+1})\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
R_N(f;\eta)(z) &= \sum_{p=0}^{N-1}z_2^{N-1-p}\left[\sum_{r=0}^{p}z_1^r(-1)^{p-r}z_2^{p-r}\Sigma_{p-r}(\eta_1,\ldots,\eta_p)\right]\Delta_p(r_N(\zeta,z))\\
&= \sum_{r=0}^{N-1}z_1^r z_2^{N-1-r}\sum_{p=r}^{N-1}(-1)^{p-r}\Sigma_{p-r}(\eta_1,\ldots,\eta_p)\Delta_p(r_N(\zeta,z))\,,
\end{aligned}$$

where $\Sigma_r(\eta_1,\ldots,\eta_p)=\sum_{1\leq j_1<\cdots<j_r\leq p}\eta_{j_1}\cdots\eta_{j_r}$. Thus

$$\begin{aligned}
\frac{1}{k_1!}\frac{\partial^{k_1}}{\partial z_1^{k_1}}|_{z=0}[R_N(f;\eta)(z)] &= \frac{1}{k_1!}\frac{\partial^{k_1}}{\partial z_1^{k_1}}|_{z=0}\left[z_1^{N-1}\Delta_{N-1}(r_N(\zeta,z))\right]\\
&= \Delta_{N-1,(\eta_{N-1},\ldots,\eta_1)}\left(\zeta\mapsto\frac{1}{k_1!}\frac{\partial^{k_1}}{\partial z_1^{k_1}}|_{z=0}\left[z_1^{N-1}r_N(\zeta,z)\right]\right)(\eta_N)\,.
\end{aligned}$$

Since $k_1 \geq N$, one has for all $\zeta \in \mathbb{C}$

$$\begin{aligned}\frac{1}{k_1!}\frac{\partial^{k_1}}{\partial z_1^{k_1}}|_{z=0}\left[z_1^{N-1} r_N(\zeta,z)\right] &= \frac{1}{(k_1-N+1)!}\frac{\partial^{k_1-N+1}}{\partial z_1^{k_1-N+1}}|_{z=0}\left[\sum_{m\geq N}\sum_{k+l=m} a_{k,l}\zeta^k\left(\frac{z_2+\overline{\zeta} z_1}{1+|\zeta|^2}\right)^{m-N+1}\right] \\ &= \sum_{k+l=k_1} a_{k,l}\zeta^k\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{k_1-N+1},\end{aligned}$$

and the proof is achieved.

√

As an application, we get the following result.

Lemma 10. *For all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ and $p \geq 0$, $q \geq 1$, one has*

$$\left|\Delta_{p,(\eta_p,\dots,\eta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q \sum_{k+l=p+q} a_{k,l}\zeta^k\right](\eta_{p+1})\right| \leq \sup_{z\in\overline{B_2(0,\sqrt{2})}} |R_{p+1}(f;\eta)(z)|\,.$$

Proof. By applying Lemma 9 with $k_1 = p+q$, $N = p+1$ (that is possible since $p+q \geq p+1$) and by Cauchy's formula, one has

$$\begin{aligned}\Delta_{p,(\eta_p,\dots,\eta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q \sum_{k+l=p+q} a_{k,l}\zeta^k\right](\eta_{p+1}) &= \frac{1}{(p+q)!}\frac{\partial^{p+q}}{\partial z_1^{p+q}}|_{z=0}[R_{p+1}(f;\eta)(z)] \\ &= \frac{1}{(2i\pi)^2}\int_{|\zeta_1|=|\zeta_2|=1}\frac{R_{p+1}(f;\eta)(\zeta_1,\zeta_2)\,d\zeta_1\wedge d\zeta_2}{\zeta_1^{p+q+1}\zeta_2}\,.\end{aligned}$$

The lemma follows by estimating the last integral on the closed ball $\overline{B_2(0,\sqrt{2})}$.

√

We finish this part with this result that gives the necessity of the estimate (1.9) in the general case.

Lemma 11. *Let be $\{\eta_j\}_{j\geq 1}$ such that, for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, $R_N(f;\eta)$ is uniformly bounded on any compact subset of $\mathbb{C}^2$. Then the estimate (1.9) from Theorem 1 is satisfied.*

Proof. Let fix any $f \in \mathcal{O}\left(\mathbb{C}^2\right)$. In particular, $\forall\, p \geq 0$, $\sup_{z\in\overline{B_2(0,\sqrt{2})}} |R_{p+1}(f;\eta)(z)| \leq M(f)$. Then for all $p \geq 0$ and $q \geq 1$, one has by Lemma 10

$$\left|\Delta_{p,(\eta_p,\dots,\eta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q \sum_{k+l=p+q} a_{k,l}\zeta^k\right](\eta_{p+1})\right| \leq M(f)\,.$$

Now let consider any function entire on $\mathbb{C}$, $w(\zeta) = \sum_{n\geq 0} b_n\zeta^n$, and set $f_w(z) := w(z_2)$. Then $f_w \in \mathcal{O}\left(\mathbb{C}^2\right)$ and for all $p \geq 0$, $q \geq 1$,

$$\sum_{k+l=p+q} a_{k,l}(f_w)\zeta^k = a_{0,p+q}(f_w) = b_{p+q}\,.$$

It follows with the choice of f_w that $\forall\, p \geq 0$, $\forall\, q \geq 1$,

$$|b_{p+q}|\left|\Delta_{p,(\eta_p,\dots,\eta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right| \leq M(h) := M\left(f_w\right),$$

then

$$\sup_{p\geq 0,\, q\geq 1}\left\{|b_{p+q}|^{\frac{1}{p+q}}\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\left(\frac{\bar{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right|^{\frac{1}{p+q}}\right\} \quad < \quad +\infty\,.$$

Since $w \in \mathcal{O}(\mathbb{C})$, $\limsup_{n\to\infty}|b_n|^{1/n} = 0$. Conversely, if $(\varepsilon_n)_{n\geq 1}$ is any sequence that converges to 0, the function $w_\varepsilon(\zeta) := \sum_{n\geq 1}\varepsilon_n^n\zeta^n$ is entire on $\mathbb{C}$ and

$$\sup_{p\geq 0,\, q\geq 1}\left\{|\varepsilon_{p+q}|\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\left(\frac{\bar{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right|^{\frac{1}{p+q}}\right\} \quad < \quad +\infty\,. \tag{3.4}$$

Now the estimate (1.9) that we want to prove is equivalent to the following one:

$$R_\eta \quad := \quad \sup_{p\geq 0,\, q\geq 1}\left\{\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\left(\frac{\bar{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right|^{\frac{1}{p+q}}\right\} \;<\; +\infty\,. \tag{3.5}$$

Indeed, if $q=0$, $p\geq 1$, one has $\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}(\zeta\mapsto 1)(\eta_{p+1})\right| = 0$ and $|\Delta_0(\zeta\mapsto 1)(\eta_1)| = 1 = R_\eta^0$, then the proof will be achieved.

The estimate (3.5) will be an application of the Banach-Steinhaus Theorem. First, let consider the (complex) space $\mathcal{C}_0$ of the sequences $(\varepsilon_n)_{n\geq 1}$ that tend to 0 as n tends to infinity, with the supremum norm. Then $(\mathcal{C}_0, \|\cdot\|_\infty)$ is a Banach space as a subspace of the set of the bounded sequences (with the same norm). On the other hand, let consider the family of linear forms $(\lambda_m)_{m\geq 1}$ on $\mathcal{C}_0$ defined as:

$$\begin{array}{rccl}\lambda_m\;: & \mathcal{C}_0 & \to & \mathbb{C}\\ & (\varepsilon_n)_{n\geq 1} & \mapsto & \varepsilon_m \displaystyle\max_{p\geq 0, q\geq 1, p+q=m}\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\left(\frac{\bar{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right|^{\frac{1}{p+q}}.\end{array}$$

Then λ_m is well-defined and for all $m\geq 1$, its operator norm is

$$\|\lambda_m\| \quad = \quad \max_{p\geq 0, q\geq 1, p+q=m}\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\left(\frac{\bar{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right|^{\frac{1}{p+q}} \;<\; +\infty\,.$$

Now one has from the above estimate (3.4) that $\forall\,\varepsilon \in \mathcal{C}_0$, $\sup_{m\geq 1}|\lambda_m(\varepsilon)| < +\infty$. It follows by the Banach-Steinhaus Theorem that

$$\sup_{p\geq 0,\, q\geq 1}\left\{\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\left(\frac{\bar{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right|^{\frac{1}{p+q}}\right\} \;=\; \sup_{m\geq 1}\|\lambda_m\| \quad < \quad +\infty\,,$$

and this proves the estimate (3.5).

$\surd$

3.3. **On the equivalence when $\{\eta_j\}_{j\geq 1}$ is bounded.** In this part we deal with the equivalence between the convergence of the interpolation formula $E_N(f;\eta)$ and the validity of the estimate (1.9), i.e. we give the proof of Theorem 1. First, we assume that the set $\{\eta_j\}_{j\geq 1}$ is bounded,

$$\|\eta\|_\infty \;:=\; \sup_{j\geq 1}|\eta_j| \quad < \quad +\infty\,, \tag{3.6}$$

and satisfies (1.9). We begin with the following result that is a little stronger consequence.

Lemma 12. *There is $R'_\eta \geq 1$ such that, for all $p, q, s \geq 0$ with $0 \leq s \leq q$, one has*

$$\left| \Delta_{p,(\eta_p,\dots,\eta_1)} \left[\zeta \mapsto \frac{\overline{\zeta}^s}{(1+|\zeta|^2)^q} \right] (\eta_{p+1}) \right| \;\leq\; {R'_\eta}^{p+q} .$$

Proof. Set

$$\begin{cases} R \;=\; \max\left(1, R_\eta\right) , \\ Q \;=\; \max(3, R_\eta) , \\ S \;=\; 3\max(1, \|\eta\|_\infty) \end{cases} \tag{3.7}$$

and

$$R'_\eta \;:=\; [\max(R,Q,S)]^2 \;=\; \left[\max\left(3, 3\|\eta\|_\infty, R_\eta\right)\right]^2 . \tag{3.8}$$

In order to prove the lemma, it will suffice to prove the following estimate: $\forall\, p, q, s \geq 0$ with $s \leq q$,

$$\left| \Delta_{p,(\eta_p,\dots,\eta_1)} \left[\zeta \mapsto \frac{\overline{\zeta}^s}{(1+|\zeta|^2)^q} \right] (\eta_{p+1}) \right| \;\leq\; R^p Q^q S^{q-s} . \tag{3.9}$$

Indeed, we will get for all $p, q, s \geq 0$ with $s \leq q$,

$$\left| \Delta_{p,(\eta_p,\dots,\eta_1)} \left(\frac{\overline{\zeta}^s}{(1+|\zeta|^2)^q} \right) (\eta_{p+1}) \right| \;\leq\; R^p Q^q S^q \;\leq\; {R'_\eta}^{p+q} .$$

The estimate (3.9) will be proved by induction on $p+q-s \geq 0$. If $p+q-s = 0$ then since $p, q-s \geq 0$, necessarily $p = 0$ and $s = q \geq 0$. One has by the Cauchy-Schwarz inequality

$$\left| \Delta_0 \left(\frac{\overline{\zeta}^q}{(1+|\zeta|^2)^q} \right) (\eta_1) \right| \;=\; \left| \frac{\overline{\eta_1}^q}{(1+|\eta_1|^2)^q} \right| \leq \left(\frac{\sqrt{1+|\eta_1|^2}}{1+|\eta_1|^2} \right)^q \;\leq\; R^0 Q^q S^0 .$$

If $p+q-s = 1$, then either $p = 1$ and $q = s \geq 0$, or $p = 0$ and $0 \leq s = q-1$. In the first case, since $\{\eta_j\}_{j\geq 1}$ satisfies (1.9),

$$\left| \Delta_{1,\eta_1} \left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2} \right)^q \right] (\eta_2) \right| \;\leq\; R_\eta^{1+q} \;\leq\; R^1 Q^q S^0 .$$

In the second case, one has for all $q \geq 1$

$$\left| \Delta_0 \left(\frac{\overline{\zeta}^{q-1}}{(1+|\zeta|^2)^q} \right) (\eta_1) \right| \;=\; \frac{1}{1+|\eta_1|^2} \left| \frac{\overline{\eta_1}}{1+|\eta_1|^2} \right|^{q-1} \;\leq\; 1 \;\leq\; R^0 Q^q S^1 .$$

Now let be $m \geq 1$ and assume that (3.9) is true for all $p, q, s \geq 0$ with $s \leq q$ and such that $p+q-s \leq m$. Let consider $p, q, s \geq 0$ with $s \leq q$ and such that $p+q-s = m+1$. One has different cases:

- if $p = 0$ then

$$\left| \Delta_0 \left(\frac{\overline{\zeta}^s}{(1+|\zeta|^2)^q} \right) (\eta_1) \right| \;=\; \left| \frac{\overline{\eta_1}}{(1+|\eta_1|^2)} \right|^s \frac{1}{(1+|\eta_1|^2)^{q-s}} \;\leq\; 1 \;\leq\; R^0 Q^q S^{q-s} ;$$

- if $s = q$ then by (1.9)

$$\left| \Delta_{p,(\eta_p,\dots,\eta_1)} \left[\frac{\overline{\zeta}^q}{(1+|\zeta|^2)^q} \right] (\eta_{p+1}) \right| \;\leq\; R_\eta^{p+q} \;\leq\; R^p Q^q S^0 ;$$

- otherwise $p \geq 1$ and $0 \leq s \leq q-1$ (in particular $q \geq 1$). On one hand, one has by Lemmas 3 and 4

$$\Delta_{p,(\eta_p,\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s+1}\zeta}{(1+|\zeta|^2)^q}\right)(\eta_{p+1}) =$$

$$= \sum_{r=0}^{p}\Delta_{r,(\eta_r,\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s+1}}{(1+|\zeta|^2)^q}\right)(\eta_{r+1})\Delta_{p-r,(\eta_p,\dots,\eta_{r+1})}(\zeta\mapsto\zeta)(\eta_{p+1})$$

$$= \Delta_{p-1,(\eta_{p-1},\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s+1}}{(1+|\zeta|^2)^q}\right)(\eta_p)\times 1 + \Delta_{p,(\eta_p,\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s+1}}{(1+|\zeta|^2)^q}\right)(\eta_{p+1})\times\eta_{p+1}\,.$$

On the other hand,

$$\Delta_{p,(\eta_p,\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s}|\zeta|^2}{(1+|\zeta|^2)^q}\right)(\eta_{p+1}) =$$

$$= \Delta_{p,(\eta_p,\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s}}{(1+|\zeta|^2)^{q-1}}\right)(\eta_{p+1}) - \Delta_{p,(\eta_p,\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s}}{(1+|\zeta|^2)^q}\right)(\eta_{p+1})\,.$$

Thus

$$\Delta_{p,(\eta_p,\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s}}{(1+|\zeta|^2)^q}\right)(\eta_{p+1}) = \Delta_{p,(\eta_p,\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s}}{(1+|\zeta|^2)^{q-1}}\right)(\eta_{p+1})$$
$$- \Delta_{p-1,(\eta_{p-1},\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s+1}}{(1+|\zeta|^2)^q}\right)(\eta_p) - \eta_{p+1}\Delta_{p,(\eta_p,\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s+1}}{(1+|\zeta|^2)^q}\right)(\eta_{p+1})\,.$$

Since $s \leq q-1$ and $(p-1)+q-(s+1) \leq p+(q-1)-s = p+q-(s+1) = m$, by induction and (3.7) it follows that

$$\left|\Delta_{p,(\eta_p,\dots,\eta_1)}\left(\frac{\overline{\zeta}^{s}}{(1+|\zeta|^2)^q}\right)(\eta_{p+1})\right| \leq R^{p-1}Q^{q-1}S^{q-s-1}\left(R+Q+\|\eta\|_\infty RQ\right)$$
$$\leq R^{p-1}Q^{q-1}S^{q-s-1}\left(R\frac{SQ}{3}+Q\frac{RS}{3}+\frac{S}{3}RQ\right)\,,$$

and this proves (3.9).

$\surd$

In the following the constant R_η will mean R'_η from Lemma 12. One can deduce as a consequence the next result.

Lemma 13. *For all $p,q \geq 0$ and $z \in \mathbb{C}^2$,*

$$\left|\Delta_{p,(\eta_p,\dots,\eta_1)}\left[\zeta\mapsto\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right| \leq R_\eta^p(2R_\eta\|z\|)^q\,.$$

Proof. Indeed, Lemma 12 yields

$$\left|\Delta_{p,(\eta_p,\dots,\eta_1)}\left[\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right| \leq \sum_{u=0}^{q}\frac{q!}{u!\,(q-u)!}|z_2|^{q-u}|z_1|^u\left|\Delta_{p,(\eta_p,\dots,\eta_1)}\left(\frac{\overline{\zeta}^u}{(1+|\zeta|^2)^q}\right)(\eta_{p+1})\right|$$
$$\leq \|z\|^q\sum_{u=0}^{q}\frac{q!}{u!\,(q-u)!}R_\eta^{p+q} = \|z\|^q 2^q R_\eta^{p+q}\,.$$

$\surd$

Now we need the following combinatorial result.

Lemma 14. *For all* $n, p \geq 0$,

$$A_n^p \quad := \quad card\,\{(l_1, \ldots, l_p) \in \mathbb{N}^p,\, n \geq l_1 \geq l_2 \geq \cdots \geq l_p \geq 0\} \;=\; \frac{(n+p)!}{n!\,p!}\,.$$

Proof. First, we admit that if $p = 0$ and $n \geq 0$, then $A_n^0 = 1$. Next, if $n = 0$ and $p \geq 1$, then $A_n^p = 1$. Finally, if $n \geq 1$ and $p = 1$, then $A_n^p = n + 1$.

So one can assume that $n \geq 1$, $p \geq 2$ and prove this result by induction on $n + p \geq 2$. If $n + p = 2$, then we are already done. One can consider $n + p \geq 3$ and claim that

$$A_n^p \quad = \quad A_{n-1}^p \;+\; A_n^{p-1}\,.$$

Indeed, for any $(l_1, \ldots, l_p)$ such that $n \geq l_1 \geq \cdots \geq l_p \geq 0$, either $l_1 = n$ or $l_1 \leq n-1$. Then

$$\begin{aligned} A_n^p \quad &= \quad card\left\{(l_2, \ldots, l_p) \in \mathbb{N}^{p-1},\; n \geq l_2 \geq \cdots \geq l_p \geq 0\right\} \\ &\quad + card\,\{(s_1, \ldots, s_p) \in \mathbb{N}^p,\; n-1 \geq s_1 \geq \cdots \geq s_p \geq 0\} \;=\; A_n^{p-1} + A_{n-1}^p\,. \end{aligned}$$

This proves the claim and the lemma follows by applying the induction hypothesis to A_n^{p-1} and A_{n-1}^p.

$\surd$

Now Lemmas 13 and 14 lead to the following result.

Lemma 15. *Let be* $f \in \mathcal{O}\,(B_2(0, r_0))$ *and set*

$$\varepsilon_\eta \quad := \quad \frac{1}{2\sqrt{2}\,(1 + \|\eta\|_\infty)^2 R_\eta^2}\,.$$

For all $N \geq 1$, $p = 0, \ldots, N-1$ *and* $z \in \mathbb{C}^2$, $r < r_0$ *such that* $\|z\| < \varepsilon_\eta r$, *one has*

$$\left| \Delta_{p,(\eta_p,\ldots,\eta_1)} \left[\zeta \mapsto \sum_{m \geq N} \left(\frac{z_2 + \overline{\zeta} z_1}{1 + |\zeta|^2} \right)^{m-N+1} \sum_{k+l=m} a_{k,l} \zeta^k \right] (\eta_{p+1}) \right| \quad \leq$$

$$\leq \quad \frac{8\|f\|_r R_\eta^2 \|z\|}{\|\eta\|_\infty (1 - 2\sqrt{2}\,\|\eta\|_\infty R_\eta^2 \|z\|/r)} \left(\frac{\sqrt{2}\,\|\eta\|_\infty R_\eta}{r} \right)^N \left(\frac{R_\eta (1 + \|\eta\|_\infty)}{\|\eta\|_\infty} \right)^p,$$

with

$$\|f\|_r \quad := \quad \sup_{z \in B_2(0,r)} |f(z)|\,.$$

Proof. First, for all $z \in \mathbb{C}^2$ and $m \geq N$, one has by Lemma 3

$$\Delta_{p,(\eta_p,\ldots,\eta_1)} \left[\left(\frac{z_2 + \overline{\zeta} z_1}{1 + |\zeta|^2} \right)^{m-N+1} \sum_{k+l=m} a_{k,l} \zeta^k \right] (\eta_{p+1}) \quad = \tag{3.10}$$

$$= \quad \sum_{v=0}^{p} \Delta_{v,(\eta_v,\ldots,\eta_1)} \left[\left(\frac{z_2 + \overline{\zeta} z_1}{1 + |\zeta|^2} \right)^{m-N+1} \right] (\eta_{v+1}) \Delta_{p-v,(\eta_p,\ldots,\eta_{v+1})} \left(\sum_{k+l=m} a_{k,l} \zeta^k \right) (\eta_{p+1})\,.$$

Next, for all $0 \leq v \leq p$ and $m \geq N$ $(> p)$, one has by Lemmas 4 and 14

$$\left| \Delta_{p-v,(\eta_p,\ldots,\eta_{v+1})} \left(\sum_{k=0}^{m} a_{k,m-k} \zeta^k \right) (\eta_{p+1}) \right| \quad \leq$$

$$\begin{aligned}
&\leq \sum_{k=p-v}^{m} |a_{k,m-k}| \sum_{l_1=0}^{k-p+v} |\eta_{v+1}|^{k-p+v-l_1} \cdots \sum_{l_{p-v}=0}^{l_{p-v-1}} |\eta_p|^{l_{p-v-1}-l_{p-v}} |\eta_{p+1}|^{l_{p-v}} \\
&\leq \sum_{k=p-v}^{m} |a_{k,m-k}| \|\eta\|_\infty^{k-p+v} \frac{k!}{(p-v)!\,(k-p+v)!}\,.
\end{aligned}$$

On the other hand, for all r_1, $r_2 > 0$, let consider the bidisc

$$D_2(0,(r_1,r_2)) \quad = \quad D_1(0,r_1) \times D(0,r_2) \; = \; \left\{ z \in \mathbb{C}^2,\; |z_1| < r_1,\; |z_2| < r_2 \right\}.$$

Since, for all $r < r_0$, $\overline{D_2(0,(r/\sqrt{2}, r/\sqrt{2}))} \subset \overline{B_2(0,r)} \subset B_2(0,r_0)$ and $f \in \mathcal{O}\left(B_2(0,r_0)\right)$, one has

$$|a_{k,l}| \quad = \quad \left| \frac{1}{(2i\pi)^2} \int_{|\zeta_1|=|\zeta_2|=r/\sqrt{2}} \frac{f(\zeta_1,\zeta_2)\, d\zeta_1 \wedge d\zeta_2}{\zeta_1^{k+1}\zeta_2^{l+1}} \right| \leq \frac{\|f\|_r}{(r/\sqrt{2})^{k+l}}\,.$$

Thus by Cauchy's formula,

$$\left| \Delta_{p-v,(\eta_p,\ldots,\eta_{v+1})} \left(\sum_{k=0}^{m} a_{k,m-k}\zeta^k \right) (\eta_{p+1}) \right| \quad \leq$$

$$\begin{aligned}
&\leq \quad \|f\|_r \left(\frac{\sqrt{2}}{r}\right)^m \frac{1}{(p-v)!} \frac{\partial^{p-v}}{\partial t^{p-v}}|_{t=\|\eta\|_\infty} \left[\sum_{k=0}^{m} t^k \right] \;=\; \|f\|_r \left(\frac{\sqrt{2}}{r}\right)^m \frac{1}{2i\pi} \int_{|t|=R_\eta} \frac{\sum_{k=0}^m t^k\, dt}{(t-\|\eta\|_\infty)^{p-v+1}} \\
&\leq \quad \|f\|_r \left(\frac{\sqrt{2}}{r}\right)^m \frac{R_\eta^{m+1}-1}{(R_\eta - \|\eta\|_\infty)^{p-v+1}(1-1/R_\eta)} \;\leq\; 2\|f\|_r \frac{R_\eta}{\|\eta\|_\infty} \left(\frac{R_\eta\sqrt{2}}{r}\right)^m \frac{1}{\|\eta\|_\infty^{p-v}}\,,
\end{aligned}$$

since by (3.8), $R_\eta \geq \max(2\|\eta\|_\infty, 2)$. Then (3.10) and Lemma 13 yield

$$\left| \Delta_{p,(\eta_p,\ldots,\eta_1)} \left[\left(\frac{z_2 + \bar{\zeta} z_1}{1+|\zeta|^2} \right)^{m-N+1} \sum_{k+l=m} a_{k,l}\zeta^k \right] (\eta_{p+1}) \right| \quad \leq$$

$$\begin{aligned}
&\leq \frac{2\|f\|_r R_\eta}{\|\eta\|_\infty} \left(\frac{R_\eta\sqrt{2}}{r}\right)^m (2R_\eta\|z\|)^{m-N+1} \frac{1}{\|\eta\|_\infty^p} \sum_{v=0}^{p} (R_\eta\|\eta\|_\infty)^v \\
&\leq \frac{2\|f\|_r R_\eta}{\|\eta\|_\infty (2R_\eta\|z\|)^{N-1}} \left(\frac{2\sqrt{2}\,\|\eta\|_\infty R_\eta^2\|z\|}{r} \right)^m \frac{1}{\|\eta\|_\infty^p} \frac{(R_\eta(1+\|\eta\|_\infty))^{p+1}-1}{R_\eta(1+\|\eta\|_\infty)-1} \\
&\leq \frac{4\|f\|_r R_\eta}{\|\eta\|_\infty (2R_\eta\|z\|)^{N-1}} \left(\frac{2\sqrt{2}\,\|\eta\|_\infty R_\eta^2\|z\|}{r} \right)^m \left(\frac{R_\eta(1+\|\eta\|_\infty)}{\|\eta\|_\infty} \right)^p.
\end{aligned}$$

Now by the definition of ε_η from the statement of the lemma, for all $\|z\| < \varepsilon_\eta r_0$, there is $r < r_0$ such that one still has $\|z\| < \varepsilon_\eta r \leq \dfrac{r}{2\sqrt{2}\,\|\eta\|_\infty R_\eta^2}$, then for all $N \geq 1$ and $p = 0, \ldots, N-1$,

$$\left| \Delta_{p,(\eta_p,\ldots,\eta_1)} \left[\sum_{m\geq N} \left(\frac{z_2 + \bar{\zeta} z_1}{1+|\zeta|^2} \right)^{m-N+1} \sum_{k+l=m} a_{k,l}\zeta^k \right] (\eta_{p+1}) \right| \quad \leq$$

$$\leq \frac{4\|f\|_r R_\eta}{\|\eta\|_\infty (2R_\eta\|z\|)^{N-1}} \left(\frac{R_\eta(1+\|\eta\|_\infty)}{\|\eta\|_\infty}\right)^p \sum_{m\geq N} \left(\frac{2\sqrt{2}\,\|\eta\|_\infty R_\eta^2\|z\|}{r}\right)^m$$
$$\leq \frac{8\|f\|_r R_\eta^2\|z\|}{\|\eta\|_\infty (1-2\sqrt{2}\,\|\eta\|_\infty R_\eta^2\|z\|/r)} \left(\frac{\sqrt{2}\,\|\eta\|_\infty R_\eta}{r}\right)^N \left(\frac{R_\eta(1+\|\eta\|_\infty)}{\|\eta\|_\infty}\right)^p.$$

$\surd$

We can finally give the proof of Theorem 1.

Proof. The proof of *(1)* $\Rightarrow$ *(2)* is exactly Lemma 8, as well as *(2)* $\Rightarrow$ *(3)* follows by Lemma 11.

(3) $\Rightarrow$ *(1)*: $f \in \mathcal{O}(B_2(0,r_0))$ and K any compact subset of $B_2(0,\varepsilon_\eta r_0)$ being fixed, there is $r_K < r_0$ such that one still has $\sup_{z\in K}\|z\| < \varepsilon_\eta r_K$, $\forall\, z \in K$. It follows from (3.3) and Lemma 15 that for all $N \geq 1$,

$$|R_N(f;\eta)(z)| \leq$$
$$\leq \frac{8\|f\|_{r_K} R_\eta^2\|z\|\,(\sqrt{2}\,\|\eta\|_\infty R_\eta/r_K)^N}{\|\eta\|_\infty (1-2\sqrt{2}\,\|\eta\|_\infty R_\eta^2\|z\|/r_K)} \sum_{p=0}^{N-1} |z_2|^{N-1-p}\|z\|^p \prod_{j=1}^{p}\sqrt{1+|\eta_j|^2}\left(\frac{R_\eta(1+\|\eta\|_\infty)}{\|\eta\|_\infty}\right)^p$$
$$\leq \frac{8\|f\|_{r_K} R_\eta^2}{\|\eta\|_\infty (1-2\sqrt{2}\,\|\eta\|_\infty R_\eta^2\|z\|/r_K)} \left(\frac{\sqrt{2}\,\|\eta\|_\infty R_\eta\|z\|}{r_K}\right)^N \frac{((1+\|\eta\|_\infty)^2 R_\eta/\|\eta\|_\infty)^N - 1}{(1+\|\eta\|_\infty)^2 R_\eta/\|\eta\|_\infty - 1}$$
$$\leq \frac{16\|f\|_{r_K} R_\eta}{(1+\|\eta\|_\infty)^2(1-2\sqrt{2}\,\|\eta\|_\infty R_\eta^2\|z\|/r_K)} \left(\frac{\sqrt{2}\,R_\eta^2(1+\|\eta\|_\infty)^2\|z\|}{r_K}\right)^N.$$

Since

$$\sup_{z\in K}\|z\| < \varepsilon_\eta r_K = \frac{1}{2}\frac{r_K}{\sqrt{2}\,(1+\|\eta\|_\infty)^2 R_\eta^2},$$

then

$$\sup_{z\in K}|R_N(f;\eta)(z)| \leq \frac{16\|f\|_{r_K} R_\eta}{(1+\|\eta\|_\infty)^2(1-2\sqrt{2}\,\|\eta\|_\infty R_\eta^2 \sup_{z\in K}\|z\|/r_K)} \frac{1}{2^N} \xrightarrow[N\to\infty]{} 0 \tag{3.11}$$

and the proof of Theorem 1 is achieved.

$\surd$

We finish this section with the proof of Corollary 1 that gives some precision for the convergence of $E_N(f;\eta)$ to f as well as the uniform property.

Proof. The case of $\mathcal{O}(B_2(0,r_0))$ follows from Lemma 7 and the estimate (3.11) of the last proof above. In addition, this also proves the case for $\mathcal{O}(\mathbb{C}^2)$ by applying again Lemma 7 on one hand, Lemma 8 and the estimate (3.2) from its proof on the other hand.

$\surd$

4. On the generalization to the case of any set $\{\eta_j\}_{j\geq 1}$

In this part, we will give the proof of Theorem 2. First, let consider $\mathcal{U}(2,\mathbb{C})$ to be the group of 2×2 matrices that are isometric with respect to the Hermitian structure of $\mathbb{C}^2$. We need to consider the action of (a subset of) $\mathcal{U}(2,\mathbb{C})$ on the complex lines $\{z_1 - \eta_j z_2 = 0\}$, $j \geq 1$ and $\mathcal{O}(B_2(0,r_0))$ (resp. $\mathcal{O}(\mathbb{C}^2)$).

4.1. **On the rotation of the lines.** Let fix any $\eta^c \notin \{\eta_j\}_{j\geq 1}$ and consider the matrix defined as

$$U_{\eta^c} \;:=\; \frac{1}{\sqrt{1+|\eta^c|^2}}\begin{pmatrix}\overline{\eta^c} & 1\\ 1 & -\eta^c\end{pmatrix}. \tag{4.1}$$

$U_{\eta^c}\in\mathcal{U}(2,\mathbb{C})$, i.e.

$$U^{\star}_{\eta^c} \;=\; U^{-1}_{\eta^c} \;=\; \frac{1}{\sqrt{1+|\eta^c|^2}}\begin{pmatrix}\eta^c & 1\\ 1 & -\overline{\eta^c}\end{pmatrix} \;=\; U_{\overline{\eta^c}} \tag{4.2}$$

and

$$\begin{cases} U_{\eta^c}(\{z_1-\eta^c z_2=0\}) \;=\; \{z_2=0\}\,,\\ U^{\star}_{\eta^c}(\{z_2=0\}) \;=\; \{z_1-\eta^c z_2=0\}\,.\end{cases}$$

We remind the definition of $\theta_j = h_{\eta^c}(\eta_j)$ associated to η^c (Introduction, (1.10)):

$$\forall\, j\geq 1,\ \theta_j \;=\; \frac{1+\overline{\eta^c}\eta_j}{\eta_j-\eta^c}\,,$$

and we give the proof of this preliminar result.

Lemma 16. *Let be $f\in\mathcal{O}\left(B_2(0,r_0)\right)$ (resp. $f\in\mathcal{O}\left(\mathbb{C}^2\right)$). For all $N\geq 1$ and $z\in B_2(0,r_0)$ (resp. $z\in\mathbb{C}^2$),*

$$R_N(f;\eta)(z) \;=\; R_N\left(f\circ U^{\star}_{\eta^c};\theta\right)\left(U_{\eta^c}z\right)\,.$$

Proof. It is sufficient to consider the case of $f\in\mathcal{O}\left(B_2(0,r_0)\right)$ with $z\in B_2(0,r_0)$. We set $w = U_{\eta^c}z$. Since U_{η^c} is unitary, then $R_N\left(f\circ U^{\star}_{\eta^c};\theta\right)$ is well-defined (and so is $E_N\left(f\circ U^{\star}_{\eta^c};\theta\right)(U_{\eta^c}z)$). On the other hand, $z = U_{\overline{\eta^c}}w$ and $\eta_j = \dfrac{1+\eta^c\theta_j}{\theta_j-\overline{\eta^c}}$ (notice that $\theta_j\neq\overline{\eta^c}$, $\forall\, j\geq 1$, since η_j is supposed to be finite). First, one has for all $p=1,\ldots,N$,

$$\begin{aligned}
\prod_{j=1,j\neq p}^{N}\frac{z_1-\eta_j z_2}{\eta_p-\eta_j} \;&=\; \prod_{j=1,j\neq p}^{N}\frac{\dfrac{\eta^c w_1+w_2}{\sqrt{1+|\eta^c|^2}} - \dfrac{1+\eta^c\theta_j}{\theta_j-\overline{\eta^c}}\dfrac{w_1-\overline{\eta^c}w_2}{\sqrt{1+|\eta^c|^2}}}{\dfrac{1+\eta^c\theta_p}{\theta_p-\overline{\eta^c}} - \dfrac{1+\eta^c\theta_j}{\theta_j-\overline{\eta^c}}}\\
&=\; \left(\frac{\theta_p-\overline{\eta^c}}{\sqrt{1+|\eta^c|^2}}\right)^{N-1}\prod_{j=1,j\neq p}^{N}\frac{-(1+|\eta^c|^2)w_1+(1+|\eta^c|^2)\theta_j w_2}{(1+|\eta^c|^2)\theta_j-(1+|\eta^c|^2)\theta_p}\\
&=\; \left(\frac{\theta_p-\overline{\eta^c}}{\sqrt{1+|\eta^c|^2}}\right)^{N-1}\prod_{j=1,j\neq p}^{N}\frac{w_1-\theta_j w_2}{\theta_p-\theta_j}\,.
\end{aligned}$$

Next, one has

$$\begin{aligned}
\frac{z_2+\overline{\eta_p}z_1}{1+|\eta_p|^2} \;&=\; \frac{\dfrac{w_1-\overline{\eta^c}w_2}{\sqrt{1+|\eta^c|^2}} + \dfrac{1+\overline{\eta^c}\overline{\theta_p}}{\overline{\theta_p}-\eta^c}\dfrac{\eta^c w_1+w_2}{\sqrt{1+|\eta^c|^2}}}{1+\dfrac{1+\eta^c\theta_p}{\theta_p-\overline{\eta^c}}\dfrac{1+\overline{\eta^c}\overline{\theta_p}}{\overline{\theta_p}-\eta^c}}\\
&=\; \frac{\theta_p-\overline{\eta^c}}{\sqrt{1+|\eta^c|^2}}\frac{(1+|\eta^c|^2)\overline{\theta_p}w_1+(1+|\eta^c|^2)w_2}{(1+|\theta_p|^2)(1+|\eta^c|^2)} \;=\; \frac{\theta_p-\overline{\eta^c}}{\sqrt{1+|\eta^c|^2}}\frac{w_2+\overline{\theta_p}w_1}{1+|\theta_p|^2}\,.
\end{aligned}$$

It follows that, for all $N \geq 1$,

$$\begin{aligned} R_N(f;\eta)(z) &= \sum_{p=1}^{N} \prod_{j=1, j\neq p}^{N} \frac{z_1 - \eta_j z_2}{\eta_p - \eta_j} \sum_{k+l\geq N} a_{k,l}\eta_p^k \left(\frac{z_2 + \eta_p z_1}{1+|\eta_p|^2}\right)^{k+l-N+1} \\ &= \sum_{p=1}^{N} \left(\frac{\theta_p - \overline{\eta^c}}{\sqrt{1+|\eta^c|^2}}\right)^{N-1} \prod_{j=1, j\neq p}^{N} \frac{w_1 - \theta_j w_2}{\theta_p - \theta_j} \times \\ &\quad \times \sum_{k+l\geq N} a_{k,l} \left(\frac{1+\eta^c\theta_p}{\theta_p - \overline{\eta^c}}\right)^k \left(\frac{\theta_p - \overline{\eta^c}}{\sqrt{1+|\eta^c|^2}}\right)^{k+l-N+1} \left(\frac{w_2 + \overline{\theta_p} w_1}{1+|\theta_p|^2}\right)^{k+l-N+1} \\ &= \sum_{p=1}^{N} \prod_{j=1, j\neq p}^{N} \frac{w_1 - \theta_j w_2}{\theta_p - \theta_j} \sum_{m\geq N} \left(\frac{w_2 + \overline{\theta_p} w_1}{1+|\theta_p|^2}\right)^{m-N+1} \sum_{k+l=m} a_{k,l} \frac{(1+\eta^c\theta_p)^k(\theta_p - \overline{\eta^c})^l}{(\sqrt{1+|\eta^c|^2})^m}. \end{aligned}$$

Now notice that for all $m \geq 0$ and all k, l with $k + l = m$, one has

$$(\eta^c\theta_p + 1)^k(\theta_p - \overline{\eta^c})^l = \sum_{u+v=m} b_{u,v}^{(k,l)} \theta_p^u,$$

where $b_{u,v}^{(k,l)}$ is the coefficient of X^uY^v of the polynomial $(\eta^c X + Y)^k(X - \overline{\eta^c}Y)^l$ (that is homogeneous with total degree $k + l = m$). It follows that

$$\sum_{k+l=m} a_{k,l}(\eta^c\theta_p + 1)^k(\theta_p - \overline{\eta^c})^l = \sum_{u+v=m} \theta_p^u \sum_{k+l=m} a_{k,l} b_{u,v}^{(k,l)}.$$

Now we claim that, for all $u, v \geq 0$, $\frac{1}{(\sqrt{1+|\eta^c|^2})^m} \sum_{k+l=m} a_{k,l} b_{u,v}^{(k,l)}$ is the Taylor coefficient of $w_1^u w_2^v$ of the function $f \circ U_{\eta^c}^{\star}$. Indeed,

$$\begin{aligned} \left(f \circ U_{\eta^c}^{\star}\right)(w) &= \sum_{k,l\geq 0} a_{k,l} \left(\frac{\eta^c w_1 + w_2}{\sqrt{1+|\eta^c|^2}}\right)^k \left(\frac{w_1 - \overline{\eta^c} w_2}{\sqrt{1+|\eta^c|^2}}\right)^l \\ &= \sum_{m\geq 0} \frac{1}{(\sqrt{1+|\eta^c|^2})^m} \sum_{k+l=m} a_{k,l}(\eta^c w_1 + w_2)^k(w_1 - \overline{\eta^c} w_2)^l \\ &= \sum_{m\geq 0} \frac{1}{(\sqrt{1+|\eta^c|^2})^m} \sum_{k+l=m} a_{k,l} \sum_{u+v=m} b_{u,v}^{(k,l)} w_1^u w_2^v \\ &= \sum_{u,v\geq m} w_1^u w_2^v \left[\frac{1}{(\sqrt{1+|\eta^c|^2})^m} \sum_{k+l=m} a_{k,l} b_{u,v}^{(k,l)}\right]. \end{aligned}$$

The claim follows by the uniqueness of the Taylor expansion of $f \circ U_{\eta^c}^{\star}$. Finally,

$$R_N(f;\eta)(z) = \sum_{p=1}^{N} \prod_{j\neq p} \frac{w_1 - \theta_j w_2}{\theta_p - \theta_j} \sum_{m\geq N} \left(\frac{w_2 + \overline{\theta_p} w_1}{1+|\theta_p|^2}\right)^{m-N+1} \sum_{u+v=m} \theta_p^u \frac{1}{(\sqrt{1+|\theta_p|^2})^m} \sum_{k+l=m} a_{k,l} b_{u,v}^{(k,l)},$$

that is exactly the formula $R_N(f \circ U_{\eta^c}^{\star};\theta)(w)$. The lemma is proved.

$\surd$

4.2. **Proof of part (1) from Theorem 2.**

Proof. By Definition 1, since $E_N(\cdot;\eta)$ converges, then in particular for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, the interpolation formula $E_N(f;\eta)$ converges to f, uniformly on any compact subset of $\mathbb{C}^2$.

Let fix any $\eta^c \notin \{\eta_j\}_{j\geq 1}$, $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ and let consider the associated $\theta_j = h_{\eta^c}(\eta_j)$, $j \geq 1$. By Lemmas 7 and 16, one has that $[R_N(f;\theta)] \circ U_{\eta^c} = R_N(f \circ U_{\eta^c};\eta)$ converges to zero uniformly on any compact subset of $\mathbb{C}^2$, then also $R_N(f;\theta)$ since U_{η^c} is an isometry. By applying Lemma 7 again, it follows that $E_N(f;\theta)$ converges to f, uniformly on any compact subset of $\mathbb{C}^2$. Thus the assertion is a consequence of Lemma 11 (since there is no condition for the set $\{\theta_j\}_{j\geq 1}$, see Remark 3.1).

$\surd$

4.3. **Proof of part (2) from Theorem 2.** First, we give the proof of the following result that will be useful in order to prove the second part of Theorem 2.

Lemma 17. *Let $\{\eta_j\}_{j\geq 1}$ be such that the interpolation formula $E_N(f;\eta)$ converges to f, uniformly on any compact subset of $\mathbb{C}^2$ and also on any f in any compact subset of $\mathcal{O}\left(\mathbb{C}^2\right)$. Then the estimate (1.12) from Theorem 2 is satisfied, i.e. $\exists\, R_\eta$, $\forall\, p,q \geq 0$,*

$$sup_{\eta^c\notin\{\eta_j\}_{j\geq 1}} \left| \Delta_{p,(h_{\eta^c}(\eta_p),\ldots,h_{\eta^c}(\eta_1))} \left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](h_{\eta^c}(\eta_{p+1}))\right| \quad\leq\quad R_\eta^{p+q}\,.$$

Proof. First, it follows by Lemma 7 that $R_N(\cdot;\eta)$ converges to 0, uniformly on any compact subsets of $\mathbb{C}^2$ and $\mathcal{O}\left(\mathbb{C}^2\right)$ repectively. Next, let fix any $\eta^c \notin \{\eta_j\}_{j\geq 1}$ and set $\theta_j := h_{\eta^c}(\eta_j)$, $\forall\, j \geq 1$. On the other hand, let fix any $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ and consider its Taylor expansion $f(z) = \sum_{k,l\geq 0} a_{k,l} z_1^k z_2^l$. By applying Lemma 10 to the θ_j's and f, one has for all $p \geq 0$, $q \geq 1$,

$$\begin{aligned}\left|\Delta_{p,(\theta_p,\ldots,\theta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q \sum_{k+l=p+q} a_{k,l}\zeta^k\right](\theta_{p+1})\right| \quad&\leq\quad \sup_{z\in B_2(0,\sqrt{2})} |R_{p+1}(f;\theta)(z)| \\ &=\quad \sup_{z\in B_2(0,\sqrt{2})} |R_{p+1}(f\circ U_{\eta^c};\eta)(z)|\,,\end{aligned}$$

the last equality coming from Lemma 16 and the fact that U_{η^c} is unitary. Since the family $\{f \circ U_{\eta^c}\}_{\eta^c\notin\{\eta_j\}}$ is a relatively compact subset of $\mathcal{O}\left(\mathbb{C}^2\right)$, it follows by hypothesis that

$$sup_{\eta^c\notin\{\eta_j\}}\left[\sup_{p\geq 0,\, z\in B_2(0,\sqrt{2})} |R_{p+1}(f\circ U_{\eta^c};\eta)(z)|\right] \quad\leq\quad M(f) \;<\; +\infty\,,$$

thus

$$sup_{\eta^c\notin\{\eta_j\},p\geq 0,q\geq 1}\left|\Delta_{p,(h_{\eta^c}(\eta_p),\ldots,h_{\eta^c}(\eta_1))}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q \sum_{k+l=p+q} a_{k,l}\zeta^k\right](h_{\eta^c}(\eta_{p+1}))\right| < +\infty\,.$$

Now the end of the proof uses the same argument from Lemma 11: first, we deduce that for all sequence $(\varepsilon_n)_{n\geq 1}$ that converges to 0,

$$\sup_{\eta^c\notin\{\eta_j\}_{j\geq 1},p\geq 0,\, q\geq 1} |\varepsilon_{p+q}| \left|\Delta_{p,(h_{\eta^c}(\eta_p),\ldots,h_{\eta^c}(\eta_1))}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](h_{\eta^c}(\eta_{p+1}))\right|^{\frac{1}{p+q}} \quad<\quad +\infty\,.$$

By replacing $(\lambda_m)_{m\geq 1}$ with $(\lambda_{m,\eta^c})_{m\geq 1,\eta^c\notin\{\eta_j\}_{j\geq 1}}$, where

$$\begin{aligned}\lambda_{m,\eta^c}\;:\quad &\mathcal{C}_0 \;\to\; \mathbb{C}\\ &(\varepsilon_n)_{n\geq 1} \;\mapsto\; \varepsilon_m \max_{p\geq 0,q\geq 1,p+q=m}\left|\Delta_{p,(h_{\eta^c}(\eta_p),\ldots,h_{\eta^c}(\eta_1))}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](h_{\eta^c}(\eta_{p+1}))\right|^{\frac{1}{p+q}}\end{aligned}$$

and

$$\|\lambda_{m,\eta^c}\| \quad = \quad \max_{p\geq 0, q\geq 1, p+q=m} \left| \Delta_{p,(h_{\eta^c}(\eta_p),\ldots,h_{\eta^c}(\eta_1))} \left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2} \right)^q \right] (h_{\eta^c}(\eta_{p+1})) \right|^{\frac{1}{p+q}},$$

the last above estimate means that for all $\varepsilon \in \mathcal{C}_0$, $sup_{m\geq 1, \eta^c \notin \{\eta_j\}_{j\geq 1}} |\lambda_{m,\eta^c}(\varepsilon)| < +\infty$. It follows by the Banach-Steinhaus Theorem that $sup_{m\geq 1, \eta^c \notin \{\eta_j\}_{j\geq 1}} \|\lambda_{m,\eta^c}\| < +\infty$, i.e.

$$\sup_{\eta^c \notin \{\eta_j\}_{j\geq 1}, p\geq 0, q\geq 1, p+q=m} \left| \Delta_{p,(h_{\eta^c}(\eta_p),\ldots,h_{\eta^c}(\eta_1))} \left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2} \right)^q \right] (h_{\eta^c}(\eta_{p+1})) \right|^{\frac{1}{p+q}} \quad < \quad +\infty$$

and the lemma is proved.

$\surd$

Now we can give the proof of part (2) from Theorem 2.

Proof. 2a) $\Rightarrow$ 2b): Once again, by Definition 1 and Lemma 7, for all $f \in \mathcal{O}(\mathbb{C}^2)$, $R_N(f;\eta)$ converges to 0, uniformly on any compact subset of $\mathbb{C}^2$. On the other hand, since $\{\eta_j\}_{j\geq 1}$ is not dense, then

$$\exists\, \eta_\infty \in \mathbb{C},\ \exists\, \varepsilon_\infty > 0,\ \forall\, j \geq 1,\ |\eta_j - \eta_\infty| \geq \varepsilon_\infty \tag{4.3}$$

(w.l.o.g. one can assume that $\eta_\infty \neq 0$). Let consider $\theta_j = h_{\eta_\infty}(\eta_j)$, $\forall\, j \geq 1$. We claim that $\{\theta_j\}_{j\geq 1}$ is bounded. Indeed, if $|\eta_j| \leq 2|\eta_\infty|$, then by (4.3)

$$|\theta_j| \;=\; \frac{|1+\overline{\eta_\infty}\eta_j|}{|\eta_j - \eta_\infty|} \;\leq\; \frac{1+2|\eta_\infty|^2}{\varepsilon_\infty} \;<\; +\infty\,.$$

Otherwise $|\eta_j| > 2|\eta_\infty|$ then

$$|\theta_j| \;=\; \frac{|\overline{\eta_\infty}+1/\eta_j|}{|1-\eta_\infty/\eta_j|} \;\leq\; \frac{|\eta_\infty|+1/(2|\eta_\infty|)}{1-1/2} \;=\; 2\left(|\eta_\infty|+1/(2|\eta_\infty|)\right) \;<\; +\infty\,.$$

Now by Lemma 16 applied to $\eta^c := \eta_\infty$, it follows that for all $f \in \mathcal{O}(\mathbb{C}^2)$, $R_N(f;\theta) = [R_N(f\circ U_{\eta_\infty};\eta)] \circ U^\star_{\eta_\infty}$ also converges to 0 uniformly on any compact subset (and so does $E_N(f;\theta)$ to f by Lemma 7). In particular, Corollary 1 can be deduced to the (bounded) set $\{\theta_j\}_{j\geq 1}$ that satisfies one of the equivalent conditions of Theorem 1. As a consequence, the convergence of $E_N(f;\theta)$ to f is also uniform on any f belonging to any relatively compact subset of $\mathcal{O}(\mathbb{C}^2)$. By applying Lemmas 7 and 16 again, $E_N(f;\eta)$ converges to f too, uniformly on any compact subset and on any f belonging to any relatively compact subset of $\mathcal{O}(\mathbb{C}^2)$. Finally, an application of Lemma 17 leads to the required estimate (1.12).

2b) $\Rightarrow$ 2c): It is an immediate consequence since $\mathbb{C} \setminus \overline{\{\eta_j\}_{j\geq 1}}$ is not empty.

2c) $\Rightarrow$ 2a): Once again, let consider $\theta_j = h_{\eta_\infty}(\eta_j)$, $\forall\, j \geq 1$ as above. Then the bounded set $\{\theta_j\}_{j\geq 1}$ satisfies (1.9). Let fix any $r_0 > 0$. It follows by Theorem 1 that there is $\varepsilon_\theta > 0$ such that, for all $f \in \mathcal{O}(B_2(0,r_0))$, $E_N(f;\theta)$ converges to f uniformly on any compact subset of $B_2(0,\varepsilon_\theta r_0)$. By applying Lemmas 7 and 16, so does $E_N(f;\eta)$ to f uniformly on any compact subset of $B_2(0,\varepsilon_\theta r_0)$ for any $f \in \mathcal{O}(B_2(0,r_0))$. Finally, one can deduce by Lemma 8 that $E_N(\cdot;\eta)$ converges in the meaning of Definition 1.

In addition, as it has been specified above, the statement of Corollary 1 is still valid as a consequence of one of these equivalent conditions.

$\surd$

5. About the geometric criterion

5.1. **Proof of Theorem 3 when $\{\eta_j\}_{j\geq 1}$ is bounded.** Before giving the proof of this result, we want to prove the following lemma that has been claimed in Introduction and that will be usefull in all the following.

Lemma 18. *The set $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves if and only if it can locally holomorphically interpolate the conjugate function, i.e. for all $\zeta \in \overline{\{\eta_j\}_{j\geq 1}}$, there are a neighborhood V of ζ and $g \in \mathcal{O}(V)$ such that*

$$\overline{\eta_j} \quad = \quad g(\eta_j)\,,\ \forall\, \eta_j \in V\,. \tag{5.1}$$

Proof. Let be $\zeta_0 \in \overline{\{\eta_j\}_{j\geq 1}}$. If there are $V_{\zeta_0} \in \mathcal{V}(\zeta_0)$, $g \in \mathcal{O}(V_{\zeta_0})$ such that $\overline{\eta_j} = g(\eta_j)$, $\forall\, \eta_j \in V$, then any η_j is in the zero set of $h(x,y) := x - iy - g(x+iy)$. h is real-analytic in the neighborhood of $(x_0,y_0) := (\Re\,\zeta_0, \Im\,\zeta_0)$ and regular since $\nabla h(x_0,y_0) = (1 - \dfrac{\partial g}{\partial x}(\zeta_0), -i - i\dfrac{\partial g}{\partial x}(\zeta_0)) \neq 0$.

Conversely, assume that there are $V_{\zeta_0} \in \mathcal{V}(\zeta_0)$ and a real-analytic curve $\mathcal{C}_{\zeta_0}$ such that $\{\eta_j\}_{j\geq 1} \cap V_{\zeta_0} \subset \mathcal{C}_{\zeta_0}$. Let be h real-analytic on V_{ζ_0} such that $\mathcal{C}_{\zeta_0} = \{\zeta \in V_{\zeta_0},\ h(\Re\zeta, \Im\zeta) = 0\}$ with $\nabla h(x_0,y_0) \neq 0$. By considering the Taylor expansion of h on $(\Re\zeta_0, \Im\zeta_0)$ and the change of variables using the Euler's formulas for ζ, there is $\widetilde{h}(z,w)$ which is holomorphic on a neighborhood of $(\zeta_0, \overline{\zeta_0})$ such that (after reducing V_{ζ_0} if necessary), for all $\zeta \in V_{\zeta_0}$, $h(\Re\zeta, \Im\zeta) = \widetilde{h}\left(\zeta, \overline{\zeta}\right)$. Moreover, $\dfrac{\partial \widetilde{h}}{\partial \overline{\zeta}}(x_0,y_0) = \dfrac{1}{2}\left(\dfrac{\partial h}{\partial x}(x_0,y_0) + i\dfrac{\partial h}{\partial y}(x_0,y_0)\right) \neq 0$ since h is real and $\nabla h(x_0,y_0) \neq 0$. By the holomorphic implicit function Theorem, there exist a neighborhood $U_{\zeta_0} \times W_{\overline{\zeta_0}}$ of $(\zeta_0, \overline{\zeta_0})$ and $g \in \mathcal{O}(U_{\zeta_0})$ such that, $\forall\,(z,w) \in U_{(\zeta_0,\overline{\zeta_0})}$, $\widetilde{h}(z,w) = 0$ if and only if $w = g(z)$. In particular, since for all $\zeta \in \mathcal{C}_{\zeta_0}$, $0 = h(\Re\zeta, \Im\zeta) = \widetilde{h}(\zeta, \overline{\zeta})$, then after reducing V_{ζ_0} if necessary, one has $\overline{\zeta} = g(\zeta)$, $\forall\,\zeta \in V_{\zeta_0}$. It follows that $g_{|V_{\zeta_0}} \in \mathcal{O}(V_{\zeta_0})$ satisfies the required conditions for all $\eta_j \in V_{\zeta_0}$.

$\surd$

Now we will give the proof of Theorem 3 in the special case when $\{\eta_j\}_{j\geq 1}$ is bounded.

Lemma 19. *Let $\{\eta_j\}_{j\geq 1}$ be bounded and locally interpolable by real-analytic curves. Then Theorem 3 is valid in this case.*

Proof. First, it will suffice to prove that any bounded set $\{\eta\}_{j\geq 1}$ that is locally interpolable by real-analytic curves satisfies (1.9). Indeed, an application of Theorem 1 will allow to deduce that $E_N(\cdot;\eta)$ converges in the meaning of Definition 1, as well as the statement of Corollary 1.

Let fix any $\zeta_0 \in \overline{\{\eta_j\}_{j\geq 1}}$. First, by Lemma 18, $\exists\, V_{\zeta_0} \in \mathcal{V}(\zeta_0)$, $g_{\zeta_0} \in \mathcal{O}(V_{\zeta_0})$, such that $\forall\,\eta_j \in V_{\zeta_0}$, $\overline{\eta_j} = g_{\zeta_0}(\eta_j)$. In particular, $\overline{\zeta_0} = g_{\zeta_0}(\zeta_0)$. Next, since $\left|\dfrac{g_{\zeta_0}(\zeta_0)}{1+\zeta_0 g_{\zeta_0}(\zeta_0)}\right| = \left|\dfrac{\overline{\zeta_0}}{1+|\zeta_0|^2}\right| \leq \dfrac{1}{\sqrt{1+|\zeta_0|^2}} \leq 1$, then by reducing V_{ζ_0} if necessary, one has $\left|\dfrac{g_{\zeta_0}(\zeta)}{1+\zeta g_{\zeta_0}(\zeta)}\right| \leq 2$, for all $\zeta \in V_{\zeta_0}$. This proves that there is ε_{ζ_0} small enough such that

$$\begin{cases} g_{\zeta_0} \text{ and } \dfrac{g_{\zeta_0}(\zeta)}{1+\zeta g_{\zeta_0}(\zeta)} \in \mathcal{O}\left(\overline{D(\zeta_0, 2\varepsilon_{\zeta_0})}\right)\,, \\ \forall\,\eta_j \in D(\zeta_0, 2\varepsilon_{\zeta_0}),\ \overline{\eta_j} = g_{\zeta_0}(\eta_j)\,, \\ \forall\,\zeta \in D(\zeta_0, 2\varepsilon_{\zeta_0}),\ \left|\dfrac{g_{\zeta_0}(\zeta)}{1+\zeta g_{\zeta_0}(\zeta)}\right| \leq 2\,. \end{cases} \tag{5.2}$$

This last property is true for all $\zeta \in \overline{\{\eta_j\}_{j\geq 1}}$, it follows that $\overline{\{\eta_j\}_{j\geq 1}} \subset \bigcup_{\zeta\in\overline{\{\eta_j\}_{j\geq 1}}} D(\zeta,\varepsilon_\zeta)$. On the other hand, since it is a compact subset, by the Lebesgue's number Lemma, there is $\varepsilon_0 > 0$ (with $\varepsilon_0 \leq 1$) that satisfies: for all $\zeta \in \overline{\{\eta_j\}_{j\geq 1}}$, there is $\zeta' \in \overline{\{\eta_j\}_{j\geq 1}}$ such that $D(\zeta,\varepsilon_0) \subset D(\zeta',\varepsilon_{\zeta'})$.

Now the estimate (1.9) will be deduced as a consequence of the following stronger one that we will prove for all $q \geq 0$ and by induction on $p \geq 0$:

$$\left|\Delta_{p,(\eta_{i_p},\ldots,\eta_{i_1})}\left[\zeta\mapsto\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{i_{p+1}})\right| \leq \left(\frac{2}{\varepsilon_0}\right)^p 2^q, \tag{5.3}$$

where there is no condition on the choice for the $p+1$ first points. This estimate being obvious for $p=0$ and all $q\geq 0$, one can assume that $p \geq 0$ and consider $p+2$ arbitrary points from the η_j's.

First, if there are j, k with $1\leq j<k\leq p+2$ and such that $|\eta_{i_j}-\eta_{i_k}| \geq \varepsilon_0$, then by Lemma 5

$$\left|\Delta_{p+1,(\eta_{i_{p+1}},\ldots,\eta_{i_1})}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{i_{p+2}})\right| = \left|\Delta_{p+1,(\eta_{i_j},\eta_{i_{p+2}},\ldots,\widehat{\eta_{i_k}},\ldots,\widehat{\eta_{i_j}},\ldots,\eta_{i_1})}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{i_k})\right|$$

$$\begin{aligned}
&\leq \frac{\left|\Delta_{p,(\eta_{i_{p+2}},\ldots,\widehat{\eta_{i_j}},\widehat{\eta_{i_k}},\ldots\eta_{i_1})}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{i_j})\right| + \left|\Delta_{p,(\eta_{i_{p+2}},\ldots,\widehat{\eta_k},\widehat{\eta_{i_j}},\ldots,\eta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{i_k})\right|}{|\eta_{i_j}-\eta_{i_k}|}\\
&\leq \frac{2(2/\varepsilon_0)^p 2^q}{\varepsilon_0}
\end{aligned}$$

(the notation $\widehat{\eta_j}$ means that η_j has been removed), and this proves the induction.

Otherwise, one has that $|\eta_{i_j}-\eta_{i_1}| < \varepsilon_0$, $\forall\, j=2,\ldots,p+2$. By the above property, there is $\zeta_0 \in \overline{\{\eta_j\}_{j\geq 1}}$ such that $D(\eta_{j_1},\varepsilon_0)\subset D(\zeta_0,\varepsilon_{\zeta_0})$. In particular, $\varepsilon_{\zeta_0}\geq\varepsilon_0$. An application of the estimate (2.3) from Lemma 6 that is guaranteed by (5.2) and the choice of $\dfrac{g_{\zeta_0}(\zeta)}{1+\zeta g_{\zeta_0}(\zeta)} \in \mathcal{O}\left(\overline{D(\zeta_0,2\varepsilon_{\zeta_0})}\right)$ with $r=2\varepsilon_{\zeta_0}$, yields

$$\begin{aligned}
\left|\Delta_{p,(\eta_{i_p},\ldots,\eta_{i_1})}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{i_{p+1}})\right| &= \left|\Delta_{p,(\eta_{i_p},\ldots,\eta_{i_1})}\left[\left(\frac{g_{\zeta_0}(\zeta)}{1+\zeta g_{\zeta_0}(\zeta)}\right)^q\right](\eta_{i_{p+1}})\right|\\
&\leq 2\varepsilon_{\zeta_0}\sup_{|\zeta-\zeta_0|=2\varepsilon_{\zeta_0}}\left|\frac{1}{\prod_{j=1}^{p+1}(\zeta-\eta_{i_j})}\left(\frac{g_{\zeta_0}(\zeta)}{1+\zeta g_{\zeta_0}(\zeta)}\right)^q\right|\\
&\leq 2\varepsilon_{\zeta_0}\frac{1}{\varepsilon_{\zeta_0}^{p+1}}\,2^q\,,
\end{aligned}$$

the last estimate coming from (5.2) and the fact that $\eta_{i_1},\ldots,\eta_{i_{p+2}}\in D(\eta_{j_1},\varepsilon_0)\subset D(\zeta_0,\varepsilon_{\zeta_0})$. Since $\varepsilon_{\zeta_0}\geq\varepsilon_0$ and $\varepsilon_0\leq 1$, it follows that

$$\left|\Delta_{p,(\eta_{i_p},\ldots,\eta_{i_1})}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{i_{p+1}})\right| \leq \frac{2}{\varepsilon_0^p}\,2^q \leq \frac{2^{p+1}}{\varepsilon_0^{p+1}}\,2^q$$

and this proves (5.3) in the second case. The proof of the lemma is finished.

$\surd$

Remark 5.1. The proof can also be direct without using Theorem 1 (see [9]).

5.2. **Proof of Theorem 3 in the general case.** We start by giving some comments about a set $\{\eta_j\}_{j\geq 1}$ that is locally interpolable by real-analytic curves. First, in Definition 2, we do not need to assume that $\{\eta_j\}_{j\geq 1}$ is not dense. The following result specifies that it cannot be the case.

Lemma 20. *The topological closure of a set that is locally interpolable by real-analytic curves, has empty interior.*

Proof. Let assume that it is not the case. Then there are $\zeta_0 \in \mathbb{C}$ and $\varepsilon_0 > 0$ such that $D(\zeta_0, \varepsilon_0) \subset \overline{\{\eta_j\}_{j\geq 1}}$. In particular, ζ_0 cannot be isolated. Since $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves, by Lemma 18 there are $V_{\zeta_0} \in \mathcal{V}(\zeta_0)$ and $g_{\zeta_0} \in \mathcal{O}(V_{\zeta_0})$ such that, for all $\eta_j \in V$, $g_{\zeta_0}(\eta_j) = \overline{\eta_j}$. By reducing V_{ζ_0} if necessary, one can assume that $V_{\zeta_0} \subset \overline{\{\eta_j\}_{j\geq 1}}$. For any subsequence $(\eta_{j_k})_{k\geq 1}$ that converges to ζ_0 with $\eta_{j_k} \neq \zeta_0$, $\forall\, k \geq 1$, one has for all k large enough (so that $\eta_{j_k} \in V_{\zeta_0}$)

$$\frac{\overline{\eta_{j_k} - \zeta_0}}{\eta_{j_k} - \zeta_0} = \frac{g(\eta_{j_k}) - g(\zeta_0)}{\eta_{j_k} - \zeta_0} \xrightarrow[k\to\infty]{} \frac{\partial g}{\partial \zeta}(\zeta_0)\,.$$

In particular $\left|\frac{\partial g}{\partial \zeta}(\zeta_0)\right| = 1$ then $\frac{\partial g}{\partial \zeta}(\zeta_0) = e^{i\theta}$ for $\theta \in [0, 2\pi)$.

Now let set $w_p = \zeta_0 + i\, e^{-i\theta/2}/p$ with p large enough so that $w_p \in V_{\zeta_0}$. Since $\{w_p\}_{p\geq p_0} \subset V_{\zeta_0} \subset \overline{\{\eta_j\}_{j\geq 1}}$, for all $p \geq p_0$, there is $\eta_{j_p} \in \{\eta_j\}_{j\geq 1}$ such that $\eta_{j_p} \in V_{\zeta_0}$ and $\left|\eta_{j_p} - w_p\right| \leq 1/(2p^2)$ (in particular, $(\eta_{j_p})_{p\geq p_0}$ converges to ζ_0), then

$$\begin{aligned} e^{i\theta} = \frac{\partial g}{\partial \zeta}(\zeta_0) &= \lim_{p\to\infty} \frac{\overline{\eta_{j_p}} - \overline{\zeta_0}}{\eta_{j_p} - \zeta_0} = \lim_{p\to\infty} \frac{\overline{w_p} - \overline{\zeta_0} + \overline{\eta_{j_p}} - \overline{w_p}}{w_p - \zeta_0 + \eta_{j_p} - w_p} \\ &= \lim_{p\to\infty} \frac{-ie^{i\theta/2}/p + O(1/p^2)}{ie^{-i\theta/2}/p + O(1/p^2)} = -e^{i\theta}\,, \end{aligned}$$

and that is impossible.

$\surd$

The following result specifies that this geometric condition is not changed by any rotation of the lines.

Lemma 21. *Let assume that $\overline{\{\eta_j\}_{j\geq 1}}$ is locally interpolable by real-analytic curves. Then for all $\eta^c \notin \{\eta_j\}_{j\geq 1}$, so is $\{\theta_j\}_{j\geq 1}$ where $\theta_j = h_{\eta^c}(\eta_j)$, $\forall\, j \geq 1$.*

Proof. Let fix any η^c and consider the associated θ_j's. First, since h_{η^c} is homographic, in particular it is a topological isomorphism of $\overline{\mathbb{C}}$ then $\overline{\{\theta_j\}_{j\geq 1}} = h_{\eta^c}\left(\overline{\{\eta_j\}_{j\geq 1}}\right)$.

Next, let be $\zeta_0 \in \overline{\{\eta_j\}_{j\geq 1}}$ with ζ_0 finite. If $h_{\eta^c}(\zeta_0)$ is finite, then the equivalent geometric criterion claimed by Lemma 18 is satisfied on $h_{\eta^c}(\zeta_0)$ since condition (5.1) is invariant under the action of biholomorphisms. Indeed, let V be a neighborhood of ζ_0 and $g \in \mathcal{O}(V)$ which satisfy (5.1) and let recall the definition (1.10) of h_{η^c} from Introduction:

$$\begin{aligned} h_{\eta^c} : \overline{\mathbb{C}} &\to \overline{\mathbb{C}} \\ \zeta &\mapsto \frac{\overline{\eta^c}\zeta + 1}{\zeta - \eta^c}\,. \end{aligned}$$

In particular $h_{\eta^c}^{-1}(\zeta) = \frac{\eta^c\zeta + 1}{\zeta - \overline{\eta^c}} = \overline{h_{\eta^c}\left(\overline{\zeta}\right)}$ and one can see that $\overline{\theta_j} = h_{\eta^c}^{-1}\left[g\left(h_{\eta^c}^{-1}(\theta_j)\right)\right]$, $\forall\, \theta_j \in h_{\eta^c}(V)$.

If $h_{\eta^c}(\zeta_0)=\infty$, the same argument holds since the analogous function $h_{\eta^c}^{-1}\circ g\circ h_{\eta^c}^{-1}$ will be holomorphic in a neighborhood of ∞ (as specified by (1.13) from Introduction).

The case where $\zeta_0=\infty$ and $h_{\eta^c}(\infty)$ is finite (resp. $=\infty$) is analogous.

$\surd$

Now we can give the proof of Theorem 3.

Proof. Let $\{\eta_j\}_{j\geq 1}$ be a subset that is locally interpolable by real-analytic curves. If it is bounded, then Theorem 3 follows by Lemma 19.

Otherwise, we know by Lemma 20 that $\{\eta_j\}_{j\geq 1}$ cannot be dense, then there is $\eta_\infty\notin\overline{\{\eta_j\}_{j\geq 1}}$. Let consider the associated bounded subset $\{\theta_j\}_{j\geq 1}$, where $\theta_j=h_{\eta_\infty}(\eta_j)$, $j\geq 1$ (where h_{η_∞} is defined as in (1.10) from Introduction with the choice of $\eta^c:=\eta_\infty$). Then $\{\theta_j\}_{j\geq 1}$ is bounded (the justification is the same as in the proof of part (2) from Theorem 2). On the other hand, by Lemma 21, $\{\theta_j\}_{j\geq 1}$ is still locally interpolable by real-analytic curves. It follows by Lemmas 19 and 7 that there is $\varepsilon_\eta>0$ such that $R_N(f;\theta)$ converges to 0 uniformly on any compact subset of $B_2(0,\varepsilon_\eta r_0)$ (resp. $\mathbb{C}^2$), for any $f\in\mathcal{O}\left(B_2(0,r_0)\right)$ (resp. $\mathcal{O}\left(\mathbb{C}^2\right)$). Finally, by Lemma 16, this holds for $R_N(\cdot;\eta)$ (hence for $E_N(\cdot;\eta)$ by lemma 7) and this proves the theorem.

$\surd$

5.3. **A special case of equivalence with the geometric condition.** In this part we give the proof of Proposition 2 claimed in Introduction.

Proof. First, the set $\{\eta_j\}_{j\geq 1}$ is bounded since the sequence $(\eta_j)_{j\geq 1}$ is convergent. One has in addition $\overline{\{\eta_j\}_{j\geq 1}}=\{\eta_\infty\}\cup\{\eta_j\}_{j\geq 1}$. It follows that, $\zeta_0\in\overline{\{\eta_j\}_{j\geq 1}}$ being given, one can assume that $\zeta_0=\eta_\infty$ (otherwise ζ_0 is one of the η_j's that are isolated). Since $E_N(\cdot;\eta)$ converges (in the meaning of Definition 1), then in particular $E_N(f;\eta)$ converges to f on any compact subset of $\mathbb{C}^2$ and for all $f\in\mathcal{O}\left(\mathbb{C}^2\right)$. By applying Theorem 1 and the estimate (1.9) for $q=1$, it follows that there is R_η such that, $\forall\, p\geq 0$, one has

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\zeta\mapsto\frac{\overline{\zeta}}{1+|\zeta|^2}\right](\eta_{p+1})\right| \quad\leq\quad {R_\eta}^{p+1}\,. \tag{5.4}$$

For all $N\geq 1$, let consider $\mathcal{L}_N[\varphi]$, the Lagrange interpolating polynomial of $\varphi(\zeta)=\overline{\zeta}/(1+|\zeta|^2)$ (see (2.1)). We know by Lemma 2 that

$$\mathcal{L}_N[\varphi](\zeta) \quad=\quad \sum_{p=0}^{N-1}\left(\prod_{j=1}^{p}\left((\zeta-\eta_\infty)-(\eta_j-\eta_\infty)\right)\right)\Delta_{p,(\eta_p,\ldots,\eta_1)}[\zeta\mapsto\varphi(\zeta)]\,(\eta_{p+1})\ .$$

Let be p_0 such that, $\forall\, j\geq p_0$, $|\eta_j-\eta_\infty|\leq 1/(3R_\eta)$ and let consider $V=D\left(\eta_\infty,1/(3R_\eta)\right)$. One has by (5.4) for all $N\geq p_0$ and $\zeta\in V$,

$$\begin{aligned}|\mathcal{L}_N(\varphi)(\zeta)| \quad\leq\quad & \sum_{p=0}^{p_0-1}\left(\prod_{j=1}^{p}\left(|\zeta|+|\eta_j|\right)\right)\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[\varphi]\,(\eta_{p+1})\right| \\ & +\sum_{p=p_0}^{N}\left(\prod_{j=1}^{p_0}\left(|\zeta|+|\eta_j|\right)\prod_{j=p_0+1}^{p}\left(|\zeta-\eta_\infty|+|\eta_j-\eta_\infty|\right)\right)\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[\varphi]\,(\eta_{p+1})\right|\end{aligned}$$

$$\leq \sum_{p=0}^{p_0-1}\left(|\eta_\infty|+1/(3R_\eta)+\max_{1\leq j\leq p_0}|\eta_j|\right)^p R_\eta^{p+1}$$
$$+\sum_{p=p_0}^{N}\left(|\eta_\infty|+1/(3R_\eta)+\max_{1\leq j\leq p_0}|\eta_j|\right)^{p_0}\left(\frac{2}{3R_\eta}\right)^{p-p_0} R_\eta^{p+1}\,.$$

It follows that there is C_{p_0} large enough such that $\forall\, N \geq p_0$,

$$\sup_{N\geq p_0}\sup_{\zeta\in V}|\mathcal{L}_N(\varphi)(\zeta)| \;\leq\; \sup_{N\geq p_0}\left(C_{p_0}+C_{p_0}\sum_{p=p_0}^{N}(2/3)^{p-p_0}\right) \;<\; +\infty\,.$$

The sequence $(\mathcal{L}_N(\varphi))_{N\geq 1}$ of polynomials is uniformly bounded on V. By the Stiltjes-Vitali-Montel Theorem, there is a subsequence $(\mathcal{L}_{N_k}(\varphi))_{k\geq 1}$ that uniformly converges on any compact $K\subset V$ to a function g that is holomorphic on V. One has in addition for all $\eta_j\in V$,

$$g(\eta_j) \;=\; \lim_{k\to\infty, N_k\geq j}\mathcal{L}_{N_k}(\varphi)(\eta_j) \;=\; \lim_{k\to\infty, N_k\geq j}\varphi(\eta_j) \;=\; \varphi(\eta_j)\,,$$

i.e. the (nonholomorphic) function φ coincides with g on $V\cap\{\eta_j\}_{j\geq 1}$. In particular, this yields

$$\overline{\eta_j} \;=\; \frac{g(\eta_j)}{1-\eta_j g(\eta_j)}\,. \tag{5.5}$$

On the other hand,

$$|\eta_\infty g(\eta_\infty)| \;=\; \lim_{j\to\infty}|\eta_j g(\eta_j)| \;=\; \lim_{j\to\infty}|\eta_j\varphi(\eta_j)| \;=\; |\eta_\infty\varphi(\eta_\infty)| \;=\; \frac{|\eta_\infty|^2}{1+|\eta_\infty|^2} \;<\; 1\,.$$

Then by reducing V if necessary, the following function

$$\begin{array}{rcl}\widetilde{g}\,:\,V & \longrightarrow & \mathbb{C}\\ \zeta & \mapsto & \dfrac{g(\zeta)}{1-\zeta g(\zeta)}\,,\end{array}$$

is well-defined, holomorphic and satisfies by (5.5) the geometric criterion (5.1). The proof follows by applying Lemma 18.

$\surd$

Remark 5.2. One can see from this proof that, in order to prove the assertion, it suffices to assume that $E_N(f;\eta)$ converges to f on any compact subset of $\mathbb{C}^2$ and for all $f\in\mathcal{O}\left(\mathbb{C}^2\right)$ (part (2) from Theorem 1).

As a consequence we get an effective process to construct examples of sets $\{\eta_j\}_{j\geq 1}$ for which the associated interpolation formula $E_N(\cdot;\eta)$ does not converge: any convergent sequence that cannot be embedded in any real-analytic curve like in the following result.

Corollary 2. *Let consider the following sequence defined as*

$$\eta_j \;:=\; \frac{i^j}{j}\,,\;\forall\, j\geq 1\,.$$

Then the associated interpolation formula $E_N(\cdot;\eta)$ does not converge (in the meaning of Definition 1).

Proof. By Lemma 8, in order to prove that $E_N(\cdot;\eta)$ does not converge in the meaning of Definition 1, it suffices to prove that $E_N(\cdot;\eta)$ does not satisfy part (2) from Theorem 1, i.e. it is false that $E_N(f;\eta)$ converges to f uniformly on any compact subset of $\mathbb{C}^2$ and for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$. If it were true, then by Proposition 2 and Remark 5.2, the convergent sequence $(\eta_j)_{j\geq 1}$ would be locally interpolable by real-analytic curves. By Lemma 18, there would exist a neighborhood V of 0 and $g \in \mathcal{O}(V)$ that satisfy condition (5.1). In particular, one would have for all j large enough that $g(\eta_{2j}) = g((-1)^j/(2j)) = (-1)^j/(2j) = \eta_{2j}$ (resp. $g(\eta_{2j+1}) = g((-1)^j\, i/(2j+1)) = -(-1)^j\, i/(2j+1) = -\eta_{2j+1}$), hence by the uniqueness theorem, $\zeta = g(\zeta) = -\zeta$. This is impossible in any neighborhood of 0.

$\surd$

References

[1] B. Berndtsson, A formula for interpolation and division in $\mathbb{C}^n$, *Math. Ann.* **263** (1983), 399–418.
[2] C. de Boor, Divided differences, *Surveys in Approximation Theory* **1** (2005), 46–69.
[3] N. Coleff, M. Herrera, Les courants résiduels associés à une forme méromorphe (French), *Lecture Notes in Mathematics*, **633**, Springer, Berlin (1978).
[4] R.A. DeVore, G.G. Lorentz, *Constructive Approximation*, Springer-Verlag (1993).
[5] A.O. Guelfond, *Calcul des Différences Finies* (French), Dunod (1963).
[6] G.M. Henkin, A.A. Shananin, $\mathbb{C}^n$-capacity and multidimensional moment problem, *Notre Dame Math. Lectures* **12** (1992), 69–85.
[7] M. Herrera, M. Lieberman, Residues and principal values on complex spaces, *Math. Ann.* **194** (1971), 259–294.
[8] A. Irigoyen, An approximation formula for holomorphic functions by interpolation on the ball (2008), http://arxiv.org/abs/0803.4178
[9] A. Irigoyen, A criterion for the explicit reconstruction of a holomorphic function from its restrictions on lines (2010), http://arxiv.org/abs/1001.2431
[10] A. Irigoyen, Geometric conditions for the reconstruction of a holomorphic function by interpolation formula (2011), http://arxiv.org/abs/1108.0008
[11] A. Irigoyen, An application of reconstruction formulas by interpolation methods in Radon transform theory, *in progress*.
[12] B.F. Logan, L.A. Shepp, Optimal reconstruction of a function from its projections, *Duke Math. J.* **42** (1975), 645–659.
[13] A. Martineau, Équations différentielles d'ordre infini, *Bull. Soc. Math. France*, **95** (1967), 109–154.
[14] J.F. Steffensen, *Interpolation*, Williams and Wilkins, Baltimore (1927).
[15] J.F. Steffensen, Note on divided differnences, *Danske Vid. Selsk. Math.-Fys. Medd.* **17 (3)** (1939), 1–12.

Appendix: Proof of relation (1.2) from Introduction

In this part we want to prove the following result.

Proposition 3. *Let be $f \in \mathcal{O}(B_2(0,r_0))$ (resp. $f \in \mathcal{O}(\mathbb{C}^2)$) and $\sum_{k,l\geq 0} a_{k,l} z_1^k z_2^l$ its Taylor expansion. One has, for all $N \geq 1$ and $z \in B_2(0,r_0)$ (resp. $z \in \mathbb{C}^2$),*

$$f(z) \quad = \quad E_N(f;\eta)(z) - R_N(f;\eta)(z) + \sum_{k+l\geq N} a_{k,l} z_1^k z_2^l\,, \tag{5.6}$$

where

$$\begin{aligned} E_N(f;\eta)(z) \quad &:= \quad \sum_{p=1}^{N} \left(\prod_{j=p+1}^{N} (z_1 - \eta_j z_2) \right) \sum_{q=p}^{N} \frac{1+\eta_p \overline{\eta_q}}{1+|\eta_q|^2} \frac{1}{\prod_{j=p, j\neq q}^{N} (\eta_q - \eta_j)} \times \\ & \times \sum_{m\geq N-p} \left(\frac{z_2 + \overline{\eta_q} z_1}{1+|\eta_q|^2} \right)^{m-N+p} \frac{1}{m!} \frac{\partial^m}{\partial v^m}|_{v=0}[f(\eta_q v, v)] \end{aligned}$$

and

$$R_N(f;\eta)(z) := \sum_{p=1}^{N} \left(\prod_{j=1, j\neq p}^{N} \frac{z_1 - \eta_j z_2}{\eta_p - \eta_j} \right) \sum_{k+l\geq N} a_{k,l} \eta_p^k \left(\frac{z_2 + \overline{\eta_p} z_1}{1+|\eta_p|^2} \right)^{k+l-N+1}.$$

Moreover, the function $E_N(f;\eta)$ satisfies the following properties:

(1) $E_N(f;\eta) \in \mathcal{O}(B_2(0,r_0))$ *(resp. $E_N(f;\eta) \in \mathcal{O}(\mathbb{C}^2)$);*
(2) $E_N(f;\eta)$ *is an explicit formula that is constructed with the data*
$$\left\{ f_{|\{z_1=\eta_p z_2\}} \right\}_{1\leq p\leq N} \; ;$$
(3) $\forall\, p = 1, \ldots, N,$
$$E_N(f;\eta)_{|\{z_1=\eta_p z_2\}} \quad = \quad f_{|\{z_1=\eta_p z_2\}}\,;$$
(4) $\forall\, P \in \mathbb{C}[z_1, z_2]$ *with* $\deg P \leq N-1$, $E_N(P;\eta) \equiv P$.

This result allows to justify relation (1.2) that has been claimed in Introduction. Before giving its proof, we begin with the following preliminar lemma.

Lemma 22. *$E_N(f;\eta)(z)$ (resp. $R_N(f;\eta)(z)$) is well-defined for all $z \in B_2(0,r_0)$ and is holomorphic on $B_2(0,r_0)$. Similarly, if $f \in \mathcal{O}(\mathbb{C}^2)$, then $E_N(f;\eta)$, $R_N(f;\eta) \in \mathcal{O}(\mathbb{C}^2)$.*

Proof. We just consider $E_N(f;\eta)$ (the case of $R_N(f;\eta)$ is similar). It will suffice to show that the series is absotulely convergent for any $z \in B_2(0,r_0)$. First, by the Taylor expansion of f, one has that, for all $m \geq 0$ and $q = 1, \ldots, N$,

$$\frac{1}{m!} \frac{\partial^m}{\partial v^m}|_{v=0}[f(\eta_q v, v)] \;=\; \sum_{k+l\geq 0} a_{k,l} \eta_q^k \frac{1}{m!} \frac{\partial^m}{\partial v^m}|_{v=0}\left[v^{k+l}\right] \;=\; \sum_{k+l=m} a_{k,l} \eta_q^k\,. \tag{5.7}$$

For all $N \geq 1$, $p = 1, \ldots, N$, by using the Cauchy-Schwarz inequality (one can assume that $z \neq 0$ otherwise the assertion is obvious), this leads to

$$\sum_{m\geq N-p} \left| \frac{z_2 + \overline{\eta_q} z_1}{1+|\eta_q|^2} \right|^{m-N+p} \frac{1}{m!} \left| \frac{\partial^m}{\partial v^m}|_{v=0}[f(\eta_q v, v)] \right| \quad \leq$$

$$
\begin{aligned}
&\leq \sum_{m\geq N-p}\left(\frac{\|z\|\sqrt{1+|\eta_q|^2}}{1+|\eta_q|^2}\right)^{m-N+p}\sum_{k+l=m}|a_{k,l}|\;|\eta_q|^k\\
&\leq \left(\frac{\sqrt{1+|\eta_q|^2}}{\|z\|}\right)^{N-p}\sum_{k+l\geq N-p}|a_{k,l}|\left(\frac{|\eta_q|\;\|z\|}{\sqrt{1+|\eta_q|^2}}\right)^{k}\left(\frac{\|z\|}{\sqrt{1+|\eta_q|^2}}\right)^{l},
\end{aligned}
$$

This sum is finished since $f\in\mathcal{O}\left(B_2(0,r_0)\right)$ and

$$
\left\|\left(\frac{|\eta_q|\;\|z\|}{\sqrt{1+|\eta_q|^2}},\frac{\|z\|}{\sqrt{1+|\eta_q|^2}}\right)\right\| \;=\; \frac{\|z\|}{\sqrt{1+|\eta_q|^2}}\sqrt{|\eta_q|^2+1} \;=\; \|z\| \;<\; r_0\,,
$$

then one has $\left(\frac{|\eta_q|\;\|z\|}{\sqrt{1+|\eta_q|^2}},\frac{\|z\|}{\sqrt{1+|\eta_q|^2}}\right)\in B_2(0,r_0)$ and the Taylor expansion (also any partial sum) on z is finished.

Thus $E_N(f;\eta)$ is well-defined and holomorphic on $B_2(0,r_0)$. Finally, the case of $f\in\mathcal{O}\left(\mathbb{C}^2\right)$ follows by restriction on $B_2(0,r_0)$ where r_0 can be taken arbitrary large.

$\surd$

The next result is the proof of (5.6).

Lemma 23. *One has for all $N\geq 1$ and $z\in B_2(0,r_0)$,*

$$
f(z) \;=\; E_N(f;\eta)(z)-R_N(f;\eta)(z)+\sum_{k+l\geq N}a_{k,l}z_1^kz_2^l\,.
$$

Proof. We prove this equality by induction on $N\geq 1$. For $N=1$, by using (5.7), one has for all $z\in B_2(0,r_0)$,

$$
E_1(f;\eta)(z) \;=\; \sum_{k+l\geq 0}a_{k,l}\eta_1^k\left(\frac{z_2+\overline{\eta_1}z_1}{\sqrt{1+|\eta_1|^2}}\right)^{k+l}
$$

and

$$
R_1(f;\eta)(z) \;=\; \sum_{k+l\geq 1}a_{k,l}\eta_1^k\left(\frac{z_2+\overline{\eta_1}z_1}{\sqrt{1+|\eta_1|^2}}\right)^{k+l}.
$$

It follows that

$$
E_1(f;\eta)(z)-R_1(f;\eta)(z)+\sum_{k+l\geq 1}a_{k,l}z_1^kz_2^l \;=\; a_{0,0}+\sum_{k+l\geq 1}a_{k,l}z_1^kz_2^l \;=\; f(z)\,,
$$

and this proves the lemma for $N=1$.

Now we assume that the assertion is true for N, i.e. it is true for any function $f\in\mathcal{O}\left(B(0,r_0)\right)$ and any N-set of different points $\eta_1,\ldots,\eta_N$. Then we consider $E_{N+1}(f;\eta)(z)$ and $R_{N+1}(f;\eta)(z)$ (with $z\in B_2(0,r_0)$ being fixed). First, by using (5.7) and isolating the index $p=1$ in the below sum, one has

$$
\begin{aligned}
E_{N+1}(f;\eta)(z) \;&=\; \sum_{p=1}^{N+1}\left(\prod_{j=p+1}^{N+1}(z_1-\eta_jz_2)\right)\sum_{q=p}^{N+1}\frac{1+\eta_p\overline{\eta_q}}{1+|\eta_q|^2}\frac{1}{\prod_{j=p,j\neq q}^{N+1}(\eta_q-\eta_j)}\times\\
&\quad\times\sum_{k+l\geq N+1-p}a_{k,l}\eta_q^k\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{k+l-(N+1)+p}\\
&=\; A_{N+1}+B_{N+1}\,,
\end{aligned}\tag{5.8}
$$

where

$$A_{N+1} = \prod_{j=2}^{N+1}(z_1-\eta_j z_2)\sum_{q=1}^{N+1}\frac{1+\eta_1\overline{\eta_q}}{1+|\eta_q|^2}\frac{1}{\prod_{j=1,j\neq q}^{N+1}(\eta_q-\eta_j)}\sum_{k+l\geq N}a_{k,l}\eta_q^k\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{k+l-N}$$

and

$$\begin{aligned}B_{N+1} &= \sum_{p=2}^{N+1}\left(\prod_{j=p+1}^{N+1}(z_1-\eta_j z_2)\right)\sum_{q=p}^{N+1}\frac{1+\eta_p\overline{\eta_q}}{1+|\eta_q|^2}\frac{\sum_{k+l\geq N+1-p}a_{k,l}\eta_q^k\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{k+l-N-1+p}}{\prod_{j=p,j\neq q}^{N+1}(\eta_q-\eta_j)}\\ &= \sum_{p=1}^{N}\left(\prod_{j=p+1}^{N}(z_1-\eta_{j+1}z_2)\right)\sum_{q=p}^{N}\frac{1+\eta_{p+1}\overline{\eta_{q+1}}}{1+|\eta_{q+1}|^2}\frac{\sum_{k+l\geq N-p}a_{k,l}\eta_{q+1}^k\left(\frac{z_2+\overline{\eta_{q+1}}\,z_1}{1+|\eta_{q+1}|^2}\right)^{k+l-N+p}}{\prod_{j=p,j\neq q}^{N}(\eta_{q+1}-\eta_{j+1})}.\end{aligned}$$

In particular,

$$B_{N+1} = E_N(f;\eta')(z), \tag{5.9}$$

where

$$\eta' := (\eta_2,\eta_3,\dots). \tag{5.10}$$

Next, we claim that

$$\begin{aligned}R_{N+1}(f;\eta)(z) &= \sum_{q=1}^{N+1}\left(\prod_{j=1,j\neq q}^{N+1}\frac{z_1-\eta_j z_2}{\eta_q-\eta_j}\right)\sum_{k+l\geq N}a_{k,l}\eta_q^k\left(\frac{z_2+\overline{\eta_q}\,z_1}{1+|\eta_q|^2}\right)^{k+l-N}\\ &\quad -\sum_{k+l=N}a_{k,l}z_1^k z_2^l.\end{aligned}\tag{5.11}$$

Indeed, for all $z_2\neq 0$ (that will suffice since the involved functions are holomorphic), one has

$$\begin{aligned}R_{N+1}(f;\eta)(z) &= \sum_{q=1}^{N+1}\left(\prod_{j=1,j\neq q}^{N+1}\frac{z_1-\eta_j z_2}{\eta_q-\eta_j}\right)\sum_{k+l\geq N+1}a_{k,l}\eta_q^k\left(\frac{z_2+\overline{\eta_q}\,z_1}{1+|\eta_q|^2}\right)^{k+l-(N+1)-1}\\ &= \sum_{q=1}^{N+1}\left(\prod_{j=1,j\neq q}^{N+1}\frac{z_1-\eta_j z_2}{\eta_q-\eta_j}\right)\sum_{k+l\geq N}a_{k,l}\eta_q^k\left(\frac{z_2+\overline{\eta_q}\,z_1}{1+|\eta_q|^2}\right)^{k+l-N}\\ &\quad -z_2^N\sum_{k+l=N}a_{k,l}\left[\sum_{q=1}^{N+1}\left(\prod_{j=1,j\neq q}^{N+1}\frac{z_1/z_2-\eta_j}{\eta_q-\eta_j}\right)\eta_q^k\right].\end{aligned}$$

On the other hand, since $0\leq k\leq k+l=N<N+1$, then the Lagrange polynomial $\mathcal{L}_{N+1}(X^k)$ on the $N+1$ points $\eta_1,\dots,\eta_{N+1}$, is exactly X^k, hence

$$z_2^N\sum_{k+l=N}a_{k,l}\left[\sum_{q=1}^{N+1}\left(\prod_{j=1,j\neq q}^{N+1}\frac{z_1/z_2-\eta_j}{\eta_q-\eta_j}\right)\eta_q^k\right] = \sum_{k+l=N}a_{k,l}z_1^k z_2^{N-k}$$

and this proves the claim.

Finally, the last part with which we have to deal is A_{N+1}. Since for all $q=1,\dots,N+1$, one has

$$\prod_{j=2}^{N+1}(z_1-\eta_j z_2) = \prod_{j=1,j\neq q}^{N+1}(z_1-\eta_j z_2)-z_2(\eta_q-\eta_1)\prod_{j=2,j\neq q}^{N+1}(z_1-\eta_j z_2),$$

then

$$\begin{aligned}
A_{N+1} \;=\; & \sum_{q=1}^{N+1}\left(\prod_{j=1,j\neq q}^{N+1}\frac{z_1-\eta_j z_2}{\eta_q-\eta_j}\right)\frac{1+\eta_1\overline{\eta_q}}{1+|\eta_q|^2}\sum_{k+l\geq N} a_{k,l}\eta_q^k\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{k+l-N}\\
& -\sum_{q=2}^{N+1}\left(z_2\left(\eta_q-\eta_1\right)\prod_{j=2,j\neq q}^{N+1}\left(z_1-\eta_j z_2\right)\right)\frac{1+\eta_1\overline{\eta_q}}{1+|\eta_q|^2}\frac{\sum_{k+l\geq N} a_{k,l}\eta_q^k\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{k+l-N}}{\left(\eta_q-\eta_1\right)\prod_{j=2,j\neq q}^{N+1}(\eta_q-\eta_j)}\\
=\; & \sum_{q=1}^{N+1}\left(\prod_{j=1,j\neq q}^{N+1}\frac{z_1-\eta_j z_2}{\eta_q-\eta_j}\right)\frac{1+\eta_1\overline{\eta_q}}{1+|\eta_q|^2}\sum_{k+l\geq N} a_{k,l}\eta_q^k\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{k+l-N}\\
& -z_2\sum_{q=1}^{N}\left(\prod_{j=1,j\neq q}^{N}\frac{z_1-\eta_{j+1}z_2}{\eta_{q+1}-\eta_{j+1}}\right)\frac{1+\eta_1\overline{\eta_{q+1}}}{1+|\eta_{q+1}|^2}\sum_{k+l\geq N} a_{k,l}\eta_{q+1}^k\left(\frac{z_2+\overline{\eta_{q+1}}\,z_1}{1+|\eta_{q+1}|^2}\right)^{k+l-N}.
\end{aligned}$$

By applying (5.11), one has

$$\begin{aligned}
A_{N+1}-R_{N+1}(f;\eta)(z) \;=\; & \sum_{q=1}^{N+1}\left(\prod_{j=1,j\neq q}^{N+1}\frac{z_1-\eta_j z_2}{\eta_q-\eta_j}\right)\left[\frac{1+\eta_1\overline{\eta_q}}{1+|\eta_q|^2}-1\right]\sum_{k+l\geq N} a_{k,l}\eta_q^k\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{k+l-N}\\
-\, z_2\sum_{q=1}^{N}\left(\prod_{j=1,j\neq q}^{N}\frac{z_1-\eta_{j+1}z_2}{\eta_{q+1}-\eta_{j+1}}\right) & \frac{1+\eta_1\overline{\eta_{q+1}}}{1+|\eta_{q+1}|^2}\sum_{k+l\geq N} a_{k,l}\eta_{q+1}^k\left(\frac{z_2+\overline{\eta_{q+1}}\,z_1}{1+|\eta_{q+1}|^2}\right)^{k+l-N}+\sum_{k+l=N} a_{k,l}z_1^k z_2^l\,.
\end{aligned}$$

The first sum becomes

$$\begin{aligned}
& \sum_{q=2}^{N+1}\frac{z_1-\eta_1 z_2}{\eta_q-\eta_1}\left(\prod_{j=2,j\neq q}^{N+1}\frac{z_1-\eta_j z_2}{\eta_q-\eta_j}\right)\frac{\overline{\eta_q}(\eta_1-\eta_q)}{1+|\eta_q|^2}\sum_{k+l\geq N} a_{k,l}\eta_q^k\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{k+l-N} =\\
& = -\sum_{q=1}^{N}\left(\prod_{j=1,j\neq q}^{N}\frac{z_1-\eta_{j+1}z_2}{\eta_{q+1}-\eta_{j+1}}\right)\frac{\overline{\eta_{q+1}}(z_1-\eta_1 z_2)}{1+|\eta_{q+1}|^2}\sum_{k+l\geq N} a_{k,l}\eta_{q+1}^k\left(\frac{z_2+\overline{\eta_{q+1}}z_1}{1+|\eta_{q+1}|^2}\right)^{k+l-N}.
\end{aligned}$$

Since for all $q=1,\dots,N$, $\overline{\eta_{q+1}}(z_1-\eta_1 z_2)+z_2(1+\eta_1\overline{\eta_{q+1}})=z_2+\overline{\eta_{q+1}}z_1$, it follows that

$$\begin{aligned}
& A_{N+1}-R_{N+1}(f;\eta)(z) \;=\\
& = -\sum_{q=1}^{N}\left(\prod_{j=1,j\neq q}^{N}\frac{z_1-\eta_{j+1}z_2}{\eta_{q+1}-\eta_{j+1}}\right)\frac{z_2+\overline{\eta_{q+1}}z_1}{1+|\eta_{q+1}|^2}\sum_{k+l\geq N} a_{k,l}\eta_{q+1}^k\left(\frac{z_2+\overline{\eta_{q+1}}z_1}{1+|\eta_{q+1}|^2}\right)^{k+l-N}+\sum_{k+l=N} a_{k,l}z_1^k z_2^l\\
& = -R_N\left(f;\eta'\right)(z)+\sum_{k+l=N} a_{k,l}z_1^k z_2^l\,,
\end{aligned}$$

with the same definition of η' as in (5.10). We can finally deduce by applying (5.8) and (5.9) that

$$\begin{aligned}
& E_{N+1}(f;\eta)(z)-R_{N+1}(f;\eta)(z)+\sum_{k+l\geq N+1} a_{k,l}z_1^k z_2^l \;=\\
& \quad = -R_N\left(f;\eta'\right)(z)+\sum_{k+l=N} a_{k,l}z_1^k z_2^l+E_N\left(f;\eta'\right)(z)+\sum_{k+l\geq N+1} a_{k,l}z_1^k z_2^l\,,
\end{aligned}$$

that is exactly $f(z)$ by applying the induction hypothesis and the lemma follows.

$\surd$

Now we can prove Proposition 3.

Proof. The relation (5.6) is Lemma 23, property (1) is Lemma 22 and property (2) follows from the definition of $E_N(f;\eta)$.

In order to prove property (3), let fix $N \geq 1$ and p with $1 \leq p \leq N$. For all $z \in B_2(0, r_0)$ with $z_1 = \eta_p z_2$, one has

$$R_N(f;\eta)\left(\eta_p z_2, z_2\right) = z_2^{N-1} \sum_{k+l \geq N} a_{k,l} \eta_p^k z_2^{k+l-N+1} = \sum_{k+l \geq N} a_{k,l} \left(\eta_p z_2\right)^k z_2^l \,,$$

that is exactly the restriction on the complex line $\{z_1 = \eta_p z_2\}$ of the remainder part $\sum_{k+l \geq N} a_{k,l} z_1^k z_2^l$. The required property follows by (5.6).

Finally, property (4) is an immediate consequence of (5.6) since, for all k, l with $k + l \geq N > \deg P$, $a_{k,l}(P) = \dfrac{1}{k!\, l!} \dfrac{\partial^{k+l}}{\partial z_1^k \partial z_2^l}|_{z=0} \left[P\left(z\right)\right] = 0$.

$\surd$