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Stabilization of transmission coupled wave and Euler–Bernoulli equations on Riemannian manifolds by nonlinear feedbacks

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ABSTRACT

We consider the transmission system of coupling wave equations with Euler–Bernoulli equations on Riemannian manifolds. By introducing nonlinear boundary feedback controls, we establish the exponential and rational energy decay rate for the problem. Our proofs rely on the geometric multiplier method.

1. Introduction

In recent years, the control problems of transmission systems described by partial differential equations have been studied by many authors. Transmission systems appear in many practical control systems such as electromagnetic coupling, coupled chemical reactions, fluid-structure interactions, structural-acoustic systems and many other interactive physical processes. The analysis of controllability and stabilization for the transmission of hyperbolic systems, which arises in the control and suppression of noise, has been widely carried out.

For the coupled wave equations with constant coefficients (describing the wave with constant propagation speed), the stabilization and controllability results are presented in [15,16] by means of multiplier method. For the coupled wave equations with variable coefficients (describing the wave with variable propagation speed), the boundary controllability is treated in [13] and the stabilization is investigated in [5,7]. For the

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coupled Euler–Bernoulli plate equations, the uniform decay of solution is proved in [2], and by applying force and moment feedback control on different geometric domain a logarithmic decay of energy is obtained in [3] by the means of Carleman estimates. For the transmission system of coupled wave and plate equations, the stabilization problem is considered in [1] where the boundary feedback controls are applied on both the wave and plate equations and a logarithmic decay of the energy is obtained in [4] where the feedback is applied only on the boundary of the plate. For other transmission problems of hyperbolic systems, we refer to [6,11,13,17–19] for the related results.

These works offer a rich body of results regarding the transmission systems in the Euclidean setting. However, the problem on the compact Riemannian manifolds is less developed than that on Euclidean spaces. One important reason is that the trapped geodesics in the general Riemannian manifolds can preclude the effectiveness of the canonical multiplier such that it is hard to obtain the global estimates that are necessary in the stabilization and controllability analysis. On the other hand, a strong motivation for the problem on Riemannian manifolds comes from the practical applications. An example of this is the shape of protein and the non-zero curvature of membranes in the biological processes lead one to discuss the physical models on manifolds. In particular, if the plate in [1] has a curved middle surface Ω that is a part of surface in R^3, then the problem treated by [1] becomes one on the Riemannian (Ω, g), where g is the induced metric of R^3.

In the present paper we consider the stabilization of a coupled wave-plate system on Riemannian manifolds. K. Ammari and S. Nicaise presented in [1] the stabilization of the coupled wave-plate system in the Euclidean metric case (where the Gaussian curvature function on the whole domain equals zero). Inspired by [1] and [12] in which Guo and Yao investigated the uniform stabilization of a single Euler–Bernoulli plate equation on a Riemannian manifold, we consider the stabilization of the wave equation coupled by the interface with an Euler–Bernoulli plate equation on a Riemannian manifold. By introducing the nonlinear boundary feedback control on the boundary of the domain (not on the interface), we establish the uniform energy decay rate for the coupled system.

The main tool we use is the geometric multiplier method, which first appeared in [23] and subsequently in [8,10,14,22,24,25], and many others. First, we establish energy estimates for the coupled wave-plate system on Riemannian manifolds. Then, under different assumptions on the growth rate of the nonlinear feedbacks on the boundary, we obtain the exponential and polynomial energy decays of the coupled system in Theorems 2.2 and 2.3, respectively.

The content of this paper is organized as follows. We introduce the notations and state our main results in Section 2. In Section 3, we establish the energy estimates for the transmission system by the geometric multiplier method. Finally in Section 4, we present the proofs of the main results.

2. Main results

2.1. Some notations

We introduce some notations and definitions which are standard and classical in the literature, see [21]. Let M be a complete two dimensional Riemannian manifold of class C^3 with C^3-metric $g(\cdot,\cdot) = \langle\cdot,\cdot\rangle$. We shall denote it by (M, g). For each $x \in M$, M_x is the tangential space of M at x. Denote the set of all n order tensor fields on M by $T^n(M) = \bigcup_{x\in M} T^n_x(M)$, where n is a nonnegative integer. It is well known that the space $T^n_x(M)$ of n order tensor on M_x is an inner product space. Its inner product is defined by

$$\langle T_1, T_2\rangle_{T^n_x} = \sum_{i_1,i_2,\cdots,i_n=1}^{2} T_1(e_{i_1},\cdots,e_{i_n})T_2(e_{i_1},\cdots,e_{i_n}) \quad \text{at } x, \tag{2.1}$$

for any $T_1, T_2 \in T^n_x(M)$, where e_1, e_2 is an orthonormal basis of M_x for $x \in M$. For any $T \in T^2(M)$, the trace of T is defined by

$$\operatorname{tr} T = \sum_{i=1}^{2} T(e_i, e_i). \tag{2.2}$$

Further, we denote by ∇ the gradient, by D the Levi–Civita connection, by D^2 the Hessian, by $\Delta = \operatorname{div}(\nabla)$ the Laplace–Beltrami operator in the Riemannian metric g. For any vector field H on M, DH is the covariant differential of H which is a second order tensor field in the following sense:

$$DH(X,Y) = D_Y H(X) = \langle D_Y H, X\rangle \quad \text{for all } X, Y \in M_x,\ x \in M. \tag{2.3}$$

For scalar function u we have $Du = \nabla u$.

We refer the readers to [9,12,24] for further relationships.

2.2. Statement of the problem and main results

Let Ω be an open, bounded, connected subset of M with smooth boundary such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, where Ω_i, $i = 1, 2$ are two disjoint open connected bounded domains with smooth boundary. They satisfy that $\bar{\Omega}_1 \cap \bar{\Omega}_2 = S$, $\partial\Omega_1 = S \cup \Gamma_1$, and $\partial\Omega_2 = S \cup \Gamma_2$, where $S \neq \emptyset$, $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$.

We consider the wave equation in Ω_1 coupled with the Euler–Bernoulli plate equation in Ω_2 by the interface S under nonlinear boundary feedbacks v_i $(i = 1, 2, 3)$. More precisely, we consider the following system:

$$\begin{cases} \partial_t^2 u_1(x,t) - \Delta u_1(x,t) = 0, & \text{in } \Omega_1 \times (0, +\infty), \\ \partial_t^2 u_2(x,t) + \Delta^2 u_2(x,t) - (1-\mu)\delta(\mathcal{K} du_2)(x,t) = 0, & \text{in } \Omega_2 \times (0, +\infty), \\ u_i(x,0) = u_i^0(x),\ \partial_t u_i(x,0) = u_i^1(x), & \text{in } \Omega_i,\ i = 1, 2, \\ u_1 = u_2,\ B_1 u_2 = 0,\ B_2 u_2 = \partial_{\nu_1} u_1, & \text{on } S \times (0, +\infty), \\ \partial_{\nu_1} u_1 = v_1, & \text{on } \Gamma_1 \times (0, +\infty), \\ B_1 u_2 = v_2, & \text{on } \Gamma_2 \times (0, +\infty), \\ B_2 u_2 = v_3, & \text{on } \Gamma_2 \times (0, +\infty), \end{cases} \tag{2.4}$$

where $\nu_i = \nu_i(x)$ denotes the unit outward normal vector of Ω_i along $\partial\Omega_i = \Gamma_i \cup S$ for $i = 1, 2$. Here $\partial_{\nu_i} u_i = \frac{\partial u_i}{\partial \nu_i} = \langle \nu_i, Du_i\rangle$. In the above system, $\mathcal{K}$ is the Gaussian curvature function on Ω_2, $0 < \mu < \frac{1}{2}$ is the Poisson coefficient, d is the exterior derivative, δ is the formal adjoint operator of d, and the boundary operators B_1, B_2 are defined on $\partial\Omega_2 = \Gamma_2 \cup S$ as follows:

$$B_1 y = \Delta y - (1-\mu) D^2 y(\tau_2, \tau_2),$$
$$B_2 y = \partial_{\nu_2} \Delta y + (1-\mu) \frac{\partial}{\partial \tau_2}\big(D^2 y(\tau_2, \nu_2)\big) + \mathcal{K}\partial_{\nu_2} y,$$

where $D^2 y$ is the Hessian of y and τ_2 is the unit tangential vector along the boundary $\partial\Omega_2 = \Gamma_2 \cup S$.

The nonlinear functions v_i $(i = 1, 2, 3)$ are the feedbacks implemented on the boundary acting through the Euler–Bernoulli equation and the wave equation, respectively. In the present work, we will use the following nonlinear boundary feedback laws:

$$v_1 = -\alpha(x) u_1 - p(\partial_t u_1), \qquad v_2 = -\beta(x)\partial_{\nu_2} u_2 - h(\partial_{\nu_2}\partial_t u_2), \qquad v_3 = \gamma(x) u_2 + q(\partial_t u_2). \tag{2.5}$$

Remark 2.1. The term $(1-\mu)\delta(\mathcal{K} du_2)$ in system (2.4) comes from the curvedness of the Riemannian metric g. Transmission system (2.4) is considered in [1] when $M = R^2$, $\mathcal{K} = 0$ (the Euclidean metric case), and $p(s) = q(s) = h(s) = s$ (linear boundary feedbacks).

In what follows, we write system (2.4) as an abstract Cauchy problem to treat its well-posedness. Define a Hilbert space, by

$$\mathcal{H} = \{(u_1, w_1, u_2, w_2) \in H^1(\Omega_1) \times L^2(\Omega_1) \times H^2(\Omega_2) \times L^2(\Omega_2),\ u_1|_S = u_2|_S\}$$

equipped with the inner product

$$\begin{aligned} &\langle (u_1, w_1, u_2, w_2), (y_1, z_1, y_2, z_2)\rangle_{\mathcal{H}} \\ &\quad = \int_{\Omega_1} (\langle Du_1, Dy_1\rangle + w_1 z_1)\,dx + \int_{\Omega_2} (a(u_2, y_2) + w_2 z_2)\,dx \\ &\qquad + \int_{\Gamma_1} \alpha(x) u_1 y_1\,d\sigma + \int_{\Gamma_2} (\beta(x)\partial_{\nu_2} u_2 \partial_{\nu_2} y_2 + \gamma(x) u_2 y_2)\,d\sigma. \end{aligned} \tag{2.6}$$

Then we define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\begin{pmatrix} 0 & Id & 0 & 0 \\ \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & Id \\ 0 & 0 & -\Delta^2 + (1-\mu)\delta\mathcal{K}d & 0 \end{pmatrix} \tag{2.7}$$

with the domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \{(u_1, w_1, u_2, w_2) \in \mathcal{H} \mid\ & (w_1, \Delta u_1, w_2, -\Delta^2 u_2) \in \mathcal{H}, \\ & B_1 u_2 = 0,\ B_2 u_2 = \partial_{\nu_1} u_1,\ \text{on } S, \\ & \partial_{\nu_1} u_1 = -\alpha(x) u_1 - p(w_1),\ \text{on } \Gamma_1, \\ & B_1 u_2 = -\beta(x)\partial_{\nu_2} u_2 - h(\partial_{\nu_2} w_2),\ \text{on } \Gamma_2, \\ & B_2 u_2 = \gamma(x) u_2 + q(w_2),\ \text{on } \Gamma_2\}. \end{aligned} \tag{2.8}$$

By the semigroup theory, we can show that the operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ defined by (2.7) and (2.8) generates a contraction semigroup on the Hilbert space $\mathcal{H}$ endowed with the inner product (2.6). Thus we can obtain the well-posedness for problem (2.4). Here we state this proposition without proof. Its proof is similar as that in [1].

Proposition 2.1. *For all given initial data $(u_1^0, u_1^1, u_2^0, u_2^1) \in \mathcal{H}$, problem (2.4) admits a unique global weak solution*

$$(u_1, \partial_t u_1, u_2, \partial_t u_2) \in C([0,\infty); \mathcal{D}(\mathcal{A})) \cap C^1([0,\infty), \mathcal{H}). \tag{2.9}$$

We define the energy of a solution

$$u = \begin{cases} u_1, & x \in \Omega_1, \\ u_2, & x \in \Omega_2, \end{cases}$$

of system (2.4) with feedback laws (2.5) by

$$E(t)=\frac{1}{2}\left\{\int_{\Omega_1}\left(\left|\partial_t u_1(x,t)\right|^2+\left|Du_1(x,t)\right|^2\right)dx+\int_{\Omega_2}\left(\left|\partial_t u_2(x,t)\right|^2+a(u_2,u_2)\right)dx\right\}$$
$$+\frac{1}{2}\int_{\Gamma_1}\alpha(x)\left|u_1(x,t)\right|^2 d\sigma+\frac{1}{2}\int_{\Gamma_2}\left(\beta(x)\left|\partial_{\nu_2}u_2(x,t)\right|^2+\gamma(x)\left|u_2(x,t)\right|^2\right)d\sigma, \tag{2.10}$$

where the bilinear form $a(\cdot,\cdot)$ is defined on $H^2(\Omega_2)\times H^2(\Omega_2)$ by

$$a(y,z)=(1-\mu)\left\langle D^2y,D^2z\right\rangle_{T_x^2}+\mu\left(\operatorname{tr}D^2y\operatorname{tr}D^2z\right), \tag{2.11}$$

where the inner product $\langle\cdot,\cdot\rangle_{T_x^2}$ and operator $\operatorname{tr}T$ are given in (2.1) and (2.2), respectively. The following formula presents the relationship between the interior terms and boundary terms, which can be found in [12, Lemma 3.1]:

$$\int_{\Omega_2}\left[\Delta^2 y-(1-\mu)\delta(\mathcal{K}dy)\right]z\,dx=\int_{\Omega_2}a(y,z)dx-\int_{\Gamma_2\cup S}B_1y\frac{\partial z}{\partial\nu_2}d\sigma+\int_{\Gamma_2\cup S}B_2yz\,d\sigma, \tag{2.12}$$

for all $y\in H^4(\Omega_2)$ and $z\in H^2(\Omega_2)$. By using the above formula (2.12) and Eq. (2.4) we have

$$\frac{d}{dt}E(t)=-\int_{\Gamma_1}p(\partial_t u_1)\partial_t u_1 d\sigma-\int_{\Gamma_2}\left(h(\partial_{\nu_2}\partial_t u_2)\partial_{\nu_2}\partial_t u_2+q(\partial_t u_2)\partial_t u_2\right)d\sigma. \tag{2.13}$$

Here we can see, if we assume that p, q and h are nondecreasing continuous functions such that $p(0)=q(0)=h(0)=0$, it follows from (2.13) that system (2.4) with feedback law (2.5) is dissipative in the sense that the associated energy $E(t)$ is non-increasing.

To obtain the stabilization of problem (2.4), the following geometrical hypotheses are assumed:

Geometrical assumptions. Given the triple $\{\Omega_1,S,\Omega_2\}$, there exists a vector field H on Riemannian manifold (M,g) such that the following three properties hold true:

(A.1) $DH(\cdot,\cdot)$ is strictly positive definite on $\overline{\Omega}$: there exists a constant $\rho>0$ such that for all $x\in\overline{\Omega}$, for all $X\in M_x$ (the tangent space at x):

$$DH(X,X)\equiv\langle D_X H,X\rangle\geq\rho|X|^2. \tag{2.14}$$

(A.2)
$$\langle H,\nu_i\rangle\geq\theta\quad\text{on }\Gamma_i,\ i=1,2, \tag{2.15}$$
for some positive real number θ.

(A.3)
$$\langle H,\nu_i\rangle=0,\quad i=1,2\text{ on }S. \tag{2.16}$$

Remark 2.2. For any Riemannian manifold M, the existence of such a vector field H in (A.1) has been proved in [23], where some examples are given, too. See also [24]. For the Euclidean metric, taking the vector field $H=x-x_0$ and we have $DH(X,X)=|X|^2$, which means assumption (A.1) always holds true with $\rho=1$ for the Euclidean case.

Our main results can be summarized as follows:

Theorem 2.2. *We assume geometric assumptions (A.1)–(A.3) hold. Assume there exist some constants α_i, β_i and γ_i $(i = 0, 1)$ such that*

$$\alpha(x) \in L^\infty(\Gamma_1), \quad \alpha_1 \geq \alpha(x) \geq \alpha_0 > 0, \quad \text{on } \Gamma_1, \tag{2.17}$$

$$\beta(x) \in L^\infty(\Gamma_2), \quad \beta_1 \geq \beta(x) \geq \beta_0 > 0, \quad \text{on } \Gamma_2, \tag{2.18}$$

$$\gamma(x) \in L^\infty(\Gamma_2), \quad \gamma_1 \geq \gamma(x) \geq \gamma_0 > 0, \quad \text{on } \Gamma_2. \tag{2.19}$$

We assume the nonlinear feedbacks satisfy

$$p(x) \in C^0(R),\ \text{nondecreasing},\ p(0) = 0,\ p(s)s > 0,\ \text{for all } s \neq 0, \tag{2.20}$$

$$q(x) \in C^0(R),\ \text{nondecreasing},\ q(0) = 0,\ q(s)s > 0,\ \text{for all } s \neq 0, \tag{2.21}$$

$$h(x) \in C^0(R),\ \text{nondecreasing},\ h(0) = 0,\ h(s)s > 0,\ \text{for all } s \neq 0. \tag{2.22}$$

Let u be a solution of system (2.4).

(i) If there exist $L_0, L_1, L_2, L_3, L_4, L_5 > 0$ such that

$$\frac{|s|}{L_1} \leq p(s) \leq L_0|s|, \qquad \frac{|s|}{L_3} \leq q(s) \leq L_2|s|, \qquad \frac{|s|}{L_5} \leq h(s) \leq L_4|s|, \quad \text{for all } s \in R, \tag{2.23}$$

then for any given $C_0 > 1$, there exists a constant $\omega_1 > 0$ such that

$$E(t) \leq C_0 E(0) \exp(-\omega_1 t), \quad \text{for all } t \geq 0. \tag{2.24}$$

(ii) If there exist $L_0, L_1, L_2, L_3, L_4, L_5 > 0$ and $k > 1$ such that

$$\begin{aligned} &\frac{1}{L_1} \min\{|s|, |s|^k\} \leq p(s) \leq L_0|s|, \\ &\frac{1}{L_3} \min\{|s|, |s|^k\} \leq q(s) \leq L_2|s|, \\ &\frac{1}{L_5} \min\{|s|, |s|^k\} \leq h(s) \leq L_4|s|, \end{aligned} \tag{2.25}$$

for all $s \in R$, then for any given $C_0 > 1$, there exists a constant $\omega_2 > 0$, depending continuously on $E(0)$ such that

$$E(t) \leq C_0 E(0)(1 + \omega_2 t)^{-\frac{2}{k-1}}, \quad \text{for all } t \geq 0. \tag{2.26}$$

Theorem 2.3. *Assume assumptions (2.17)–(2.19) and (2.20)–(2.22) hold true. In addition, if there exist several constants L_0, L_1, L_2, L_3, L_4, $L_5 > 0$ and $k < 1$ such that*

$$\begin{aligned} &\frac{|s|}{L_1} \leq p(s) \leq L_0^k \max\{|s|, |s|^k\}, \\ &\frac{|s|}{L_3} \leq q(s) \leq L_2^k \max\{|s|, |s|^k\}, \\ &\frac{|s|}{L_5} \leq h(s) \leq L_4^k \max\{|s|, |s|^k\}, \end{aligned} \tag{2.27}$$

for all $s \in R$, then for any given $C_0 > 1$, there exists a constant $\omega_3 > 0$, depending continuously on $E(0)$ such that

$$E(t) \le C_0 E(0)(1+\omega_3 t)^{-\frac{2k}{1-k}}, \quad \text{for all } t \ge 0. \tag{2.28}$$

3. Some technical lemmas

To get some technical estimates for the solution of problem (2.4), in this section we assume that u is a strong solution. By a classical density argument we can see that these estimates hold for a weak solution.

Lemma 3.1. *(See [12, Lemma 3.1, p. 55].) Let $y, w \in H^4(\Omega_2)$. Then we have*

$$\int_{\Omega_2} \big(\Delta^2 y - (1-\mu)\delta(\mathcal{K} dy)\big) w dx = \int_{\Omega_2} a(y,w) dx - \int_{\partial\Omega_2} B_1 y \frac{\partial w}{\partial \nu_2} d\sigma + \int_{\partial\Omega_2} B_2 y w d\sigma. \tag{3.1}$$

Lemma 3.2. *Let H satisfy assumption (A.1). We have*

$$\int_{\Omega_2} a\big(y, H(y)\big) dx \ge \frac{1}{2} \int_{\partial\Omega_2} a(y,y)\langle H, \nu_2\rangle d\sigma + \rho \int_{\Omega_2} a(y,y) dx + \mathcal{L}(y), \tag{3.2}$$

where

$$\mathcal{L}(y) = \mu \operatorname{tr} D^2 l(y) - \frac{1}{4\rho} \int_{\Omega_2} \big|l(y)\big|^2_{T_x^2}, \tag{3.3}$$

which is a lower order term with respect to $a(y,y)$ and ρ is the constant given in (2.14).

Proof. Since $D^2 y$ is a symmetric second order tensor field, we have from [25, Lemma 2.7] that

$$\big\langle D^2 y, D^2\big(H(y)\big)\big\rangle_{T_x^2} = \frac{1}{2} H\big(\big|D^2 y\big|^2_{T_x^2}\big) + 2DH\big(D^2 y, D^2 y\big) + \big\langle D^2 y, l(y)\big\rangle_{T_x^2}, \tag{3.4}$$

and

$$\operatorname{tr} D^2 y \cdot \operatorname{tr} D^2\big(H(y)\big) = \frac{1}{2} H\big((\Delta y)^2\big) + 2DH(\Delta y, \Delta y) + \operatorname{tr} D^2 l(y), \tag{3.5}$$

where $l(y) = -R(Dy, \cdot, H, \cdot) - D^2 H(Dy, \cdot, \cdot)$ and "$\cdot$" denotes the position of the variable. Here R is the curvature tensor of the Levi–Civita connection D (for details see [21]).

Applying Cauchy–Schwartz's inequality, we have

$$\left| \int_{\Omega_2} \big\langle D^2 y, l(y)\big\rangle_{T_x^2} dx \right| \le \rho \int_{\Omega_2} \big|D^2 y\big|^2_{T_x^2} dx + \frac{1}{4\rho} \int_{\Omega_2} \big|l(y)\big|^2_{T_x^2} dx, \tag{3.6}$$

where ρ is given in (2.14). Combing (3.4), (3.5), (3.6) and assumption (A.1) we obtain

$$\int_{\Omega_2} a\big(y, H(y)\big) dx \ge \frac{1}{2} \int_{\partial\Omega_2} a(y,y)\langle H, \nu_2\rangle d\sigma + \rho \int_{\Omega_2} a(y,y) dx + \mu \operatorname{tr} D^2 l(y) - \frac{1}{4\rho} \int_{\Omega_2} \big|l(y)\big|^2_{T_x^2}.$$

We notice that both $\int_{\Omega_2} |l(y)|^2_{T_x^2} dx$ and $\operatorname{tr} D^2 l(y)$ are lower order terms respect to $a(y,y)$. Thus we denote

$$\mathcal{L}(y) = \mu \operatorname{tr} D^2 l(y) - \frac{1}{4\rho} \int\limits_{\Omega_2} \left| l(y) \right|^2_{T_x^2},$$

and the proof is completed. □

Lemma 3.3. *Let $y \in H^4(\Omega)$ be such that*

$$\begin{cases} \Delta^2 y \in L^2(\Omega_2), & \text{on } \Omega_2, \\ B_1 y = 0 \in L^2(S), & \text{on } S, \\ B_2 y = \chi_1 \in L^2(S), & \text{on } S, \\ B_1 y = v_2 \in L^2(\Gamma_2), & \text{on } \Gamma_2, \\ B_2 y = v_3 \in L^2(\Gamma_2), & \text{on } \Gamma_2. \end{cases} \tag{3.7}$$

Then we have

$$\begin{aligned} &- \int\limits_{\Omega_2} \big(\Delta^2 y - (1-\mu)\delta(\mathcal{K} dy)\big) H(y) dx \\ &\quad \le -\rho \int\limits_{\Omega_2} a(y,y) dx + \mathcal{L}(y) - \int\limits_S \chi_1 H(y) d\sigma \\ &\qquad + C \int\limits_{\Gamma_2} \big(|v_2|^2 + |v_3|^2 + \gamma(x)|y|^2 + \beta(x)|\partial_{\nu_2} y|^2\big) d\sigma, \end{aligned} \tag{3.8}$$

where H is the vector field defined in (2.14), $C > 0$ is a constant independent of function y, and $\mathcal{L}(y)$ defined in (3.3) is the lower order term respect to $a(y,y)$.

Proof. By Lemmas 3.1 and 3.2 we deduce that

$$\begin{aligned} &\int\limits_{\Omega_2} \big(\Delta^2 y - (1-\mu)\delta(\mathcal{K} dy)\big) H(y) dx \\ &\quad \ge \frac{1}{2} \int\limits_{\partial\Omega_2} a(y,y)\langle H, \nu_2\rangle d\sigma + \rho \int\limits_{\Omega_2} a(y,y) dx + \mathcal{L}(y) - \int\limits_{\partial\Omega_2} B_1 y \partial_{\nu_2}\big(H(y)\big) d\sigma + \int\limits_{\partial\Omega_2} B_2 y H(y) d\sigma \\ &\quad \ge \frac{\theta}{2} \int\limits_{\Gamma_2} a(y,y) d\sigma + \rho \int\limits_{\Omega_2} a(y,y) dx + \mathcal{L}(y) - \int\limits_{\Gamma_2} v_2 \partial_{\nu_2}\big(H(y)\big) d\sigma \\ &\qquad + \int\limits_S \chi_1 H(y) d\sigma + \int\limits_{\Gamma_2} v_3 H(y) d\sigma, \end{aligned} \tag{3.9}$$

where via assumptions (2.15) and (2.16) we have that

$$\frac{1}{2} \int\limits_{\partial\Omega_2} a(y,y)\langle H, \nu_2\rangle d\sigma = \frac{1}{2} \int\limits_{\Gamma_2} a(y,y)\langle H, \nu_2\rangle d\sigma \ge \frac{\theta}{2} \int\limits_{\Gamma_2} a(y,y) d\sigma.$$

Since H is a vector field, there exists a constant $C > 0$ independent of y such that

$$\big|\partial_{\nu_2}\big(H(y)\big)\big|^2 \le C\big(|\partial_{\nu_2} y|^2 + \big|D^2 y\big|^2_{T_x^2}\big).$$

Thus we have

$$\begin{aligned}\int_{\Gamma_2}\big|v_2\partial_{\nu_2}\big(H(y)\big)\big|d\sigma &\le \frac{(1-\mu)\theta}{4C}\int_{\Gamma_2}\big|\partial_{\nu_2}\big(H(y)\big)\big|^2 d\sigma + \frac{C}{(1-\mu)\theta}\int_{\Gamma_2}|v_2|^2 d\sigma \\ &\le \frac{(1-\mu)\theta}{4}\int_{\Gamma_2}\big|D^2y\big|^2_{T_x^2} d\sigma + \frac{(1-\mu)\theta}{4}\int_{\Gamma_2}\big|\partial_{\nu_2}y\big|^2 d\sigma + \frac{C}{(1-\mu)\theta}\int_{\Gamma_2}|v_2|^2 d\sigma \\ &\le \frac{\theta}{4}\int_{\Gamma_2} a(y,y)d\sigma + \frac{(1-\mu)\theta}{4}\int_{\Gamma_2}|\partial_{\nu_2}y|^2 d\sigma + \frac{C}{(1-\mu)\theta}\int_{\Gamma_2}|v_2|^2 d\sigma,\end{aligned} \tag{3.10}$$

recalling the definition of $a(y,y)$ in (2.11).

Next, we estimate the term $\int_{\Gamma_2} v_3(H(y))d\sigma$. Let σ_0 be the smallest positive constant such that

$$\int_{\Gamma_2}|\nabla y|^2 d\sigma \le \sigma_0\bigg\{\int_{\Omega}\big|D^2y\big|_{T_x^2} + \int_{\Gamma_2}\big(\beta\big|\partial_{\nu_2}y\big|^2 + \gamma|y|^2\big)d\sigma\bigg\}, \tag{3.11}$$

for any $y \in H^2(\Omega_2)$. It follows from the above inequality (3.11) that

$$\begin{aligned}\int_{\Gamma_2}\big|v_3 H(y)\big|d\sigma &\le \sup_{x\in\Gamma_2}|H|\int_{\Gamma_2}|v_3||\nabla y|d\sigma \\ &\le \frac{(1-\mu)\theta}{4\sigma_0}\int_{\Gamma_2}|\nabla y|^2 d\sigma + \frac{(\sup_{x\in\Gamma_2}|H|)^2\sigma_0}{(1-\mu)\theta}\int_{\Gamma_2}|v_3|^2 d\sigma \\ &\le \frac{\theta}{4}\int_{\Gamma_2} a(y,y)d\sigma + C\int_{\Gamma_2}\big(|v_3|^2 + \beta|\partial_{\nu_2}y|^2 + \gamma|y|^2\big)d\sigma.\end{aligned} \tag{3.12}$$

Substituting estimates (3.10) and (3.12) to inequality (3.9) yields the desired estimate (3.8). □

Lemma 3.4. *Let $w \in H^2(\Omega)$ be such that*

$$\begin{cases}\Delta w \in L^2(\Omega_1), & \text{on } \Omega_1,\\ w = \chi_2 \in L^2(S), & \text{on } S,\\ \partial_{\nu_1} w = v_1 \in L^2(\Gamma_1), & \text{on } \Gamma_1.\end{cases} \tag{3.13}$$

Then we have

$$\begin{aligned}\int_{\Omega_1}\Delta w\big(H(w)+\sigma_1 w\big)dx &\le -\int_{\Omega_1}\bigg(\rho+\sigma_1-\frac{1}{2}\operatorname{div} H(x)\bigg)|Dw|^2 + \int_S \partial_{\nu_1} w\big(H(w)+\sigma_1 w\big)d\sigma \\ &\quad + C\bigg(\int_{\Gamma_1} v_1^2 + \int_{\Gamma_1}\alpha w^2 d\sigma\bigg),\end{aligned} \tag{3.14}$$

for any constant σ_1. Here $C > 0$ is a constant independent of function w.

Proof. Using the identities in [23, Theorem 2.1] we have

$$\int_{\Omega_1} \Delta w H(w)dx = \int_{\partial\Omega_1} H(w)\partial_{\nu_1} w d\sigma - \int_{\Omega_1} DH(Dw, Dw)dx$$
$$- \frac{1}{2}\int_{\partial\Omega_1} |Dw|^2\langle H, \nu_1\rangle d\sigma + \frac{1}{2}\int_{\Omega_1} |Dw|^2 \operatorname{div} H(x)dx, \tag{3.15}$$

and

$$\int_{\Omega_1} \sigma_1 \Delta w w dx = \int_{\partial\Omega_1} \sigma_1 w \partial_{\nu_1} w d\sigma - \int_{\Omega_1} \sigma_1 |Dw|^2 dx. \tag{3.16}$$

Due to (3.15) and (3.16), we have

$$\begin{aligned}\int_{\Omega_1} \Delta w\big(H(w) + \sigma_1 w\big)dx &= \int_{\partial\Omega_1=\Gamma_1\cup S} \big(H(w) + \sigma_1 w\big)\partial_{\nu_1} w d\sigma - \frac{1}{2}\int_{\partial\Omega_1=\Gamma_1\cup S} |Dw|^2\langle H, \nu_1\rangle \\ &\quad - \int_{\Omega_1} DH(Dw, Dw)dx - \int_{\Omega_1}\left(\sigma_1 - \frac{1}{2}\operatorname{div} H(x)\right)|Dw|^2 \\ &\le \int_{\partial\Omega_1=\Gamma_1\cup S} \big(H(w) + \sigma_1 w\big)\partial_{\nu_1} w d\sigma - \frac{1}{2}\int_{\partial\Omega_1=\Gamma_1\cup S} |Dw|^2\langle H, \nu_1\rangle \\ &\quad - \int_{\Omega_1}\left(\rho + \sigma_1 - \frac{1}{2}\operatorname{div} H(x)\right)|Dw|^2. \end{aligned}\tag{3.17}$$

Now we notice that Young's inequality yields

$$\int_{\Gamma_1} \partial_{\nu_1} w H(w)d\sigma \le \epsilon\Big(\sup_{\Gamma_1} |H|\Big)\int_{\Gamma_1} |Dw|^2 d\sigma + C_\epsilon \int_{\Gamma_1} v_1^2 d\sigma, \tag{3.18}$$

and

$$\begin{aligned}\int_{\Gamma_1} \partial_{\nu_1} w \sigma_1 w d\sigma &\le \frac{1}{2}\int_{\Gamma_1} v_1^2 d\sigma + \frac{\sigma_1^2}{2}\int_{\Gamma_1} w^2 dx \\ &\le \frac{1}{2}\int_{\Gamma_1} v_1^2 d\sigma + \frac{\sigma_1^2}{2\alpha_0}\int_{\Gamma_1} \alpha w^2 dx,\end{aligned}\tag{3.19}$$

where α_0 is the positive lower bound of $\alpha(x)$ given in (2.17). Using assumptions (2.15) and (2.16) we have

$$-\frac{1}{2}\int_{\partial\Omega_1} |Dw|^2\langle H, \nu_1\rangle d\sigma = -\frac{1}{2}\int_{\Gamma_1} |Dw|^2\langle H, \nu_1\rangle d\sigma \le -\frac{\theta}{2}\int_{\Gamma_1} |Dw|^2 d\sigma. \tag{3.20}$$

Finally substituting estimates (3.18), (3.19) and (3.20) to (3.17) yields the desired estimate (3.14). □

Corollary 3.5. *Assume that geometric assumptions (A.1), (A.2) and (A.3) hold true. Let u be a strong solution of problem (2.4). Then we have*

$$\begin{aligned}&\int_{\Omega_1} \Delta u_1\big(H(u_1)+\sigma_1 u_1\big)dx - \int_{\Omega_2}\big(\Delta^2 u_2-(1-\mu)\delta(\mathcal{K}du_2)\big)\big(H(u_2)+\sigma_1 u_2\big)dx\\ &\quad\le -\int_{\Omega_1}\bigg(\rho+\sigma_1-\frac{1}{2}\operatorname{div} H(x)\bigg)|Du_1|^2-(\rho+\sigma_1)\int_{\Omega_2} a(u_2,u_2)dx + C\bigg(\int_{\Gamma_1} v_1^2+\int_{\Gamma_1}\alpha u_1^2 d\sigma\bigg)\\ &\qquad + C\int_{\Gamma_2}\big(v_2^2+v_3^2+\gamma(x)u_2^2+\beta(x)|\partial_{\nu_2}u_2|^2\big)d\sigma+\mathcal{L}(u_2), \end{aligned} \tag{3.21}$$

for any constant σ_1. Here $C>0$ is a constant independent of function u, and $\mathcal{L}(u_2)$ defined in (3.3) is the lower order term with respect to the energy.

Proof. By Lemma 3.1, we have

$$\begin{aligned}&-\int_{\Omega_2}\big(\Delta^2 u_2-(1-\mu)\delta(\mathcal{K}du_2)\big)\sigma_1 u_2 dx\\ &\quad= -\sigma_1\bigg(\int_{\Omega_2} a(u_2,u_2)dx - \int_{\partial\Omega_2=\Gamma_2\cup S} B_1u_2\partial_{\nu_2}u_2 d\sigma + \int_{\partial\Omega_2=\Gamma_2\cup S} B_2 u_2 u_2 d\sigma\bigg)\\ &\quad\le -\sigma_1\int_{\Omega_2} a(u_2,u_2)dx - \int_S B_2u_2\cdot(\sigma_1 u_1)dx\\ &\qquad + C\int_{\Gamma_2}\big(v_2^2+v_3^2+\gamma(x)u_2^2+\beta(x)|\partial_{\nu_2}u_2|^2\big)d\sigma. \end{aligned} \tag{3.22}$$

We combine inequalities (3.8), (3.14) and (3.22) to obtain the desired estimate (3.21), noticing the boundary condition on S in system (2.4). □

For the proof of the exponential and polynomial decay of the energy, we need the following lemma which plays an important role in the proof.

Lemma 3.6. *Let u be a smooth enough solution of problem (2.4). Consider $z=(z_1,z_2)$ the solution of*

$$\begin{cases}\Delta z_1(x,t)=0, & \text{in } \Omega_1\times(0,+\infty),\\ \Delta^2 z_2(x,t)-(1-\mu)\delta(\mathcal{K}dz_2(x,t))=0, & \text{in } \Omega_2\times(0,+\infty),\\ z_1=z_2, \Delta z_2=0, \partial_{\nu_2}\Delta z_2=\partial_{\nu_1}z_1, & \text{on } S\times(0,+\infty),\\ z_1=u_1, & \text{on } \Gamma_1\times(0,+\infty),\\ z_2=u_2, & \text{on } \Gamma_2\times(0,+\infty),\\ \partial_{\nu_2}z_2=\partial_{\nu_2}u_2, & \text{on } \Gamma_2\times(0,+\infty).\end{cases} \tag{3.23}$$

Introduce the bilinear form

$$b(z,y)=\int_{\Omega_1}\langle Dz_1,Dy_1\rangle dx+\int_{\Omega_2} a(z_2,y_2)dx, \tag{3.24}$$

for all $z,y\in V$, where

$$V=\big\{z=(z_1,z_2)\in H^1(\Omega_1)\times H^2(\Omega_2): z_1=z_2 \text{ on } S\big\}.$$

Then there exists a constant $\sigma_2 > 0$ *such that*

$$\int_\Omega |z|^2 dx \le \sigma_2 \left\{ \int_{\Gamma_1} \alpha \left| u_1(x,t) \right|^2 d\sigma + \int_{\Gamma_2} \left(\beta \left| \partial_{\nu_2} u_2(x,t) \right|^2 + \gamma \left| u_2(x,t) \right|^2 \right) d\sigma \right\}, \tag{3.25}$$

$$b(z,u) = b(z,z) \ge 0. \tag{3.26}$$

Proof. According to the classical theory for elliptic equations, we have

$$\int_\Omega |z|^2 dx \le C \left\{ \int_{\Gamma_1} \left| z_1(x,t) \right|^2 d\sigma + \int_{\Gamma_2} \left(\left| \partial_{\nu_2} z_2(x,t) \right|^2 + \left| z_2(x,t) \right|^2 \right) d\sigma \right\}, \tag{3.27}$$

which yields the desired estimate (3.25) as we notice assumptions (2.17), (2.18) and (2.19) on the functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$, respectively.

Next we aim to prove (3.26). Here we start with $z \in H^2(\Omega_1) \times H^4(\Omega_2)$. By Green's formula we have

$$\begin{aligned} \int_{\Omega_1} \left\langle Dz_1, D(z_1 - u_1) \right\rangle dx &= \int_{\partial\Omega_1} \partial_{\nu_1} z_1 (z_1 - u_1) d\sigma - \int_{\Omega_1} \Delta z_1 (z_1 - u_1) dx \\ &= \int_{\partial\Omega_1} \partial_{\nu_1} z_1 (z_1 - u_1) d\sigma. \end{aligned} \tag{3.28}$$

Applying Lemma 3.1 with $y = z_2$ and $w = z_2 - u_2$, we have

$$\begin{aligned} \int_{\Omega_2} a(z_2, z_2 - u_2) dx &= \int_{\Omega_2} \left(\Delta^2 z_2 - (1-\mu)\delta(\mathcal{K} dz_2) \right)(z_2 - u_2) dx \\ &\quad + \int_{\partial\Omega_2} B_1 z_2 \partial_{\nu_2}(z_2 - u_2) d\sigma - \int_{\partial\Omega_2} B_2 z_2 (z_2 - u_2) d\sigma \\ &= \int_{\partial\Omega_2} B_1 z_2 \partial_{\nu_2}(z_2 - u_2) d\sigma - \int_{\partial\Omega_2} B_2 z_2 (z_2 - u_2) d\sigma. \end{aligned} \tag{3.29}$$

Combining the above two equalities and using the boundary conditions in Eq. (3.23) we obtain $b(z, z-u) = 0$. Finally by a standard argument of density, we complete the proof of Lemma 3.6. □

For further purposes, we assume u is the weak solution of Eq. (2.4) in class (2.9). Let z be the solution of system (3.23). For some constant $r > 0$ we define

$$R(t) = \int_\Omega \partial_t u \left(H(u) + ru + \sigma_3 z \right) dx, \tag{3.30}$$

where the constants r and σ_3 are positive constants which to be determined later.

Lemma 3.7. *Assume that geometric assumptions (A.1), (A.2) and (A.3) hold true. Let* u *be a strong solution of problem (2.4). Then there exist four positive constants* C_1, C_2, C_3 *and* C_4 *such that*

$$\left| R(t) \right| \le C_1 E(t), \tag{3.31}$$

$$R'(t) \le -\frac{\rho}{2} E(t) + C_2 \int_{\Gamma_1} \left(|\partial_t u_1|^2 + p^2(\partial_t u_1) \right) d\sigma$$

$$+ C_3 \int_{\Gamma_2} \big(|\partial_t u_2|^2 + q^2(\partial_t u_2) \big) d\sigma$$

$$+ C_4 \int_{\Gamma_2} \big(|\partial_{\nu_2} \partial_t u_2|^2 + h^2(\partial_{\nu_2} \partial_t u_2) \big) d\sigma. \tag{3.32}$$

Proof. The first estimate is easy to be verified by using Cauchy–Schwartz's inequality and Lemma 3.6. A direct derivation of $R(t)$ yields

$$\begin{aligned}
R'(t) &= \int_{\Omega} u_{tt} \big(H(u) + ru + \sigma_3 z \big) dx + \int_{\Omega} u_t \big(H(u_t) + ru_t + \sigma_3 z_t \big) \\
&= \int_{\Omega} u_t \big(H(u_t) + ru_t + \sigma_3 z_t \big) + \int_{\Omega_1} \Delta u_1 \big(H(u_1) + ru_1 + \sigma_3 z_1 \big) \\
&\quad - \int_{\Omega_2} \big(\Delta^2 u_2 - (1-\mu)\delta \mathcal{K} du_2 \big) \big(H(u_2) + ru_2 + \sigma_3 z_2 \big) dx \\
&= \int_{\Omega_1} \Delta u_1 \big(H(u_1) + ru_1 \big) - \int_{\Omega_2} \big(\Delta^2 u_2 - (1-\mu)\delta \mathcal{K} du_2 \big) \big(H(u_2) + ru_2 \big) dx \\
&\quad + \sigma_3 \int_{\Omega_1} \Delta u_1 z_1 - \sigma_3 \int_{\Omega_2} \big(\Delta^2 u_2 - (1-\mu)\delta \mathcal{K} du_2 \big) z_2 dx \\
&\quad + \int_{\Omega} u_t \big(H(u_t) + ru_t + \sigma_3 z_t \big) dx \triangleq I_1 + I_2 + I_3,
\end{aligned} \tag{3.33}$$

where we denote as

$$I_1 = \int_{\Omega_1} \Delta u_1 \big(H(u_1) + ru_1 \big) - \int_{\Omega_2} \big(\Delta^2 u_2 - (1-\mu)\delta \mathcal{K} du_2 \big) \big(H(u_2) + ru_2 \big) dx, \tag{3.34}$$

$$I_2 = \sigma_3 \int_{\Omega_1} \Delta u_1 z_1 - \sigma_3 \int_{\Omega_2} \big(\Delta^2 u_2 - (1-\mu)\delta \mathcal{K} du_2 \big) z_2 dx, \tag{3.35}$$

$$I_3 = \int_{\Omega} u_t \big(H(u_t) + ru_t + \sigma_3 z_t \big) dx. \tag{3.36}$$

Now we estimate the items I_1, I_2 and I_3, respectively.

According to Corollary 3.5, we have

$$\begin{aligned}
I_1 \le & - \int_{\Omega_1} \left(\rho + r - \frac{1}{2} \operatorname{div} H \right) |Du_1|^2 - (\rho + r) \int_{\Omega_2} a(u_2, u_2) dx + C \left(\int_{\Gamma_1} v_1^2 + \int_{\Gamma_1} \alpha u_1^2 d\sigma \right) \\
& + C \int_{\Gamma_2} \big(v_2^2 + v_3^2 + \gamma(x) u_2^2 + \beta(x) |\partial_{\nu_2} u_2|^2 \big) d\sigma + \mathcal{L}(u_2).
\end{aligned} \tag{3.37}$$

Due to Green's Formula and Lemma 3.1 we have

$$I_2 = \sigma_3 \left\{ \int_{\partial\Omega_1} (\partial_{\nu_1} u_1) z_1 - \int_{\Omega_1} \langle Du_1, Dz_1 \rangle dx \right.$$

$$
\begin{aligned}
&- \int_{\Omega_2} a(u_2, z_2) + \int_{\partial\Omega_2} B_1 u_2 \partial_{\nu_2} z_2 d\sigma - \int_{\partial\Omega_2} (B_2 u_2) z_2 d\sigma \Big\} \\
&= \sigma_3 \Big\{ \int_{\Gamma_1} v_1 u_1 - b(u, z) + \int_{\Gamma_2} v_2 \partial_{\nu_2} u_2 d\sigma - \int_{\Gamma_2} v_3 u_2 d\sigma \Big\} \\
&\le \sigma_3 \Big\{ \int_{\Gamma_1} v_1 u_1 + \int_{\Gamma_2} v_2 \partial_{\nu_2} u_2 d\sigma - \int_{\Gamma_2} v_3 u_2 d\sigma \Big\},
\end{aligned} \tag{3.38}
$$

where we noticed the boundary condition on the common boundary S and estimate (3.26).

An integral by parts in space yields

$$
\int_{\Omega_i} u_{it} H(u_{it}) dx = \frac{1}{2} \int_{\partial\Omega_i} u_{it}^2 \langle H, \nu_i \rangle d\sigma - \frac{1}{2} \int_{\Omega} (\operatorname{div} H) u_{it}^2 dx. \tag{3.39}
$$

Using Cauchy–Schwartz's inequality we have

$$
\begin{aligned}
&\sigma_3 \int_{\Omega} u_t z_t dx \\
&\le \epsilon \int_{\Omega} |\partial_t u|^2 dx + \frac{\sigma_3^2}{4\epsilon} \int_{\Omega} |\partial_t z|^2 dx \\
&\le \epsilon \int_{\Omega} |\partial_t u|^2 dx + \frac{\sigma_2 \sigma_3^2}{4\epsilon} \Big\{ \int_{\Gamma_1} \alpha |\partial_t u_1|^2 d\sigma + \int_{\Gamma_2} \big(\beta |\partial_{\nu_2} \partial_t u_2|^2 + \gamma |\partial_t u_2|^2 \big) d\sigma \Big\}.
\end{aligned} \tag{3.40}
$$

Substituting inequalities (3.39) and (3.40) to (3.36) yields

$$
\begin{aligned}
I_3 &\le C_0 \int_{\Gamma_1} |\partial_t u_1|^2 d\sigma + C_0 \int_{\Gamma_2} |\partial_t u_2|^2 d\sigma + \int_{\Omega} \Big(r + \epsilon - \frac{\operatorname{div} H}{2} \Big) |\partial_t u|^2 dx \\
&\quad + \frac{\sigma_2 \sigma_3^2}{4\epsilon} \Big\{ \int_{\Gamma_1} \alpha |\partial_t u_1|^2 d\sigma + \int_{\Gamma_2} \big(\beta |\partial_{\nu_2} \partial_t u_2|^2 + \gamma |\partial_t u_2|^2 \big) d\sigma \Big\} \\
&\le \Big(C_0 + \frac{\alpha_1 \sigma_2 \sigma_3^2}{4\epsilon} \Big) \int_{\Gamma_1} |\partial_t u_1|^2 d\sigma + \Big(C_0 + \frac{\gamma_1 \sigma_2 \sigma_3^2}{4\epsilon} \Big) \int_{\Gamma_2} |\partial_t u_2|^2 d\sigma \\
&\quad + \frac{\beta_1 \sigma_2 \sigma_3^2}{4\epsilon} \int_{\Gamma_2} |\partial_{\nu_2} \partial_t u_2|^2 d\sigma.
\end{aligned} \tag{3.41}
$$

Inserting estimates (3.37), (3.38) and (3.41) to inequality (3.33) yields

$$
\begin{aligned}
R'(t) &\le - \int_{\Omega_1} \Big(\rho + r - \frac{1}{2} \operatorname{div} H \Big) |Du_1|^2 - (\rho + r) \int_{\Omega_2} a(u_2, u_2) dx + \int_{\Omega} \Big(r + \epsilon - \frac{\operatorname{div} H}{2} \Big) |\partial_t u|^2 dx \\
&\quad + C\kappa_1 \int_{\Gamma_1} v_1^2 + C \int_{\Gamma_1} \alpha(x) u_1^2 d\sigma + C\kappa_2 \int_{\Gamma_2} v_2^2 d\sigma + C\kappa_3 \int_{\Gamma_2} v_3^2 d\sigma \\
&\quad + C \int_{\Gamma_2} \big(\gamma(x) u_2^2 + \beta(x) |\partial_{\nu_2} u_2|^2 \big) d\sigma + \sigma_3 \Big\{ \int_{\Gamma_1} v_1 u_1 + \int_{\Gamma_2} v_2 \partial_{\nu_2} u_2 d\sigma - \int_{\Gamma_2} v_3 u_2 d\sigma \Big\}
\end{aligned}
$$

$$+\left(C_0+\frac{\alpha_1\sigma_2\sigma_3^2}{4\epsilon}\right)\int_{\Gamma_1}|\partial_t u_1|^2 d\sigma+\left(C_0+\frac{\gamma_1\sigma_2\sigma_3^2}{4\epsilon}\right)\int_{\Gamma_2}|\partial_t u_2|^2 d\sigma$$
$$+\frac{\beta_1\sigma_2\sigma_3^2}{4\epsilon}\int_{\Gamma_2}|\partial_{\nu_2}\partial_t u_2|^2 d\sigma+\mathcal{L}(u_2). \tag{3.42}$$

First by the definition of the energy we know

$$-\int_{\Omega_1}\left(\rho+r-\frac{1}{2}\operatorname{div}H\right)|Du_1|^2-(\rho+r)\int_{\Omega_2}a(u_2,u_2)dx+\int_{\Omega}\left(r+\epsilon-\frac{\operatorname{div}H}{2}\right)|\partial_t u|^2 dx$$
$$=-\frac{1}{2}\rho E(t)-\int_{\Omega_1}\left(\frac{3}{4}\rho+r-\frac{1}{2}\operatorname{div}H\right)|Du_1|^2$$
$$-\left(\frac{3}{4}\rho+r\right)\int_{\Omega_2}a(u_2,u_2)dx+\int_{\Omega}\left(\frac{1}{4}\rho+r+\epsilon-\frac{\operatorname{div}H}{2}\right)|\partial_t u|^2 dx$$
$$+\frac{1}{4}\rho\int_{\Gamma_1}\alpha(x)\left|u_1(x,t)\right|^2 d\sigma+\frac{1}{4}\rho\int_{\Gamma_2}\left(\beta(x)\left|\partial_{\nu_2}u_2(x,t)\right|^2+\gamma(x)\left|u_2(x,t)\right|^2\right)d\sigma$$
$$\le-\frac{\rho}{2}E(t)+\frac{\rho}{4}\int_{\Gamma_1}\alpha(x)\left|u_1(x,t)\right|^2 d\sigma$$
$$+\frac{\rho}{4}\int_{\Gamma_2}\left(\beta(x)\left|\partial_{\nu_2}u_2(x,t)\right|^2+\gamma(x)\left|u_2(x,t)\right|^2\right)d\sigma, \tag{3.43}$$

where we notice $\operatorname{div}H=\operatorname{tr}DH\ge 2\rho$ and take

$$0<\frac{1}{2}\operatorname{div}H-\frac{3}{4}\rho\le r\le\frac{1}{2}\operatorname{div}H-\frac{1}{4}\rho-\epsilon.$$

Next we use feedback laws (2.5) and assumptions (2.17)–(2.19) on the functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ that lead to

$$v_1^2\le-2\alpha_1 v_1 u_1-\alpha_1\alpha u_1^2+\frac{\alpha_1}{\alpha_0}p^2(\partial_t u_1)\quad\text{on }\Gamma_1\times(0,\infty), \tag{3.44}$$
$$v_2^2\le-2\beta_1 v_2\partial_{\nu_2}u_2-\beta_1\beta|\partial_{\nu_2}u_2|^2+\frac{\beta_1}{\beta_0}h^2(\partial_{\nu_2}\partial_t u_2)\text{ on }\Gamma_2\times(0,\infty), \tag{3.45}$$
$$v_3^2\le 2\gamma_1 v_3 u_2-\gamma_1\gamma u_2^2+\frac{\gamma_1}{\gamma_0}q^2(\partial_t u_2)\quad\text{on }\Gamma_2\times(0,\infty). \tag{3.46}$$

Using the above inequalities (3.43), (3.44), (3.45) and (3.46) in (3.42) to get

$$R'(t)\le-\frac{\rho}{2}E(t)+C\left(\frac{\rho}{4C}+1-\kappa_1\alpha_1\right)\int_{\Gamma_1}\alpha(x)u_1^2 d\sigma$$
$$+C\left(\frac{\rho}{4C}+1-\kappa_3\gamma_1\right)\int_{\Gamma_2}\gamma(x)u_2^2 d\sigma+C\left(\frac{\rho}{4C}+1-\kappa_2\beta_1\right)\int_{\Gamma_2}\beta(x)|\partial_{\nu_2}u_2|^2 d\sigma$$
$$+(\sigma_3-2C\kappa_1\alpha_1)\int_{\Gamma_1}v_1u_1+(\sigma_3-2C\kappa_2\beta_1)\int_{\Gamma_2}v_2\partial_{\nu_2}u_2 d\sigma-(\sigma_3-2C\kappa_3\gamma_1)\int_{\Gamma_2}v_3u_2 d\sigma$$

$$+\left(C_0+\frac{\alpha_1\sigma_2\sigma_3^2}{4\epsilon}\right)\int\limits_{\Gamma_1}|\partial_t u_1|^2 d\sigma + C\kappa_1\frac{\alpha_1}{\alpha_0}\int\limits_{\Gamma_1}p^2(\partial_t u_1)d\sigma$$
$$+\left(C_0+\frac{\gamma_1\sigma_2\sigma_3^2}{4\epsilon}\right)\int\limits_{\Gamma_2}|\partial_t u_2|^2 d\sigma + C\kappa_3\frac{\gamma_1}{\gamma_0}\int\limits_{\Gamma_2}q^2(\partial_t u_2)d\sigma$$
$$+\frac{\beta_1\sigma_2\sigma_3^2}{4\epsilon}\int\limits_{\Gamma_2}|\partial_{\nu_2}\partial_t u_2|^2 d\sigma + C\kappa_2\frac{\beta_1}{\beta_0}\int\limits_{\Gamma_2}h^2(\partial_{\nu_2}\partial_t u_2)d\sigma + \mathcal{L}(u_2). \tag{3.47}$$

Here we notice that both σ_3 and κ_i $(i = 1, 2, 3)$ are constants to be determined. To get rid of the integration terms about v_1u_1, $v_2\partial_\nu u_2$ and v_3u_2 we can take

$$\kappa_1 = \frac{\sigma_3}{2C\alpha_1} \geq 1,$$
$$\kappa_2 = \frac{\sigma_3}{2C\beta_1} \geq 1,$$
$$\kappa_3 = \frac{\sigma_3}{2C\gamma_1} \geq 1,$$

which means we only need to choose the constant σ_3 such that

$$\sigma_3 \geq 2C\max\{\alpha_1, \beta_1, \gamma_1\}. \tag{3.48}$$

Thus we obtain

$$R'(t) \leq -\frac{\rho}{2}E(t) + \mathcal{L}(u_2) + C\left(1-\frac{\sigma_3}{2C}+\frac{\rho}{4C}\right)\int\limits_{\Gamma_1}\alpha(x)u_1^2 d\sigma$$
$$+C\left(1-\frac{\sigma_3}{2C}+\frac{\rho}{4C}\right)\int\limits_{\Gamma_2}\gamma(x)u_2^2 d\sigma + C\left(1-\frac{\sigma_3}{2C}+\frac{\rho}{4C}\right)\int\limits_{\Gamma_2}\beta(x)|\partial_{\nu_2}u_2|^2 d\sigma$$
$$+\left(\rho+C_0+\frac{\alpha_1\sigma_2\sigma_3^2}{4\epsilon}\right)\int\limits_{\Gamma_1}|\partial_t u_1|^2 d\sigma + C\kappa_1\frac{\alpha_1}{\alpha_0}\int\limits_{\Gamma_1}p^2(\partial_t u_1)d\sigma$$
$$+\left(\rho+C_0+\frac{\gamma_1\sigma_2\sigma_3^2}{4\epsilon}\right)\int\limits_{\Gamma_2}|\partial_t u_2|^2 d\sigma + C\kappa_3\frac{\gamma_1}{\gamma_0}\int\limits_{\Gamma_2}q^2(\partial_t u_2)d\sigma$$
$$+\frac{\beta_1\sigma_2\sigma_3^2}{4\epsilon}\int\limits_{\Gamma_2}|\partial_{\nu_2}\partial_t u_2|^2 d\sigma + C\kappa_2\frac{\beta_1}{\beta_0}\int\limits_{\Gamma_2}h^2(\partial_{\nu_2}\partial_t u_2)d\sigma. \tag{3.49}$$

To get rid of the integration terms about u_1^2, u_2^2 and $|\partial_{\nu_2}u_2|^2$ we need

$$\sigma_3 \geq 2C\left(1+\frac{\rho}{4C}\right). \tag{3.50}$$

Combing (3.48) and (3.50) we take $\sigma_3 = 2C\max\{\alpha_1, \beta_1, \gamma_1, 1+\frac{\rho}{4C}\}$ to obtain

$$R'(t) \leq -\frac{\rho}{2}E(t) + C_2\int\limits_{\Gamma_1}\left(|\partial_t u_1|^2 + p^2(\partial_t u_1)\right)d\sigma$$
$$+C_3\int\limits_{\Gamma_2}\left(|\partial_t u_2|^2 + q^2(\partial_t u_2)\right)d\sigma$$

$$+ C_4 \int_{\Gamma_2} \left(|\partial_{\nu_2}\partial_t u_2|^2 + h^2(\partial_{\nu_2}\partial_t u_2)\right) d\sigma + \mathcal{L}(u_2). \tag{3.51}$$

Finally, by a compactness-uniqueness argument, the lower order term $\mathcal{L}(u_2)$ can be absorbed. Thus we get the desired estimate (3.32). □

With these lemmas we can get the desired decay rate of the energy.

4. Proof of the main results

Proof of Theorem 2.2. Our proof is similar to that in [12,20]. For the sake of completeness, we give the sketch of the proof.

For any $\varepsilon > 0$, we define the functional

$$E_\varepsilon(t) = E(t) + \varepsilon R(t)\left(E(t)\right)^{(k-1)/2}. \tag{4.1}$$

Since $E(t)$ is nonincreasing in t and by Lemma 3.7, we have

$$\left(1 - \varepsilon C_1 E(0)^{(k-1)/2}\right) E(t) \le E_\varepsilon(t) \le \left(1 + \varepsilon C_1 E(0)^{(k-1)/2}\right) E(t). \tag{4.2}$$

Thus for any given constant $C_0 > 1$, a straight computation shows that

$$C_0^{-1/2}\left(E_\varepsilon(t)\right)^{(k+1)/2} \le E(t)^{(k+1)/2} \le C_0^{1/2}\left(E_\varepsilon(t)\right)^{(k+1)/2}, \tag{4.3}$$

provided ε is small enough such that

$$\varepsilon \le \left(E(0)\right)^{(1-k)/2}\left(1 - C_0^{-1/(k+1)}\right)/C_1.$$

A simple computation leads to

$$E'_\varepsilon(t) = E'(t) + \varepsilon \frac{k-1}{2}\left(E(t)\right)^{\frac{(k-3)}{2}} R(t) E'(t) + \varepsilon \left(E(t)\right)^{\frac{(k-1)}{2}} R'(t). \tag{4.4}$$

Using assumption (2.23) or (2.25) on functions $p(s)$, $q(s)$ and $h(s)$ we deduce from (3.32) that there exists a constant $C_5 > 0$ such that

$$R'(t) \le -\frac{\rho}{2} E(t) + C_5 \int_{\Gamma_1} |\partial_t u_1|^2 d\sigma + C_5 \int_{\Gamma_2} \left(|\partial_t u_2|^2 + |\partial_{\nu_2}\partial_t u_2|^2\right) d\sigma. \tag{4.5}$$

By (2.13), (4.5) and Lemma 3.7, we have

$$\begin{aligned} E'_\varepsilon(t) \le & -\frac{\rho}{2}\varepsilon\left(E(t)\right)^{\frac{k+1}{2}} - \left(1 + \varepsilon \frac{k-1}{2} E(0)^{\frac{k-1}{2}} C_1\right) \int_{\Gamma_1} p(\partial_t u_1)\partial_t u_1 d\sigma \\ & - \left(1 + \varepsilon \frac{k-1}{2} E(0)^{\frac{k-1}{2}} C_1\right) \int_{\Gamma_2} \left(h(\partial_{\nu_2}\partial_t u_2)\partial_{\nu_2}\partial_t u_2 + q(\partial_t u_2)\partial_t u_2\right) d\sigma \\ & + \varepsilon C_5 \left(E(t)\right)^{\frac{(k-1)}{2}} \left\{ \int_{\Gamma_1} |\partial_t u_1|^2 d\sigma + \int_{\Gamma_2} |\partial_t u_2|^2 d\sigma + \int_{\Gamma_2} |\partial_{\nu_2}\partial_t u_2|^2 d\sigma \right\}. \end{aligned} \tag{4.6}$$

(i) If $k = 1$, then by (2.17)–(2.19), (2.23) and (4.6) we have

$$\begin{aligned} E_\varepsilon'(t) \le &-\frac{\rho}{2}\varepsilon E(t) - \int_{\Gamma_1} p(\partial_t u_1)\partial_t u_1 d\sigma \\ &- \int_{\Gamma_2} \big(h(\partial_{\nu_2}\partial_t u_2)\partial_{\nu_2}\partial_t u_2 + q(\partial_t u_2)\partial_t u_2\big) d\sigma \\ &+ \varepsilon C_5 \Big\{ \int_{\Gamma_1} |\partial_t u_1|^2 d\sigma + \int_{\Gamma_2} |\partial_t u_2|^2 d\sigma + \int_{\Gamma_2} |\partial_{\nu_2}\partial_t u_2|^2 d\sigma \Big\} \\ \le &-\frac{\rho}{2}\varepsilon E(t) + (-1 + \varepsilon C_5 L_1) \int_{\Gamma_1} p(\partial_t u_1)\partial_t u_1 d\sigma \\ &+ \big(-1 + \varepsilon C_5(L_3 + L_5)\big) \int_{\Gamma_2} \big(h(\partial_{\nu_2}\partial_t u_2)\partial_{\nu_2}\partial_t u_2 + q(\partial_t u_2)\partial_t u_2\big) d\sigma. \end{aligned} \tag{4.7}$$

By choosing $\varepsilon C_5(L_1 + L_3 + L_5) \le 1$, we obtain

$$E_\varepsilon'(t) \le -\frac{\rho}{2}\varepsilon E(t) \le -\frac{\rho}{2}\varepsilon C_0^{-1/2} E_\varepsilon(t). \tag{4.8}$$

Then (4.3) and (4.8) imply that

$$E(t) \le C_0 E(0) \exp\Big(-\frac{\rho}{2}\varepsilon C_0^{-1/2} t\Big) = C_0 E(0) \exp(-\omega_1 t),$$

with the constant $\omega_1 = \frac{\rho}{2}\varepsilon C_0^{-1/2}$.

(ii) If $k > 1$, by (2.17)–(2.19) and (2.25) we have

$$|s|^{k+1} \le L_1 p(s)s \quad \text{for all } |s| \le 1; \qquad |s|^2 \le L_1 p(s)s, \quad \text{for all } |s| \ge 1, \tag{4.9}$$

$$|s|^{k+1} \le L_3 q(s)s \quad \text{for all } |s| \le 1; \qquad |s|^2 \le L_3 q(s)s, \quad \text{for all } |s| \ge 1, \tag{4.10}$$

$$|s|^{k+1} \le L_5 h(s)s \quad \text{for all } |s| \le 1; \qquad |s|^2 \le L_5 h(s)s, \quad \text{for all } |s| \ge 1. \tag{4.11}$$

Using (4.9) we have

$$\begin{aligned} &\varepsilon C_5 \big(E(t)\big)^{\frac{(k-1)}{2}} \int_{\Gamma_1} |\partial_t u_1|^2 d\sigma \\ &\le \varepsilon C_5 \big(E(t)\big)^{\frac{(k-1)}{2}} \int_{\Gamma_1 \cap \{|\partial_t u_1| \ge 1\}} |\partial_t u_1|^2 d\sigma + \varepsilon C_5 \big(E(t)\big)^{\frac{(k-1)}{2}} \int_{\Gamma_1 \cap \{|\partial_t u_1| \le 1\}} |\partial_t u_1|^2 d\sigma \\ &\le \varepsilon C_5 L_1 \big(E(0)\big)^{\frac{(k-1)}{2}} \int_{\Gamma_1 \cap \{|\partial_t u_1| \ge 1\}} p(\partial_t u_1)\partial_t u_1 d\sigma \\ &\quad + \varepsilon C_5 \big(E(t)\big)^{\frac{(k-1)}{2}} \int_{\Gamma_1 \cap \{|\partial_t u_1| \le 1\}} |\partial_t u_1|^2 d\sigma. \end{aligned} \tag{4.12}$$

Now applying Young's inequality to the second term on the RHS of (4.12) we get

$$\varepsilon C_5\big(E(t)\big)^{\frac{(k-1)}{2}} \int\limits_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} |\partial_t u_1|^2 d\sigma \le \frac{\varepsilon\rho}{8} E(t)^{\frac{(k+1)}{2}} + \varepsilon\big(2\rho^{-1}C_5\big)^{\frac{(k+1)}{2}} \Bigg(\int\limits_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} |\partial_t u_1|^2 d\sigma\Bigg)^{\frac{(k+1)}{2}}. \tag{4.13}$$

But by Holder's inequality we know that

$$\Bigg(\int\limits_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} |\partial_t u_1|^2 d\sigma\Bigg)^{\frac{(k+1)}{2}} \le \Bigg(\int\limits_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} d\sigma\Bigg)^{\frac{(k-1)}{2}} \int\limits_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} |\partial_t u_1|^{k+1} d\sigma \le (\operatorname{meas}\Gamma_1)^{\frac{(k-1)}{2}} L_1 \int\limits_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} p(\partial_t u_1)\partial_t u_1 d\sigma. \tag{4.14}$$

Substituting (4.13) and (4.14) to (4.12) gives

$$\varepsilon C_5\big(E(t)\big)^{\frac{(k-1)}{2}} \int\limits_{\Gamma_1} |\partial_t u_1|^2 d\sigma \le \frac{\varepsilon\rho}{8} E(t)^{\frac{(k+1)}{2}} + \varepsilon L_1\Big\{C_5\big(E(0)\big)^{\frac{(k-1)}{2}} + (\operatorname{meas}\Gamma_1)^{\frac{(k-1)}{2}}\big(2\rho^{-1}C_5\big)^{\frac{(k+1)}{2}}\Big\} \int\limits_{\Gamma_1} p(\partial_t u_1)\partial_t u_1 d\sigma. \tag{4.15}$$

Similarly, using (4.10) and (4.11) we show that

$$\varepsilon C_5\big(E(t)\big)^{\frac{(k-1)}{2}} \int\limits_{\Gamma_2} |\partial_t u_2|^2 d\sigma \le \frac{\varepsilon\rho}{8} E(t)^{\frac{(k+1)}{2}} + \varepsilon L_3\Big\{C_5\big(E(0)\big)^{\frac{(k-1)}{2}} + (\operatorname{meas}\Gamma_2)^{\frac{(k-1)}{2}}\big(2\rho^{-1}C_5\big)^{\frac{(k+1)}{2}}\Big\} \int\limits_{\Gamma_2} q(\partial_t u_2)\partial_t u_2 d\sigma, \tag{4.16}$$

and

$$\varepsilon C_5\big(E(t)\big)^{\frac{(k-1)}{2}} \int\limits_{\Gamma_2} |\partial_{\nu_2}\partial_t u_2|^2 d\sigma \le \frac{\varepsilon\rho}{8} E(t)^{\frac{(k+1)}{2}} + \varepsilon L_5\Big\{C_5\big(E(0)\big)^{\frac{(k-1)}{2}} + (\operatorname{meas}\Gamma_2)^{\frac{(k-1)}{2}}\big(2\rho^{-1}C_5\big)^{\frac{(k+1)}{2}}\Big\} \int\limits_{\Gamma_2} h(\partial_{\nu_2}\partial_t u_2)\partial_{\nu_2}\partial_t u_2 d\sigma. \tag{4.17}$$

Plugging (4.15)–(4.17) into (4.6), it follows

$$E'_\varepsilon(t) \le -\frac{\rho\varepsilon}{8}\big(E(t)\big)^{\frac{k+1}{2}} \le -\frac{\rho\varepsilon}{8} C_0^{-1/2}\big(E_\varepsilon(t)\big)^{\frac{k+1}{2}}, \tag{4.18}$$

provided ε is chosen such that

$$\varepsilon(L_1+L_3+L_5)\left\{C_5\big(E(0)\big)^{\frac{(k-1)}{2}}+(\operatorname{meas}\Gamma_1\cup\Gamma_2)^{\frac{(k-1)}{2}}\big(2\rho^{-1}C_5\big)^{\frac{(k+1)}{2}}\right\}\le 1.$$

Combining (4.18) and (4.6), we obtain

$$E(t)\le C_0E(0)\left\{1+\varepsilon\rho\frac{k-1}{4}C_0^{-\frac{k}{k+1}}E(0)^{\frac{k-1}{2}}\right\}^{-\frac{2}{k-1}}=C_0E(0)(1+\omega_2 t)^{-\frac{2}{k-1}}\quad\text{for all } t\ge 0,\tag{4.19}$$

with the constant $\omega_2=\varepsilon\rho\frac{k-1}{4}C_0^{-\frac{k}{k+1}}E(0)^{\frac{k-1}{2}}$. $\square$

Proof of Theorem 2.3. The proof of this theorem is similar to that of Theorem 2.2 provided k is replaced by $\frac{1}{k}$. Here for the sake of completeness, we give the sketch of the proof. For any $\varepsilon>0$, we define the functional

$$E_\varepsilon(t)=E(t)+\varepsilon R(t)\big(E(t)\big)^{(1-k)/2k}.\tag{4.20}$$

Since $E(t)$ is nonincreasing in t and by Lemma 3.7, we have

$$\big(1-\varepsilon C_1E(0)^{(1-k)/2k}\big)E(t)\le E_\varepsilon(t)\le\big(1+\varepsilon C_1E(0)^{(1-k)/2k}\big)E(t).\tag{4.21}$$

Thus for any given constant $C_0>1$, a straight computation shows that

$$C_0^{-1/2}\big(E_\varepsilon(t)\big)^{(k+1)/2k}\le E(t)^{(k+1)/2k}\le C_0^{1/2}\big(E_\varepsilon(t)\big)^{(k+1)/2k},\tag{4.22}$$

provided ε is small enough such that

$$\varepsilon\le\big(E(0)\big)^{(k-1)/2k}\big(1-C_0^{-k/(k+1)}\big)/C_1.$$

A simple computation leads to

$$E_\varepsilon'(t)=E'(t)+\varepsilon\frac{1-k}{2k}\big(E(t)\big)^{\frac{(1-3k)}{2k}}R(t)E'(t)+\varepsilon\big(E(t)\big)^{\frac{(1-k)}{2k}}R'(t).\tag{4.23}$$

Using assumption (2.27) on functions $p(s)$, $q(s)$ and $h(s)$ we deduce from (3.32) that

$$R'(t)\le-\frac{\rho}{2}E(t)+C_6\left(\int_{\Gamma_1}p^2(\partial_t u_1)d\sigma+\int_{\Gamma_2}\big(q^2(\partial_t u_2)+h^2(\partial_{\nu_2}\partial_t u_2)\big)d\sigma\right),\tag{4.24}$$

with the constant $C_6=\max\{C_2(1+L_1^2),C_3(1+L_3^2),C_4(1+L_5^2)\}$.

By (2.13), (4.24) and Lemma 3.7, we have

$$\begin{aligned}E_\varepsilon'(t)\le{}&-\frac{\rho}{2}\varepsilon\big(E(t)\big)^{\frac{1+k}{2k}}+\left(-1+\varepsilon\frac{1-k}{2k}E(0)^{\frac{1-k}{2k}}C_1\right)\int_{\Gamma_1}p(\partial_t u_1)\partial_t u_1 d\sigma\\&+\left(-1+\varepsilon\frac{1-k}{2k}E(0)^{\frac{1-k}{2k}}C_1\right)\int_{\Gamma_2}\big(h(\partial_{\nu_2}\partial_t u_2)\partial_{\nu_2}\partial_t u_2+q(\partial_t u_2)\partial_t u_2\big)d\sigma\\&+\varepsilon C_6\big(E(t)\big)^{\frac{(1-k)}{2k}}\left\{\int_{\Gamma_1}p^2(\partial_t u_1)d\sigma+\int_{\Gamma_2}\big(q^2(\partial_t u_2)+h^2(\partial_{\nu_2}\partial_t u_2)\big)d\sigma\right\}.\end{aligned}\tag{4.25}$$

By (2.17)–(2.19) and (2.27) we have

$$\left|p(s)\right|^{(k+1)/k} \le L_0 p(s)s \quad \text{for all } |s| \le 1; \qquad \left|p(s)\right|^2 \le L_0^k p(s)s, \quad \text{for all } |s| \ge 1, \tag{4.26}$$

$$\left|q(s)\right|^{(k+1)/k} \le L_2 q(s)s \quad \text{for all } |s| \le 1; \qquad \left|q(s)\right|^2 \le L_2^k q(s)s, \quad \text{for all } |s| \ge 1, \tag{4.27}$$

$$\left|h(s)\right|^{(k+1)/k} \le L_4 h(s)s \quad \text{for all } |s| \le 1; \qquad \left|h(s)\right|^2 \le L_4^k h(s)s, \quad \text{for all } |s| \ge 1. \tag{4.28}$$

Using (4.26) we have

$$\begin{aligned}
&\varepsilon C_6\big(E(t)\big)^{\frac{(1-k)}{2k}} \int_{\Gamma_1} p^2(\partial_t u_1)d\sigma \\
&\le \varepsilon C_6\big(E(t)\big)^{\frac{(1-k)}{2k}} \int_{\Gamma_1\cap\{|\partial_t u_1|\ge 1\}} p^2(\partial_t u_1)d\sigma + \varepsilon C_6\big(E(t)\big)^{\frac{(1-k)}{2k}} \int_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} p^2(\partial_t u_1)d\sigma \\
&\le \varepsilon C_6 L_0^k\big(E(0)\big)^{\frac{(1-k)}{2k}} \int_{\Gamma_1\cap\{|\partial_t u_1|\ge 1\}} p(\partial_t u_1)\partial_t u_1 d\sigma \\
&\quad + \varepsilon C_6\big(E(t)\big)^{\frac{(1-k)}{2k}} \int_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} p^2(\partial_t u_1)d\sigma.
\end{aligned} \tag{4.29}$$

Now applying Young's inequality to the second term on the RHS of (4.29) we get

$$\begin{aligned}
&\varepsilon C_6\big(E(t)\big)^{\frac{(1-k)}{2k}} \int_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} p^2(\partial_t u_1)d\sigma \\
&\le \frac{\varepsilon\rho}{8}E(t)^{\frac{(k+1)}{2}} + \varepsilon\big(2\rho^{-1}C_6\big)^{\frac{(k+1)}{2k}} \Bigg(\int_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} p^2(\partial_t u_1)d\sigma\Bigg)^{\frac{(k+1)}{2k}}.
\end{aligned} \tag{4.30}$$

But by Holder's inequality we know that

$$\begin{aligned}
\Bigg(\int_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} p^2(\partial_t u_1)d\sigma\Bigg)^{\frac{(k+1)}{2k}} &\le \Bigg(\int_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} d\sigma\Bigg)^{\frac{(1-k)}{2k}} \int_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} \big|p(\partial_t u_1)\big|^{\frac{k+1}{k}} d\sigma \\
&\le (\operatorname{meas}\Gamma_1)^{\frac{(1-k)}{2k}} L_0 \int_{\Gamma_1\cap\{|\partial_t u_1|\le 1\}} p(\partial_t u_1)\partial_t u_1 d\sigma.
\end{aligned} \tag{4.31}$$

Substituting (4.30) and (4.31) to (4.29) gives

$$\begin{aligned}
&\varepsilon C_6\big(E(t)\big)^{\frac{(1-k)}{2k}} \int_{\Gamma_1} p^2(\partial_t u_1)d\sigma \\
&\le \frac{\varepsilon\rho}{8}E(t)^{\frac{(k+1)}{2k}} \\
&\quad + \varepsilon\Big\{C_6 L_0^k\big(E(0)\big)^{\frac{(1-k)}{2k}} + L_0(\operatorname{meas}\Gamma_1)^{\frac{(1-k)}{2k}}\big(2\rho^{-1}C_6\big)^{\frac{(k+1)}{2k}}\Big\} \int_{\Gamma_1} p(\partial_t u_1)\partial_t u_1 d\sigma.
\end{aligned} \tag{4.32}$$

Similarly, using (4.27) and (4.28) we show that

$$\begin{aligned}
&\varepsilon C_6\big(E(t)\big)^{\frac{(1-k)}{2k}}\int_{\Gamma_2} q^2(\partial_t u_2)d\sigma \\
&\quad\le \frac{\varepsilon\rho}{8}E(t)^{\frac{(k+1)}{2k}} \\
&\qquad + \varepsilon\big\{C_6L_2^k\big(E(0)\big)^{\frac{(1-k)}{2k}} + L_2(\operatorname{meas}\Gamma_2)^{\frac{(1-k)}{2k}}\big(2\rho^{-1}C_6\big)^{\frac{(k+1)}{2k}}\big\}\int_{\Gamma_2} q(\partial_t u_2)\partial_t u_2 d\sigma,
\end{aligned}\tag{4.33}$$

and

$$\begin{aligned}
&\varepsilon C_6\big(E(t)\big)^{\frac{(1-k)}{2k}}\int_{\Gamma_2} h^2(\partial_{\nu_2}\partial_t u_2)d\sigma \\
&\quad\le \frac{\varepsilon\rho}{8}E(t)^{\frac{(k+1)}{2k}} \\
&\qquad + \varepsilon\big\{C_6L_4^k\big(E(0)\big)^{\frac{(1-k)}{2k}} + L_4(\operatorname{meas}\Gamma_2)^{\frac{(1-k)}{2k}}\big(2\rho^{-1}C_6\big)^{\frac{(k+1)}{2k}}\big\}\int_{\Gamma_2} h(\partial_{\nu_2}\partial_t u_2)\partial_{\nu_2}\partial_t u_2 d\sigma.
\end{aligned}\tag{4.34}$$

Plugging (4.32)–(4.34) into (4.25), it follows

$$E_\varepsilon'(t) \le -\frac{\rho\varepsilon}{8}\big(E(t)\big)^{\frac{k+1}{2k}} \le -\frac{\rho\varepsilon}{8}C_0^{-1/2}\big(E_\varepsilon(t)\big)^{\frac{k+1}{2k}},\tag{4.35}$$

provided ε is chosen such that

$$\varepsilon\bigg\{\bigg(C_6\big(L_0^k+L_2^k+L_4^k\big)+\frac{1-k}{2k}C_1\bigg)\big(E(0)\big)^{\frac{(1-k)}{2k}} + (L_0+L_2+L_4)(\operatorname{meas}\Gamma_1\cup\Gamma_2)^{\frac{(1-k)}{2k}}\big(2\rho^{-1}C_6\big)^{\frac{(k+1)}{2k}}\bigg\} \le 1.$$

Combining (4.35) and (4.25), we obtain

$$E(t) \le C_0E(0)\bigg\{1+\varepsilon\rho\frac{1-k}{4k}C_0^{-\frac{1}{k+1}}E(0)^{\frac{1-k}{2k}}\bigg\}^{-\frac{2k}{1-k}} = C_0E(0)(1+\omega_3 t)^{-\frac{2k}{1-k}} \quad \text{for all } t\ge 0,\tag{4.36}$$

with the constant $\omega_3 = \varepsilon\rho\frac{1-k}{4k}C_0^{-\frac{1}{k+1}}E(0)^{\frac{1-k}{2k}}$. $\square$

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