



Towards optimal regularity for the fourth-order thin film equation in \mathbb{R}^N : Graveleau-type focusing self-similarity \star



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ABSTRACT

An approach to some “optimal” (more precisely, non-improvable) regularity of solutions of the *thin film equation*

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N,$$

where $n \in (0, 2)$ is a fixed exponent, with smooth compactly supported initial data $u_0(x)$, in dimensions $N \geq 2$ is discussed. Namely, a precise exponent for the Hölder continuity with respect to the spatial radial variable $|x|$ is obtained by construction of a Graveleau-type focusing self-similar solution. As a consequence, optimal regularity of the gradient ∇u in certain L^p spaces, as well as a Hölder continuity property of solutions with respect to x and t , are derived, which cannot be obtained by classic standard methods of integral identities–inequalities. Several profiles for the solutions in the cases $n = 0$ and $n > 0$ are also plotted. In general, we claim that, even for arbitrarily small $n > 0$ and positive analytic initial data $u_0(x)$, the solutions $u(x, t)$ cannot be better than $C_x^{2-\varepsilon}$ -smooth, where $\varepsilon(n) = O(n)$ as $n \rightarrow 0$.

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1. Introduction: TFE-4 and known related regularity results

This paper is devoted to an “optimal regularity” analysis for a fourth-order quasilinear degenerate evolution equation of the parabolic type, called the *thin film equation* (TFE-4), with a fixed exponent $n > 0$,

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

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where $n \in (0, 2)$ is a fixed exponent, and bounded, sufficiently smooth, and compactly supported initial data u_0 with an arbitrary dimension $N \geq 2$.

Moreover, we assume $u_0 = u_0(|x|)$ to be radially symmetric, so solutions $u = u(|x|, t)$ do the same. These initial conditions could be relaxed (for example $u_0 \in L^1 \cap L^\infty$, etc.). However, for simplicity, we have chosen those initial conditions, since the focusing phenomenon to be studied here exists in any reasonable functional class of initial data, including radial ones.

The main result obtained here is an approach to the so-called optimal regularity of solutions of the TFE-4 for the N -dimensional case with $N \geq 2$, ascertaining an “optimal” Hölder continuity exponent in x and t .

For the one-dimensional case, similar results were obtained in [7], establishing the Hölder continuity in $C^{0, \frac{1}{2}}$ with respect to the variable x and in $C^{0, \frac{1}{5}}$ with respect to t . However, in the N -dimensional case, the optimal regularity has been unsolved besides the interest of the specialised mathematical community; see [8,10,15,16]. In many of those works it is also assumed that the solutions are non-negative for the Cauchy problem (CP). However, as we recently proved in the CP settings, the solutions for the TFE-4 (1.1) are oscillatory and sing-changing for all not that large values of $n > 0$; see [2,13] for recent results and references therein.

One of the difficulties concerning optimal regularity for the equation (1.1) is due to the fact that this equation (1.1) is *not fully divergent*, so the whole “global” regularity theory for the TFE-4 can use just a couple of well-known integral identities/inequalities. Indeed, as discussed in [2] an integral identities argument, as that performed by Bernis–Friedman in 1D, fails to show the Hölder continuity in higher dimensions; see [2] for details and reasons for these issues. Consequently, we claim that, for such “partially” (and/or fully non-) divergent equations, a different approach should be put in charge.

1.1. Methodology: Graveleau-type focusing similarity solutions

Here, working on the radially symmetric problem of TFE-4, such that

$$u(x, t) = u_*(r, t), \quad \text{with } r = |x|,$$

we base our analysis on the *focusing argument* performed, firstly, by Graveleau [14] in a formal sense, and later justified in Aronson–Graveleau [4] for a class of non-negative solutions (via the Maximum Principle, MP) of the *classic porous medium equation* (PME-2):

$$v_t = \Delta(v^m) \quad \text{in } \mathbb{R}^N \times (-\infty, 0), \quad \text{with an exponent } m > 1. \quad (1.2)$$

Actually, in comparison with (1.1), $m = 1 + n > 1$. In particular, the authors in [4] obtained, after constructing a one-parameter family of self-similar solutions, that,

$$\text{when } N \geq 2, \quad \text{there exists } 0 < \mu = \mu(m, N) < 1,$$

and a radial self-similar solution $v_*(r, t)$ to the focusing problem and with the focusing time $t = 0^-$, such that¹

$$v_*(r, 0^-) = C_* r^\mu, \quad \text{where } C_* \text{ is an arbitrary constant.} \quad (1.3)$$

This finished a long discussion concerning optimal regularity for the PME-2 after seminar results by Caffarelli–Friedman [9] (see [MathSciNet](#) for many further extensions) proving, earlier, Hölder continuity of $v(x, t)$ in a very general setting.

¹ We do not put further details how to get (1.3), since we will explain this shortly for the TFE-4.

The above results of Aronson and Graveleau showed that such a Hölder continuity is, indeed, optimal, and even the corresponding Hölder exponent $\mu = \mu(m, N)$ from (1.3) can be evaluated, at least, numerically for any $m > 1$ and $N \geq 2$. Indeed, the blow-up singularity (1.3) shows, actually, not an *optimal* regularity for such PME-2 flows, but the *non-improvable* one, in the sense that, for the Hölder exponent of solutions of (1.2),

$$\text{a non-improvable regularity} \quad \Rightarrow \quad \text{Hölder exponent} \not\geq \mu. \tag{1.4}$$

Fortunately, as we have mentioned above, Caffarelli–Friedman earlier results of the 1980s had been already available at that time, so, together with (1.4), those proved that a proper Hölder regularity of any non-negative solutions of the PME-2 is optimal.

These Aronson–Graveleau focusing (singularity-blow-up) ideas for the PME-2 (1.2) and other second-order quasilinear parabolic equations (e.g. for the p -Laplacian one) got further development and extensions in a number of papers devoted to various important aspects of porous medium flows; see references and results in [3,5].

More precisely talking about such Graveleau-type similarity approach, from the point of view of applications, in the focusing problem, one assumes that, initially, there is a non-empty compact set K in the complement of the $\text{supp } v_0$, where v vanishes. In other words, there is a hole in the support of the initial value $v(x, 0) = v_0(x) \geq 0$, and, in finite time T , this hole disappears, making the solution v to become positive along the boundary of K and eventually at all points of K . Basically, due to the finite propagation property that these equations possess (the porous medium equation and the thin film equation). Thus, as the flow evolves the liquid enters K and eventually reaches all points of K at the instant T . We are then interested in how the solution for the TFE-4 (1.1) behaves near the focusing time T . We again suppose (as in the Graveleau’s argument for PME-2) the focusing time $t = 0^-$.

However, in our case, with the TFE-4 (1.1) and $n > 0$, but close to zero, we need to take into consideration the existence of oscillatory solutions of changing sign.

Furthermore, for the TFE-4 (1.1), there are not still any general regularity results even in the radial setting in \mathbb{R}^N . Consequently, we will follow the lines and the logic of (1.4), i.e. we will estimate a certain *non-improvable* Hölder exponent for radial (and, hence, all other) solutions.

To this end, we use those previous singularity (blow-up) ideas from the PME-2 (1.2) to establish some properties for the self-similar solutions of the radially symmetric problem of the TFE-4 (1.1) and, eventually, some optimal regularity information about its more general solutions. Indeed, as we will see through the analysis performed in this paper, to obtain such an “optimal” (non-improvable) regularity for the TFE-4 (1.1), it seems that we must solve a nonlinear focusing problem for that TFE-4 that will be derived from the associated self-similar equation.

To be precise, we first apply the focusing ideas performed by Aronson–Graveleau [4] for the PME-2 (1.2) to the TFE-4 (1.1). To do so, we will work on the space of radially self-similar solutions with $r = |x| > 0$ of the form

$$u_*^\pm(r, t) = (\pm t)^\alpha f(y), \quad y = \frac{r}{(\pm t)^\beta} \quad \text{for} \quad \pm t > 0, \quad \text{where} \quad \beta = \frac{1+\alpha n}{4} > 0. \tag{1.5}$$

Thus, these solutions will solve an associated self-similar equation, or *nonlinear eigenvalue problem*,

$$-\frac{1}{y^{N-1}} \left[y^{N-1} |f|^n \left(\frac{1}{y^{N-1}} (y^{N-1} f')' \right)' \right]' \pm \beta y f' \mp \alpha f = 0,$$

with $-$ or $+$ depending on if we are analysing the focusing phenomena or the defocusing phenomena. In other words, before the focusing time or after (in our situation $t = 0$).

As usual, the unknown *a priori* α represents the nonlinear eigenvalues supporting the reason to call that equation a *nonlinear eigenvalue problem*.

Crucial in our analysis will be to impose some radiation-type conditions, or some minimal and maximal growth at infinity, i.e. as $y \rightarrow +\infty$. This actually allows us, among other things, to determine the existence of a discrete family of nonlinear eigenvalues denoted by $\{\alpha_k\}$.

Moreover, this behaviour at infinity leads us to have the solutions of the self-similar TFE-4 bounded at infinity by functions of the form

$$f(y) = Cy^\mu(1 + o(1)) \quad C \in \mathbb{R} \quad \text{and} \quad \mu = \mu(\alpha, n) := \frac{4\alpha}{1+\alpha n} > 0.$$

Thus, we ascertain a limit at the focusing point

$$u_*(r, t) \rightarrow Cr^\mu \quad \text{as} \quad t \rightarrow 0^-, \quad \text{with} \quad \mu = \frac{\alpha}{\beta},$$

uniformly on compact subsets (see details in Section 3). However, this does not provide yet information about the influence of the dimension on the behaviour of the solutions at infinity.

To get to that point, we need to find those nonlinear eigenvalues² $\{\alpha_k\}$ associated with a discrete family of nonlinear eigenfunctions $\{f_k\}$. Here, we have done this analysis via a homotopy deformation as $n \rightarrow 0^+$ (Sections 4 and 5) to the linear problem with $n = 0$,

$$u_t = -\Delta^2 u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_-, \tag{1.6}$$

and with those patterns occurring at $n = 0$ acting as branching points of nonlinear eigenfunctions of the Cauchy problem (1.1) when n is close to zero. In particular, according to our analysis, when $n = 0$, it follows that

$$\alpha_k(0) = \alpha_k = \frac{k}{2}, \quad k = 1, 2, 3, \dots$$

As an observation, those values of the parameter α when $n = 0$ might be written as a perturbation of the eigenvalues for the fourth-order operator

$$\mathbf{B}^* f \equiv -\frac{1}{y^{N-1}} \left[y^{N-1} \left(\frac{1}{y^{N-1}} (y^{N-1} f')' \right)' \right]' - \frac{1}{4} y f' = \lambda f,$$

analysed in full detail in [11]; see further comments below.

Hence, we perform a homotopy deformation from the TFE-4 (1.1) to the parabolic bi-harmonic equation (1.6), for which we can apply again a similar logic in ascertaining Graveleau-type “focusing solutions”. Indeed, we get a minimal and maximal growth for the radial self-similar solutions of this linear problem (1.6). Moreover, since we know that there exists a discrete family of eigenvalues for the corresponding self-similar equation associated with (1.6), using this branching/homotopy argument we obtain the desired family of values for the parameter α from which we will have a family of radial self-similar solutions emanating at $n = 0$ from the self-similar solutions of the linear problem (1.6). Note also that the family of eigenfunctions for the linear problem (1.6) is a complete set of generalised Hermite polynomials with finite oscillatory properties [11].

In this paper, we also support this analysis numerically, performing a shooting procedure from $y = 0$ to $y = +\infty$ that gives us those linear eigenvalues α_k and the profile of the corresponding eigenfunctions.

Additionally, we show some numerical analysis that provided us, with very difficult to ascertain, profiles of the nonlinear eigenfunctions $\{f_k\}$ with $n > 0$ and sufficiently close to zero.

² Since we obtain those nonlinear eigenvalues as a perturbation from a family of linear eigenvalues when the parameter n is close to zero, we will use indistinctively this notation for the family of linear or nonlinear eigenvalues. To be precise, the nonlinear eigenvalues correspond to the family $\{\alpha_k(n)\}$ and the linear eigenvalues to the family $\{\alpha_k(0)\}$.

This numerical analysis supports the conjecture that by homotopy continuity the properties for the profile when $n = 0$ remain valid for $n > 0$ and sufficiently close to zero. As seen above, we have fixed the interval $n \in (0, 2)$, however, the numerics suggest that those properties are lost from $n = 0.8$ or 0.9 . In fact, if $n \geq 2$ we just have nonnegative solutions as shown in [7] so that, this focusing argument is not possible. Thus, we restrict this analysis to $n \in (0, 2)$, or more precisely, of $n > 0$ sufficiently close to zero.

Finally, we obtain such an “optimal” (non-improvable) regularity for the TFE-4 (1.1) proving that there is Hölder continuity with a specific coefficient

$$\mu_k = \frac{4\alpha_k}{1 + n\alpha_k} \quad \text{and} \quad \nabla u(x, t) \in L^p_{\text{loc}}(\mathbb{R}^N) \quad \text{with} \quad p < p^*(n, N) = \frac{N}{1 - \mu_k},$$

having such regularisation depending on the dimension N , as we claimed and were looking for. Consequently, if $n > 0$ small, we find that

$$u_*(r, t) \in C_r^{2-\varepsilon(n)}, \quad \text{with} \quad \varepsilon(n) = n\mu + O(n^2). \tag{1.7}$$

It turns out that, thanks to our branching analysis, there holds $\mu > 0$, so that, for $n > 0$, the regularity condition (1.7) becomes obviously worse. Note that, if $\mu < 0$, we have a better regularity than for $n = 0$ which is impossible.

In general, we obtain (and the numerics suggest that as well) that there exist several focusing singularities in the radial geometry of the type $C^{2k-\varepsilon}$ with $k = 1, 2, 3, \dots$. In particular, for the values of α_k obtained here, it follows that for $k = 1$, $\alpha_1 = \frac{1}{2}$ is the minimal and crucial one, since it seems to be the only degenerate case, i.e. $f_1(0) = 0$. All others satisfy

$$f_k(0) > 0, \quad K = 1, 2, 3, \dots, \quad \text{i.e. the TFE-4 for } n > 0 \text{ is not degenerate}$$

initially, but, indeed, will be eventually, at the focusing time $t = T^- (= 0^-)$. Therefore, we can conclude that slight changes in the parameter α destroy the regularity, supporting the fact that these results are valid only when n is sufficiently close to zero. Indeed, via branching analysis we find that

$$\alpha_k(n) = \alpha_k + \mu_k n + O(n^2),$$

so that we have the regularity condition

$$u_*(r, t) \in C^{2k-\mu_k n+O(n^2)}.$$

Note that, even for $\alpha_2 = 1$, the analytic positive solution becomes, at $t = T^-$, $C^{4-\varepsilon}$, i.e. not classical solutions in C^4 . Hence, for the minimal $\alpha_1 = \frac{1}{2}$ it is $C^{2-\varepsilon}$, which is much worse.

Remark. As mentioned above, we also observe that, for the PME-2 (1.2), one needs to have such a non-empty compact hole to apply the Maximum Principle. However, we believe that, for the TFE-4 (1.1) and the value $\alpha_1 = \frac{1}{2}$, that provides us with the minimal regularity $C^{2-\varepsilon(n)}$, there is not such a hole. We do not have a rigorous justification of it but the numerics presented in this paper suggest it that way.

In relation to the Hölder regularity with respect to the temporal variable t , assuming the radially self-similar solutions of the form (1.5) we find that

$$|u(0, t) - u(0, 0)| = (-t)^\alpha f(0).$$

Hence, the Hölder’s exponent for the variable t when it is very close to $t = 0$ cannot be bigger than

$$\alpha_1(n) = 1/2 + O(n).$$

Therefore, we provide with an estimation from above for the Hölder continuity with respect to the t variable. Moreover, this estimation improves the Hölder’s exponent obtained by Bernis–Friedman [7] showing that theirs was actually not optimal since it was $1/8$.

1.2. TFE-4 problem settings

This setting is well-known nowadays (though many things were not fully proved in a general case), so, below, we omit many details; see surveys [15,16]. Note that, for some values of $n > 0$ (not that large), focusing similarity solutions to be constructed do not exhibit finite interfaces, so the resulting optimal regularity results are true for any FBP and/or Cauchy problem settings. Principal differences between the CP and the standard FBP settings for the TFE-4 (1.1) are explained in [12,13].

We recall that the solutions are assumed to satisfy the following zero contact angle boundary conditions:

$$\begin{cases} u = 0, & \text{zero-height,} \\ \nabla u = 0, & \text{zero contact angle,} \\ -\mathbf{n} \cdot \nabla(|u|^n \Delta u) = 0, & \text{conservation of mass (zero-flux)} \end{cases} \tag{1.8}$$

at the singularity surface (interface) $\Gamma_0[u]$, which is the lateral boundary of

$$\text{supp } u \subset \mathbb{R}^N \times \mathbb{R}_+, \quad N \geq 1,$$

where \mathbf{n} stands for the unit outward normal to $\Gamma_0[u]$, which is assumed to be sufficiently smooth (the treatment of such hypotheses is not any goal of this paper). For smooth interfaces, the condition on the flux can be read as

$$\lim_{\text{dist}(x, \Gamma_0[u]) \downarrow 0} -\mathbf{n} \cdot \nabla(|u|^n \Delta u) = 0.$$

Next, we denote by

$$M(t) := \int u(x, t) \, dx \tag{1.9}$$

the mass of the solution, where integration is performed over the support. Then, differentiating $M(t)$ with respect to t and applying the divergence theorem, we have that

$$J(t) := \frac{dM}{dt} = - \int_{\Gamma_0 \cap \{t\}} \mathbf{n} \cdot \nabla(|u|^n \Delta u).$$

The mass is conserved if $J(t) \equiv 0$, which is satisfied by the flux condition in (1.8).

2. Radial self-similar solutions: focusing and defocusing cases

Now, we construct the operators and specific solutions in order to apply the ideas performed by Aronson–Graveleau [4] for the PME-2 (1.2) to the TFE-4 (1.1).

Thus, thanks to the scaling invariant property of these nonlinear parabolic equations, we now construct radially self-similar solutions, i.e. in terms of $r = |x| > 0$, with a still unknown value of the parameter $\alpha > 0$ (clearly, it must be positive, as shown below)

$$u_*^\pm(r, t) = (\pm t)^\alpha f(y), \quad y = \frac{r}{(\pm t)^\beta} \quad \text{for } \pm t > 0, \quad \text{where } \beta = \frac{1+\alpha n}{4} > 0. \tag{2.1}$$

Here, by the time-translation, we ascribe the blow-up or *focusing* time to $T = 0^-$. We then simultaneously consider two cases:

- (i) **Focusing** (i.e., Graveleau-type) similarity solutions, which play a key role, corresponding to $(-t)$ in (2.1) and the singular blow-up limit as $t \rightarrow 0^-$, and
- (ii) **Defocusing** similarity solutions, with $(+t)$ in (2.1), playing a secondary role as extensions of the previous ones for $t > 0$, i.e., corresponding to the not-that-singular (or, at least, less singular) limit $t \rightarrow 0^+$. Actually, these defocusing solutions are well-posed solutions of the Cauchy problem for the TFE-4 for $t > 0$ with initial data $u_*(r, 0^-)$ obtained from the previous blow-up limit as $t \rightarrow 0^-$.

Substituting (2.1) into (1.1) we arrive at the similarity profiles $f(y)$ satisfying the following *nonlinear eigenvalue problems*:

$$\mathbf{B}_n^\pm(\alpha, f) \equiv -\nabla_y \cdot (|f|^n \nabla_y \Delta_y f) \pm \beta y \cdot \nabla_y f \mp \alpha f = 0 \quad \text{for } y > 0; \quad f'(0) = f'''(0) = 0, \tag{2.2}$$

where ∇_y and Δ_y stand for the radial gradient and the radial Laplacian. In (2.2), we present two symmetry conditions at the origin (which should be modified if $f(y) \equiv 0$ near the origin, i.e. it contains a “zero hole” nearby). As we will see with the numerical analysis performed in Section 4 of this paper, we can choose other conditions depending on if either $f = 0$ or $f \neq 0$.

To complete these nonlinear eigenvalue settings one needs extra “radiation-type” (or growth-type) conditions at infinity to be introduced next.

Indeed, we actually find radially similarity profiles f depending on the single variable y . The operator of the equation (2.2) is then

$$\mathbf{A}_n^\pm(\alpha, f) \equiv -\frac{1}{y^{N-1}} [y^{N-1} |f|^n (\frac{1}{y^{N-1}} (y^{N-1} f')')']' \pm \beta y f' \mp \alpha f = 0, \tag{2.3}$$

denoting the radial nonlinear operator as

$$\mathbf{R}_n(\alpha, f) = \frac{1}{y^{N-1}} [y^{N-1} |f|^n (\frac{1}{y^{N-1}} (y^{N-1} f')')']'. \tag{2.4}$$

Thus, here, $\alpha > 0$ is a parameter, which, in fact, stands in both cases for admitted real *nonlinear eigenvalues* to be determined, in the focusing Graveleau-type case, via the solvability of the corresponding *nonlinear eigenvalue problem*, accomplished with some special (radiation-type) conditions at infinity, to be introduced shortly.

Remark 2.1. In general, with respect to the similarity profiles described above we note that there are two main types of self-similar solutions. For solutions of the first kind the similarity variable y can be determined *a priori* from dimensional considerations and conservation laws, such as the conservation of mass (1.9) or momentum.

For solutions of the second kind the exponent β (and by relations the exponent α) in the similarity variable must be obtained along with the solution by solving a nonlinear eigenvalue problem of the form (2.2).

The first published examples of self-similar solutions of second kind are due to G. Guderley in 1942 [17] studying imploding shock waves, although the term was introduced by Ya.B. Zel’dovich in 1956 [18]. Further examples might be found in the theory of the collapse of bubbles in compressible fluids or in works on gas motion under an impulsive load; see Barenblatt [6] for an extensive work on this matter.

3. Minimal growth at infinity (a “nonlinear radiation condition”) for Graveleau-type profiles

Here, we consider the blow-up problem (2.3)₋, i.e. with the lower signs. One concludes that, in order to get a possible discrete set of eigenvalues $\alpha > 0$, some extra conditions denoted by nonlinear radiation condition on the behaviour (in particular, growth) of $f(y)$ as $y \rightarrow +\infty$ must be imposed.

Obviously, such a “radiation-type” condition (we use a standard term from dispersion theory) just follows from analysing all possible types of such a behaviour, which can be admitted by the ODE (2.3)₋, which is not that a difficult problem. There are two cases:

- (I) *More difficult: there is a “zero hole” for $f(y)$ near the origin.* Finite interfaces for the TFE-4 are well known in the case of the standard FBP setting and also in the Cauchy one (see [12,13] and references therein). Then we need to look for profiles $f(y)$ which vanish at finite $y = y_0 > 0$ and describe asymptotics of the general solution satisfying zero contact angle and zero flux conditions as $y \rightarrow y_0^+$

$$f(y) \rightarrow 0, \quad f'(y) \rightarrow 0, \quad -|f|^n f'''(y) \rightarrow 0.$$

This case causes a difficult problem, since the oscillatory behaviour of $f(y)$ close to the interface is quite tricky for the CP [12], while, for the FBP, it is much better understood.

- (II) *More standard and easier: $f(0) \neq 0$.* This case also includes the border possibility $y_0 = 0$, i.e. when $f(0) = 0$, but $f \not\equiv 0$ in any arbitrarily small neighbourhood of $y = 0$.

3.1. Minimal and maximal growth at infinity

This is key for our regularity analysis. Our radial ODE (2.3)₋ admits two kinds of asymptotic behaviour at infinity, i.e. when $y_0 \rightarrow +\infty$. For the operator $\mathbf{A}_n^-(\alpha, f)$, we state the following result:

Proposition 3.1. *For any $\alpha > 0$, the ODE (2.3)₋, with the operator $\mathbf{A}_n^-(\alpha, f)$ possesses:*

- (i) *Solutions $f(y)$ with a minimal growth*

$$f(y) = Cy^\mu(1 + o(1)) \quad \text{as } y \rightarrow +\infty, \quad C \in \mathbb{R}, \tag{3.1}$$

where

$$\mu = \mu(\alpha, n) = \frac{4\alpha}{1+\alpha n} > 0. \tag{3.2}$$

- (ii) *Moreover, there exist solutions of (2.3)₋ with a maximal growth*

$$f(y) \sim y^{\mu_0}, \quad \text{as } y \rightarrow +\infty, \quad \text{where } \mu_0 = \frac{4}{n}. \tag{3.3}$$

Remark.

- It is important to mention that the expression (3.3) for solutions of the maximal growth does not include the “additional or extra” corresponding *oscillatory component* $\varphi(s)$, with $s = \ln y$, and shows just the growth behaviour of its “envelope” with algebraic growth. Such an oscillatory maximal behaviour will be introduced and studied later.
- Note also that, as $n \rightarrow 0^+$, we have that

$$\frac{4}{n} \rightarrow +\infty, \tag{3.4}$$

which corresponds to an exponential oscillatory growth for the linear problem occurring for $n = 0$; see the next section.

- The difference between (3.1) and (3.3), which implies such terms as minimal and maximal growth at infinity, is obvious:

$$\text{for any } \alpha > 0, \quad \frac{4}{n} > \frac{\alpha}{\beta} = \frac{4\alpha}{1+\alpha n}. \tag{3.5}$$

Proof. This follows from a balancing of linear and nonlinear operators in this ODE, though a rigorous justification is rather involved and technical. A formal derivation is surely standard and easy:

- (i) In this first case, we assume a linear asymptotic as $y \rightarrow +\infty$ (assuming simple radial behaviour $f \sim y^\mu$)

$$\frac{1+\alpha n}{4} y f' - \alpha f + \dots = 0 \implies f(y) \sim y^\mu, \quad \text{where } \mu = \frac{4\alpha}{1+\alpha n} > 0. \tag{3.6}$$

Since this behaviour is asymptotically linear, eventually, we get an arbitrary constant $C \neq 0$ in (3.1). A full justification of existence of such orbits is straightforward, since the nonlinear term in the ODE is then negligible. So that in the equivalent integral equation though a singular term, it produces a negligible perturbation.

- (ii) On the other hand, the solutions are also bounded by a maximal growth, which in this case comes from non-linear asymptotics that balance all three operators

$$\mathbf{R}_n(f) + \frac{1+\alpha n}{4} y f' - \alpha f = 0 \implies f(y) \sim y^{\frac{4}{n}} \quad \text{as } y \rightarrow +\infty, \tag{3.7}$$

where we again indicate the envelope behaviour of this oscillatory bundle. A justification here, via Banach’s contraction principle is even easier, but rather technical, so we omit details. \square

Overall this allows us to formulate such a condition at infinity, which now takes a clear “minimal nature” such that solutions f are now bounded at infinity by a function

$$f(y) = C y^\mu (1 + o(1)), \quad \text{with } \mu = \frac{4\alpha}{1+\alpha n} > 0. \tag{3.8}$$

Obviously, thus we just need a global solution $f(y)$ of our ODE (2.3)₋ in \mathbb{R}_+ , satisfying the minimal growth (3.1). Indeed, for such profiles, there exists a finite limit at the focusing (blow-up) point:

$$u_*(r, t) \rightarrow C r^\mu \quad \text{as } t \rightarrow 0^-, \quad \text{with } \mu = \frac{\alpha}{\beta}, \tag{3.9}$$

uniformly on compact intervals in $r = |x| \geq 0$. One can see that, for any maximal profile as in (3.3), the limit as in (3.9) is infinite, so that such similarity solutions do not leave a finite trace as $t \rightarrow 0^-$.

However, the above proposition leaves aside the principle question on the dimensions of the corresponding minimal and maximal bundles as $y \rightarrow +\infty$, which is not a straightforward problem. Note that the latter is actually supposed to determine the strategy of a well posed shooting of possible solutions of the above focusing problem.

Indeed, the main problem is how to find those admitted values of nonlinear eigenvalues $\{\alpha_k\}_{k \geq 1}$ (possibly and hopefully, a discrete set), for which $f = f_k(y)$ exist producing finite limits as in (3.9).

To clarify those issues, we consider the much simpler linear case when $n = 0$ and, subsequently, pass to the limit as $n \rightarrow 0^+$ in (2.3). This analysis will provide us, eventually, with some qualitative information about the solutions, at least when n is very close to zero.

4. The linear problem: discretisation of α for $n = 0$

For $n = 0$, the TFE-4 (1.1) becomes the classic *bi-harmonic equation*

$$u_t = -\Delta^2 u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_-. \tag{4.1}$$

Of course, solutions of (4.1) are analytic in both x and t , so any focusing for it makes no sense. However, we will show that (following a similar philosophy to the one above) Graveleau-type “focusing solutions” for (4.1) are rather helpful to predict some properties of true blow-up self-similar solutions of (1.1), at least, for small $n > 0$.

4.1. Maximal and minimal bundles

Thus, first we consider the same “focusing” solutions (2.1)₋ for (4.1), that take a simpler form

$$u_*(r, t) = (-t)^\alpha f(y), \quad y = \frac{r}{(-t)^{1/4}} \quad (\beta = \frac{1}{4}). \tag{4.2}$$

Then, the corresponding linear radial ODE (2.3) takes also a simpler form

$$\mathbf{A}_{0,y}^-(\alpha, f) \equiv -\frac{1}{y^{N-1}} [y^{N-1} (\frac{1}{y^{N-1}} (y^{N-1} f')')]' - \frac{1}{4} y f' + \alpha f = 0. \tag{4.3}$$

We first calculate the solutions of (4.3) with a maximal behaviour. Those are exponentially growing solutions of the form

$$f(y) \sim e^{ay^\gamma} \quad \text{as } y \rightarrow +\infty \implies a^3 = -\frac{1}{4} (\frac{3}{4})^3. \tag{4.4}$$

This characteristic equation gives two roots with $\text{Re}(\cdot) > 0$:

$$a_{1,2} = \frac{3}{4} 4^{-\frac{1}{3}} [\frac{1}{2} \pm i \frac{\sqrt{3}}{2}] \equiv c_0 \pm i c_1 \tag{4.5}$$

and one negative root (this actually goes to the bundle of minimal solutions; see below)

$$a_3 = -\frac{3}{4} 4^{-\frac{1}{3}} < 0. \tag{4.6}$$

Hence, the bundle of maximal solutions is oscillatory as $y \rightarrow +\infty$ (including the multiplicative algebraic factor),

$$f(y) \sim y^{-\frac{2}{3}(N+2\alpha)} e^{c_0 y^{4/3}} [C_1 \cos(c_1 y^{\frac{4}{3}}) + C_2 \sin(c_1 y^{\frac{4}{3}})], \quad C_{1,2} \in \mathbb{R}. \tag{4.7}$$

In fact, obviously, this bundle is 3D since it includes the 1D sub-bundle of exponentially decaying solutions with the exponent (4.6). However, in a shooting procedure performed at the end of this section, we intend to get rid of just two coefficients in (4.7)

$$C_1 = C_2 = 0.$$

Indeed, this numerical evidence shows that those coefficients vanish at certain values of the parameter α that we will determine below specifically. Moreover, we provide an explanation in the next subsection.

On the other hand, the minimal behaviour (3.1) now reads

$$f(y) = C y^{4\alpha} (1 + o(1)), \quad C \in \mathbb{R} \quad (\text{with } \frac{\alpha}{\beta} = 4\alpha). \tag{4.8}$$

The whole bundle of such minimal solutions is 2D. Besides the parameter C in (4.8), it includes a 1D sub-bundle of exponentially decaying solutions with the exponent (4.6), so that, overall, the minimal solutions compose a 2D family:

$$f(y) \sim Cy^{4\alpha}(1 + o(1)) + De^{a_3y^{4/3}}(1 + o(1)), \quad C, D \in \mathbb{R}. \tag{4.9}$$

Justification of both behaviours is, indeed, simpler for such a standard linear ODE problem.

Finally, passing to the limit as in (3.9) then gives, for minimal solutions, a finite “focusing trace”:

$$u_*(r, t) \rightarrow Cr^{4\alpha} \quad \text{as } t \rightarrow 0^-. \tag{4.10}$$

We explain why those trivial results are so important in what follows. The conclusion from (4.10) is straightforward: since $u_*(r, 0^-)$ must be analytic then we have the following.

Proposition 4.1. *For the above linear “focusing eigenvalue problem” (4.3), there exist not more than a countable set of admissible eigenvalues $\{\alpha_k\}_{k \geq 1}$ given by*

$$\alpha_k = \frac{k}{2}, \quad k = 1, 2, 3, \dots \tag{4.11}$$

Proof. The function $r^{4\alpha}$ is analytic at $r = 0$ only for values α_k from (4.11), i.e. 4α must be a real even integer, say $2k$. Note also that, in the present problem, all functions are analytic, so that, automatically, the set of roots $\{\alpha_k\}$ is discrete with a possible accumulation point at infinity only. \square

Remark. Solving the *linear eigenvalue problem*:

$$\mathbf{A}_{0,y}^-(\alpha_k, f_k) = 0 \quad \text{in } \mathbb{R}, \quad f_k \in L^2_\rho(\mathbb{R}),$$

then, it seems that the nonlinear eigenvalue problem (2.3)₋ formally reduces to the classic linear eigenvalue problem (4.3) at $n = 0$, providing us with another reason to call (2.3) a *nonlinear eigenvalue problem*.

Also, the values of the parameter α can be written as shifting from the eigenvalues of the eigenvalue problem

$$\mathbf{B}^* f \equiv -\frac{1}{y^{N-1}} \left[y^{N-1} \left(\frac{1}{y^{N-1}} (y^{N-1} f') \right)' \right]' - \frac{1}{4} y f' = \lambda f,$$

analysed in [11] and whose discrete spectrum takes the form

$$\sigma(\mathbf{B}^*) = \left\{ \lambda_k = -\frac{k}{4}; k = 0, 1, \dots \right\}.$$

Hence,

$$\alpha_k = \lambda_k - \frac{k+1}{4}, \quad \text{with } k = 1, 2, 3, \dots,$$

having a countable family of eigenvalues for the problem (4.3).

Moreover, note that $\mathbf{A}_{0,y}^-(\alpha, f)$ (4.3) is a non-symmetric linear operator, which is bounded from $H^4_\rho(\mathbb{R})$ to $L^2_\rho(\mathbb{R})$ with the exponential weight given by (4.7), i.e.,

$$\rho(y) = e^{-c_0|y|^{4/3}}, \quad c_0 > 0 \quad \text{small and defined by (4.5).}$$

4.2. Well-posed shooting procedure

We now discuss a practical procedure to obtain the linear eigenvalues α_k . We perform standard shooting from $y = 0$ to $y = +\infty$ for the ODE (4.3) by posing four conditions at the origin:

- Either

$$f(0) = 1 \quad (\text{normalisation}), \quad f''(0) = 0, \quad f'(0) = f'''(0) = 0 \quad (\text{symmetry}); \tag{4.12}$$

- or

$$f(0) = 0, \quad f''(0) = 1 \quad (\text{normalisation}), \quad f'(0) = f'''(0) = 0 \quad (\text{symmetry}). \tag{4.13}$$

These two sets of conditions correspond to partitioning of the eigenfunctions into two subsets with the corresponding properties at the origin of

$$\text{either } f(0) = 1, \quad f''(0) = 0, \quad \text{or } f(0) = 0, \quad f''(0) = 1,$$

together with symmetry conditions. Explicitly, the first four eigenfunctions are

$$\begin{aligned} k = 1 : \quad & \alpha_1 = \frac{1}{2}, \quad f_1(y) = \frac{1}{2}y^2, \\ k = 2 : \quad & \alpha_2 = 1, \quad f_2(y) = 1 + \frac{1}{8N(N+2)}y^4, \\ k = 3 : \quad & \alpha_3 = \frac{3}{2}, \quad f_3(y) = \frac{1}{2}y^2 + \frac{1}{48(N+2)(N+4)}y^6, \\ k = 4 : \quad & \alpha_4 = 2, \quad f_4(y) = 1 + \frac{1}{4N(N+2)}y^4 + \frac{1}{192N(N+2)(N+4)(N+6)}y^8, \end{aligned}$$

where the above properties are immediately apparent.

Numerically we may illustrate the appearance of the eigenfunctions and their eigenvalues through consideration of an Initial Value Problem (IVP). The linear ODE (4.3) is solved numerically using ode15s (with tight error tolerances $\text{AbsTol} = \text{RelTol} = 10^{-13}$) and subject to either (4.12) or (4.13) as initial conditions. The far-field behaviour (4.7) is extracted from the numerical solution. The oscillatory component in the far-field behaviour is revealed by considering the scaled solution

$$f(y)y^{\frac{2}{3}(N+2\alpha)}e^{-c_0y^{4/3}}. \tag{4.14}$$

A least squares fitting of this function to the remaining oscillatory component in (4.7) over a suitable interval for large y , allows the determination of the constants $C_{1,2}$. The large y interval taken was typically [250, 300]. The scaled function (4.14) is shown in Fig. 1 for each of the two types of initial conditions in the $N = 1, 2$ and 3 cases. Two selected values of the parameter α are taken in each case. The figures illustrate convergence to the oscillatory part of the far-field behaviour, with the extracted least squares estimates of the constants $C_{1,2}$ shown inset for each of the two α values. Fig. 2 shows the variation of the far-field constants $C_{1,2}$ with α in the $N = 1, 2, 3$ cases and the two initial conditions. The constants $C_{1,2}$ both vanish precisely at the eigenvalues, when $\alpha = \alpha_k$, the first five being shown in Fig. 2 in each N case. The magnitude of the constants $C_{1,2}$ grow rapidly as α increases, making determination of further eigenvalues more difficult.

This approach demonstrates the validity of numerical determination of the eigenvalues and eigenfunctions by choosing α to minimise the far-field behaviour (4.7).

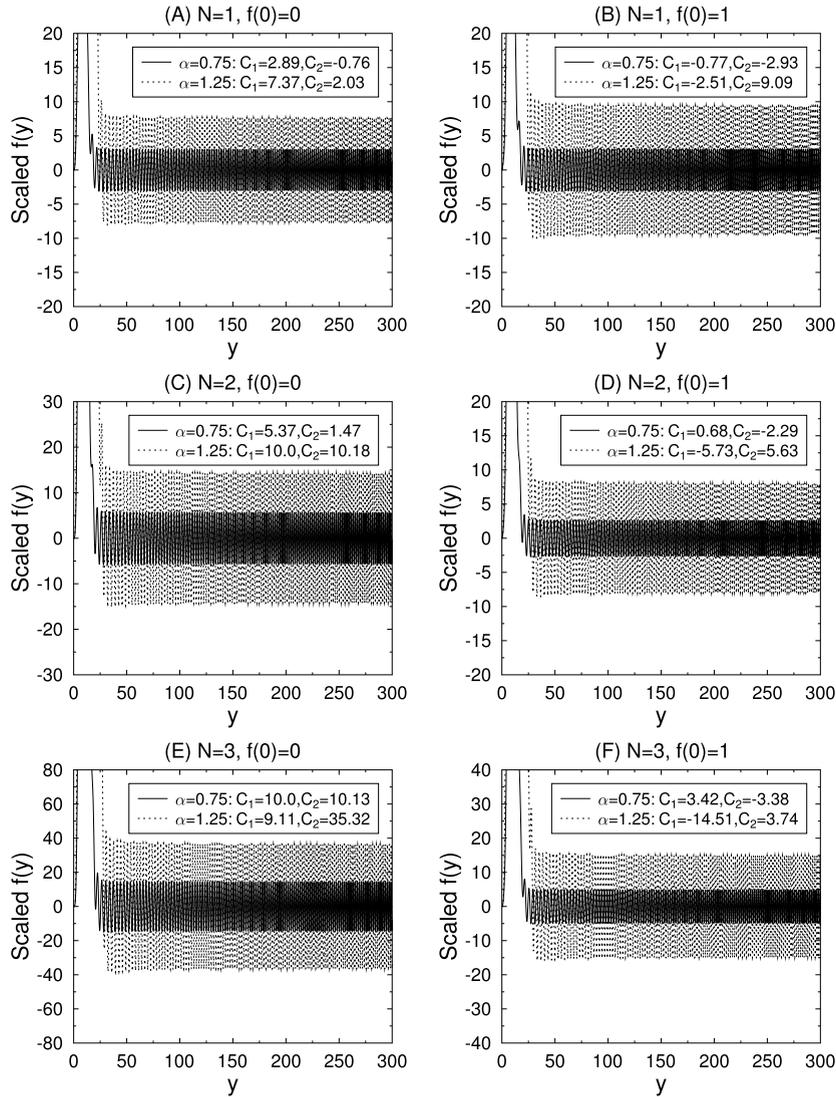


Fig. 1. Numerical illustration of the oscillatory component in the far-field behaviour (4.7) in the linear case. In each dimensional case $N = 1, 2, 3$ scaled profiles (4.14) are shown for two selected values of the parameter α for each initial condition (4.12) and (4.13). The extracted least squares values of the far-field constants $C_{1,2}$ are stated inset in each figure.

5. A “homotopic” transition to small $n > 0$: some key issues

Next, we are going to use the above linear results to predict true nonlinear eigenvalues $\alpha_k(n)$ for the TFE-4, at least, for sufficiently small $n > 0$. By continuity (to be discussed later on as a continuous deformation after applying a homotopic argument) we now know that

$$\alpha_k(0) = \frac{k}{2}, \quad k = 1, 2, 3, \dots \tag{5.1}$$

Another important conclusion: since, for $n = 0$, any $f_k(0) \neq 0$ (this could happen only accidentally, with a probability 0), we can also expect that

$$\text{for small } n > 0, \quad f_k(0) \neq 0, \tag{5.2}$$

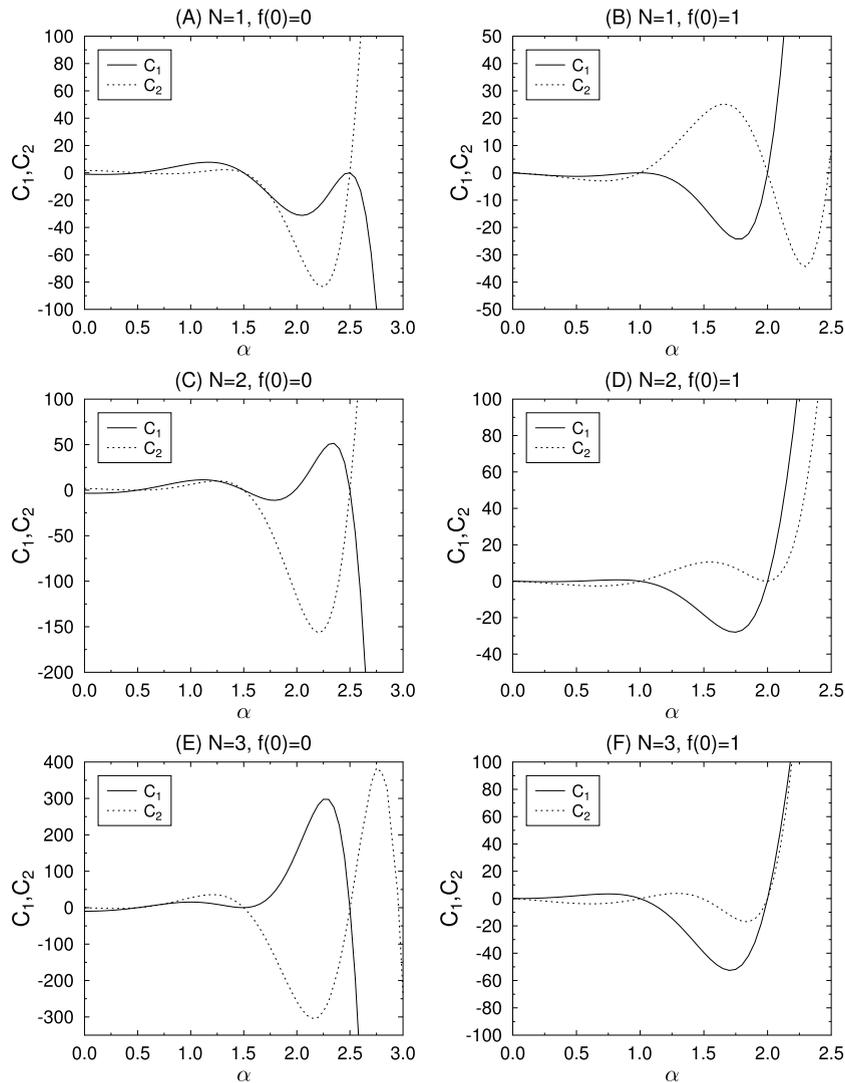


Fig. 2. Numerical determination of the far-field constants $C_{1,2}$ for (4.7) in the stated dimensional and initial condition cases. Shown are their variation with the parameter α . The coincident zeros correspond to the vanishing of the far-field maximal bundle yielding the eigenvalues of the linear problem $\alpha = \alpha_k$.

meaning that, in this case, we do not need to perform a shooting from the interface point, with a quite tricky behaviour nearby.

Also, we conclude from (4.7) that:

$$\text{for small } n > 0, \text{ maximal bundle is 2D and oscillatory as } y \rightarrow +\infty. \tag{5.3}$$

Those properties will aid progress on the nonlinear eigenvalue problem.

5.1. Passing to the limit $n \rightarrow 0^+$ in the nonlinear eigenvalue problem

Subsequently, we perform a homotopy deformation from the self-similar equation (2.3)₋ (with lower signs) of the TFE-4 (1.1) to the linear radial ODE (4.3) corresponding to the *classical bi-harmonic parabolic equation* (4.1). In particular we construct a continuous deformation from the radial equation (2.3)₋ to the linear equation (4.3) for which we know solutions explicitly.

It is clear that the CP for the *bi-harmonic equation* (4.1) is well-posed and has a unique solution given by the convolution

$$u(x, t) = b(x - \cdot, t) * u_0(\cdot), \tag{5.4}$$

where $b(x, t)$ is the fundamental solution of the operator $D_t + \Delta^2$.

Due to our analysis is possible to establish a connection between the radial solutions of (1.1) and the corresponding to (4.1) via the self-similar associated equations when $n \rightarrow 0^+$. To this end we apply the Lyapunov–Schmidt method to ascertain qualitative properties of the self-similar equation (2.3)_– following a similar analysis as the one carried out in [1].

Thus, as we already know, the operator $\mathbf{A}_{0,y}^-(\alpha, f)$ defined by (4.3) produces a countable family of eigenvalues

$$\alpha_k \equiv \alpha_k(0) = \frac{k}{2}, \quad \text{with } k = 0, 1, \dots$$

Note also that, (4.3) admits a complete and closed set of eigenfunctions being *generalised Hermite polynomials*, which exhibit finite oscillatory properties.

This oscillatory issue seems to be crucial. In fact, in [12] it was observed that a similar analysis of blow-up patterns for a TFE-4 like (1.1) did not detect any stable oscillatory behaviour of solutions near the interfaces of the radially symmetric associated equation. Hence, all the blow-up patterns turned out to be nonnegative, which is a specific feature of the PDE under consideration therein. However, this does not mean that blow-up similarity solutions of the CP do not change sign near the interfaces or inside the support. Actually, it was pointed out that local sign-preserving property could be attributed only to the blow-up ODE and not to the whole PDE (1.1). Hence, the possibility of having oscillatory solutions cannot be ruled out for every case.

5.2. Branching/bifurcation analysis

Now, we construct a continuous deformation such that the patterns occurring for the nonlinear eigenvalue problem (2.3)_– are homotopically connected to the ones of the equation (4.3). Thus, we assume for small $n > 0$ in (2.3) the following expansions:

$$\alpha_k(n) := \alpha_k + \mu_{1,k}n + o(n), \quad |f|^n \equiv |f|^n = e^{n \ln |f|} := 1 + n \ln |f| + o(n), \tag{5.5}$$

where the last one is assumed to be understood in a weak sense. The second expansion cannot be interpreted pointwise for oscillatory changing sign solutions $f(y)$, though now these functions are assumed to have *finite* number of zero surfaces (as the generalised Hermite polynomials for $n = 0$ do). Indeed, as discussed in [1] for (2.3) this is true if the zeros are transversal.

Furthermore, in order to apply the Lyapunov–Schmidt branching analysis we suppose the expansion

$$f = \sum_{|\beta|=k} c_\beta f_\beta + V_k, \quad \text{for every } k \geq 1, \tag{5.6}$$

under the natural “normalising” constraint

$$\sum_{|\beta|=k} c_\beta = 1. \tag{5.7}$$

Moreover, we write

$$\{f_\beta\}_{|\beta|=k} = \{f_1, \dots, f_{M_k}\},$$

as the natural basis of the M_k -dimensional eigenspace, with $M_k \geq 1$, such that

$$f_k = \sum_{|\beta|=k} c_\beta f_\beta \quad \text{and} \quad V_k = \sum_{|\beta|>k} c_\beta f_\beta,$$

with $V_k \in Y_k$ where Y_k is the complementary invariant subspace of corresponding kernel. In particular, we consider the expansion

$$V_k := n\Phi_{1,k} + o(n).$$

Thus, applying the Fredholm alternative, we obtain the existence of a number of branches emanating from the solutions (α_k, f_k) at the value of the parameter $n = 0$. We can guarantee that the first profile is unique but for the rest there could be more than branch of solutions emanating at $n = 0$. The number of branches will depend on the dimension of the eigenspace so that the dimension is somehow involved; see [1] for any further comments and a detailed analysis of a similar branching analysis.

Remark. Furthermore, when $n \rightarrow 0^+$ we have a few different profiles of $f(y)$. For the PME-2 (1.2) the Graveleau profiles are always unique by the Maximum Principle but not for the TFE-4 (1.1). However, for $\alpha = \frac{1}{2}$ the only profile is $f(y) = y^2$.

The previous discussion can be summarised in the following lemma.

Lemma 5.1. *The patterns occurring for equation (4.3), i.e. the ones for (2.3)₋ for $n = 0$, act as branching points of nonlinear eigenfunctions of the Cauchy problem for the operator (2.3)₋, at least when the parameter n is sufficiently close to zero.*

Remark. It turns out that using classical branching theory “nonlinear eigenfunctions” $f(y)$ of changing sign, which satisfies the *nonlinear eigenvalue problem* (2.3)₋ (with an extra “radiation-minimal-like” condition at infinity) at least, for sufficiently small $n > 0$, can be connected with eigenfunctions f_k of the linear problem (4.3).

6. Non-improvable regularity for the TFE-4

The following main result is a straightforward consequence of our focusing self-similar analysis.

Theorem 6.1. *Let, for a fixed $n > 0$, the nonlinear eigenvalue problem (2.3)₋ have a nontrivial solution (eigenfunction) $f_k(y)$ for some eigenvalue $\alpha_k > 0$, i.e. there exists a self-similar focusing solution of the problem (2.2)₋ exhibiting the finite-time trace (3.9). Then:*

- (i) *For the general Cauchy problem for the TFE-4 (1.1), even in the radial setting, the Hölder continuity exponent of solutions cannot exceed³*

$$\mu_k(n, N) = \frac{\alpha_k}{\beta_k} \equiv \frac{4\alpha_k}{1+n\alpha_k}. \tag{6.1}$$

- (ii) *Let there exist $\mu_k < 1$, i.e. $\alpha_k < \frac{1}{4-n}$, for $n \in (0, 2)$.⁴ Then, for the TFE-4 (1.1),*

$$\nabla u_*(r, 0^-) \in L^p_{\text{loc}}(\mathbb{R}^N) \quad \text{iff} \quad p \in [1, p_*), \quad p_*(n, N) = \frac{N}{1-\mu_k}, \tag{6.2}$$

³ We mean $C^{l+\varepsilon}$, if $\mu_k \geq 1$, that, not that surprisingly, happens for all small $n > 0$.

⁴ Actually, for smaller n 's; note that for every larger n 's solutions of the TFE-4 are known to be strictly positive [7].

so that, for (even radial) solutions of (1.1), in general, for any $t > 0$,

$$\nabla u(x, t) \notin L_{\text{loc}}^{p_*}(\mathbb{R}^N). \tag{6.3}$$

Remark. Basically $|\nabla u_*(r, 0^-)| \in L_{\text{loc}}^p(\mathbb{R}^N)$ if and only if

$$\int_{\mathbb{R}^N} |\nabla u_*(r, 0^-)|^p = C \int_0^1 r^{N-1} (r^{\mu_k-1})^p < \infty.$$

Thus, since $0 < \mu_* < 1$ we find that to have that integral bounded we need that, evaluating the coefficients in the second integral

$$p < p_* \equiv p_*(n, N) = \frac{N}{1-\mu_k(n, N)}.$$

Remark. Of course, it can happen that $\mu_k > 1$, so focusing does not supply us with a truly Hölder continuous focusing trace but rather than a $C^{l+\varepsilon}$ -trace. Actually, by continuity in n , exactly this happens for $n > 0$ small, where $\mu_1(0, N) = 2$.

We expect that the minimal value of μ_k in (6.1) is attained at $k = 1$, but cannot prove this for all $n > 0$ (for small ones, this is obvious). Note again that, for all large $n \geq 2$, such a focusing is not possible in principle, since the solutions must remain strictly positive for all times [7].

Thus, this optimal (non-improvable) regularity result for the TFE-4 (and, seems for many other parabolic equations) depends on the solvability of the nonlinear eigenfunction focusing problem.

Note that in the previous section we showed a homotopy connection between that nonlinear eigenvalue problem and the linear problem at $n = 0$ (4.3). For those reasons the analysis performed in those two previous sections is crucial in ascertaining qualitative information about the solutions of the TFE-4.

Furthermore, we can also ascertain the Hölder continuity with respect to the temporal variable t . This fact comes directly from the non-improvable Hölder’s exponent we have already obtained above. Indeed, assuming the radial self-similar solutions of the form (1.5) for the focusing, one easily finds that

$$|u(0, t) - u(0, 0)| = (-t)^\alpha f(0).$$

Hence, the t -Hölder’s exponent close to $t = 0$ cannot be bigger than α .

Therefore, the focusing solution given here provides us with optimal Hölder estimations for the variables x and t .

7. Oscillatory structure of maximal solutions for $n > 0$

Let us now consider $n \in (0, 2)$. For the ODE (2.3), we will try the same *ansatz* as in [13]. Namely, we will find solutions of the maximal type, with the envelope as in (3.3). We introduce a corresponding *oscillatory* component as follows:

$$f(y) = y^\mu \varphi(s), \quad s = \ln y, \quad \mu = \frac{4}{n} \quad \text{for } y \gg 1. \tag{7.1}$$

Then,

$$\begin{aligned} f' &= y^{\mu-1}(\dot{\varphi} + \mu\varphi), & f'' &= y^{\mu-2}[\ddot{\varphi} + (\mu-1)\dot{\varphi} + \mu(\mu-1)\varphi], \\ f''' &= y^{\mu-3} [\dddot{\varphi} + 3(\mu-1)\ddot{\varphi} + (3\mu^2 - 6\mu + 2)\dot{\varphi} + \mu(\mu^2 - 3\mu + 2)\varphi], \end{aligned}$$

where $' = d/dy$ and $\dot{} = d/ds$. Substituting these expressions into (2.3) yields the following fourth-order homogeneous ODE for $\varphi(s)$

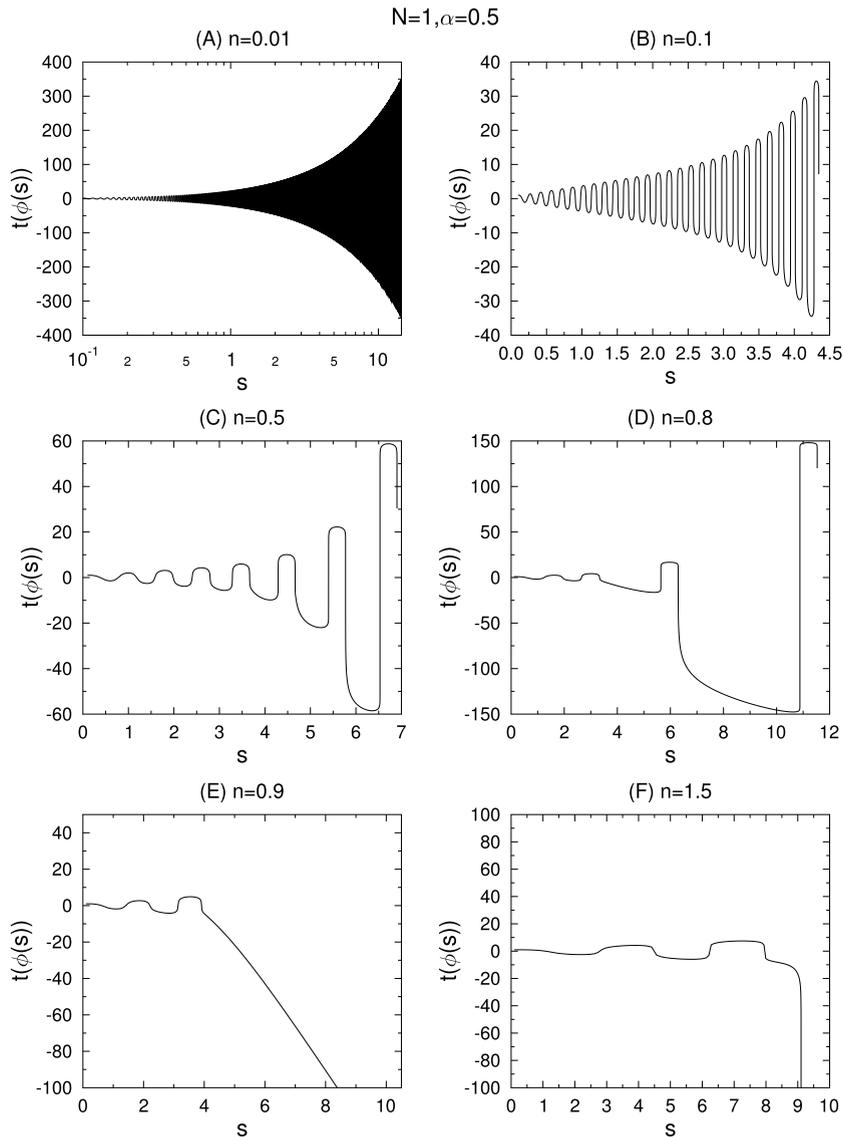


Fig. 3. Illustrative numerical profiles of transformed $\phi(s)$ via (7.3). Shown is the parameter case $N = 1, \alpha = 0.5$ for selected n .

$$\begin{aligned}
 & \ddot{\varphi} + 2(N - 4 + 2\mu)\ddot{\varphi} + (6\mu^2 + 6(N - 4)\mu + 11 + (N - 1)(N - 9))\dot{\varphi} \\
 & + 2(2\mu + N - 4)(\mu^2 + (N - 4)\mu + 2 - N)\dot{\varphi} + \mu(\mu - 2)(\mu^2 + 2(N - 3)\mu + 3 + (N - 1)(N - 5))\varphi \\
 & + n \left(\frac{\dot{\varphi}}{\varphi} + \mu \right) \left[\ddot{\varphi} + (N - 4 + 3\mu)\dot{\varphi} + (3\mu^2 + 2(N - 4)\mu + 4 - 2N)\dot{\varphi} + \mu(\mu - 2)(N - 2 + \mu)\varphi \right] \\
 & + \left(\frac{1}{4}(1 + n\alpha)(\dot{\varphi} + \mu\varphi) - \alpha\varphi \right) |\varphi|^{-n} = 0.
 \end{aligned}
 \tag{7.2}$$

We mention that this equation is autonomous and thus may be reduced to third-order, although we do not utilise this reduction here. Figs. 3 and 4 illustrate the periodic nature of the solutions for ϕ , at least for n small enough. Since the oscillations occur over such a large range, we use the following transformation to allow the oscillations to be visible on the plots,

$$t(\phi(s)) = \begin{cases} \ln \phi(s) + 1, & \text{if } \phi(s) > 1, \\ \phi(s), & \text{if } -1 < \phi(s) < 1, \\ -\ln(-\phi(s)) - 1, & \text{if } \phi(s) < -1. \end{cases}
 \tag{7.3}$$

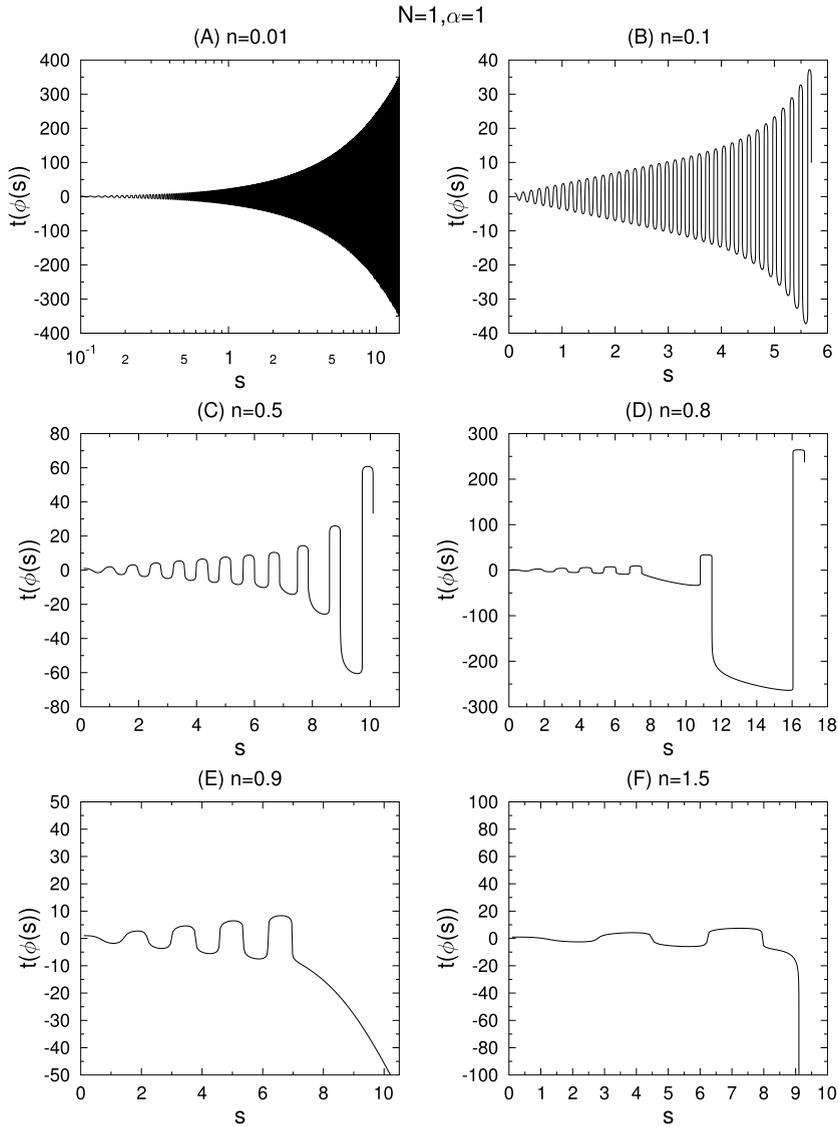


Fig. 4. Illustrative numerical profiles of transformed $\phi(s)$ via (7.3). Shown is the parameter case $N = 1, \alpha = 1$ for selected n .

Plotted are numerical solutions for ϕ in the two cases $\alpha = 0.5, 1$ with $N = 1$. The subplots illustrate the change in the profile behaviour with n . The numerics support the proposition that the family (7.1) is 2D, composed of:

- (i) a 1D stable manifold $\varphi_*(s)$,
- and
- (ii) a phase shift $\varphi_*(s + s_0)$ for any $s_0 \in \mathbb{R}$.

As we have seen earlier, these two properties are true for $n = 0$, and, hence, by continuity, we conjecture remain true for small $n > 0$.

However, the periodic exponential structure, with the linear ODE for $n = 0$ is replaced by a more difficult one (7.1) for $n > 0$. Nevertheless, the periodic nature of such a behaviour appears universal, and is expected to remain for larger n . The numerics though do suggest that it is lost (in a homoclinic–heteroclinic bifurcation) for n near to 0.8 or 0.9.

To reconcile with the known behaviour in the linear case, we now study the behaviour of the periodic solutions for small $n > 0$. To reveal the limiting oscillatory behaviour as $n \rightarrow 0$, we keep the leading terms of the coefficients in (7.2) as $\mu \rightarrow \infty$ to obtain

$$\ddot{\varphi} + 4\mu\ddot{\varphi} + 6\mu^2\ddot{\varphi} + 4\mu^3\dot{\varphi} + \mu^4\varphi + \frac{1}{4}(\dot{\varphi} + \mu\varphi)|\varphi|^{-n} = 0. \tag{7.4}$$

We now rescale as follows

$$\varphi(s) = \mu^{-\frac{3}{n}}\hat{\varphi}(\hat{s}), \quad s = \frac{\hat{s}}{\mu}, \tag{7.5}$$

leading to

$$\hat{\varphi}'''' + 4\hat{\varphi}''' + 6\hat{\varphi}'' + 4\hat{\varphi}' + \hat{\varphi} + \frac{1}{4}(\hat{\varphi}' + \hat{\varphi})|\hat{\varphi}|^{-n} = 0, \tag{7.6}$$

where here $\dot{}$ denotes $d/d\hat{s}$. For $n = 0$, the equation (7.6) becomes linear,

$$\hat{\varphi}'''' + 4\hat{\varphi}''' + 6\hat{\varphi}'' + 4\hat{\varphi}' + \hat{\varphi} + \frac{1}{4}(\hat{\varphi}' + \hat{\varphi}) = 0, \tag{7.7}$$

with the characteristic equation

$$(m + 1) \left((m + 1)^3 + \frac{1}{4} \right) = 0,$$

for exponential solutions $\hat{\varphi}(\hat{s}) = e^{m\hat{s}}$. The roots of the characteristic equation are linked to those of the controlling factor in (4.4) being

$$m + 1 = \frac{4}{3}a_1, \quad \frac{4}{3}a_2, \quad \frac{4}{3}a_3 \quad \text{and} \quad 0.$$

Consequently we obtain the dominant asymptotic behaviour

$$\hat{\varphi}(\hat{s}) \sim e^{(-1+\frac{4}{3}c_0)\hat{s}} \left[\hat{A}_1 \cos\left(\frac{4}{3}c_1\hat{s}\right) + \hat{A}_2 \cos\left(\frac{4}{3}c_1\hat{s}\right) \right] \quad \text{as} \quad \hat{s} \rightarrow \infty, \tag{7.8}$$

for arbitrary constants $\hat{A}_{1,2}$. Denoting the independent variable by \hat{y} rather than y for convenience, we thus have the small n behaviour

$$f(\hat{y}) \sim \hat{y}^{\frac{4}{n}} \left(\frac{4}{n}\right)^{-\frac{3}{n}} \hat{\varphi}\left(\frac{4}{n} \ln(\hat{y})\right). \tag{7.9}$$

This may be reconciled with the expression in (4.7) through the identifications

$$\hat{y} = e^{\frac{3n}{16}y^{4/3}}, \quad \hat{A}_{1,2} = \left(\frac{4}{n}\right)^{\frac{3}{n}} C_{1,2}.$$

8. Nonlinear eigenfunctions by shooting

The eigenfunctions for $n > 0$ may be obtained via shooting. The conditions (4.12) or (4.13) are again used as initial conditions and α determined by capturing the growth (3.8) for sufficiently large y values. This growth behaviour may be imposed by

$$\text{minimising} \quad \mu y f' - \alpha f \quad \text{at} \quad y \quad \text{values,}$$

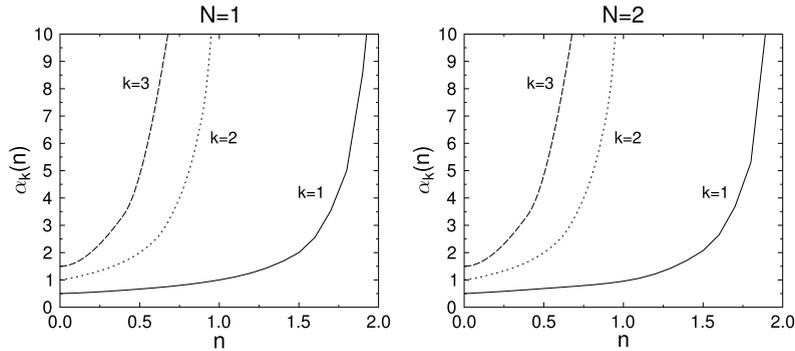


Fig. 5. Numerical determination of the eigenvalues $\alpha_k(n)$ for the first three branches $k = 1, 2, 3$ of eigenfunctions.

typically chosen to be around 40. Standard regularisation of the term $|f(y)|^n$ is required in the form $(f^2 + \delta^2)^{\frac{n}{2}}$ with δ taken relatively small.

Fig. 5 shows the eigenvalues of the first three branches $k = 1, 2, 3$. These numerically extend the $n = 0$ eigenvalues of Proposition 4.1 to $n > 0$.

As in the $n = 0$ linear case, the eigenvalues remain the same irrespective of the spatial dimension N , at least to the accuracy of the numerical calculations that were performed.

Furthermore, it is worth remarking that along each branch i.e. fixed k , the exponent μ of the far-field behaviour of the eigenfunctions

$$f \sim Cy^\mu,$$

remains quite close to its value in the linear case i.e.,

$$\mu(n) \approx \mu(0).$$

Thus, as an approximation we have

$$\alpha_k(n) \approx \alpha_k(0)/(1 - \alpha_k(0)n),$$

which seems to be a reasonable approximation at least

$$\text{for } n < 1/\alpha_k(0).$$

The corresponding eigenfunctions are thus similar to those in the linear case.

An alternative approach to determining the eigenfunctions in both the linear $n = 0$ and nonlinear $n > 0$ cases, is through minimisation of the oscillatory maximal profile to obtain the non-oscillatory minimal profile for $y \gg 1$.

In principle we have two parameters α and, say $\nu = f''(0)$ or $f(0)$, depending on which branch of eigenfunctions is considered. In the linear case, using (4.7), we are required to satisfy two algebraic equations with analytic functions:

$$\begin{cases} C_1(\alpha, \nu) = 0, \\ C_2(\alpha, \nu) = 0. \end{cases} \tag{8.1}$$

Therefore, we arrive at a well-posed 2 – 2 shooting problem, which cannot have more than a countable pairs of solutions (as mentioned already). This approach is practical for the linear $n = 0$ case, as the two parameter form of the maximal bundle is known explicitly as given in (4.7) and was essentially pursued in

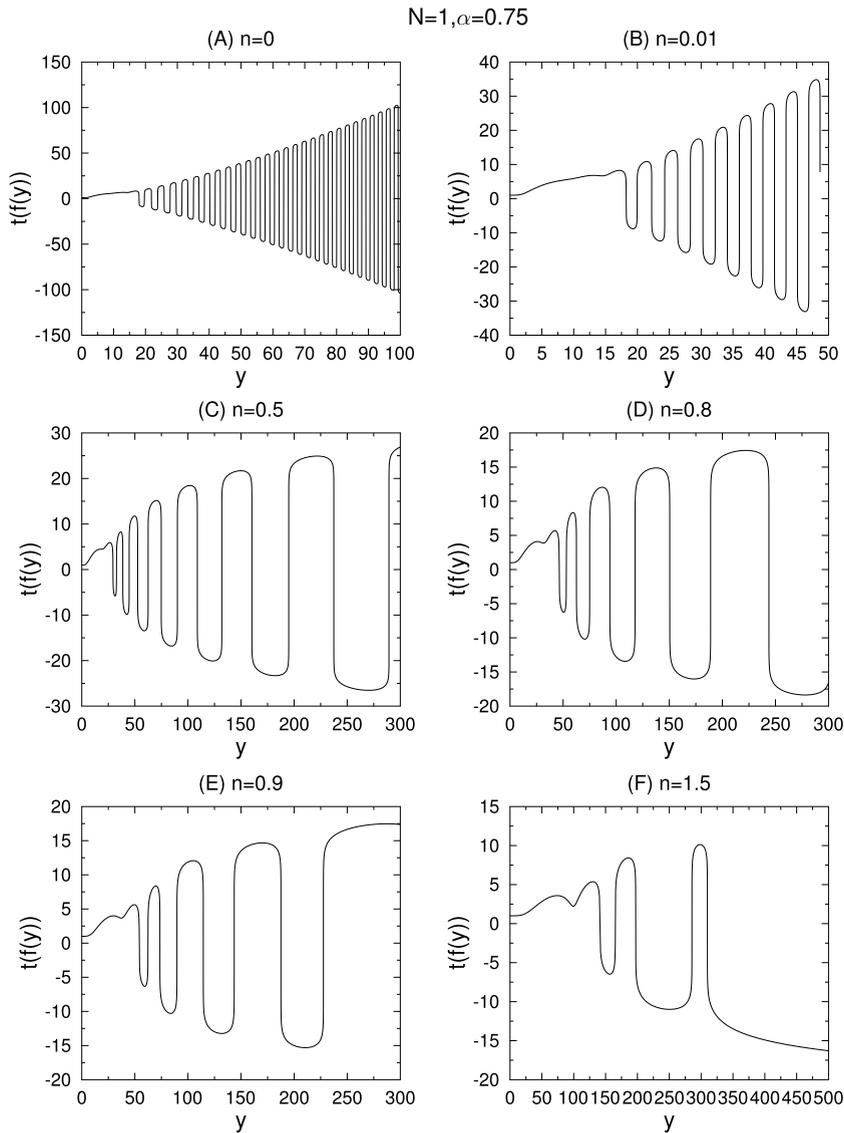


Fig. 6. Illustrative numerical profiles of transformed $\phi(s)$ via (7.3). Shown is the parameter case $N = 1, \alpha = 1$ for selected n .

Section 4.2. However, lacking such an explicit expression for $n > 0$, limits this approach in the nonlinear case.

Finally it is worth presenting some numerical experiments showing the behaviour of the maximal profiles. Fig. 6 shows profiles in the case $N = 1, \alpha = 0.75$ for selected n . Since the oscillatory profiles occur over such large ranges, the transformed profile using (7.3) is depicted.

The figures suggest the loss of a pure oscillatory structure as n increases, which is compatible with the behaviour seen for ϕ in Figs. 3 and 4.

We conjecture that this is highly suggestive of a homoclinic–heteroclinic bifurcation. The precise values of n though at which this occurs is difficult to determine with sufficient accuracy.

9. After-focusing self-similar extension

As usual and as we have observed in many similar problems on a (unique) extension of a solution after blow-up, this is much easier. In fact, after focusing at $t = 0^-$, we arrive at a well-posed CP (or TFE, that

does not matter since $u_0(r) > 0$, no interfaces are available) for the TFE-4 (1.1) with initial data (3.9) (with $\mu = \mu_k$ when possible), already satisfying the necessary minimal growth at infinity.

Therefore, there exists a self-similar solution (2.1)₊ with $f(y)$ satisfying the ODE (2.3)₊, with already fixed “eigenvalue” $\alpha = \alpha_k$. Because of changing the signs in front of the linear terms, this changes the dimension of the stable/unstable manifolds as $y \rightarrow +\infty$ (in particular, the unstable manifold becomes 1D, similar to $n = 0$).

To explain the latter, as usual consider the case $n = 0$ (and hence small $n > 0$). Then, with the change of sign in the two linear terms in (4.3), we arrive at a different characteristic equation for a :

$$f(y) \sim e^{ay^\gamma} \quad \text{as } y \rightarrow +\infty \implies a^3 = +\frac{1}{4} \left(\frac{3}{4}\right)^3. \tag{9.1}$$

This characteristic equation gives two stable roots with $\text{Re}(\cdot) < 0$:

$$b_{1,2} = \frac{3}{4} 4^{-\frac{1}{3}} \left[-\frac{1}{2} \pm i \frac{\sqrt{3}}{2} \right] \equiv c_0 \pm i c_1 \tag{9.2}$$

and one positive root maximal solution (see below)

$$b_3 = \frac{3}{4} 4^{-\frac{1}{3}} > 0.$$

Hence, the bundle of maximal solutions is oscillatory as $y \rightarrow +\infty$ is just 1D, so we arrive at an *undetermined* problem for ν, α , which satisfy just a single algebraic equation (unlike two in (8.1)). Indeed, this makes the α -spectrum continuous and solvability for any $\alpha > 0$.

This allows us to get a unique extension profile $f(y)$ for such an *a priori* fixed eigenvalue α_k . In other words, the focusing extension problem is not an eigenvalue one, since a proper α_k has been fixed by the previous focusing blow-up evolution.

It might be said, using standard terminology from linear operator theory, that for the sign + the spectrum of this problem becomes continuous, unlike the discrete one for -. Therefore, we do not study this much easier problem anymore, especially, since it has nothing to do with our goal: to detect a non-improvable regularity for the TFE-4 via blow-up focusing.

Appendix A. The limit $n \rightarrow 0$ for maximal solutions

We consider here the singular limit $n \rightarrow 0^+$ for the equation in (3.7), written explicitly here as

$$\frac{1}{y^{N-1}} \left[y^{N-1} |f|^n \left(\frac{1}{y^{N-1}} (y^{N-1} f')' \right)' \right]' + \frac{1}{4} (1 + \alpha n) y f' - \alpha f = 0.$$

The non-uniform solution in this limit comprises two regions, an *Inner region* where $1 \ll y < O(n^{-3/4})$ with $|f|^n \sim 1$, and an *Outer region* $y = O(n^{-3/4})$ where $\ln |f| = O(1/n)$. The labelling of these regions as inner and outer becomes apparent during the course of the scalings.

In the inner region $1 \ll y < O(n^{-3/4})$, we obtain at leading order in n the linear ODE (4.3). Posing $f \sim f_0$ with $|f_0|^n \sim 1$, we obtain

$$\frac{1}{y^{N-1}} \left[y^{N-1} \left(\frac{1}{y^{N-1}} (y^{N-1} f_0')' \right)' \right]' + \frac{1}{4} (1 + \alpha n) y f_0' - \alpha f_0 = 0. \tag{A.1}$$

The far-field behaviour of (A.1) may be determined using a WKB expansion in the form

$$f_0(y) \sim A(y) e^{\phi(y)} \quad \text{as } y \rightarrow +\infty \tag{A.2}$$

which gives

$$(\phi')^3 = \frac{1}{4}y, \quad 3yA' + 2(N + 2\alpha - 1 - 12(\phi')^2\phi'')A = 0. \tag{A.3}$$

The required solutions to (A.3) take the form

$$\phi(y) = ay^{4/3}, \quad A(y) = ky^{-\frac{2}{3}(N+2\alpha)}, \tag{A.4}$$

where a satisfies the cubic equation in (4.4) and k an arbitrary constant. Thus, the dominant behaviour for large y is

$$f_0(y) \sim y^{-\frac{2}{3}(N+2\alpha)} \left(k_1 \exp \{ a_1 y^{\frac{4}{3}} \} + k_2 \exp \{ a_2 y^{\frac{4}{3}} \} \right), \tag{A.5}$$

with $a_{1,2}$ as given in (4.5) and $k_{1,2}$ complex constants chosen so that the expression is real as stated in (4.7).

This inner solution breaks down when $y = O(n^{-3/4})$, where $\ln |f_0| = O(1/n)$. This suggests the consideration of an outer region with scaling $Y = n^{-\frac{3}{4}}y$. In $Y = O(1)$, we have

$$\frac{n^3}{Y^{N-1}} \frac{d}{dY} \left[Y^{N-1} |f|^n \frac{d}{dY} \left(\frac{1}{Y^{N-1}} \frac{d}{dY} \left(Y^{N-1} \frac{df}{dY} \right) \right) \right] + \frac{1}{4}(1 + \alpha n)Y \frac{df}{dY} - \alpha f = 0. \tag{A.6}$$

Rather than posing a multiple-scales ansatz directly, it is more convenient to work in complex form and we may instead consider

$$f(Y) \sim e^{b(Y)/n} B(Y) \quad \text{as } n \rightarrow 0, \tag{A.7}$$

where b is complex in order to match with the inner solution expression (A.5). Thus, at $O(1/n)$ in (A.6) we obtain

$$|e^b| (b')^3 + \frac{1}{4}Y = 0, \tag{A.8}$$

whilst at $O(1)$ we have

$$3\frac{B'}{B} + \left(1 - \frac{\alpha}{4} + \ln |B|\right) b' + 6\frac{b''}{b'} + \frac{2((N-1)+2\alpha)}{Y} = 0, \tag{A.9}$$

where $'$ denotes $\frac{d}{dY}$ and the approximation

$$|f|^n \sim |e^b| (1 + n \ln |B|) \tag{A.10}$$

has been used. The solution to (A.8) that matches with (A.2) and (A.4) is

$$b(Y) = a \frac{3}{c_0} \ln \left(1 + \frac{c_0}{3} Y^{4/3} \right). \tag{A.11}$$

The amplitude $B(Y)$ is determined from (A.9) using the solution (A.11). We obtain

$$\ln |B| = \frac{1}{(1 + \frac{c_0}{3} Y^{4/3})} \left(|k| - \frac{2}{3}(N + 2\alpha) \ln Y - \frac{c_0}{12}(2N + 3\alpha - 4)Y^{4/3} \right),$$

after matching to (A.2) with (A.4).

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