



A Cheeger finiteness theorem for Finsler manifolds



Wei Zhao^{a,*}, Yibing Shen^b

^a Department of Mathematics, East China University of Science and Technology, Shanghai 200237, PR China

^b School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, PR China

ARTICLE INFO

Article history:

Received 17 May 2015

Available online 1 September 2015

Submitted by H.R. Parks

Keywords:

Finsler manifold

Finiteness theorem

Injectivity radius

Center of mass

ABSTRACT

In this paper, we establish a Cheeger finiteness theorem for Berwald manifolds. Moreover, a Cheeger type estimate on injectivity radii for Finsler manifolds is obtained and the existence of the center of mass of a Berwald manifold is proved.

© 2015 Elsevier Inc. All rights reserved.

1. Introduction

Finiteness theorems are theorems giving bounds on certain geometrical quantities such that the family of manifolds admitting metrics which satisfy the bounds is finite up to homotopy equivalence, or homeomorphism, or diffeomorphism. A famous finiteness result in Riemannian geometry is Cheeger's finiteness theorem [7,10,14], which says that given $n \in \mathbb{N}$ and $k, D, V > 0$, there are only finitely many diffeomorphism types among compact Riemannian n -manifolds (M^n, g) satisfying $|\mathbf{K}_M| \leq k$, $\text{diam}(M) \leq D$ and $\text{Vol}(M) \geq V$. Note that according to [10] and [7, Theorem 2.1], the assumptions of Cheeger's theorem eliminate the collapsing case. Also refer to [1,6,15] for more details.

Finsler metrics are just Riemannian metrics without quadratic restriction. It is a natural problem that whether an analogue of Cheeger's theorem still holds in the Finslerian case. A Finsler metric F on a manifold M is called Berwaldian if there is a linear Berwald connection such that all the tangent spaces of M are linearly isometric to a common Minkowski space. Berwald metrics are a kind of important Finsler metrics, which are more general than Riemannian and locally Minkowskian metrics and therefore can be viewed as a median between Riemannian metrics and general Finsler metrics (cf. [22,3]). It is also noticeable that there are infinitely many Berwald metrics which are neither Riemannian nor locally Minkowskian.

* Corresponding author.

E-mail addresses: szhao_wei@yahoo.com (W. Zhao), yibingshen@zju.edu.cn (Y. Shen).

Example 1. (See [3].) Set $M = \mathbb{S}^2 \times \mathbb{S}^1$. Let α be the canonical Riemannian product metric on M and let t be the usual spherical coordinate of \mathbb{S} . Then

$$F_\epsilon := \alpha + \epsilon dt, \text{ for } \epsilon \in (0, 1),$$

is a family of Berwald(–Randers) metrics globally defined on M with non-negative flag curvature.

Example 2. (See [8].) Given an arbitrary smooth function $f : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ with $f(\lambda s, \lambda t) = \lambda f(s, t)$ for all $\lambda > 0$, let (M_i, g_i) , $i = 1, 2$ be two arbitrary Riemannian manifolds. Then

$$F(x, y) := \sqrt{f[g_1(x_1, y_1), g_2(x_2, y_2)]}$$

is a Berwald metric on $M_1 \times M_2$, where $x = (x_1, x_2) \in M$ and $y = y_1 \oplus y_2 \in T_x(M_1 \times M_2) \cong T_{x_1}M_1 \oplus T_{x_2}M_2$.

More generally, Z. Szabó in [22,23] points out that any Berwald metric is a perturbed-Cartesian product of Riemannian, Minkowski and such non-Riemannian metrics which can be constructed on irreducible symmetric manifolds of rank > 1 . Refer to [3,8,19,22,23,27] for more interesting examples. We also remark that there is a long existing problem concerning Berwald metrics, that is, whether or not any Landsberg metric is a Berwald metric. Recently, Ben-Shen [4], Z. Shen [20] and Z. Szabó [24,25] have given affirmative answers under additional assumptions, but the problem is still open in the general case. Hence, the class of Berwald metrics is a large and interesting class, and the study on Berwald manifolds will enhance our understanding of Finsler geometry. The main purpose of this paper is to study the finiteness problem above in the Berwaldian case.

In [22], Z. Szabó finds an important relationship between Berwald metrics and Riemannian metrics, that is, there always exists a Riemannian metric such that its Levi-Civita connection coincides the Chern connection of a given Berwald manifold. For this reason, the curvature properties of a Berwald metric are similar to ones of a Riemannian metric (cf. [3,19]). Even so, geometric-topological properties of Berwald manifolds are much different from those of Riemannian manifolds. For example, a compact manifold M with a family of Riemannian metrics $\{g_m\}$ such that all (M, g_m) satisfy the assumptions of Cheeger’s theorem cannot collapse. However, this is not true for Berwald metrics.

Example 3. Define a sequence of Berwald metrics on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ by

$$F_m := \alpha + \left(1 - \frac{1}{m}\right) \beta, \quad m \geq 1,$$

where α is the canonical Riemannian product metric on \mathbb{T}^2 , and β is a parallel 1-form on \mathbb{T}^2 with $\|\beta\|_\alpha = 1$. Then $\{(\mathbb{T}^2, F_m)\}_m$ satisfy

$$\mathbf{K}_m = 0, \quad \text{diam}_m(\mathbb{T}^2) \leq 2(\sqrt{2} + 1)\pi, \quad \mu_m(\mathbb{T}^2) = 4\pi^2,$$

where μ_m is the Holmes–Thompson volume of (\mathbb{T}^2, F_m) . However, the injectivity radius $i_m(\mathbb{T}^2) \rightarrow 0$ as $m \rightarrow +\infty$.

Before explaining the reason of the collapsing above, we introduce the uniformity constant Λ_F of a Finsler manifold (M, F) , which is a non-Riemannian geometric quantity (cf. [9]). The uniformity constant is a scalar, which is defined by

$$\Lambda_F := \sup_{X, Y, Z \in SM} \frac{g_X(Y, Y)}{g_Z(Y, Y)},$$

where g is the fundamental tensor of F . Clearly, $\Lambda_F \geq 1$ with equality if and only if F is Riemannian. Roughly speaking, the injectivity radius is inversely proportional to the uniformity constant. And a direct calculation yields that the uniformity constants Λ_m in [Example 1](#) satisfy

$$\Lambda_m \geq (2m-1)^2 \rightarrow \infty, \text{ as } m \rightarrow \infty,$$

which makes (\mathbb{T}^2, F_m) collapse. On the other hand, if a compact n -dimensional manifold with a family of Berwald metrics $\{F_m\}$ satisfies $\Lambda_m \leq \Lambda$, $|\mathbf{K}_m| \leq k$, $\text{diam}_m(M) \leq D$ and $\mu_m(M) \geq V$, then (M, F_m) cannot collapse. In fact, we shall establish the following Cheeger type estimate on the injectivity radius

$$i_m \geq \frac{1}{1 + \Lambda^{\frac{1}{2}}} \min \left\{ \frac{(1 + \Lambda^{-\frac{1}{2}})\pi}{\sqrt{k}}, \frac{(n-1)V}{c_{n-2}\Lambda^{\frac{3n}{2}}\mathfrak{s}_{-k}^{n-1}(D)} \right\}.$$

See [Theorem 3.4](#) below for an estimate on general Finsler manifolds.

Inspired by the observation above, we investigate the diffeomorphism types of compact Berwald n -manifolds with a uniform upper bound on the uniformity constants and the assumptions of Cheeger's theorem. Then we shall establish the following

Theorem 1.1. *Given $n \in \mathbb{N}$, $\Lambda \geq 1$, $k \geq 0$ and $V, D > 0$, there exist only finitely many diffeomorphism classes of compact Berwald n -manifolds (M, F) satisfying*

$$\Lambda_F \leq \Lambda, \quad |\mathbf{K}_M| \leq k, \quad \mu(M) \geq V, \quad \text{diam}(M) \leq D,$$

where $\mu(M)$ is either the Busemann–Hausdorff volume or the Holmes–Thompson volume of M .

It is easy to see that [Theorem 1.1](#) implies the original Cheeger's theorem. Our main tool is a generalized Peter's lemma (see [Section 6](#) below). It should be remarked that with additional assumptions on T-curvature and Riemannian curvature, our method works for general Finsler metrics. This will be discussed somewhere else. We also remark that Gromov in [\[10\]](#) obtains Cheeger's theorem by a remarkable theorem, which is called Gromov's convergence theorem in [\[12\]](#). However, this theorem cannot be extended to the Finsler setting (even to the Berwald setting), because it requires the uniformity constant to be almost equal to 1. Refer to [\[19,21,26,28\]](#) for other finiteness theorems in Finsler geometry.

The arrangement of contents of this paper is as follows. In [Section 2](#), we brief some necessary definitions and properties concerned with Finsler geometry. In [Section 3](#), the Finslerian versions of Klingenberg's theorem and Cheeger's estimate on injectivity radii are established. In [Section 4](#), we estimate the convex radius and study the center of mass of a Berwald manifold. [Theorem 1.1](#) is proved in [Section 5](#) by a generalized Peter's lemma, and the latter is proved in [Section 6](#). In [Appendix A](#), we give some estimates for Jacobi fields on Finsler manifolds. In [Appendix B](#), we study the parallel transformations on a Berwald manifold.

2. Preliminaries

In this section, we recall some definitions and properties about Finsler manifolds. See [\[3,19\]](#) for more details.

Let (M, F) be a (connected) Finsler n -manifold with Finsler metric $F : TM \rightarrow [0, \infty)$. Define $S_x M := \{y \in T_x M : F(x, y) = 1\}$ and $SM := \cup_{x \in M} S_x M$. Let $(x, y) = (x^i, y^i)$ be local coordinates on TM . Define

$$\begin{aligned} \ell^i &:= \frac{y^i}{F}, \quad g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, & A_{ijk}(x, y) &:= \frac{F}{4} \frac{\partial^3 F^2(x, y)}{\partial y^i \partial y^j \partial y^k}, \\ \gamma_{jk}^i &:= \frac{1}{2} g^{il} \left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right), & N_j^i &:= (\gamma_{jk}^i \ell^j - A_{jk}^i \gamma_{rs}^k \ell^r \ell^s) \cdot F. \end{aligned}$$

The Chern connection ∇ is defined on the pulled-back bundle π^*TM and its forms are characterized by the following structure equations:

- (1) Torsion freeness: $dx^j \wedge \omega_j^i = 0$;
- (2) Almost g -compatibility: $dg_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_j^k = 2\frac{A_{ijk}}{F}(dy^k + N_l^k dx^l)$.

From above, it's easy to obtain $\omega_j^i = \Gamma_{jk}^i dx^k$, and $\Gamma_{jk}^i = \Gamma_{kj}^i$. It should be remarked that $\Gamma_{kj}^i = \Gamma_{kj}^i(x, y)$ is a local smooth function on SM . In particular, F is called a Berwald metric if $\partial\Gamma_{kj}^i/\partial y^s = 0$.

The curvature form of the Chern connection is defined as

$$\Omega_j^i := d\omega_j^i - \omega_j^k \wedge \omega_k^i =: \frac{1}{2}R_{jk}^i dx^k \wedge dx^l + P_{jk}^i dx^k \wedge \frac{dy^l + N_s^l dx^s}{F}.$$

Given a non-zero vector $V \in T_x M$, the flag curvature $K(y, V)$ on $(x, y) \in TM \setminus 0$ is defined as

$$\mathbf{K}(y, V) := \frac{V^i y^j R_{jikt} y^l V^k}{g_y(y, y)g_y(V, V) - [g_y(y, V)]^2},$$

where $R_{jikt} := g_{is}R_{jk}^s$.

The reversibility λ_F and the uniformity constant Λ_F of (M, F) are defined as

$$\lambda_F := \sup_{X \in SM} \frac{F(-X)}{F(X)}, \quad \Lambda_F := \sup_{X, Y, Z \in SM} \frac{g_X(Y, Y)}{g_Z(Y, Y)}.$$

Clearly, $\lambda_F \geq 1$ with equality if and only if F is reversible, and $\Lambda_F \geq 1$ with equality if and only if F is Riemannian. In particular, $\lambda_F \leq \sqrt{\Lambda_F}$.

The Legendre transformation $\mathcal{L} : TM \rightarrow T^*M$ is defined by

$$\mathcal{L}(Y) = \begin{cases} 0, & Y = 0, \\ g_Y(Y, \cdot), & Y \neq 0. \end{cases}$$

For each $x \in M$, the Legendre transformation is a smooth diffeomorphism from $T_x M \setminus \{0\}$ to $T_x^* M \setminus \{0\}$.

The average Riemannian metric \tilde{g} induced by F is defined by

$$\tilde{g}(X, Y) = \frac{1}{\text{Vol}(x)} \int_{S_x M} g_y(X, Y) d\nu_x(y), \quad \forall X, Y \in T_x M,$$

where $\text{Vol}(x) = \int_{S_x M} d\nu_x(y)$, and $d\nu_x$ is the Riemannian volume form of $S_x M$ induced by F .

3. The injectivity radius of a Finsler manifold

Let (M, F) be a compact Finsler manifold. There exist two points p and q such that

$$i_M = d(p, q) = d(p, \text{Cut}_p).$$

Let $\gamma_y(t)$, $0 \leq t \leq d(p, q)$ be a normal minimal geodesic from p to q . Then q is the cut point of p along γ_y . If q is not the first conjugate point of p along γ_y , [3, Proposition 8.2.1] implies that there exists another distinct normal minimal geodesic $\gamma_w(t)$, $0 \leq t \leq d(p, q)$ from p to q . If F is reversible, [19, Lemma 12.2.5] yields $\dot{\gamma}_y(d(p, q)) = -\dot{\gamma}_w(d(p, q))$ and hence, Klingenberg's theorem [13] still holds in the reversible Finsler case. If F is nonreversible, then the first variation formula [3, p. 123] together with [17, Lemma 9.3] yields

that $\mathcal{L}_{\dot{\gamma}_y}(d(p, q)) = -\mathcal{L}_{\dot{\gamma}_w}(d(p, q))$. However, one cannot deduce $\dot{\gamma}_y(d(p, q)) = -\dot{\gamma}_w(d(p, q))$, since the Legendre transformation is non-linear. Hence, it is an interesting and important question that whether Klingenberg's theorem holds in the nonreversible case.

In the following, we use the methods of Rademacher in [16–18] to establish a Finslerian version of Klingenberg's theorem.

Theorem 3.1. *Let (M, F) be a compact Finsler manifold with $\mathbf{K}_M \leq k$. Then*

$$i_M \geq \min \left\{ \frac{\pi}{\lambda_F \sqrt{k}}, \frac{1}{1 + \lambda_F} \text{the shortest simple closed geodesic in } M \right\}.$$

In particular, the equality holds if F is reversible (i.e., $\lambda_F = 1$).

Proof. As in [16,17], set

$$\tilde{i}_p := \inf \{ \tilde{d}(p, q) : q \in \text{Cut}_p \}, \quad \tilde{i}_M := \inf_{p \in M} \tilde{i}_p,$$

where $\tilde{d}(p, q) := \frac{1}{2}(d(p, q) + d(q, p))$. Clearly,

$$\frac{1 + \lambda_F^{-1}}{2} i_M \leq \tilde{i}_M \leq \frac{(1 + \lambda_F)}{2} i_M. \quad (3.1)$$

Given a closed geodesic $c(t)$, $0 \leq t \leq 1$, let $c(t_0)$ denote the cut point of $c(0)$ along c . Then we have

$$L(c) = d(c(0), c(t_0)) + L(c|_{[t_0, 1]}) \geq d(c(0), c(t_0)) + d(c(t_0), c(0)) \geq 2\tilde{i}_M. \quad (3.2)$$

Hence, (3.1) together with (3.2) implies that in order to prove the theorem, we just need to show that there exists a simple closed geodesic c with $L(c) = 2\tilde{i}_M$ in the case of $\tilde{i}_M < \frac{(1 + \lambda_F^{-1})\pi}{2\sqrt{k}}$.

Suppose $\tilde{i}_M < \frac{(1 + \lambda_F^{-1})\pi}{2\sqrt{k}}$. We now construct a simple geodesic loop c (based at $c(0)$) with $L(c) = 2\tilde{i}_M$. Since M is compact, there is a point $p \in M$ with $\tilde{i}_p = \tilde{i}_M$. Let q be the point in Cut_p with $\tilde{d}(p, q) = \tilde{i}_p$. Since

$$d(p, q) \leq \frac{2}{(1 + \lambda_F^{-1})} \tilde{d}(p, q) = \frac{2}{(1 + \lambda_F^{-1})} \tilde{i}_p < \frac{\pi}{\sqrt{k}},$$

q is not the conjugate point of p . Thus, there exist two distinct normal minimal geodesics $c_1(t)$ and $c_2(t)$, $t \in [0, d(p, q)]$ from p to q . Let $c_3(t)$, $t \in [0, d(q, p)]$ be a normal minimal geodesic from q to p .

The proof of [17, Lemma 9.4] implies that either $c_1 * c_3$ or $c_2 * c_3$ is smooth at q . Without loss of generality, we assume that $c_1 * c_3$ is smooth at q and therefore, $c_1 * c_3$ is a geodesic loop based at p though q with $L(c) = d(p, q) + d(q, p) = 2\tilde{d}(p, q) = 2\tilde{i}_M$. We now show that $c(t) = c_1 * c_3(t)$, $t \in [0, 2\tilde{i}_M]$ is a closed geodesic.

The continuity of the cut value [3, Proposition 8.4.1] implies that for a small positive number $\varepsilon (< d(p, q))$, there exists $t_\varepsilon \in (d(p, q), 2\tilde{i}_M)$ such that $q_\varepsilon = c(t_\varepsilon)$ is a cut point of $p_\varepsilon = c(\varepsilon)$ along $c(t)$. Hence,

$$\begin{aligned} 2\tilde{i}_M &\leq 2\tilde{d}(p_\varepsilon, q_\varepsilon) \leq d(p_\varepsilon, q_\varepsilon) + d(q_\varepsilon, p) + d(p, p_\varepsilon) \\ &= d(p_\varepsilon, q) + d(q, q_\varepsilon) + d(q_\varepsilon, p) + d(p, p_\varepsilon) \\ &= d(p, p_\varepsilon) + d(p_\varepsilon, q) + d(q, q_\varepsilon) + d(q_\varepsilon, p) \\ &= d(p, q) + d(q, p) = L(c) = 2\tilde{i}_M, \end{aligned} \quad (3.3)$$

which implies that (see (3.3))

$$d(q_\varepsilon, p_\varepsilon) = d(q_\varepsilon, p) + d(p, p_\varepsilon) = L(c_3|_{[t_\varepsilon - d(p, q), d(q, p)]}) + L(c_1|_{[0, \varepsilon]}).$$

That is, $c = c_1 * c_3$ is smooth at p . \square

In [19], Shen introduces T-curvature, which is an important non-Riemannian quantity. However, the definition of the bound on T-curvature seems a little complicated. For convenience, we give a new definition of the bound on T-curvature. Also refer to [19, 27] for more details.

Definition 3.2. Given $y, v \in T_x M$ with $y \neq 0$, define the T-curvature \mathbf{T} as

$$\mathbf{T}_y(v) := g_y(\nabla_v^V V, y) - g_y(\nabla_v^Y V, y),$$

where V (resp. Y) is a vector field with $V_x = v$ (resp. $Y_x = y$). Set

$$\mathbf{T}_p := \sup_{y, v \in S_p M} |\mathbf{T}_y(v)|, \quad \mathbf{T}_M := \sup_{p \in M} \mathbf{T}_p.$$

Clearly, for a compact Finsler manifold, \mathbf{T}_M is finite. And $\mathbf{T}_M = 0$ if and only if F is Berwaldian. By the proof of [27, Theorem 1.1], we have the following result.

Theorem 3.3. Let (M, F) be a compact Finsler n -manifold with $\mathbf{K}_M \geq k$, $\mathbf{T}_M \leq \tau$, $\Lambda_F \leq \Lambda$ and $\text{diam}(M) \leq D$. Then for any simple closed geodesic γ ,

$$L(\gamma) \geq \frac{\mu(M)}{c_{n-2} \Lambda^{\frac{3n}{2}} \left[\frac{s_k^{n-1} \left(\min \left\{ D, \frac{\pi}{2\sqrt{k}} \right\} \right)}{n-1} + \Lambda^{\frac{1}{2}} \tau \int_0^D s_k^{n-1}(t) dt \right]},$$

where $\mu(M)$ is either the Busemann–Hausdorff volume or the Holmes–Thompson volume of M and $c_{n-2} := \text{Vol}(\mathbb{S}^{n-2})$.

Theorem 3.1 together with Theorem 3.3 then yields the following Cheeger type estimate.

Theorem 3.4. Let (M, F) be a compact Finsler n -manifold with $|\mathbf{K}_M| \leq k$, $\mathbf{T}_M \leq \tau$, $\Lambda_F \leq \Lambda$, $\text{diam}(M) \leq D$ and $\mu(M) \geq V$, where $\mu(M)$ is either the Busemann–Hausdorff volume or the Holmes–Thompson volume of M . Then

$$i_M \geq \frac{1}{1 + \Lambda^{\frac{1}{2}}} \min \left\{ \frac{(1 + \Lambda^{-\frac{1}{2}})\pi}{\sqrt{k}}, \frac{\mu(M)}{c_{n-2} \Lambda^{\frac{3n}{2}} \left[\frac{s_{-k}^{n-1}(D)}{n-1} + \Lambda^{\frac{1}{2}} \tau \int_0^D s_{-k}^{n-1}(t) dt \right]} \right\}.$$

Remark 1. By Theorem 3.1 and the standard arguments (see [1, 15]), one can show the following result, which is an extension of the results in [13, 17, 18].

Let (M, F) be an even-dimensional, compact Finsler manifold with $0 < \mathbf{K}_M \leq k$.

(1) If M is orientable, then

$$i_M \geq \frac{\text{Conj}_M}{\lambda_F} \geq \frac{\pi}{\lambda_F \sqrt{k}}.$$

In particular, if F is reversible, then $i_M = \text{Conj}_M$.

(2) If M is not orientable, then

$$i_M \geq \frac{\pi}{\lambda_F(1 + \lambda_F)\sqrt{k}}.$$

4. The convex radius of a Berwald manifold

Recall that a subset $A \subset M$ is called *strongly (geodesically) convex* if for any $p, q \in A$, there exists a geodesic γ_{pq} such that γ_{pq} is the unique minimizer in M from p to q , and γ_{pq} is the only geodesic contained in A from p to q .

Definition 4.1. Let (M, F) be a forward complete Finsler manifold. The convexity radius at a point $x \in M$ is defined by

$$\text{Conv}_x := \sup\{r > 0 : B_x^+(s) \text{ is strongly convex for any } s < r\}.$$

And the convexity radius of (M, F) is defined by $\text{Conv}(M, F) := \inf_{x \in M} \text{Conv}_x$.

In [19], Shen estimates convexity radii in the reversible Finslerian case. Here, we give an estimate on the convexity radius of a Berwald manifold.

Theorem 4.2. Let (M, F) be a forward complete Berwald manifold with $\mathbf{K}_M \leq k$, $i_M \geq \varsigma$ and $\lambda_F \leq \lambda$. Then

$$\text{Conv}(M, F) \geq \min \left\{ \frac{\pi}{2\sqrt{k}}, \frac{\varsigma}{\lambda(1 + \lambda)} \right\}.$$

Proof. Choosing an arbitrary point $x \in M$ and any $r \in (0, \min\{\frac{\pi}{2\sqrt{k}}, \frac{\varsigma}{\lambda(1+\lambda)}\})$, we now show that $B_x^+(r)$ is strictly convex.

For each two points $p_1, p_2 \in B_x^+(r)$, let $\gamma_{p_1 p_2}(t)$, $0 \leq t \leq l$ denote a normal minimal geodesic from p_1 to p_2 . Since

$$l = L(\gamma_{p_1 p_2}) \leq d(p_1, x) + d(x, p_2) \leq \lambda \cdot d(x, p_1) + d(x, p_2) < \varsigma/\lambda,$$

$\gamma_{p_1 p_2}$ is the unique minimal geodesic from p_1 to p_2 and hence, $\rho(\cdot) := d(x, \cdot)$ is smooth on $\gamma_{p_1 p_2}([0, l]) - \{x\}$.

Fix a point $p \in B_x^+(r)$ and set

$$\text{Co}_p := \{q \in B_x^+(r) : \gamma_{pq}([0, l]) \subset B_x^+(r)\}.$$

We first prove that Co_p is an open subset of $B_x^+(r)$. For any sequence $\{q_n\} \subset B_x^+(r) - \text{Co}_p$ converging to some point $q \in B_x^+(r)$, there exists $t_n \in (0, l)$ such that $\rho(\gamma_{pq_n}(t_n)) \geq r$ for each n . Since $\{\gamma_{pq_n}\}$ is uniformly bounded, by the Arzelà–Ascoli theorem [5], we can assume that $\{\gamma_{pq_n}\}$ converges to the minimal geodesic γ_{pq} and $t_n \rightarrow t_0$. Clearly, $\rho(\gamma_{pq}(t_0)) \geq r$, which implies $q \in B_x^+(r) - \text{Co}_p$. Hence, $B_x^+(r) - \text{Co}_p$ is a closed subset of $B_x^+(r)$.

Secondly, we prove that Co_p is a closed subset of $B_x^+(r)$. It suffices to show $\partial \text{Co}_p \subset \text{Co}_p$. Given any point $q \in \partial \text{Co}_p$, the argument is divided into the following two cases:

Case 1. Suppose $x \notin \gamma_{pq}$. Then $\rho \circ \gamma_{pq}(t)$ is smooth, and the Hessian comparison theorem [19] implies that $\frac{d^2}{dt^2} \rho \circ \gamma_{pq}(t) \geq 0$, which implies that

$$\rho \circ \gamma_{pq}(t) \leq \max\{\rho \circ \gamma_{pq}(0), \rho \circ \gamma_{pq}(1)\} < r.$$

Hence, $\gamma_{pq}([0, 1]) \subset B_p^+(r)$ and therefore, $q \in \text{Co}_p$.

Case 2. If there exists $t_0 \in [0, l]$ such that $\gamma_{pq}(t_0) = x$, then for $t \in [t_0, l]$,

$$\rho(\gamma_{pq}(t)) = L(\gamma_{pq}|_{[t_0, t]}) \leq L(\gamma_{pq}|_{[t_0, l]}) = \rho(q) < r.$$

On the other hand, one can easily find a number $s_0 \in [0, t_0]$ with $\rho(\gamma_{pq}(t)) < r$ for all $t \in [s_0, t_0]$. And the argument of [Case 1](#) implies that $\gamma_{pq}([0, s_0]) \subset B_p^+(r)$. Hence, $\gamma_{pq}([0, l]) \subset B_p^+(r)$ and therefore, $q \in \text{Co}_p$.

Since $x \in \text{Co}_p$, we have $\text{Co}_p = B_x^+(r)$ and hence, $B_x^+(r)$ is strictly convex. \square

Remark 2. Denote by \mathbf{T}_M^s the upper bound of T-curvature in the sense of Shen [\[19\]](#). Using the argument above, one can obtain an estimate on the convexity radius of a general Finsler manifold. More precisely, let (M, F) be a forward complete Finsler manifold with $\mathbf{K}_M \leq k$, $\mathbf{i}_M \geq \varsigma$, $\lambda_F \leq \lambda$ and $\mathbf{T}_M^s \leq \xi$. Then

$$\text{Conv}(M, F) \geq \min \left\{ \mathfrak{v}, \frac{\varsigma}{\lambda(1 + \lambda)} \right\},$$

where \mathfrak{v} is the first positive zero of the following equation

$$\mathfrak{s}'_k(t) - \xi \cdot \mathfrak{s}_k(t) = 0.$$

This estimate coincides with Shen's result [\[19, Theorem 15.2.1\]](#) in the reversible case.

Proposition 4.3. *Let (M, F) be a forward complete Berwald manifold with $\mathbf{K}_M \leq k$ and $\mathbf{i}_M \geq \varsigma$. Set $l := \min\{\pi/(2\sqrt{k}), \varsigma\}$. Given any $x \in M$ and any $0 < r < l$, if a geodesic γ is tangent to the forward sphere $S_x^+(r) = \partial B_x^+(r)$ at q , then there exists a small neighborhood U_q of q such that $U_q \cap \gamma$ is outside $\overline{B_x^+(r)} - \{q\}$.*

Proof. Suppose that γ is a normal geodesic. Let $\rho(\cdot) := d(x, \cdot)$. Clearly, for any $p \in B_x^+(r) - \{x\}$, $\rho(p)$ is smooth. Set $\gamma(t_0) := q$. Since $\nabla \rho$ is the normal vector field along $S_p^+(r)$, we have

$$\left. \frac{d}{dt} \right|_{t=t_0} \rho(\gamma(t)) = g_{\nabla \rho}(\nabla \rho, \dot{\gamma}(t_0)) = 0. \quad (4.1)$$

$\rho \circ \gamma(t_0) = r < l$ together with Hessian comparison theorem yields that there is a small number $\epsilon > 0$ such that $\rho \circ \gamma(t) < l$ and $\frac{d^2}{dt^2} \rho(\gamma(t)) > 0$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. This together with [\(4.1\)](#) implies that $\rho \circ \gamma$ has a minimum at t_0 and hence, the proposition follows. \square

In the rest of this section, we assume that $(A, d\mathbf{m})$ is a measure space of volume 1, and (M, F) is a forward complete Berwald n -manifold. Given $p \in M$ and $r > 0$, any measurable map $f : A \rightarrow B_p^+(r)$ is called a *mass distribution* on $B_p^+(r)$. Define a vector field V on $\overline{B_p^+(r)}$ by

$$V(x) := - \int_A \exp_x^{-1} f(a) d\mathbf{m}(a).$$

Then we have the following theorem. Refer to [\[11\]](#) for the results of the center of mass in the Riemannian case.

Theorem 4.4. *Let (M, F) be a forward complete Berwald n -manifold with $|\mathbf{K}_M| \leq k$, $\Lambda_F \leq \Lambda$, and $\mathbf{i}_M \geq \varsigma$. There exists a constant $\mathfrak{r} = \mathfrak{r}(n, k, \Lambda, \varsigma) > 0$ such that for each $0 < r < \mathfrak{r}$, each $p \in M$ and each mass distribution $f : A \rightarrow B_p^+(r)$, there exists a unique point $q \in B_p^+(r)$ with $V(q) = 0$. q is called the center of mass $\mathcal{C}(f)$ of f .*

In particular, $V(q)$ is differentiable and the map V_{*q} is non-degenerate at $q = \mathcal{C}(f)$, where $V_* : TB_p^+(r) \rightarrow TB_p^+(r)$ is defined by

$$V_*(X) = X^i \frac{\partial V^k}{\partial x^i} \frac{\partial}{\partial x^k}, \quad \forall X = X^i \frac{\partial}{\partial x^i}$$

and (x^i, y^i) is a local coordinate system of $TB_p^+(r)$.

Proof. Let $\mathfrak{r} := \frac{1}{2\Lambda} \min\{\frac{\pi}{2\sqrt{k}}, \frac{\varsigma}{1+\sqrt{\Lambda}}, \mathfrak{t}, \frac{1}{40\Lambda^2}\}$, where \mathfrak{t} is as in [Lemma Appendix A.7](#). Given $p \in M$ and f , we consider $V(x)$ defined on $B_p^+(r)$.

Step 1. First, we show that for each $x \in \partial B_p^+(r)$, $V(x)$ is a nonzero outward vector. For each $a \in A$, set $X_a := -\exp_x^{-1} f(a)$. It is easy to see that the geodesic $\gamma_{X_a}(t)$, $t \in (-\epsilon, 0)$ is contained in $B_p^+(r)$ and $\gamma_{X_a}(t)$, $t \in (0, \epsilon)$ is outside $B_p^+(r)$. Set $\rho(\cdot) = d(p, \cdot)$. Thus, [Proposition 4.3](#) yields

$$\left. \frac{d}{dt} \right|_{t=0} \rho(\gamma_{X_a}(t)) = g_{\nabla \rho(x)}(\nabla \rho(x), X_a) \geq 0.$$

Hence,

$$g_{\nabla \rho(x)}(\nabla \rho(x), V(x)) = g_{\nabla \rho(x)}\left(\nabla \rho(x), \int_A X_a d\mathbf{m}(a)\right) > 0,$$

which implies that $V(x)$ is a nonzero outward vector.

Step 2. Now we show that V has only isolated singularities in $B_p^+(r)$. Given $a \in A$ and a geodesic $\gamma(s)$, $s \in [0, 1]$ in $B_p^+(r)$, consider the geodesic variation

$$\sigma_a(t, s) = \exp_{\gamma(s)}(t-1)X_a, \quad t \in [0, 1],$$

where $X_a := -\exp_{\gamma(s)}^{-1} f(a)$. Clearly, $\sigma_a(0, s) = f(a)$ and $\sigma_a(s, 1) = \gamma(s)$. Note that

$$U_{s;a}(t) = \frac{\partial}{\partial s} \sigma_a(t, s)$$

is a Jacobi field with $U_{s;a}(0) = 0$ and $U_{s;a}(1) = \dot{\gamma}(s)$. Set

$$T_{s;a}(t) := \frac{\partial}{\partial t} \sigma_a(t, s) = (\exp_{\gamma(s)})_{*(t-1)X_a} X_a.$$

It is easy to see that

$$U'_{s;a}(1) = \nabla_{T_{s;a}} U_{s;a} = \nabla_{U_{s;a}} T_{s;a} = \nabla_{U_{s;a}} X_a.$$

Thus, [Lemma Appendix A.7](#) together with the equalities above implies that

$$\begin{aligned} \|\dot{\gamma}(s) - \nabla_{\dot{\gamma}(s)} V\|_{\dot{\gamma}(s)} &= \left\| \int_A (\dot{\gamma}(s) - \nabla_{U_{s;a}} X_a) d\mathbf{m}(a) \right\|_{\dot{\gamma}(s)} \\ &\leq \int_A \frac{1}{20} \|\dot{\gamma}(s)\|_{\dot{\gamma}(s)} d\mathbf{m}(a) = \frac{1}{20} \|\dot{\gamma}(s)\|_{\dot{\gamma}(s)}, \end{aligned} \quad (4.2)$$

which implies that V has only isolated singularities. Here $\|\cdot\|_{\dot{\gamma}(s)} := \sqrt{g_{\dot{\gamma}(s)}(\cdot, \cdot)}$.

Step 3. We now show that $V(x)$ has exactly one singularity in $B_p^+(r)$. Since $B_p^+(r)$ is contractible and V is an outward vector field along the boundary, the sum of index of V in $B_p^+(r)$ is $+1$, which implies that V has at least one isolated singularity in $B_p^+(r)$.

On the other hand, for each isolated singularity z in $B_p^+(r)$, let $\gamma(s)$ be a geodesic from z . Eq. (4.2) implies that

$$\left. \frac{d}{ds} \right|_{s=0} g_{\dot{\gamma}(s)}(\dot{\gamma}(s), V(\gamma(s))) = g_{\dot{\gamma}(0)}(\dot{\gamma}(0), \nabla_{\dot{\gamma}} V) > 0,$$

which yields a small number $l > 0$ such that V is outward along $\partial B_z^+(l)$. The Poincaré–Hopf theorem then implies that the index of V at z is $+1$ and therefore, V has exactly one zero in $B_p^+(r)$.

Step 4. From above, one can see that $V(x)$ is differentiable at every point $x \in B_p^+(r)$, and $\nabla_X V \neq 0$ for any $X \in T_{\mathcal{C}(f)} M - \{0\}$. Let (x^i, y^i) be a coordinate system of $TB_p^+(r)$ and let $\gamma(t)$, $t > 0$ be a smooth curve from $\mathcal{C}(f)$ with $\dot{\gamma}(0) = X$. Thus,

$$\begin{aligned} 0 \neq \nabla_X V &= \left[\frac{dV^i}{dt} + \Gamma_{jk}^i(\gamma(0)) \dot{\gamma}^j(0) V^k(\mathcal{C}(f)) \right] \frac{\partial}{\partial x^i} \\ &= \left. \frac{\partial V^i}{\partial x^k} \right|_{\mathcal{C}(f)} \dot{\gamma}^k(0) \frac{\partial}{\partial x^i} = V_{*\mathcal{C}(f)}(X), \end{aligned}$$

which implies that $V_{*\mathcal{C}(f)}$ is nonsingular. \square

5. A Cheeger finiteness theorem for Berwald manifolds

Given $n \in \mathbb{N}$, $\Lambda \geq 1$, $\varsigma > 0$ and $k \geq 0$, let $\mathfrak{r} := \mathfrak{r}(n, k, \Lambda, \varsigma)$ and $\mathfrak{C} := \mathfrak{C}(n, k, \Lambda)$ be defined as in Theorem 4.4 and Lemma Appendix B.3, respectively.

Definition 5.1. We say a triple $(R, \varepsilon_1, \varepsilon_2)$ satisfies Condition (Δ) if

$$\begin{aligned} (1) \quad & 0 < R \leq \min \left\{ \frac{\mathfrak{r}}{40 \cdot \Lambda^4}, \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2 \right\}, \\ (2) \quad & 0 < \varepsilon_1 \leq \frac{R}{12\Lambda^3}, \quad \varepsilon_2 > 0, \\ (3) \quad & \frac{(1 - kR^2)}{\Lambda^5} - \mathcal{C}_3(n, k, \Lambda, R, \varepsilon_2) - \frac{2^{2n+6} \Lambda^{4n+6}}{\mathfrak{s}_k(\sqrt{\Lambda}R)} \varepsilon_1 > 0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{C}_0(k, \Lambda) &:= \sup \left\{ t > 0 : \frac{\mathfrak{s}_{-k}(3\Lambda^{\frac{5}{2}}t)}{3\Lambda^{\frac{5}{2}}t} \leq 2 \right\}, \\ \mathcal{C}_1(n, k, \Lambda) &:= \sup \left\{ t > 0 : \frac{\int_0^{\Lambda t} \mathfrak{s}_{-k}^{n-1}(s) ds}{\int_0^{\frac{t}{4\Lambda}} \mathfrak{s}_k^{n-1}(s) ds} \leq 2(4\Lambda^2)^n \right\}, \\ \mathcal{C}_2(k, \Lambda) &:= \sup \left\{ t > 0 : \frac{t}{\mathfrak{s}_{-k}(t)} \frac{\mathfrak{s}_k(\Lambda^{\frac{3}{2}}t)}{\mathfrak{s}_{-k}(\sqrt{\Lambda}t)} \geq 1 - kt^2 \right\}, \\ \mathcal{C}_3(n, k, \Lambda, R, \varepsilon_2) &:= \frac{6\Lambda^3 R}{\mathfrak{s}_k(\sqrt{\Lambda}R)} \left(\frac{\mathfrak{s}_{-k}(\sqrt{\Lambda}R)}{\sqrt{\Lambda}R} - 1 \right) \frac{\mathfrak{s}_{-k}(\sqrt{\Lambda}R)}{\mathfrak{s}_k(\sqrt{\Lambda}R)} + 30\Lambda^3 \mathfrak{C}(n, k, \Lambda) R^2 + \Lambda \varepsilon_2. \end{aligned}$$

In the following, we assume that $(R, \varepsilon_1, \varepsilon_2)$ is given and satisfies Condition (Δ) .

Definition 5.2. Given $N \in \mathbb{N}$, we say a compact Berwald n -manifold (M, F) satisfies Condition $(1-N)$ if

- (1) $\Lambda_F \leq \Lambda$, $|\mathbf{K}_M| \leq k$, $\mathbf{i}_M \geq \varsigma$, $\text{diam}(M) \geq D$;
- (2) M can be covered by N convex balls of radius $R/(2\Lambda^{\frac{3}{2}})$

$$\{B_{p_\alpha}^+(R/(2\Lambda^{\frac{3}{2}})) : \alpha = 1, \dots, N\}$$

and such balls $\{B_{p_\alpha}^+(R/(4\Lambda^2))\}_{\alpha=1}^N$ are disjoint.

Let (M_i, F_i) , $i = 1, 2$ be two Berwald n -manifolds satisfying Condition $(1-N)$. Let $\{B_{p_\alpha}^+(R/(2\Lambda^{\frac{3}{2}})) : \alpha = 1, \dots, N\}$ be the forward convex balls of M_i as in Definition 5.2.

Let $(\mathbb{R}^n, \|\cdot\|)$ be a standard Euclidean space. For each i , denote by $\|\cdot\|_i$ the average Riemannian norm on M_i induced by F_i , which yields a linear isometry $u_\alpha^i : (\mathbb{R}^n, \|\cdot\|) \rightarrow (T_{p_\alpha^i} M_i, \|\cdot\|_i)$ for each $\alpha \in \{1, \dots, N\}$ such that for

$$\frac{1}{\sqrt{\Lambda}} \leq \frac{F_i(u_\alpha^i(X))}{\|X\|} \leq \sqrt{\Lambda}, \quad \forall X \in \mathbb{R}^n.$$

Clearly, $u_\alpha^i : \overline{\mathcal{B}_0(R)} \rightarrow \overline{\mathcal{B}_{p_\alpha^i}^+(\sqrt{\Lambda}R)}$, where $\mathcal{B}_0(R)$ (resp. $\mathcal{B}_{p_\alpha^i}^+(R)$) denotes the ball of radius R centered at the origin in $(\mathbb{R}^n, \|\cdot\|)$ (resp. $(T_{p_\alpha^i} M_i, F_i)$).

Set

$$\phi_\alpha^i := \exp_{p_\alpha} \circ u_\alpha^i : \overline{\mathcal{B}_0(R)} \rightarrow \overline{\mathcal{B}_{p_\alpha^i}^+(\sqrt{\Lambda}R)}.$$

Clearly, $\phi_\alpha^i(\overline{\mathcal{B}_0(R)}) \supset \mathcal{B}_{p_\alpha^i}^+(R/\sqrt{\Lambda})$. In particular, if $\phi_\alpha^i(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^i(\overline{\mathcal{B}_0(R)}) \neq \emptyset$, the triangle inequality then yields that

$$\phi_\beta^i(\overline{\mathcal{B}_0(R)}) \subset \phi_\alpha^i(\overline{\mathcal{B}_0(3\Lambda^2 R)}), \quad \phi_\alpha^i(\overline{\mathcal{B}_0(R)}) \subset \phi_\beta^i(\overline{\mathcal{B}_0(3\Lambda^2 R)}).$$

Hence, for any α, β with $\phi_\alpha^i(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^i(\overline{\mathcal{B}_0(R)}) \neq \emptyset$, we can define a map

$$\mathfrak{f}_{\beta\alpha}^i := (\phi_\beta^i)^{-1} \circ \phi_\alpha^i : \overline{\mathcal{B}_0(R)} \rightarrow \overline{\mathcal{B}_0(3\Lambda^2 R)}.$$

The following lemma follows from Lemma Appendix A.1 directly.

Lemma 5.3. *There exists a constant $\mathcal{C} = \mathcal{C}(\varsigma, k, \Lambda)$ such that for any α, β with $\phi_\alpha^i(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^i(\overline{\mathcal{B}_0(R)}) \neq \emptyset$, we have*

$$\|\mathfrak{f}_{\beta\alpha}^i\|_{C^1} \leq \mathcal{C}.$$

For any two α, β with $\phi_\alpha^i(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^i(\overline{\mathcal{B}_0(R)}) \neq \emptyset$, there exists a unique minimal geodesic $\gamma_{\alpha\beta}^i(t)$, $0 \leq t \leq 1$ from p_α^i to p_β^i . Let $P_{\alpha\beta}^i$ denote the parallel transformation along $\gamma_{\alpha\beta}^i$ from $T_{p_\alpha^i} M_i$ to $T_{p_\beta^i} M_i$. Define a linear isomorphism $\mathfrak{g}_{\beta\alpha}^i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\mathfrak{g}_{\beta\alpha}^i := (u_\beta^i)^{-1} P_{\alpha\beta}^i u_\alpha^i.$$

Since F_i is Berwaldian, $F_i(P_{\alpha\beta}^i Y) = F_i(Y)$. Thus, one has the following result.

Lemma 5.4. For any α, β with $\phi_\alpha^i(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^i(\overline{\mathcal{B}_0(R)}) \neq \emptyset$, we have

$$\|\mathfrak{g}_{\beta\alpha}^i\|_0 := \sup_{X \neq 0} \frac{\|\mathfrak{g}_{\beta\alpha}^i X\|}{\|X\|} \leq \Lambda.$$

By the Arzelà–Ascoli theorem, one can easily show the following lemma.

Lemma 5.5. Let \mathcal{C} be as in Lemma 5.3 and let $(\mathbb{R}^n, \|\cdot\|)$ be a Euclidean space. Set

$$H_1 := \{f : \overline{\mathcal{B}_0(R)} \rightarrow \overline{\mathcal{B}_0(3\Lambda^2 R)} : f \text{ is a embedding map with } \|f\|_{C_1} \leq \mathcal{C}\},$$

$$H_2 := \{f : (\mathbb{R}^n, \|\cdot\|) \rightarrow (\mathbb{R}^n, \|\cdot\|) : f \text{ is a linear map with } \|f\|_0 \leq \Lambda\}.$$

Then H_1 and H_2 are totally bounded. That is, for each $\varepsilon > 0$, H_i , $i = 1, 2$ can be covered by a finite number of balls of radius ε .

Definition 5.6. Given $N \in \mathbb{N}$, we say two compact Berwald n -manifolds (M_i, F_i) , $i = 1, 2$ satisfy Condition (2- N) if

- (1) (M_i, F_i) , $i = 1, 2$ satisfy Condition (1- N);
- (2) $\phi_\alpha^1(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^1(\overline{\mathcal{B}_0(R)}) \neq \emptyset \Leftrightarrow \phi_\alpha^2(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^2(\overline{\mathcal{B}_0(R)}) \neq \emptyset$, for all $\alpha, \beta \in \{1, \dots, N\}$.

The following result is a generalized Peter’s lemma, which will be proved in next section. Also refer to [14] for the original Peter’s lemma.

Lemma 5.7. Let $(R, \varepsilon_1, \varepsilon_2)$ be a triple satisfying Condition (Δ) and let (M_i, F_i) , $i = 1, 2$ be two closed Berwald manifolds satisfying Condition (2- N). Suppose that for any $\alpha, \beta \in \{1, \dots, N\}$ with $\phi_\alpha^i(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^i(\overline{\mathcal{B}_0(R)}) \neq \emptyset$, we have

$$\begin{aligned} \|\mathfrak{f}_{\beta\alpha}^1 - \mathfrak{f}_{\beta\alpha}^2\|_{C_1} &\leq \varepsilon_1, \\ \|\mathfrak{g}_{\beta\alpha}^1 - \mathfrak{g}_{\beta\alpha}^2\|_0 &\leq \varepsilon_2. \end{aligned}$$

Then M_1 and M_2 are diffeomorphic.

By Lemma 5.7, we now show the following theorem.

Theorem 5.8. Given $n \in \mathbb{N}$, $\Lambda \geq 1$, $k \geq 0$ and $V, D > 0$, there exist only finitely many diffeomorphism classes of compact Berwald n -manifolds (M, F) satisfying

$$\Lambda_F \leq \Lambda, \quad |\mathbf{K}_M| \leq k, \quad \mu(M) \geq V, \quad \text{diam}(M) \leq D, \quad (5.1)$$

where $\mu(M)$ is either the Busemann–Hausdorff volume or the Holmes–Thompson volume of M .

Proof. Theorem 3.4 yields a positive constant $\varsigma = \varsigma(n, \Lambda, k, V, D)$ such that if a compact Berwald n -Finsler manifold (M, F) satisfies (5.1), then $\mathbf{i}_M \geq \varsigma$. Let $(R, \varepsilon_1, \varepsilon_2)$ be a triple defined as in Definition 5.1, i.e., $(R, \varepsilon_1, \varepsilon_2)$ satisfies Condition (Δ) .

Suppose the theorem is not true. Then there exists an infinite sequence $\{(M_s, F_s)\}$ satisfying (5.1), but $\{(M_s, F_s)\}$ are not diffeomorphic mutually.

For each s , let $\{B_{p_s^+}^+(R/(4\Lambda^2))\}_{\alpha=1}^{N_s}$ denote the maximal family of disjoint balls of radius $R/(4\Lambda^2)$ in M_s . The volume comparison theorem [28] then implies

$$N_s \leq \frac{\mu(M_s)}{\min_{\alpha} \mu(B_{p_{\alpha}^s}^+(R/(4\Lambda^2)))} \leq C_0(n, \Lambda, D, R, k) =: N_0.$$

It is not hard to check that $\{B_{p_{\alpha}^s}^+(R/(2\Lambda^{\frac{3}{2}}))\}_{\alpha=1}^{N_s}$ cover M_s . Since $\{(M_s, F_s)\}$ is an infinite sequence and N_0 is a finite number, there must be a subsequence $\{(M_{s_L}, F_{s_L})\}$ such that all $N_{s_L} \equiv N_1 \leq N_0$. That is, for each L , the number of the maximal family of disjoint balls of radius $R/(4\Lambda^2)$ in M_{s_L} is equal to N_1 . Thus, all the elements in $\{(M_{s_L}, F_{s_L})\}$ satisfy Condition (1- N_1).

Since N_1 is finite, there must be a subsequence $\{(M_K, F_K)\}$ of $\{(M_{s_L}, F_{s_L})\}$ such that for any $(M_{K_1}, F_{K_1}), (M_{K_2}, F_{K_2}) \in \{(M_K, F_K)\}$,

$$\phi_{\alpha}^{K_1}(\overline{\mathcal{B}_0(R)}) \cap \phi_{\beta}^{K_1}(\overline{\mathcal{B}_0(R)}) \neq \emptyset \Leftrightarrow \phi_{\alpha}^{K_2}(\overline{\mathcal{B}_0(R)}) \cap \phi_{\beta}^{K_2}(\overline{\mathcal{B}_0(R)}) \neq \emptyset,$$

for all $\alpha, \beta \in \{1, \dots, N_1\}$. That is, all the elements in $\{(M_K, F_K)\}$ satisfy Condition (2- N_1).

Lemma 5.5 yields that H_i can be covered by a finite number (say A_i) of ε_i -balls, $i = 1, 2$. Hence, for each K , $\mathfrak{f}_{\beta\alpha}^K = (\phi_{\beta}^K)^{-1} \circ \phi_{\alpha}^K \in H_1$ is in some ε_1 -ball. Since N_1, A_1 are finite and $\{(M_K, F_K)\}$ is an infinite sequence, there exists a subsequence $\{(M_{K'}, F_{K'})\}$ such that for any $(M_{K'_1}, F_{K'_1}), (M_{K'_2}, F_{K'_2}) \in \{(M_{K'}, F_{K'})\}$, we have

$$\|\mathfrak{f}_{\beta\alpha}^{K'_1} - \mathfrak{f}_{\beta\alpha}^{K'_2}\|_{C_1} \leq \varepsilon_1,$$

for any $\alpha, \beta \in \{1, \dots, N_1\}$ with $\phi_{\alpha}^{K'_1}(\overline{\mathcal{B}_0(R)}) \cap \phi_{\beta}^{K'_1}(\overline{\mathcal{B}_0(R)}) \neq \emptyset$.

Likewise, there exists a subsequence $\{(M_{K''}, F_{K''})\}$ of $\{(M_{K'}, F_{K'})\}$ such that for any $(M_{K''_1}, F_{K''_1}), (M_{K''_2}, F_{K''_2}) \in \{(M_{K''}, F_{K''})\}$, we have

$$\|\mathfrak{f}_{\beta\alpha}^{K''_1} - \mathfrak{f}_{\beta\alpha}^{K''_2}\|_{C_1} \leq \varepsilon_1, \quad \|\mathfrak{g}_{\beta\alpha}^{K''_1} - \mathfrak{g}_{\beta\alpha}^{K''_2}\|_0 \leq \varepsilon_2,$$

for any $\alpha, \beta \in \{1, \dots, N_1\}$ with $\phi_{\alpha}^{K''_1}(\overline{\mathcal{B}_0(R)}) \cap \phi_{\beta}^{K''_1}(\overline{\mathcal{B}_0(R)}) \neq \emptyset$.

Lemma 5.7 then implies that $\{(M_{K''}, F_{K''})\}$ are diffeomorphic mutually, which contradicts the definition of $\{(M_s, F_s)\}$. \square

6. Proof of Lemma 5.7

Before proving **Lemma 5.7**, we recall some notations used in Section 5:

$\|\cdot\|$ denotes a Euclidean norm on \mathbb{R}^n and $\|\cdot\|_i := \sqrt{\tilde{g}_i(\cdot, \cdot)}$ denotes the average Riemannian norm induced by F_i . In particular, for each α , $u_{\alpha}^i : (\mathbb{R}, \|\cdot\|) \rightarrow (T_{p_{\alpha}^i} M_i, \|\cdot\|_i)$ is a natural isometry. Given $X \in TM_i - \{0\}$, $\|\cdot\|_X := \sqrt{g_{iX}(\cdot, \cdot)}$, where g_i is the fundamental tensor induced by F_i .

The proof of Lemma 5.7.

Step 1. For each α , define a map $\mathcal{F}_{\alpha} := \phi_{\alpha}^2 \circ (\phi_{\alpha}^1)^{-1} : \phi_{\alpha}^1(\overline{\mathcal{B}_0(R)}) \rightarrow \phi_{\alpha}^2(\overline{\mathcal{B}_0(R)})$. Given $p \in \phi_{\alpha}^1(\overline{\mathcal{B}_0(R)}) \cap \phi_{\beta}^1(\overline{\mathcal{B}_0(R)})$, we now estimate $d(\mathcal{F}_{\alpha}(p), \mathcal{F}_{\beta}(p))$. Since $(\phi_{\alpha}^1)^{-1}(p) \in \overline{\mathcal{B}_0(R)}$,

$$\|\mathfrak{f}_{\beta\alpha}^1 \circ (\phi_{\alpha}^1)^{-1}(p) - \mathfrak{f}_{\beta\alpha}^2 \circ (\phi_{\alpha}^1)^{-1}(p)\|_{C_1} \leq \varepsilon_1. \quad (6.1)$$

Note that

$$\mathfrak{f}_{\beta\alpha}^1 \circ (\phi_{\alpha}^1)^{-1}(p) = (\phi_{\beta}^2)^{-1} \mathcal{F}_{\beta}(p), \quad \mathfrak{f}_{\beta\alpha}^2 \circ (\phi_{\alpha}^1)^{-1}(p) = (\phi_{\beta}^2)^{-1} \circ \mathcal{F}_{\alpha}(p).$$

Hence, (6.1) implies that

$$\begin{aligned}
& F\left(\exp_{p_\beta^2}^{-1}(\mathcal{F}_\beta(p)) - \exp_{p_\beta^2}^{-1}(\mathcal{F}_\alpha(p))\right) \\
& \leq \sqrt{\Lambda} \left\| u_\beta^{-1} \circ \exp_{p_\beta^2}^{-1}(\mathcal{F}_\beta(p)) - u_\beta^{-1} \circ \exp_{p_\beta^2}^{-1}(\mathcal{F}_\alpha(p)) \right\| \\
& = \sqrt{\Lambda} \left\| (\phi_\beta^2)^{-1} \mathcal{F}_\beta(p) - (\phi_\beta^2)^{-1} \circ \mathcal{F}_\alpha(p) \right\| \leq \sqrt{\Lambda} \varepsilon_1.
\end{aligned} \tag{6.2}$$

Clearly, $\mathcal{F}_\beta(p) \in \phi_\beta^2(\overline{\mathcal{B}_0(R)})$, that is, $d(p_\beta^2, \mathcal{F}_\beta(p)) \leq \sqrt{\Lambda}R$. Since $\phi_\alpha^1(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^1(\overline{\mathcal{B}_0(R)}) \neq \emptyset$, we have $\phi_\alpha^2(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^2(\overline{\mathcal{B}_0(R)}) \neq \emptyset$ and $\mathcal{F}_\alpha(p) \in \phi_\beta^2(\overline{\mathcal{B}_0(3\Lambda^2 R)})$.

Let $\gamma(t)$, $t \in [0, 1]$ be a curve from $\mathcal{F}_\alpha(p)$ to $\mathcal{F}_\beta(p)$ such that $\exp_{p_\beta^2}^{-1}(\gamma(t))$ is a straight line. Since $\|\exp_{p_\beta^2}^{-1}(\gamma(t))\|_2 \leq 3\Lambda^2 R$, we have $\gamma(t) \in \overline{B_{p_\beta^2}^+(3\Lambda^{5/2}R)}$ and

$$\max_{t \in [0, 1]} \frac{\mathfrak{s}_{-k}(d(p_\beta^2, \gamma(t)))}{d(p_\beta^2, \gamma(t))} \leq \frac{\mathfrak{s}_{-k}(3\Lambda^{5/2}R)}{3\Lambda^{5/2}R}.$$

Now [Lemma Appendix A.2](#) and (6.2) implies that

$$d(\mathcal{F}_\alpha(p), \mathcal{F}_\beta(p)) \leq \frac{\mathfrak{s}_{-k}(3\Lambda^{\frac{5}{2}}R)}{3\Lambda R} \varepsilon_1 < 3\Lambda^{\frac{3}{2}} \varepsilon_1.$$

Step 2. Let $\eta : \mathbb{R}^+ \rightarrow [0, 1]$ be a smooth function with $|\eta'| \leq 4$ and

$$\eta(r) = \begin{cases} 1, & 0 \leq r \leq \frac{1}{2}, \\ (0, 1), & \frac{1}{2} < r < 1, \\ 0, & r \geq 1. \end{cases}$$

Given $p \in M_1$, set

$$\eta_\alpha(p) := \eta\left(\frac{\|(\phi_\alpha^1)^{-1}(p)\|}{R}\right), \quad \psi_\alpha(p) := \frac{\eta_\alpha(p)}{\sum_\alpha \eta_\alpha(p)}.$$

Thus, $\eta_\alpha(p) > 0$ (or $\psi_\alpha(p) > 0$) if and only if $p \in \phi_\alpha^1(\mathcal{B}_0(R))$.

Given $p \in M_1$, we define a vector field on M_2 by

$$V_p(x) := \sum_{\alpha=1}^N \psi_\alpha(p) \cdot (\exp_x^{-1} \mathcal{F}_\alpha(p)) = \sum_{\alpha \in N'_p} \psi_\alpha(p) \cdot (\exp_x^{-1} \mathcal{F}_\alpha(p)),$$

where $N'_p := \{\alpha : \psi_\alpha(p) \neq 0\} = \{\alpha : p \in \phi_\alpha^1(\mathcal{B}_0(R))\}$. Clearly, if $\alpha, \beta \in N'_p$, we have $\phi_\alpha^1(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^1(\overline{\mathcal{B}_0(R)}) \neq \emptyset$. By [Step 1](#), we have

$$d(\mathcal{F}_\alpha(p), \mathcal{F}_\beta(p)) < 3\Lambda^{\frac{3}{2}} \varepsilon_1, \quad d(\mathcal{F}_\beta(p), \mathcal{F}_\alpha(p)) < 3\Lambda^{\frac{3}{2}} \varepsilon_1.$$

Hence, one can find a forward ball of radius $3\Lambda^{\frac{3}{2}} \varepsilon_1$, say $B_2(3\Lambda^{\frac{3}{2}} \varepsilon_1)$, such that $\mathcal{F}_\alpha(p) \in B_2(3\Lambda^{\frac{3}{2}} \varepsilon_1)$ for all $\alpha \in N'_p$. Define a mass distribution $f_p : N'_p \rightarrow B_2(3\Lambda^{\frac{3}{2}} \varepsilon_1)$ by $f_p(\alpha) := \mathcal{F}_\alpha(p)$. The measure \mathbf{m}_p on N'_p is defined by $\mathbf{m}_p(\alpha) = \psi_\alpha(p)$. Then

$$V_p(x) = \int_{\alpha \in N'_p} \exp_x^{-1}(f_p(\alpha)) d\mathbf{m}_p(\alpha). \tag{6.3}$$

It follows from [Theorem 4.4](#) that there exists a unique $x_p \in B_2(3\Lambda^{\frac{3}{2}}\varepsilon_1)$ such that $V_p(x_p) = 0$. Now we define a map $\mathcal{F} : M_1 \rightarrow M_2$ by $\mathcal{F}(p) = x_p$. It is not hard to check that \mathcal{F} is well-defined.

Set

$$\mathcal{V}(p, x) := \left(\sum_{\alpha=1}^N \eta_{\alpha}(p) \right) \cdot V_p(x) = \sum_{\alpha=1}^N \eta_{\alpha}(p) \cdot (\exp_x^{-1} \mathcal{F}_{\alpha}(p)).$$

Note that $\mathcal{V}(p, x)$ is C^1 and $\mathcal{V}(p, \mathcal{F}(p)) = 0$ (cf. [Theorem 4.4](#)). The implicit function theorem then yields that

$$[d\mathcal{F}] = -(D_2\mathcal{V})^{-1} \cdot D_1\mathcal{V}, \quad (6.4)$$

where $D_i\mathcal{V}$ denotes the differential matrix of \mathcal{V} respect to the i -th variable, i.e.,

$$D_1\mathcal{V} := \left(\frac{\partial \mathcal{V}}{\partial p} \right), \quad D_2\mathcal{V} := \left(\frac{\partial \mathcal{V}}{\partial x} \right).$$

Note that

$$D_2\mathcal{V} = \left(\sum_{\alpha=1}^N \eta_{\alpha}(p) \right) \cdot \left(\frac{\partial V_p}{\partial x} \right).$$

[Theorem 4.4](#) then implies that $D_2\mathcal{V}$ is non-singular at $x = \mathcal{F}(p)$. Hence, (6.4) is well-defined.

We will show that \mathcal{F} is an imbedding. Eq. (6.4) implies that it is equivalent to show that $D_1\mathcal{V}|_{(p, \mathcal{F}(p))}$ is non-singular. Note that $\mathcal{V}(p, x) \in T_x M_2$ for any fixed x . Let $\gamma(t)$, $t \in (-\epsilon, \epsilon)$ be a smooth curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = X$. Thus,

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{V}(\gamma(t), x) = \mathcal{D}_1\mathcal{V}|_{(p, x)}(X) \in T_{\mathcal{V}(p, x)}(T_x M_2) \cong T_x M_2.$$

In the following, **we always set** $x = \mathcal{F}(p)$. Clearly, $D_1\mathcal{V}|_{(p, x)}$ is non-singular if and only if

$$\begin{aligned} 0 &\neq \left. \frac{d}{dt} \right|_{t=0} \mathcal{V}(\gamma(t), x) \\ &= \sum_{\alpha=1}^N \left[\left. \frac{d\eta_{\alpha}(\gamma(t))}{dt} \right|_{t=0} \cdot Y_{\alpha}(p) + \eta_{\alpha}(p) \cdot (\exp_x^{-1})_{* \mathcal{F}_{\alpha}(p)} \left. \frac{d\mathcal{F}_{\alpha}(\gamma(t))}{dt} \right|_{t=0} \right] \\ &= \sum_{\alpha=1}^N [\langle X, d\eta_{\alpha}|_p \rangle \cdot Y_{\alpha}(p) + \eta_{\alpha}(p) \cdot \langle \langle X, dY_{\alpha} \rangle \rangle|_p], \end{aligned} \quad (6.5)$$

where

$$Y_{\alpha}(p) := \exp_x^{-1} \mathcal{F}_{\alpha}(p) \in T_x M_2, \quad \langle \langle X, dY_{\alpha} \rangle \rangle|_p := (\exp_x^{-1})_{* \mathcal{F}_{\alpha}(p)} \left. \frac{d\mathcal{F}_{\alpha}(\gamma(t))}{dt} \right|_{t=0}.$$

The proof of (6.5) is divided into three steps, i.e., [Step 3–Step 5](#).

Step 3. We first estimate

$$\text{I} := \sum_{\alpha=1}^N \langle X, d\eta_{\alpha}|_p \rangle \cdot Y_{\alpha}(p).$$

Note that $d\eta_\alpha|_p \neq 0$ if and only if

$$\frac{R}{2} < \|(\phi_\alpha^1)^{-1}(p)\| < R \Leftrightarrow p \in \phi_\alpha^1(\mathcal{B}_0(R)) - \phi_\alpha^1(\overline{\mathcal{B}_0(R/2)}).$$

Thus,

$$I = \sum_{\alpha \in N_p''} \langle X, d\eta_\alpha|_p \rangle \cdot Y_\alpha(p),$$

where

$$N_p'' := \left\{ \alpha : p \in \phi_\alpha^1(\mathcal{B}_0(R)) - \phi_\alpha^1(\overline{\mathcal{B}_0(R/2)}) \right\} \subset N_p'.$$

Recall that $\{B_{p_\alpha^1}^+(R/(4\Lambda))\}_{\alpha=1}^N$ are disjoint. Thus, we have

$$\#N_p'' \leq \frac{\mu(B_p^+(\Lambda R))}{\min_{\alpha \in N_p''} \mu(B_{p_\alpha^1}^+(R/(4\Lambda)))} \leq \Lambda^{2n} \frac{\int_0^{\Lambda R} \mathfrak{s}_{-k}^{n-1}(t) dt}{\int_0^{\frac{R}{4\Lambda}} \mathfrak{s}_k^{n-1}(t) dt} \leq 2^{2n+1} \Lambda^{4n}. \quad (6.6)$$

For each $\alpha \in N_p''$, set $Z(t) := \exp_{p_\alpha^1}^{-1}(\gamma(t)) \in T_{p_\alpha^1} M_1$. Clearly, $F_1(Z(0)) = d(p_\alpha^1, p) \in (\frac{R}{2\sqrt{\Lambda}}, \sqrt{\Lambda}R)$ and $X = \left(\exp_{p_\alpha^1} \right)_{*d(p_\alpha^1, p)} \dot{Z}(0)$. Hence, it follows from [Lemma Appendix A.1](#) that

$$F_1(\dot{Z}(0)) \leq \Lambda \frac{d(p_\alpha^1, p)}{\mathfrak{s}_k(d(p_\alpha^1, p))} F_1(X) \leq \frac{\Lambda^{\frac{3}{2}} R}{\mathfrak{s}_k(\sqrt{\Lambda}R)} F_1(X),$$

which implies

$$\begin{aligned} |\langle X, d\eta_\alpha|_p \rangle| &\leq \frac{4}{R} \left| \frac{d}{dt} \Big|_{t=0} \|Z(t)\|_1 \right| = \frac{4}{R} \left| \frac{\frac{d}{dt} \Big|_{t=0} \|Z(t)\|_1^2}{2\|Z(0)\|_1} \right| \\ &\leq \frac{4}{R} \frac{\|Z(0)\|_1 \|\dot{Z}(0)\|_1}{\|Z(0)\|_1} \leq \frac{4\Lambda^2}{\mathfrak{s}_k(\sqrt{\Lambda}R)} F_1(X). \end{aligned} \quad (6.7)$$

Since $\alpha \in N_p'' \subset N_p'$, [Step 2](#) yields

$$F_2(Y_\alpha(p)) = d(x, \mathcal{F}_\alpha(p)) < 6\Lambda^2 \varepsilon_1,$$

which together with [\(6.6\)](#) and [\(6.7\)](#) furnishes that

$$F_2(I) = \sum_{\alpha \in N_p''} F_2(\langle X, d\eta_\alpha|_p \rangle \cdot Y_\alpha(p)) \leq \frac{2^{2n+6} \Lambda^{4n+5} \varepsilon_1}{\mathfrak{s}_k(\sqrt{\Lambda}R)} F_1(X).$$

Step 4. We now estimate

$$II := \sum_{\alpha=1}^N \eta_\alpha(p) \cdot \langle \langle X, dY_\alpha|_p \rangle \rangle = \sum_{\alpha \in N_p'} \eta_\alpha(p) \cdot \langle \langle X, dY_\alpha|_p \rangle \rangle.$$

Given $\alpha \in N_p'$. Since $(u_\alpha^2 \circ (u_\alpha^1)^{-1})_* = u_\alpha^2 \circ (u_\alpha^1)^{-1}$, for each $Z \in TM_1$, we have

$$F_2((u_\alpha^2 \circ (u_\alpha^1)^{-1})_* Z) \geq \frac{1}{\sqrt{\Lambda}} \|(u_\alpha^1)^{-1} Z\| \geq \frac{1}{\Lambda} F_1(Z). \quad (6.8)$$

Recall that

$$Y_\alpha(p) = \exp_x^{-1} \mathcal{F}_\alpha(p) = \exp_x^{-1} \circ \exp_{p_\alpha^2} \circ u_\alpha^2 \circ (u_\alpha^1)^{-1} \circ \exp_{p_\alpha^1}^{-1}(p).$$

Thus, (6.8) together with Lemma Appendix A.1 yields that

$$\begin{aligned} F_2(\langle\langle X, dY_\alpha|_p \rangle\rangle) &= F_2\left(\exp_{x*}^{-1} \circ \exp_{p_\alpha^2*} \circ (u_\alpha^2 \circ (u_\alpha^1)^{-1})_* \circ \exp_{p_\alpha^1*}^{-1} X\right) \\ &\geq \frac{d(x, \mathcal{F}_\alpha(p))}{\Lambda \mathfrak{s}_{-k}(d(x, \mathcal{F}_\alpha(p)))} F_2\left(\exp_{p_\alpha^2*} \circ (u_\alpha^2 \circ (u_\alpha^1)^{-1})_* \circ \exp_{p_\alpha^1*}^{-1} X\right) \\ &\geq \frac{d(x, \mathcal{F}_\alpha(p))}{\Lambda^4 \mathfrak{s}_{-k}(d(x, \mathcal{F}_\alpha(p)))} \frac{\mathfrak{s}_k(F_2(u_\alpha^2 \circ (u_\alpha^1)^{-1} \circ \exp_{p_\alpha^1}^{-1}(p)))}{F_2(u_\alpha^2 \circ (u_\alpha^1)^{-1} \circ \exp_{p_\alpha^1}^{-1}(p))} \frac{d(p_\alpha^1, p)}{\mathfrak{s}_{-k}(d(p_\alpha^1, p))} F_1(X) \\ &\geq \frac{1}{\Lambda^5} \frac{R}{\mathfrak{s}_{-k}(R)} \frac{\mathfrak{s}_k(\Lambda^{\frac{3}{2}} R)}{\mathfrak{s}_{-k}(\sqrt{\Lambda} R)} F_1(X) \geq \frac{1}{\Lambda^5} (1 - kR^2) F_1(X). \end{aligned} \quad (6.9)$$

Since $\{B_{p_\alpha^1}^+(R/(2\Lambda^{\frac{3}{2}}))\}_{\alpha=1}^N$ is a covering, there exists β such that $d(p_\beta^1, p) < R/(2\sqrt{\Lambda})$, which implies that $\|(\phi_\beta^1)^{-1}(p)\| < R/2$. Hence, $\beta \in N'_p$ and $\sum_{\alpha \in N'_p} \eta_\alpha(p) \geq \eta_\beta(p) = 1$. We claim that

$$F_2(\langle\langle X, dY_\beta|_p \rangle\rangle) - \sup_{\alpha \in N'_p} F_2(\langle\langle X, dY_\beta|_p - dY_\alpha|_p \rangle\rangle) > 0, \quad (6.10)$$

which will be proved in Step 5. Here,

$$F_2(\langle\langle X, dY_\beta|_p - dY_\alpha|_p \rangle\rangle) := F_2(\langle\langle X, dY_\beta|_p \rangle\rangle - \langle\langle X, dY_\alpha|_p \rangle\rangle).$$

By (6.9) and (6.10), we have

$$\begin{aligned} F_2(\text{II}) &= F_2\left(\sum_{\alpha \in N'_p} \eta_\alpha(p) \cdot \langle\langle X, dY_\alpha|_p \rangle\rangle\right) \\ &\geq \sum_{\alpha \in N'_p} \eta_\alpha(p) \cdot F_2(\langle\langle X, dY_\beta|_p \rangle\rangle) - \sum_{\alpha \in N'_p} \eta_\alpha(p) \cdot F_2(\langle\langle X, dY_\beta|_p - dY_\alpha|_p \rangle\rangle) \\ &\geq \left(\sum_{\alpha \in N'_p} \eta_\alpha(p)\right) \left[F_2(\langle\langle X, dY_\beta|_p \rangle\rangle) - \sup_{\alpha \in N'_p} F_2(\langle\langle X, dY_\beta|_p - dY_\alpha|_p \rangle\rangle)\right] \\ &\geq F_2(\langle\langle X, dY_\beta|_p \rangle\rangle) - \sup_{\alpha \in N'_p} F_2(\langle\langle X, dY_\beta|_p - dY_\alpha|_p \rangle\rangle) \\ &\geq \frac{(1 - kR^2)}{\Lambda^5} F_1(X) - \sup_{\alpha \in N'_p} F_2(\langle\langle X, dY_\beta|_p - dY_\alpha|_p \rangle\rangle). \end{aligned} \quad (6.11)$$

In Step 5, we will show (6.10) and estimate (6.11).

Step 5. To estimate $F_2(\langle\langle X, dY_\beta|_p - dY_\alpha|_p \rangle\rangle)$ for $\alpha, \beta \in N'_p$, we just need to estimate the following three items

$$(1) \quad F_2 \left(\langle \langle X, dY_\alpha|_p \rangle \rangle - P_{\mathcal{F}_\alpha(p),x} \circ P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \circ u_\alpha^2 \circ (u_\alpha^1)^{-1} \circ P_{p,p_\alpha^1} X \right); \quad (6.12)$$

$$(2) \quad F_2 \left(P_{\mathcal{F}_\alpha(p),x} \circ P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \circ u_\alpha^2 \circ (u_\alpha^1)^{-1} \circ P_{p,p_\alpha^1} X - \langle \langle X, dY_\alpha|_p \rangle \rangle \right); \quad (6.13)$$

$$(3) \quad F_2 \left(P_{\mathcal{F}_\alpha(p),x} \circ P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \circ u_\alpha^2 \circ (u_\alpha^1)^{-1} \circ P_{p,p_\alpha^1} X \right. \\ \left. - P_{\mathcal{F}_\beta(p),x} \circ P_{p_\beta^2, \mathcal{F}_\beta(p)} \circ u_\beta^2 \circ (u_\beta^1)^{-1} \circ P_{p,p_\beta^1} X \right). \quad (6.14)$$

Here, $P_{p,q}$ denotes the parallel transformation along the normal minimal geodesic from p to q .

We first estimate (6.12) and (6.13). Given $\alpha \in N'_p$, set $s_1 := d(p_\alpha^1, p)$ and $s_2 := d(p_\alpha^2, \mathcal{F}_\alpha(p))$. Clearly, there exists $Y \in T_{p_\alpha^1} M_1$ such that

$$\left(\exp_{p_\alpha^1} \right)_{*\rho_\alpha^1(p) \cdot \nabla \rho_\alpha^1(p)} Y = X,$$

where $\rho_\alpha^1(\cdot) := d(p_\alpha^1, \cdot)$. Now let

$$\overline{X} := u_\alpha^2 \circ (u_\alpha^1)^{-1} \circ P_{p,p_\alpha^1}(X) \in T_{p_\alpha^2} M_2, \quad \overline{Y} := u_\alpha^2 \circ (u_\alpha^1)^{-1}(Y) \in T_{p_\alpha^2} M_2,$$

$$l := d(x, \mathcal{F}_\alpha(p)), \quad J_Y(s_1) := \left(\exp_{p_\alpha^1} \right)_{*s_1 \nabla \rho_\alpha^1(p)}(s_1 Y) \in T_p M_1,$$

$$J_{\overline{Y}}(s_2) := \left(\exp_{p_\alpha^2} \right)_{*s_2 \nabla \rho_\alpha^2(\mathcal{F}_\alpha(p))}(s_2 \overline{Y}) \in T_{\mathcal{F}_\alpha(p)} M_2.$$

Note that there exists $Z \in T_x M_2$ with

$$(\exp_x)_{*ly} Z = \frac{1}{s_2} J_{\overline{Y}}(s_2) = \left(\exp_{p_\alpha^2} \right)_{*s_2 \nabla \rho_\alpha^2(\mathcal{F}_\alpha(p))}(\overline{Y}),$$

where $y := \nabla \rho_x(\mathcal{F}_\alpha(p))$ and $\rho_x(\cdot) := d(x, \cdot)$. Thus, we have

$$\begin{aligned} Z &= (\exp_x)_{*ly}^{-1} \left(\exp_{p_\alpha^2} \right)_{*s_2 \nabla \rho_\alpha^2(\mathcal{F}_\alpha(p))}(\overline{Y}) \\ &= (\exp_x)_{*ly}^{-1} \left(\exp_{p_\alpha^2} \right)_{*s_2 \nabla \rho_\alpha^2(\mathcal{F}_\alpha(p))} u_\alpha^2 \circ (u_\alpha^1)^{-1} \left(\exp_{p_\alpha^1} \right)_{*\rho_\alpha^1(p) \cdot \nabla \rho_\alpha^1(p)}^{-1} X \\ &= \langle \langle X, dY_\alpha|_p \rangle \rangle. \end{aligned}$$

Since F_i is Berwaldian,

$$\begin{aligned} (6.12) &\leq F_2 \left(P_{\mathcal{F}_\alpha(p),x} \left(\frac{1}{s_2} J_{\overline{Y}}(s_2) \right) - P_{\mathcal{F}_\alpha(p),x} \circ P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \overline{X} \right) \\ &\quad + F_2 \left(Z - P_{\mathcal{F}_\alpha(p),x} \left(\frac{1}{s_2} J_{\overline{Y}}(s_2) \right) \right) \\ &\leq F_2 \left(\frac{1}{s_2} J_{\overline{Y}}(s_2) - P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \overline{Y} \right) + F_2 (P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \overline{Y} - P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \overline{X}) \\ &\quad + F_2 \left(Z - P_{\mathcal{F}_\alpha(p),x} \left(\frac{1}{s_2} J_{\overline{Y}}(s_2) \right) \right) \\ &\leq \sqrt{\Lambda} \left\| \frac{1}{s_2} J_{\overline{Y}}(s_2) - P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \overline{Y} \right\|_{T_1} + F_2 (\overline{Y} - \overline{X}) \\ &\quad + \sqrt{\Lambda} \left\| Z - P_{x, \mathcal{F}_\alpha(p)}^{-1} \left(\frac{1}{s_2} J_{\overline{Y}}(s_2) \right) \right\|_{T_2} \end{aligned} \quad (6.15)$$

where T_1 is the velocity of the normal geodesic from p_α^2 to $\mathcal{F}_\alpha(p)$, and T_2 is the velocity of the normal geodesic from x to $\mathcal{F}_\alpha(p)$.

Since F_i is Berwaldian, $P_{p,p_\alpha}^{-1} = P_{p_\alpha,p}^{-1}$ and $P_{x,\mathcal{F}_\alpha(p)}^{-1} = P_{\mathcal{F}_\alpha(p),x}$. And it is easy to see that $s_1 \leq \sqrt{\Lambda}R$, $s_2 \leq \sqrt{\Lambda}R$ and $l < R$. Thus, Lemma Appendix A.1 together with Lemma Appendix A.5 and Corollary Appendix A.6 yields

$$\left\| \frac{1}{s_2} J_{\bar{Y}}(s_2) - P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \bar{Y} \right\|_{T_1} \leq \frac{\Lambda^2 R}{\mathfrak{s}_k(\sqrt{\Lambda}R)} \left(\frac{\mathfrak{s}_{-k}(\sqrt{\Lambda}R)}{\sqrt{\Lambda}R} - 1 \right) F_1(X), \quad (6.16)$$

$$F_2(\bar{Y} - \bar{X}) \leq \frac{\Lambda^2 R}{\mathfrak{s}_k(\sqrt{\Lambda}R)} \left(\frac{\mathfrak{s}_{-k}(\sqrt{\Lambda}R)}{\sqrt{\Lambda}R} - 1 \right) F_1(X), \quad (6.17)$$

$$\left\| Z - P_{x, \mathcal{F}_\alpha(p)}^{-1} \left(\frac{1}{s_2} J_{\bar{Y}}(s_2) \right) \right\|_{T_2} \leq \frac{\Lambda^2 R}{\mathfrak{s}_k(R)} \left(\frac{\mathfrak{s}_{-k}(R)}{R} - 1 \right) \frac{\mathfrak{s}_{-k}(\sqrt{\Lambda}R)}{\mathfrak{s}_k(\sqrt{\Lambda}R)} F_1(X). \quad (6.18)$$

By (6.15), (6.16), (6.17) and (6.18), we have

$$(6.12) \leq \frac{3\Lambda^3 R}{\mathfrak{s}_k(\sqrt{\Lambda}R)} \left(\frac{\mathfrak{s}_{-k}(\sqrt{\Lambda}R)}{\sqrt{\Lambda}R} - 1 \right) \frac{\mathfrak{s}_{-k}(\sqrt{\Lambda}R)}{\mathfrak{s}_k(\sqrt{\Lambda}R)} \cdot F_1(X). \quad (6.19)$$

Similarly, one can show

$$(6.13) \leq \frac{3\Lambda^3 R}{\mathfrak{s}_k(\sqrt{\Lambda}R)} \left(\frac{\mathfrak{s}_{-k}(\sqrt{\Lambda}R)}{\sqrt{\Lambda}R} - 1 \right) \frac{\mathfrak{s}_{-k}(\sqrt{\Lambda}R)}{\mathfrak{s}_k(\sqrt{\Lambda}R)} \cdot F_1(X). \quad (6.20)$$

We now estimate (6.14). Given $\alpha, \beta \in N'_p$, set

$$\begin{aligned} X_\alpha &:= P_{p,p_\alpha^1} X \in T_{p_\alpha^1} M_1, \quad X_\beta := P_{p,p_\beta^1} X \in T_{p_\beta^1} M_1, \quad X'_\alpha := P_{p_\alpha^1,p_\beta^1} X_\alpha \in T_{p_\beta^1} M_1, \\ \bar{X}_\beta &:= u_\beta^2 \circ (u_\beta^1)^{-1}(X_\beta) \in T_{p_\beta^2} M_2, \quad \bar{X}_\alpha := u_\alpha^2 \circ (u_\alpha^1)^{-1}(X_\alpha) \in T_{p_\alpha^2} M_2, \\ \bar{X}'_\alpha &:= P_{p_\beta^2,p_\alpha^2} \circ u_\beta^2 \circ (u_\beta^1)^{-1}(X'_\alpha) \in T_{p_\alpha^2} M_2, \quad \bar{X}'_\beta := P_{p_\beta^2,p_\alpha^2} \circ u_\beta^2 \circ (u_\beta^1)^{-1}(X_\beta) \in T_{p_\alpha^2} M_2. \end{aligned}$$

Thus, we have

$$\begin{aligned} (6.14) &\leq F_2(P_{\mathcal{F}_\alpha(p),x} \circ P_{p_\alpha^2,\mathcal{F}_\alpha(p)} \circ \bar{X}_\alpha - P_{\mathcal{F}_\alpha(p),x} \circ P_{p_\alpha^2,\mathcal{F}_\alpha(p)} \circ \bar{X}'_\alpha) \\ &\quad + F_2(P_{\mathcal{F}_\alpha(p),x} \circ P_{p_\alpha^2,\mathcal{F}_\alpha(p)} \circ \bar{X}'_\alpha - P_{\mathcal{F}_\beta(p),x} \circ P_{p_\beta^2,\mathcal{F}_\beta(p)} \circ \bar{X}_\beta) \\ &\leq F_2(\bar{X}_\alpha - \bar{X}'_\alpha) + F_2(P_{\mathcal{F}_\alpha(p),x} \circ P_{p_\alpha^2,\mathcal{F}_\alpha(p)} \bar{X}'_\alpha - P_{\mathcal{F}_\alpha(p),x} \circ P_{p_\alpha^2,\mathcal{F}_\alpha(p)} \bar{X}'_\beta) \\ &\quad + F_2(P_{\mathcal{F}_\alpha(p),x} \circ P_{p_\alpha^2,\mathcal{F}_\alpha(p)} \bar{X}'_\beta - P_{\mathcal{F}_\beta(p),x} \circ P_{p_\beta^2,\mathcal{F}_\beta(p)} \bar{X}_\beta) \\ &= F_2(\bar{X}_\alpha - \bar{X}'_\alpha) + F_2(\bar{X}'_\alpha - \bar{X}'_\beta) \\ &\quad + F_2(P_{\mathcal{F}_\alpha(p),x} \circ P_{p_\alpha^2,\mathcal{F}_\alpha(p)} \bar{X}'_\beta - P_{\mathcal{F}_\beta(p),x} \circ P_{p_\beta^2,\mathcal{F}_\beta(p)} \bar{X}_\beta) \end{aligned} \quad (6.21)$$

Firstly, we have

$$\begin{aligned} F(\bar{X}_\alpha - \bar{X}'_\alpha) &= F(P_{p_\beta^2,p_\alpha^2}^{-1} \circ u_\alpha^2 \circ (u_\alpha^1)^{-1}(X_\alpha) - u_\beta^2 \circ (u_\beta^1)^{-1}(X'_\alpha)) \\ &\leq \sqrt{\Lambda} \left\| (u_\beta^2)^{-1} \circ P_{p_\alpha^2,p_\beta^2} \circ u_\alpha^2 \circ (u_\alpha^1)^{-1}(X_\alpha) - (u_\beta^1)^{-1}(X'_\alpha) \right\| \\ &= \sqrt{\Lambda} \left\| \mathfrak{g}_{\beta\alpha}^2((u_\alpha^1)^{-1}(X_\alpha)) - \mathfrak{g}_{\beta\alpha}^1((u_\alpha^1)^{-1}(X_\alpha)) \right\| \leq \Lambda \cdot \varepsilon_2 \cdot F_1(X). \end{aligned} \quad (6.22)$$

Secondly, [Lemma Appendix B.3](#) yields

$$F_2(\overline{X'_\alpha} - \overline{X'_\beta}) \leq \Lambda \cdot F_1(X'_\alpha - X'_\beta) \leq \mathfrak{C}(n, k, \Lambda) \cdot \Lambda^3 \cdot F_1(X) \cdot R^2, \quad (6.23)$$

where $\mathfrak{C}(n, k, \Lambda)$ is the constant as in [Lemma Appendix B.3](#). Since $\alpha, \beta \in N'_p$, $\phi_\alpha^2(\overline{\mathcal{B}_0(R)}) \cap \phi_\beta^2(\overline{\mathcal{B}_0(R)}) \neq \emptyset$ and hence, $d(p_\alpha^2, p_\beta^2) < 2\Lambda R$. By [Lemma Appendix B.3](#) again, we have

$$\begin{aligned} & F_2 \left(P_{\mathcal{F}_\alpha(p), x} \circ P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \overline{X'_\beta} - P_{\mathcal{F}_\beta(p), x} \circ P_{p_\beta^2, \mathcal{F}_\beta(p)} \overline{X'_\beta} \right) \\ & \leq F_2 \left(P_{\mathcal{F}_\alpha(p), x} \circ P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \circ P_{p_\beta^2, p_\alpha^2} \overline{X'_\beta} - P_{p_\beta^2, x} \overline{X'_\beta} \right) \\ & \quad + F_2 \left(P_{p_\beta^2, x} \overline{X'_\beta} - P_{\mathcal{F}_\beta(p), x} \circ P_{p_\beta^2, \mathcal{F}_\beta(p)} \overline{X'_\beta} \right) \\ & \leq F_2 \left(P_{\mathcal{F}_\alpha(p), x} \circ P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \circ P_{p_\beta^2, p_\alpha^2} \overline{X'_\beta} - P_{p_\beta^2, x} \overline{X'_\beta} \right) + 4\mathfrak{C}(n, k, \Lambda) \cdot \Lambda^{\frac{5}{2}} \cdot F_1(X) \cdot R^2 \\ & \leq F_2 \left(P_{\mathcal{F}_\alpha(p), x} \circ P_{p_\alpha^2, \mathcal{F}_\alpha(p)} \circ P_{p_\beta^2, p_\alpha^2} \overline{X'_\beta} - P_{\mathcal{F}_\alpha(p), x} \circ P_{p_\beta^2, \mathcal{F}_\alpha(p)} \overline{X'_\beta} \right) \\ & \quad + F_2 \left(P_{\mathcal{F}_\alpha(p), x} \circ P_{p_\beta^2, \mathcal{F}_\alpha(p)} \overline{X'_\beta} - P_{p_\beta^2, x} \overline{X'_\beta} \right) + 4\mathfrak{C}(n, k, \Lambda) \cdot \Lambda^{\frac{5}{2}} \cdot F_1(X) \cdot R^2 \\ & \leq 29\Lambda^3 \cdot \mathfrak{C}(n, k, \Lambda) \cdot R^2 \cdot F_1(X). \end{aligned} \quad (6.24)$$

Now by (6.21), (6.22), (6.23) and (6.24), we obtain

$$(6.14) \leq [30\Lambda^3 \mathfrak{C}(n, k, \Lambda) R^2 + \Lambda \varepsilon_2] F_1(X). \quad (6.25)$$

The triangle inequality then yields

$$\begin{aligned} & F_2(\langle \langle X, dY_\beta|_p - dY_\alpha|_p \rangle \rangle) \leq (6.12)_\beta + (6.14)_{\beta\alpha} + (6.13)_\alpha \\ & \leq (6.19)_\beta + (6.25)_{\beta\alpha} + (6.20)_\alpha = \mathcal{C}_3(n, k, \Lambda, R, \varepsilon_2) F_1(X), \end{aligned}$$

which together with (6.9) and (6.11) yields (6.10) and

$$F_2(\text{II}) \geq \left[\frac{(1 - kR^2)}{\Lambda^5} - \mathcal{C}_3(n, k, \Lambda, R, \varepsilon_2) \right] \cdot F_1(X). \quad (6.26)$$

[Step 3](#) furnishes that

$$F_2(-\text{I}) \leq \sqrt{\Lambda} \cdot F_2(\text{I}) \leq \frac{2^{2n+6} \Lambda^{4n+5+\frac{1}{2}}}{\mathfrak{s}_k(\sqrt{\Lambda}R)} \varepsilon_1 F_1(X). \quad (6.27)$$

Thus, (6.5) together with (6.26) and (6.27) yields

$$\begin{aligned} & F_2 \left(\left. \frac{d}{dt} \right|_{t=0} \mathcal{V}(\gamma(t), x) \right) = F_2(\text{I} + \text{II}) \geq F_2(\text{II}) - F_2(-\text{I}) \\ & \geq \left[\frac{(1 - kR^2)}{\Lambda^5} - \mathcal{C}_3(n, k, \Lambda, R, \varepsilon_2) - \frac{2^{2n+6} \Lambda^{4n+6}}{\mathfrak{s}_k(\sqrt{\Lambda}R)} \varepsilon_1 \right] \cdot F_1(X) > 0, \end{aligned}$$

which implies that \mathcal{F}_* is nonsingular (see [Step 2](#)).

Step 6. Since \mathcal{F} is a local diffeomorphism, $\mathcal{F} : M_1 \rightarrow M_2$ is a covering projection (cf. [3, Theorem 9.2.1]).

Let $\mathcal{G} : M_2 \rightarrow M_1$ be the map constructed as \mathcal{F} . Given any point $p \in M_1$, there exists a point $p_\alpha^1 \in M_1$ such that $d(p_\alpha^1, p) < R/(2\Lambda^{\frac{3}{2}})$, which implies

$$d(\mathcal{F}_\alpha(p_\alpha^1), \mathcal{F}_\alpha(p)) = F_2(u_\alpha^2 \circ (\phi_\alpha^1)^{-1}(p)) \leq \Lambda d(p_\alpha^1, p) < R/(2\sqrt{\Lambda}). \quad (6.28)$$

Since $\alpha \in N'_p$, $d(\mathcal{F}_\alpha(p), \mathcal{F}(p)) < R/(2\sqrt{\Lambda})$, which together with (6.28) yields

$$d(p_\alpha^2, \mathcal{F}(p)) = d(\mathcal{F}_\alpha(p_\alpha^1), \mathcal{F}(p)) < R/\sqrt{\Lambda},$$

that is, $\mathcal{F}(p) \in \phi_\alpha^2(\overline{\mathcal{B}_0(R)})$. Set $\mathcal{G}_\alpha := \phi_\alpha^1 \circ (\phi_\alpha^2)^{-1} : \phi_\alpha^2(\overline{\mathcal{B}_0(R)}) \rightarrow \phi_\alpha^1(\overline{\mathcal{B}_0(R)})$. The same argument as before yields that

$$d(p_\alpha^1, \mathcal{G} \circ \mathcal{F}(p)) \leq d(\mathcal{G}_\alpha(p_\alpha^2), \mathcal{G}_\alpha(\mathcal{F}(p))) + d(\mathcal{G}_\alpha(\mathcal{F}(p)), \mathcal{G}(\mathcal{F}(p))) < 2\sqrt{\Lambda}R,$$

and therefore, $\mathcal{G} \circ \mathcal{F}(p) \in B_p^+(3\sqrt{\Lambda}R)$. Likewise, one can show $\mathcal{F} \circ \mathcal{G}(q) \in B_q^+(3\sqrt{\Lambda}R)$. That is, both $\mathcal{G} \circ \mathcal{F}$ and $\mathcal{F} \circ \mathcal{G}$ map every point to a convex neighborhood of itself and hence, they are homotopic to the identity. Now we conclude that \mathcal{F} and \mathcal{G} are diffeomorphisms. \square

Acknowledgments

This work was supported by National Natural Science Foundation of China (No. 11501202), Tian Yuan Foundation (No. 11426108) and the Fundamental Research Funds for the Central Universities.

Appendix A. Some estimates for Jacobi fields

In this section, we always assume that (M, F) is a compact Finsler n -manifold with $\Lambda_F \leq \Lambda$ and $|\mathbf{K}_M| \leq k$. Given $y \in SM$, we use $\gamma_y(t)$ to denote the normal geodesic with $\dot{\gamma}_y(0) = y$.

The following lemma follows from the Finslerian version of the Rauch theorem [3, Theorem 9.6.1] directly.

Lemma Appendix A.1. *For any $y \in S_p M$ and $X \in T_p M - \{0\}$, we have*

$$\frac{\mathfrak{s}_k(t)}{t} \leq \frac{\|(\exp_p)_{*ty} X\|_T}{\|X\|_T} \leq \frac{\mathfrak{s}_{-k}(t)}{t}, \quad t \in \left[0, \frac{\pi}{2\sqrt{k}}\right],$$

where $\|\cdot\|_T := g_T(\cdot, \cdot)$ and $T := \dot{\gamma}_y(t)$.

Lemma Appendix A.2. *Given three points $p, q, x \in M$. Let $\gamma(s)$, $s \in [0, 1]$ be a smooth curve from p to q such that $d(x, \gamma(s)) < \min\{i_M, \frac{\pi}{2\sqrt{k}}\}$ for all s . Set $P := \exp_x^{-1}(p)$ and $Q := \exp_x^{-1}(q)$.*

(1) *Suppose that $\gamma(s)$ is a minimal geodesic from p to q . Then*

$$\frac{1}{\Lambda} \min_{s \in [0, 1]} \frac{\mathfrak{s}_k(d(x, \gamma(s)))}{d(x, \gamma(s))} F(Q - P) \leq d(p, q).$$

(2) *Suppose that $\exp_x^{-1}(\gamma(s))$ is a straight line from P to Q . Then*

$$d(p, q) \leq \Lambda \max_{s \in [0, 1]} \frac{\mathfrak{s}_{-k}(d(x, \gamma(s)))}{d(x, \gamma(s))} F(Q - P).$$

Proof. For each $s \in [0, 1]$, there exists $V_s \in T_x M$ such that $\exp_x V_s = \gamma(s)$. We define a geodesic variation

$$\sigma(t, s) := \exp_p(tV_s), \quad (t, s) \in [0, 1] \times [0, 1].$$

Set

$$T := \frac{\partial \sigma}{\partial t} = (\exp_x)_{*tV_s} V_s, \quad U := \frac{\partial \sigma}{\partial s} = (\exp_x)_{*tV_s} (t\dot{V}_s),$$

where $\dot{V}_s := \frac{dV_s}{ds}$. It follows from [Lemma Appendix A.1](#) that

$$\frac{1}{\Lambda} \min_{s \in [0, 1]} \frac{s_k(d(x, \gamma(s)))}{d(x, \gamma(s))} \int_0^1 F(\dot{V}_s) ds \leq \int_0^1 F(U(1, s)) ds \leq \Lambda \max_{s \in [0, 1]} \frac{s_{-k}(d(x, \gamma(s)))}{d(x, \gamma(s))} \int_0^1 F(\dot{V}_s) ds.$$

(1) Suppose that $\gamma(s)$ is a minimal geodesic from p to q . Note that $U(1, s) = \dot{\gamma}(s)$. Hence,

$$F(Q - P) \leq \int_0^1 F(\dot{V}_s) ds, \quad \int_0^1 F(U(1, s)) ds = d(p, q).$$

(2) Suppose that $\exp_x^{-1}(\gamma(s))$ is a straight line from P to Q . Thus,

$$F(Q - P) = \int_0^1 F(\dot{V}_s) ds, \quad \int_0^1 F(U(1, s)) ds \geq d(p, q). \quad \square$$

Recall the definition of curvature operator \mathcal{R} of a Finsler manifold (cf. [\[28\]](#)): Given $p \in M$ and $y \in S_p M$. Let $P_{t;y}$ denote the parallel transformation along the geodesic $\gamma_y(t)$ from $T_p M$ to $T_{\gamma_y(t)} M$. The curvature operator \mathcal{R} is defined by

$$\mathcal{R}(t; y) := P_{t;y}^{-1} \circ R_T \circ P_{t;y} : y^\perp \rightarrow y^\perp,$$

where $R_T := R_T(\cdot, T)T$ and $y^\perp := \{W \in T_p M : g_y(y, W) = 0\}$.

The following result follows from $|\mathbf{K}_M| \leq k$ directly.

Lemma Appendix A.3. *Set*

$$\|\mathcal{R}(t; y)\| := \sup_{X \in y^\perp - \{0\}} \frac{\|\mathcal{R}(t; y)X\|_y}{\|X\|_y},$$

where $\|\cdot\|_y := \sqrt{g_y(\cdot, \cdot)}$. Thus, $\|\mathcal{R}(t; y)\| \leq k$.

Using [Lemma Appendix A.3](#) and the same argument as in [\[6, Theorem IX. 4.1, Corollary IX. 4.3\]](#), one can show that

Lemma Appendix A.4. *Consider the vector equation of $\eta(t) \in y^\perp$:*

$$\eta'' + \mathcal{R}(t, y)\eta = 0.$$

If $\eta(0) = 0$, then

$$\|\eta(s) - s\eta'(0)\|_y \leq \|\eta'(0)\|_y \cdot (\mathfrak{s}_{-k}(s) - s)$$

for all $s > 0$, where $\|\cdot\|_y := \sqrt{g_y(\cdot, \cdot)}$.

Let $\mathcal{A}(t, y)$ be the solution of the matrix (or linear transformation) ordinary differential equation on y^\perp :

$$\begin{cases} \mathcal{A}'' + \mathcal{R}(t; y)\mathcal{A} = 0, \\ \mathcal{A}(0; y) = 0, \\ \mathcal{A}'(0; y) = \mathcal{I}. \end{cases}$$

Then $P_{t;y}\mathcal{A}(t, y)X = (\exp_p)_{*ty}tX$, for any $X \in y^\perp$. Now we have the following

Lemma Appendix A.5. *Given $y \in S_pM$ and $X \in T_pM$, we have*

$$\|(\exp_p)_{*ty}X - P_{t;y}X\|_T \leq \left(\frac{\mathfrak{s}_{-k}(t)}{t} - 1\right) \|X\|_T,$$

where $T := \dot{\gamma}_y(t)$ and $\|\cdot\|_T := \sqrt{g_T(\cdot, \cdot)}$.

Proof. If $X = ky$, then $(\exp_p)_{*ty}X = P_{t;y}X$. Hence, it suffices to show the lemma in the case of $X \in y^\perp$. Set $\eta := \mathcal{A}(t; y)X$. Clearly, $\eta(0) = 0$ and $\eta'(0) = X$. By [Lemma Appendix A.4](#), we have

$$\|\mathcal{A}(t; y)X - tX\|_T \leq (\mathfrak{s}_{-k}(t) - t) \|X\|_T.$$

It should be noted that $\|W\|_T = \|P_{t;y}W\|_T$ for any $W \in T_pM$. Hence,

$$\left\| \frac{P_{t;y}\mathcal{A}(t; y)X}{t} - P_{t;y}X \right\|_T \leq \left(\frac{\mathfrak{s}_{-k}(t)}{t} - 1 \right) \|X\|_T. \quad \square$$

Corollary Appendix A.6. *Given $y \in S_pM$ and $Y \in T_{\dot{\gamma}_y(t)}M$, where $0 \leq t < \frac{\pi}{2\sqrt{k}}$. Then*

$$\|(\exp_p)_{*ty}^{-1}Y - P_{t;y}^{-1}Y\|_T \leq \frac{t}{\mathfrak{s}_k(t)} \left(\frac{\mathfrak{s}_{-k}(t)}{t} - 1 \right) \|Y\|_T,$$

where $T := \dot{\gamma}_y(t)$ and $\|\cdot\|_T := \sqrt{g_T(\cdot, \cdot)}$.

Proof. Since $0 \leq t < \frac{\pi}{2\sqrt{k}}$, there exists a unique $X \in T_pM$ such that

$$Y = (\exp_p)_{*ty}X.$$

Then [Lemma Appendix A.1](#) together with [Lemma Appendix A.5](#) yields that

$$\begin{aligned} \frac{\mathfrak{s}_k(t)}{t} \|X - P_{t;y}^{-1}Y\|_T &\leq \|(\exp_p)_{*ty}(X - P_{t;y}^{-1}Y)\|_T = \|Y - (\exp_p)_{*ty}P_{t;y}^{-1}Y\|_T \\ &= \|P_{t;y}P_{t;y}^{-1}Y - (\exp_p)_{*ty}P_{t;y}^{-1}Y\|_T \\ &\leq \left(\frac{\mathfrak{s}_{-k}(t)}{t} - 1 \right) \|P_{t;y}^{-1}Y\|_T = \left(\frac{\mathfrak{s}_{-k}(t)}{t} - 1 \right) \|Y\|_T. \quad \square \end{aligned}$$

Lemma Appendix A.7. Let $\gamma(t)$, $t \geq 0$ be a normal geodesic. Then there exists a positive constant $\mathfrak{t} = \mathfrak{t}(n, k, \Lambda)$ such that for any Jacobi field $J(t)$ along γ with $J(0) = 0$, we have

$$\|J(t) - tJ'(t)\|_T \leq \frac{1}{20\Lambda} \|J(t)\|_T, \quad t \in [0, \mathfrak{t}],$$

where $T := \dot{\gamma}(t)$ and $\|\cdot\|_T := \sqrt{g_T(\cdot, \cdot)}$.

Proof. Clearly, we have

$$\frac{d}{dt} g_T(J(t) - tJ'(t), J(t) - tJ'(t)) \leq 2\|tJ''(t)\|_T \cdot \|J(t) - tJ'(t)\|_T,$$

which implies that

$$\frac{d}{dt} \|J(t) - tJ'(t)\|_T \leq \|tJ''(t)\|_T = \|tR_T(J, T)T\|_T.$$

Lemma Appendix A.3 implies that

$$\|R_T(J, T)T\|_T = \|R_T \circ P_{t;y} \mathcal{A}J(0)\|_T = \|\mathcal{R}(t; y) \mathcal{A}J(0)\|_T \leq k \cdot \|J(t)\|_T.$$

From above, we obtain

$$\frac{d}{dt} \|J(t) - tJ'(t)\|_T \leq kt \|J(t)\|_T. \quad (\text{A.1})$$

The Rauch comparison theorem yields

$$\|J(t)\|_T \leq \|J'(0)\|_T \cdot \mathfrak{s}_{-k}(t),$$

for $t \in [0, \frac{\pi}{2\sqrt{k}}]$. Eq. (A.1) then furnishes

$$\frac{d}{dt} \|J(t) - tJ'(t)\|_T \leq k \|J'(0)\|_T \cdot t \mathfrak{s}_{-k}(t),$$

which implies that

$$\begin{aligned} \|J(t) - tJ'(t)\|_T &\leq \frac{1}{\sqrt{k}} \cdot \|J'(0)\|_T \cdot \left[\sqrt{k} t \cosh \sqrt{k} t - \sinh \sqrt{k} t \right] \\ &\leq \frac{\sqrt{k} t \cosh \sqrt{k} t - \sinh \sqrt{k} t}{\sqrt{k} \cdot \mathfrak{s}_k(t)} \cdot \|J(t)\|_T. \end{aligned}$$

Since

$$\lim_{t \rightarrow 0^+} \frac{\sqrt{k} t \cosh \sqrt{k} t - \sinh \sqrt{k} t}{\sqrt{k} \cdot \mathfrak{s}_k(t)} = 0,$$

there exists some $\mathfrak{t} = \mathfrak{t}(n, k, \Lambda) \in (0, \frac{\pi}{2\sqrt{k}})$ such that for $t \in [0, \mathfrak{t}]$,

$$0 < \frac{\sqrt{k} t \cosh \sqrt{k} t - \sinh \sqrt{k} t}{\sqrt{k} \cdot \mathfrak{s}_k(t)} < \frac{1}{20\Lambda}. \quad \square$$

Appendix B. Some estimates on Berwald manifolds

In this section, we always assume that (M, F) is a Berwald manifold with $|\mathbf{K}_M| \leq k$ and $\Lambda_F \leq \Lambda$.

Lemma Appendix B.1. *Given X, Y, W and $T \in S_p M$, we have*

$$|R_T(X, Y, T, W)| \leq \frac{2}{3} \Lambda^{\frac{3}{2}} k (1 + \sqrt{\Lambda})^2.$$

Proof. Lemma Appendix A.3 yields that

$$|R_U(X, Y, T, X)| = |g_U(R(T, X)X, Y)| \leq \|R(T, X)X\|_U \cdot \|Y\|_U \leq \Lambda^{\frac{3}{2}} \cdot k, \quad (\text{B.1})$$

where $\|\cdot\|_U := \sqrt{g_U(\cdot, \cdot)}$. A direct calculation shows that

$$\begin{aligned} 6R_T(X, Y, T, W) &= -R_T(W + X, Y, W + X, T) + R_T(W - X, Y, W - X, T) \\ &\quad - R_T(T - X, Y, T - X, W) + R_T(T + X, Y, T + X, W). \end{aligned}$$

Then (B.1) furnishes that

$$\begin{aligned} |R_T(W + X, Y, W + X, T)| &= \left| R_T \left(\frac{W + X}{F(W + X)}, Y, \frac{W + X}{F(W + X)}, T \right) \right| F^2(W + X) \\ &\leq \Lambda^{\frac{3}{2}} k [F(W) + F(X)]^2 = 4\Lambda^{\frac{3}{2}} k. \\ |R_T(W - X, Y, W - X, T)| &= \left| R_T \left(\frac{W - X}{F(W - X)}, Y, \frac{W - X}{F(W - X)}, T \right) \right| F^2(W - X) \\ &\leq \Lambda^{\frac{3}{2}} k [F(W) + F(-X)]^2 = \Lambda^{\frac{3}{2}} k (1 + \sqrt{\Lambda})^2. \end{aligned}$$

Hence, we obtain

$$|R_T(X, Y, T, W)| \leq \frac{2}{3} \Lambda^{\frac{3}{2}} k (1 + \sqrt{\Lambda})^2. \quad \square$$

Lemma Appendix B.2. *Let $Y(t)$ be a smooth vector field along a constant speed geodesic $\gamma(t)$. Then*

$$\frac{d}{dt} \|Y(t)\| \leq \|\nabla_T Y\|,$$

where ∇ is the Chern connection, $T := \dot{\gamma}(t)$ and $\|\cdot\|$ is the norm induced by the average Riemannian metric \tilde{g} .

Proof. Denote by P_t the parallel transportation along γ from $T_{\gamma(0)}M$ to $T_{\gamma(t)}M$. Choose a basis $\{e_i\}$ for $T_{\gamma(0)}M$. Then $E_i(t) := P_t e_i$, $1 \leq i \leq n$, is a basis of $T_{\gamma(t)}M$. For any $w \in T_{\gamma(0)}M - \{0\}$, we have

$$\frac{d}{dt} g_{(\gamma(t), P_t w)}(E_i(t), E_j(t)) = \frac{2}{F(P_t w)} A_{(\gamma(t), P_t w)}(E_i(t), E_j(t), \nabla_{\dot{\gamma}} P_t w) = 0. \quad (\text{B.2})$$

Since (M, F) is a Berwald manifold,

$$P_t(B_{\gamma(0)}M) = B_{\gamma(t)}M, \quad \text{Vol}(x) = \text{const},$$

where $B_x M := \{y \in T_x M : F(x, y) < 1\}$ and $\text{Vol}(x)$ is the Riemannian volume of $S_x M$ (see [19, Lemma 5.3.2] and [2]). Denote by (y^i) (resp. (z^i)) the corresponding coordinate system in $T_{\gamma(0)} M$ (resp. $T_{\gamma(t)} M$) with respect to $\{e_i\}$ (resp. $\{E_i\}$). Thus, $z^i \circ P_t = y^i$. Now (B.2) together with Stokes' formula yields

$$\begin{aligned} \tilde{g}_{\gamma(t)}(E_i(t), E_j(t)) &= \frac{n}{\text{Vol}(\gamma(t))} \int_{v \in B_{\gamma(t)} M} g_{(\gamma(t), v)}(E_i(t), E_j(t)) dz^1 \wedge \cdots \wedge dz^n \\ &= \frac{n}{\text{Vol}(\gamma(t))} \int_{w \in B_{\gamma(0)} M} g_{(\gamma(t), P_t w)}(P_t e_i, P_t e_j) P_t^* dz^1 \wedge \cdots \wedge P_t^* dz^n \\ &= \frac{n}{\text{Vol}(\gamma(0))} \int_{w \in B_{\gamma(0)} M} g_{(\gamma(0), w)}(e_i, e_j) dy^1 \wedge \cdots \wedge dy^n = \tilde{g}_{\gamma(0)}(e_i, e_j), \end{aligned}$$

which implies that

$$2\|Y\| \frac{d}{dt} \|Y\| = \frac{d}{dt} \tilde{g}_{\gamma(t)}(Y(t), Y(t)) = 2\tilde{g}_{\gamma(t)}(\nabla_T Y, Y) \leq 2\|\nabla_T Y\| \cdot \|Y\|. \quad \square$$

Remark 3. For the Busemann–Hausdorff measure, the S-curvature of a Berwald manifold always vanishes (see [19]). The same argument as above implies that for the Holmes–Thompson measure, the S-curvature of a Berwald manifold also vanishes.

Lemma Appendix B.3. *Given three points p_1, p_2 and p_3 in M , let $\sigma_{ij}(t)$, $0 \leq t \leq 1$ denote the minimizing constant speed geodesic from p_i to p_j . We construct a geodesic variation $\sigma(s, t) : [0, 1] \times [0, 1] \rightarrow M$:*

- (1) $\sigma(s, 0) = p_1$ and $\sigma(s, 1) = p_3$;
- (2) Let p_4 be the mid point in σ_{13} , that is, $d(p_1, p_4) = d(p_4, p_3)$. Let $\sigma_{24}(s)$, $s \in [0, 1]$ be the minimal geodesic from p_2 to p_4 . For each $s \in [0, 1]$, let $\sigma_s(t)$, $t \in [0, \frac{1}{2}]$ be a constant geodesic from p_1 to $\sigma_{24}(s)$, and $\sigma_s(t)$, $t \in [\frac{1}{2}, 1]$ be a constant geodesic from $\sigma_{24}(s)$ to p_3 . Hence, $\sigma_s(t)$, $t \in [0, 1]$ is a piecewise geodesic from p_1 to p_3 .

Suppose that $\triangle_{p_1 p_2 p_3} \subset B_{p_1}^+(R)$, where $R < \min\{i_M, \frac{\pi}{8\sqrt{k\Lambda}}\}$. Given a vector in $X \in T_{p_1} M$, set $X_{13} := P_{\sigma_{13}} X$ and $X_{123} := P_{\sigma_{23}} P_{\sigma_{12}} X$, where $P_{\sigma_{ij}}$ is the parallel translation along σ_{ij} . Then there exists a positive number $\mathfrak{C}(n, k, \Lambda)$ such that

$$F(X_{123} - X_{13}) \leq \mathfrak{C}(n, k, \Lambda) \cdot F(X) \cdot R^2.$$

Proof.

Step 1. Set $T := \sigma_* \frac{\partial}{\partial t}$, $U := \sigma_* \frac{\partial}{\partial s}$. It should be noted that U is a Jacobi field. Since $\triangle_{p_1 p_2 p_3} \subset B_{p_1}^+(R)$, we have $F(T) \leq 4R\sqrt{\Lambda} < \frac{\pi}{2\sqrt{k}}$. Clearly,

$$U\left(s, \frac{1}{2}\right) = \frac{d}{ds} \sigma\left(s, \frac{1}{2}\right) = \frac{d}{ds} \sigma_{24}(s), \quad d(p_2, p_4) = F\left(\frac{d}{ds} \sigma_{24}(s)\right).$$

Hence,

$$F\left(U\left(s, \frac{1}{2}\right)\right) < 2R\sqrt{\Lambda}. \quad (\text{B.3})$$

Note that for each fixed $s \in [0, 1]$, there exists $Y_s \in T_{p_1}M$ such that

$$U(s, t) = (\exp_{p_1})_{*2tT(s,0)} tY_s, \quad t \in \left[0, \frac{1}{2}\right].$$

It follows from [Lemma Appendix A.1](#) that

$$F(tY_s) \leq \frac{t\Lambda F(T)}{\mathfrak{s}_k(tF(T))} F(U(s, t)),$$

which together with [\(B.3\)](#) then yields that

$$F(Y_s) \leq 2\Lambda \frac{\frac{1}{2}F(T)}{\mathfrak{s}_k(\frac{1}{2}F(T))} F\left(U\left(s, \frac{1}{2}\right)\right) < \frac{2\pi\Lambda^{\frac{3}{2}}}{\sqrt{k}} R.$$

Using [Lemma Appendix A.1](#) again, we obtain that for $s \in [0, 1]$ and $t \in [0, \frac{1}{2}]$,

$$F(U(s, t)) \leq \Lambda \frac{\mathfrak{s}_{-k}(tF(T))}{tF(T)} F(tY_s) \leq 2\mathfrak{s}_{-k} \left(\frac{\pi}{2\sqrt{k}} \right) \Lambda^{\frac{5}{2}} R. \quad (\text{B.4})$$

By considering the revised metric $\tilde{F}(y) := F(-y)$, the same argument then yields that [\(B.4\)](#) holds for $s \in [0, 1]$ and $t \in [\frac{1}{2}, 1]$.

Step 2. Let $X_t(s) := P_{\sigma_s(t)}X$ denote the vector field on $\sigma([0, 1] \times [0, 1])$ induced by the parallel transformation along $\sigma_s(t)$. Thus, for any fixed $t \in [0, 1]$, we have

$$\nabla_s X_t := \nabla_U X_t = \left[\frac{dX_t^i}{ds} + X_t^j \Gamma_{jk}^i U^k \right] \frac{\partial}{\partial x^i}.$$

Since $X_0(s) = X$ and $X_1(s) \in T_{p_3}M$, it is easy to see that

$$\lim_{t \rightarrow 0^+} \nabla_s X_t = 0, \quad \lim_{t \rightarrow 1^-} \nabla_s X_t = \frac{dX_1}{ds}(s). \quad (\text{B.5})$$

[Lemma Appendix B.2](#) together with [\(B.5\)](#) implies

$$\begin{aligned} \|\nabla_s X_t\|_{t=1-\epsilon} &= \int_0^{1-\epsilon} \frac{d}{dt} \|\nabla_s X_t\| dt \leq \int_0^{1-\epsilon} \|\nabla_T \nabla_U X_t\| dt \\ &= \int_0^{1-\epsilon} \|R(T, U)X_t\| dt \leq \sqrt{\Lambda} \int_0^{1-\epsilon} \|R(T, U)X_t\|_T dt, \end{aligned} \quad (\text{B.6})$$

where $\|\cdot\|$ is the norm induced by the average Riemannian metric \tilde{g} .

Let $\{e_i\}$ be a g_T -orthonormal basis for a fixed tangent space. Set $Z := R(T, U)X_t$. [Lemma Appendix B.1](#) together with [\(B.4\)](#) furnishes

$$\begin{aligned} \|Z\|_T &\leq \sum_i \|g_T(Z, e_i) \cdot e_i\|_T = \sum_i |R_T(X_t, e_i, T, U)| \\ &= \sum_i \left| R_T \left(\frac{X_t}{F(X_t)}, \frac{e_i}{F(e_i)}, \frac{T}{F(T)}, \frac{U}{F(U)} \right) \right| \cdot F(X_t) \cdot F(e_i) \cdot F(T) \cdot F(U) \\ &\leq C_1 \cdot F(X) \cdot R^2, \end{aligned}$$

where $C_1 = C_1(n, k, \Lambda)$ is a constant. Eq. (B.6) then implies

$$\limsup_{t \rightarrow 1^-} \|\nabla_s X_t\|_U \leq C_2 \cdot F(X) \cdot R^2, \quad (\text{B.7})$$

where $C_2 = C_2(n, k, \Lambda)$ is a constant.

Using (B.5) and (B.7), we have

$$\begin{aligned} F(X(1, 1) - X(0, 1)) &\leq \int_0^1 F\left(\frac{d}{ds}X_1(s)\right) ds \\ &= \int_0^1 \lim_{t \rightarrow 1^-} F(\nabla_s X_t) ds \leq \sqrt{\Lambda} \int_0^1 \limsup_{t \rightarrow 1^-} \|\nabla_s X_t\| ds \leq C_3 \cdot F(X) \cdot R^2, \end{aligned}$$

where $C_3 = \sqrt{\Lambda} \cdot C_2$. \square

References

- [1] U. Abresch, W.T. Meyer, Injectivity radius estimates and sphere theorems, in: K. Grove, et al. (Eds.), *Comparison Geometry*, in: Math. Sci. Res. Inst. Publ., vol. 30, Cambridge, 1997, pp. 1–47.
- [2] D. Bao, S.S. Chern, A note on the Gauss–Bonnet theorem for Finsler spaces, *Ann. of Math.* 143 (1996) 233–252.
- [3] D. Bao, S.S. Chern, Z. Shen, *An Introduction to Riemannian–Finsler Geometry*, Grad. Texts in Math., vol. 200, Springer-Verlag, 2000.
- [4] L. Ben, Z. Shen, On a class of weakly Landsberg metrics, *Sci. China Ser. A* 50 (2007) 75–85.
- [5] D. Burago, Y. Burago, S. Ivanov, *A Course in Metric Geometry*, American Mathematical Society, 2001.
- [6] I. Chavel, *Riemannian Geometry – A Modern Introduction*, Academic Press, New York, 1984.
- [7] J. Cheeger, Finiteness theorems for Riemannian manifolds, *Amer. J. Math.* 92 (1970) 61–74.
- [8] S.S. Chern, Z. Shen, *Riemann–Finsler Geometry*, World Scientific Publishers, 2005.
- [9] D. Egloff, Uniform Finsler Hadamard manifolds, *Ann. Inst. Henri Poincaré* 66 (1997) 323–357.
- [10] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, with appendices by M. Katz, P. Pansu, and S. Semmes, based on *Structures Métriques des Variétés Riemanniennes*, J. Lafontaine, P. Pansu (Eds.), *Progr. Math.*, vol. 152, Birkhäuser, Boston, Basel, 1999, translated by S.M. Bates.
- [11] H. Karcher, Riemannian center of mass and mollifier smoothing, *Comm. Pure Appl. Math.* 30 (1977) 509–541.
- [12] A. Katsuda, Gromov’s convergence theorem and its application, *Nagoya Math. J.* 100 (1985) 11–48.
- [13] W. Klingenberg, Contributions to Riemannian geometry in the large, *Ann. of Math.* 69 (1959) 654–666.
- [14] S. Peters, Cheeger’s finiteness theorem for diffeomorphism classes of Riemannian manifolds, *J. Reine Angew. Math.* 349 (1984) 77–82.
- [15] P. Petersen, *Riemannian Geometry*, second edition, Grad. Texts in Math., vol. 171, Springer, New York, 2006.
- [16] H. Rademacher, A sphere theorem for non-reversible Finsler metrics, *Math. Ann.* 328 (2004) 373–387.
- [17] H. Rademacher, Nonreversible Finsler metrics of positive flag curvature, in: *A Sampler of Riemann–Finsler Geometry*, Cambridge Univ. Press, Cambridge, 2004, pp. 261–302.
- [18] H. Rademacher, The length of a shortest geodesic loop, *C. R. Math.* 346 (13) (2008) 763–765.
- [19] Z. Shen, *Lectures on Finsler Geometry*, World Sci., Singapore, 2001.
- [20] Z. Shen, On a class of Landsberg metrics in Finsler geometry, *Canad. J. Math.* 61 (2009) 1357–1374.
- [21] Y. Shen, W. Zhao, Some results on fundamental groups and Betti numbers of Finsler manifolds, *Internat. J. Math.* 23 (2012), 16 pp.
- [22] Z. Szabó, Positive definite Berwald spaces, *Tensor (N.S.)* 35 (1981) 25–39.
- [23] Z. Szabó, Berwald metrics constructed by Chevalley’s polynomials, *arXiv:math.DG/0601522*, 2006.
- [24] Z. Szabó, All regular Landsberg metrics are always Berwald, *Ann. Global Anal. Geom.* 34 (4) (2008) 381–386.
- [25] Z. Szabó, Correction to “All regular Landsberg metrics are always Berwald”, *Ann. Global Anal. Geom.* 35 (3) (2009) 227–230.
- [26] L. Yuan, W. Zhao, Some formulas of Santaló type in Finsler geometry and its applications, *Publ. Math. Debrecen* 87 (1–2) (2015).
- [27] W. Zhao, A lower bound for the length of closed geodesics on a Finsler manifold, *Canad. Math. Bull.* 75 (2014) 194–208.
- [28] W. Zhao, Y. Shen, A universal volume comparison theorem for Finsler manifolds and related results, *Canad. J. Math.* 65 (2013) 1401–1435.