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Non-Haar MRA on local fields of positive characteristic

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ABSTRACT

We propose a simple method to construct integral periodic mask and corresponding scaling step functions that generate non-Haar orthogonal MRA on the local field $F^{(s)}$ of positive characteristic p. To construct this mask we use two new ideas. First, we consider local field as vector space over the finite field $GF(p^s)$. Second, we construct scaling function by arbitrary tree that has p^s vertices. By fixed prime number p there exist $p^{s(p^s-2)}$ such trees.

1. Introduction

The first results on the wavelet analysis on local fields were obtained by Chinese mathematicians Huikun Jiang, Dengfeng Li, and Ning Jin in the paper [8]. They introduced the notion of MRA on local fields, for the fields $F^{(s)}$ of positive characteristic p proved some simple properties, and gave an algorithm for constructing wavelets for a known scaling function. Using these results they constructed MRA and corresponding wavelets for the case when a scaling function is the characteristic function of unit ball $\mathcal{D}$. Such MRA is called usually "Haar MRA" and corresponding wavelets – "Haar wavelets". In [11] wavelet frame on local field are constructed, a necessary condition and sufficient conditions for wavelet frame on local fields are given too. Biswaranjan Behera and Qaiser Jahan [2] constructed the wavelet packets associated with MRA on local fields of positive characteristic. In the article [3] the same authors proved that a function $\varphi \in L^2(F^{(s)})$ is a scaling function for MRA in $L^2(F^{(s)})$ if and only if

$$\sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = 1 \text{ for a.e. } \xi \in \mathcal{D}, \tag{1.1}$$

$$\lim_{j\to\infty} |\hat{\varphi}(\mathfrak{p}^j \xi)| = 1 \text{ for a.e. } \xi \in F^{(s)}, \tag{1.2}$$

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and there exists an integral periodic function $m_0 \in L^2(\mathcal{D})$ such that

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi) \text{ for a.e. } \xi \in F^{(s)} \tag{1.3}$$

where $\{u(k)\}$ is the set of shifts, $\mathfrak{p}$ is a prime element. B. Behera and Q. Jahan [4] proved also if the translates of the scaling functions of two multiresolution analyses are biorthogonal, then the associated wavelet families are also biorthogonal. So, to construct MRA on a local field $F^{(s)}$ we must construct an integral periodic mask m_0 with conditions (1.1)–(1.3). To solve this problem using prime element methods developed in [16] is not simple. Currently there are no effective methods for constructing such masks and scaling functions. In articles [2–4,8,11] only Haar wavelets are obtained.

In this paper, we propose a simple method to construct integral periodic masks and corresponding scaling step functions that generate non-Haar orthogonal MRA. To construct this mask we use two new ideas. First, we consider local field as vector space over the finite field $GF(p^s)$. Second, we construct a scaling function by arbitrary tree that have p^s nodes. For fixed prime number p there exist $p^{s(p^s-2)}$ such trees.

By $s = 1$ the additive group $F^{(1)+}$ is a Vilenkin group. Issues of constructing of MRA and wavelets on Vilenkin groups may be found in [5,6,12–15].

The simplest example of a local field of characteristic zero is the field of p-adic numbers. Issues of constructing MRA and wavelets on the field of p-adic numbers can be found in [1,9,10].

The paper is organized as follows. We consider local field $F^{(s)}$ as a vector space over the finite field $GF(p^s)$. Therefore, in Section 2, we recall some concepts and facts from the theory of finite fields and define the local field $F^{(s)}$ of positive characteristic p as a set of infinite sequences $a = (\mathbf{a}_j)$, where $\mathbf{a}_j \in GF(p^s)$.

In Section 3 we prove that local field $F^{(s)}$ is a vector space over finite field $GF(p^s)$.

In Section 4 we prove that the set X of all characters of local field $\mathbb{F}^{(s)}$ also form a vector space over finite field $GF(p^s)$ with product as internal operation and powering as external operation. We define Rademacher functions, find a general view of characters, and prove a basic property of Rademacher functions.

In Section 5 we discuss the refinable equation and its mask.

In Section 6 we consider refinable equation

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1})$$

with step mask m_0 and find a necessary and sufficient condition under which an integral periodic function m_0 is a mask of some refinement equation.

In Section 7 we define (N, M) elementary sets. We prove if $E \subset F^{(s)}$ is (N, M) elementary set and $|\hat{\varphi}(\chi)| = \mathbf{1}_E(\chi)$ on X then the system of shifts $(\varphi(x \dot{-} h))_{h\in H_0}$ is an orthonormal system.

In Section 8 we reduce the problem of construction of step refinable function to construction of some tree. We consider some special class of refinable functions $\varphi(\chi)$ for which $|\hat{\varphi}(\chi)|$ is a characteristic function of a set. We introduce such concepts as "a set generated by a tree" and "a refinable step function generated by a tree" and prove, that every rooted tree containing p^s nodes generates a refinable step function that generate an orthogonal MRA on local field $F^{(s)}$. For $p = s = 2$ we give an example of a refinable step function that generate non-Haar MRA.

Using the results of the article [8] we can construct now corresponding wavelets. This example shows that MRA on local field gives an effective method to construct multidimensional step wavelets.

2. Preliminaries

We will consider two objects: Vilenkin groups and local fields. Let p be a prime number. Vilenkin group $(\mathfrak{G}, \dot{+})$ consists of sequences

$$a = (a_n)_{n\in\mathbb{Z}} = (\dots, a_{n-1}, a_n, a_{n+1}, \dots),\ a_j = \overline{0, p-1},$$

in which only a finite number of terms with negative numbers are nonzero. The operation $\dot{+}$ is defined as component wise addition modulo p, i.e.

$$a \dot{+} b = (a_n) \dot{+} (b_n) = ((a_n + b_n) \mathrm{mod}\, p)_{n\in\mathbb{Z}}.$$

The topology in $\mathfrak{G}$ is determined by subgroups

$$\mathfrak{G}_n = \{a \in \mathfrak{G} : \ a = (\dots, 0_{n-1,} a_{n-1}, a_n, a_{n+1}, \dots)\}.$$

The equality

$$\rho(a,b) = \begin{cases} \frac{1}{p^n}; & a_m \neq b_n, \ a_j = b_j \text{ for } j < n \\ 0; & a_j = b_j \text{ for } j \in \mathbb{Z} \end{cases}$$

is the non-Archimedian distance on $(\mathfrak{G}, \dot{+})$. If μ is the Haar measure on $\mathfrak{G}$ then $\mu(\mathfrak{G}_n \dot{+} g) = \mu\mathfrak{G}_n = \frac{1}{p^n}$, $n \in \mathbb{Z}$. The dilation operator $\mathcal{A}$ is defined by the equation

$$\mathcal{A}(a) = (b_n)_{n\in\mathbb{Z}}, \ \ b_n = a_{n+1}.$$

It is evident that $\mathcal{A}\mathfrak{G}_n = \mathfrak{G}_{n-1}$ and $\int\limits_{\mathfrak{G}} f(\mathcal{A}u)\, d\mu = \frac{1}{p} \int\limits_{\mathfrak{G}} f(x)\, d\mu$.

By a local field we will mean a field K which is locally compact, non-discrete and totally disconnected. We will consider local fields with positive characteristic only. By Pontrjagin–Kovalsky theorem [7] such field is isomorphic to the set $K_L(z)$ of formal Laurent series

$$\sum_{n=N}^{\infty} a_n z^n \tag{2.1}$$

with $\mathbf{a}_n \in GF(p^s)$ where $s \in \mathbb{N}$ and p is a prime number. Local field of positive characteristic is denote $F^{(s)}$.

Let $GF(p)$ be a ring (field) of residue class on modulo p. The finite field $GF(p^s)$ consist of vectors $\mathbf{a} = (a^{(0)}, a^{(1)}, \dots, a^{(s-1)})$, where $a^{(j)} \in GF(p)$. The addition operation $(\mathbf{a}) \dot{+} (\mathbf{b})$ is defined coordinate-wise i.e.

$$\mathbf{a} \dot{+} \mathbf{b} = (a^{(j)} + b^{(j)}) \,\mathrm{mod}\, p)_{j=0}^{s-1}.$$

To define a product $\mathbf{ab}$ it is necessary to represent vectors $\mathbf{a}$ and $\mathbf{b}$ as polynomials

$$\mathbf{a} = \sum_{j=0}^{s-1} a^{(j)} t^j, \mathbf{b} = \sum_{j=0}^{s-1} b^{(j)} t^j$$

and multiply these polynomials over the field $GF(p)$. We obtain the polynomial

$$Q(t) = \sum_{j=0}^{s-1} \sum_{k=0}^{s-1} a^{(j)} b^{(k)} t^{j+k} = \sum_{l=0}^{2s-2} t^l \sum_{k,j:\, k+j=l} a^{(j)} b^{(k)}$$

in which coefficients $\beta_l = \sum\limits_{k,j:\, k+j=l} a^{(j)} b^{(k)}$ are calculated in the field $GF(p)$. Then we take a prime polynomial $p_s(t)$ of degree s and divide polynomial $Q(t)$ by $p_s(t)$ over the field $GF(p)$

$$Q(t) = p_s(t) q(t) + H(t).$$

Coefficients $b_0, b_1, \dots, b_{s-1}$ of this rest $H(t)$ are components of product $\mathbf{ab}$. It is know that a prime polynomial $p_s(t)$ over the field $GF(p)$ exists but not only one. A prime polynomial $p_s(t)$ can be found by exhaustion.

We return to local fields. The sum and product of Laurent series (2.1) are defined in the standard way, i.e. if

$$a = \sum_{j=k}^{\infty} \mathbf{a}_j t^j, \; b = \sum_{j=k}^{\infty} \mathbf{b}_j t^j$$

then

$$a \dot{+} b = \sum_{j=k}^{\infty} (\mathbf{a}_j \dot{+} \mathbf{b}_j) t^j, \; \mathbf{a}_j \dot{+} \mathbf{b}_j = ((a_j^{(l)} + b_j^{(l)}) \bmod p)_{l=0}^{s-1} \tag{2.2}$$

$$ab = \sum_{l=2k}^{\infty} t^l \sum_{j,\nu:\, j+\nu=l} \mathbf{a}_j \mathbf{b}_\nu. \tag{2.3}$$

Topology in $F^{(s)}$ is given by neighborhood basis of zero

$$F_n^{(s)} = \left\{ a = \sum_{j=n}^{\infty} \mathbf{a}_j t^j \, : \, \mathbf{a}_j \in GF(p^s) \right\}.$$

If

$$a = \sum_{j=n}^{\infty} \mathbf{a}_j t^j, \; \mathbf{a}_n \neq 0$$

then we put by definition $\|a\| = \frac{1}{p^{sn}}$. Consequently

$$F_n^{(s)} = \left\{ x \in F^{(s)} \, : \, \|x\| \leq \frac{1}{p^{sn}} \right\}.$$

By $F^{(s)+}$ denote the additive group of field $F^{(s)}$. Neighborhoods $F_n^{(s)}$ are compact subgroups of group $F^{(s)+}$. We will denote them as $F_n^{(s)+}$. The next properties are fulfilled

1) $\dots \subset F_1^{(s)+} \subset F_0^{(s)+} \subset F_{-1}^{(s)+} \subset \dots$
2) $F_n^{(s)+}/F_{n+1}^{(s)+} \cong GF(p^s)$ and $\sharp(F_n^{(s)+}/F_{n+1}^{(s)+}) = p^s$.

Therefore we will assume that a local field $F^{(s)}$ of positive characteristic consists of infinite sequences

$$a = (\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots,), \; \mathbf{a}_j = (a_j^{(0)}, a_j^{(1)}, \dots, a_j^{(s-1)}) \in GF(p^s)$$

in which only finite number of element $\mathbf{a}_j$ with negative numbers are nonzero. The sum and product are defined as

$$a \dot{+} b = (\mathbf{a}_j \dot{+} \mathbf{b}_j)_{i\in\mathbb{Z}}, \; \mathbf{a}_j \dot{+} \mathbf{b}_j = (a_j^{(\nu)} + b_j^{(\nu)} \bmod p)_{\nu=0}^{s-1}, \tag{2.4}$$

$$ab = (\sum_{i,j: i+j=l} (\mathbf{a}_i \mathbf{b}_j))_{l\in\mathbb{Z}} \tag{2.5}$$

In this case

$$\|a\| = \|(\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots)\| = \frac{1}{p^{sn}} \text{ if } \mathbf{a}_n \neq \mathbf{0},$$

$$F_n^{(s)} = \{a = (\mathbf{a}_j)_{j\in\mathbb{Z}} : \ \mathbf{a}_j \in GF(p^s); \ \mathbf{a}_j = \mathbf{0} \ \forall j < n\},$$

$$\dots \subset F_1^{(s)} \subset F_0^{(s)} \subset F_{-1}^{(s)} \subset \dots,$$

$F_n^{(s)}$ are compact subgroups in $F^{(s)+}$ and $\sharp(F_n^{(s)}/F_{n+1}^{(s)}) = p^s$.

It follows that $F^{(1)+}$ is a Vilenkin group. The converse is true also: in Vilenkin group $(\mathfrak{G}, \dot{+})$ we can define product by (2.5). With such operation $(\mathfrak{G}, \dot{+}, \cdot)$ will be a field. Since $F^{(1)+}$ is a Vilenkin group, it follows that

1) $\int\limits_{\mathfrak{G}_0{}^{\perp}} (\chi, x)\, d\nu(\chi) = \mathbf{1}_{\mathfrak{G}_0}(x)$,
2) $\int\limits_{\mathfrak{G}_0} (\chi, x)\, d\mu(x) = \mathbf{1}_{\mathfrak{G}_0^{\perp}}(\chi)$,
3) $\int\limits_{\mathfrak{G}_n^{\perp}} (\chi, x)\, d\nu(\chi) = p^n \mathbf{1}_{\mathfrak{G}_n}(x)$,
4) $\int\limits_{\mathfrak{G}_n} (\chi, x)\, d\mu(x) = \frac{1}{p^n} \mathbf{1}_{\mathfrak{G}_n^{\perp}}(\chi)$

where $\mathfrak{G}_n = F_n^{(1)+}$.

From the definition of $F^{(s)}$ it follows that additive group $F^{(s)+}$ is also Vilenkin group $\mathfrak{G}$ and $F_n^{(s)+} = \mathfrak{G}_{ns}$.

3. Local field of positive characteristic as vector space over a finite field

Let $(\mathfrak{G}, \dot{+})$ be a Vilenkin group. We can define the multiplication operation on a number $\lambda \in GF(p)$ by the equation

$$a\lambda = \underbrace{a \dot{+} a \dot{+} \dots \dot{+} a}_{\lambda}.$$

Define the modulus of λ as

$$|\lambda| = \begin{cases} 1, & \lambda \neq 0, \\ 0, & \lambda = 0, \end{cases}$$

and the norm of $a \in \mathfrak{G}$ by the equation

$$\|a\| = p^{-n} \tag{3.1}$$

if

$$a = (\dots 0_{n-1} a_n a_{n+1} \dots), n \in \mathbb{Z}, a_j \in Z_p, a_n \neq 0.$$

Since $GF(p)$ is a field, it follows that $(\mathfrak{G}, \dot{+})$ is a vector space over the field $GF(p)$ and the equation (3.1) defines a norm in $(\mathfrak{G}, \dot{+}, \cdot\lambda)$.

Now we consider local field $F^{(s)}$ with positive characteristic p. Its elements are infinite sequences

$$a = (\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots), \mathbf{a}_j \in GF(p^s)$$

where

$$\mathbf{a}_j = (a_j^{(0)}, a_j^{(1)}, \dots, a_j^{(s-1)}),\ a_j^{(\nu)} \in Z_p.$$

Let $\lambda \in GF(p^s)$. By the definition $\|a\| = \frac{1}{p^{sn}}$ if $\mathbf{a}_n \neq \mathbf{0}$. Since

$$\begin{aligned}\lambda a &= (\dots \mathbf{0}_{-1}, \lambda, \mathbf{0}_1, \dots) \cdot (\dots \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots) = \\ &= (\lambda + \mathbf{0}x + \mathbf{0}x^2 + \dots)(\mathbf{a}_n x^n + \mathbf{a}_{n+1} x^{n+1} + \dots) = \lambda \mathbf{a}_n x^n + \lambda \mathbf{a}_{n+1} x^{n+1} + \dots = \\ &= (\dots \mathbf{0}_{n-1}, \lambda \mathbf{a}_n, \lambda \mathbf{a}_{n+1}, \dots)\end{aligned}$$

it follows that the product $\lambda \mathbf{a}$ is defined coordinate wise. With such operations $F^{(s)}$ is a vector space. If we define the modulus $|\lambda|$ by the equation

$$|\lambda| = \begin{cases} 1, & \lambda \neq 0, \\ 0, & \lambda = 0, \end{cases}$$

and norm $\|a\|$ by the equation

$$\|a\| = \frac{1}{p^{sn}},\ \mathbf{a}_n \neq 0 \tag{3.2}$$

then we can consider the field $F^{(s)}$ as a vector normalized space under the field $GF(p^s)$.

For brevity we denote $K := F^{(s)}$, $K_n := F_n^{(s)}$. Take an element $g \in K_1 \setminus K_2$ and fix it. It is known [16] that any element $a \in K$ may be written in the form

$$a = \sum_{n \in \mathbb{Z}} \lambda_n g^n,\ \lambda_n \in U,$$

where U is a fixed full set of coset representatives of K_1 in K_0. We can prove a more general statement.

Theorem 3.1. *Let $(g_n)_{n\in\mathbb{Z}}$ be a fixed basic sequence in K, i.e. $g_n \in K_n \setminus K_{n+1}$. Any element $a \in K$ may by written as sum of the series*

$$a = \sum_{n \in \mathbb{Z}} \overline{\lambda}_n g_n,\ \overline{\lambda}_n \in GF(p^s). \tag{3.3}$$

Proof. Let $a \in K$. If $a = 0$ then the equation (3.3) is evident. Let $a \neq 0$. Then exists $n \in \mathbb{Z}$ such that $a \in K_n^+ \setminus K_{n+1}^+$. It means that

$$a = (\dots \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots),\ \mathbf{a}_j \in GF(p^s), \mathbf{a}_n \neq \mathbf{0}.$$

Show that there exists $\overline{\lambda}_n \in GF(p^s)$ such that

$$a = \overline{\lambda}_n g_n \dot{+} \alpha_{n+1},\ \alpha_{n+1} \in K_{n+1}.$$

Indeed, since $g_n \in K_n \setminus K_{n+1}$ it follows that

$$g_n = (\dots \mathbf{0}_{n-1}, \mathbf{g}_n^{(n)}, \mathbf{g}_{n+1}^{(n)}, \dots),\ \mathbf{g}_n^{(n)} \neq \mathbf{0}.$$

Since $GF(p^s)$ is a field, it follows there exists $\overline{\lambda}_n \in GF(p^s)$ such that $\overline{\lambda}_n \mathbf{g}_n^{(n)} = \mathbf{a}_n$. Therefore

$$\overline{\lambda}_n g_n = (\dots \mathbf{0}_{n-1}, \overline{\lambda}_n \mathbf{g}_n^{(n)}, \overline{\lambda}_n \mathbf{g}_{n+1}^{(n)} \dots) = (\dots \mathbf{0}_{n-1}, \mathbf{a}_n, \tilde{\mathbf{a}}_{n+1} \dots).$$

Consequently

$$a \dot{-} \overline{\lambda}_n g_n = (\dots \mathbf{0}_{n-1}, \mathbf{0}_n, \mathbf{a}_{n+1} - \tilde{\mathbf{a}}_{n+1} \dots) = \alpha_{n+1} \in K_{n+1}^{+},$$

i.e. $a = \overline{\lambda}_n g_n \dot{+} \alpha_{n+1}$. Continuing this process, we obtain (3.3). □

Corollary. *If $g \in K_1 \setminus K_2$ then $g^n \in K_n \setminus K_{n+1}$. Therefore we can take $g_n = g^n$ in the equation (3.3).*

Definition 3.1. The operator

$$\mathcal{A} : a = \sum_{n \in \mathbb{Z}} \overline{\lambda}_n g_n \longmapsto \sum_{n \in \mathbb{Z}} \overline{\lambda}_n g_{n-1}$$

is called a dilation operator.

Remark 1. If $g_n = g^n$ and $a = \sum\limits_{n \in \mathbb{Z}} \overline{\lambda}_n g^n$ then $ag^{-1} = \sum\limits_{n \in \mathbb{Z}} \overline{\lambda}_n g^{n-1}$. So the dilation operation may be defined by equation $\mathcal{A}x = g^{-1}x$.

Remark 2. Since additive group $F^{(s)+}$ is Vilenkin group $\mathfrak{G}$ with $F_n^{(s)+} = \mathfrak{G}_{ns}$ it follows that $\mathcal{A}K_n = K_{n-1}$ and $\int\limits_{K^+} f(\mathcal{A}u)\, d\mu = \frac{1}{p^s} \int\limits_{K^+} f(x)\, d\mu$.

4. Set of characters as vector space over a finite field

Since $F^{(s)+}$ is a Vilenkin group it follows that the set of characters is a locally compact zero-dimensional group with product as group operation

$$(\chi\varphi)(a) = \chi(a) \cdot \varphi(a).$$

Denote the set of characters as X. We want to find the explicit form of characters. Let us define the character r_n in the following way. If

$$a = (\dots, \mathbf{0}_{k-1}, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots), \mathbf{a}_j \in GF(p^s)$$

and

$$\mathbf{a}_j = (a_j^{(0)}, a_j^{(1)}, \dots, a_j^{(s-1)}),\ a_j^{(\nu)} \in GF(p)$$

then $r_n(a) = e^{\frac{2\pi i}{p} a_k^{(l)}}$, where $n = ks + l$, $0 \le l < s$.

Lemma 4.1. *Any character $\chi \in X$ can be expressed uniquely as product*

$$\chi = \prod_{n=-\infty}^{+\infty} r_n^{\alpha_n} \quad (\alpha_n = \overline{0, p-1}), \tag{4.1}$$

in which the number of factors with positive numbers are finite.

Proof. Let

$$x = (\dots, \mathbf{0}, \mathbf{x}_j, \dots, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots),\ \mathbf{x}_k = (x_{ks+0}, x_{ks+1}, \dots, x_{ks+(s-1)})$$

Since $F^{(s)+}$ is a Vilenkin group, it follows that functions $r_{ks+l}(x) = e^{\frac{2\pi i}{p}x_{ks+l}}$, are Rademacher functions on $F^{(s)+}$. Therefore any character χ may by expressed in the form (4.1). $\square$

Definition 4.1. Write the character χ as

$$\chi = \prod_{k\in\mathbb{Z}} r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}}$$

and denote

$$\mathbf{r}_k^{\mathbf{a}_k} := r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}},$$

where $\mathbf{a}_k = (a_k^{(0)}, a_k^{(1)}, \dots, a_k^{(s-1)}) \in GF(p^s)$. The function $\mathbf{r}_k = \mathbf{r}_k^{(1,0,\dots,0)}$ is called Rademacher function.

Definition 4.2. Assume by the definition

$$(\mathbf{r}_k^{\mathbf{a}_k})^{\mathbf{b}_k} := \mathbf{r}_k^{\mathbf{a}_k \mathbf{b}_k},\ \mathbf{a}_k, \mathbf{b}_k \in GF(p^s).$$

In this case

$$\mathbf{r}_k^{\mathbf{a}_k} = (\mathbf{r}_k^{(1,0,\dots,0)})^{\mathbf{a}_k} = \mathbf{r}_k^{(a_k^{(0)}, a_k^{(1)}, \dots, a_k^{(s-1)})} = r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}}.$$

Therefore we can write χ as the product

$$\chi = \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k}. \tag{4.2}$$

Definition 4.3. Define $\chi^{\mathbf{b}}$, $\mathbf{b} \in GF(p^s)$ as

$$\chi^{\mathbf{b}} := \prod_{k\in\mathbb{Z}} (\mathbf{r}_k^{\mathbf{a}_k})^{\mathbf{b_k}}.$$

Lemma 4.2. *Let $\mathbf{r}_k$ be a Rademacher function. Then*

$$\mathbf{r}_k^{\mathbf{u}\dot{+}\mathbf{v}} = \mathbf{r}_k^{\mathbf{u}} \cdot \mathbf{r}_k^{\mathbf{v}},\ \mathbf{u}, \mathbf{v} \in GF(p^s).$$

Proof. Using the definition of Rademacher functions we have for $x = (x_k^{(l)})$

$$(\mathbf{r}_k^{\mathbf{u}}\mathbf{r}_k^{\mathbf{v}}, x) = (\mathbf{r}_k^{\mathbf{u}}, x)(\mathbf{r}_k^{\mathbf{v}}, x) = \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} u_{ks+l}^{(l)} x_k^{(l)}} \cdot \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} v_{ks+l}^{(l)} x_k^{(l)}} =$$

$$= \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p}(u_{ks+l}^{(l)} + v_{ks+l}^{(l)}) x_k^{(l)}} = (\mathbf{r}_k^{\mathbf{u}\dot{+}\mathbf{v}}, x). \qquad \square$$

Theorem 4.1. *The set of characters of the field $F^{(s)}$ is a vector space $(X,\ *,\ \cdot^{GF(p^s)})$ under the finite field $GF(p^s)$ with product as interior operation and powering as exterior operation.*

Proof. 1) Check $\chi^{\mathbf{u}\dot{+}\mathbf{v}} = \chi^{\mathbf{u}}\chi^{\mathbf{v}}$ for $\mathbf{u}, \mathbf{v} \in GF(p^s)$. Let

$$\chi^{\mathbf{u}} = \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k\mathbf{u}}, \quad \chi^{\mathbf{v}} = \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k\mathbf{v}}.$$

Using Lemma 4.2 we obtain

$$\chi^{\mathbf{u}}\chi^{\mathbf{v}} = \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k\mathbf{u}}\mathbf{r}_k^{\mathbf{a}_k\mathbf{v}} = \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k(\mathbf{u}\dot{+}\mathbf{v})} = \chi^{\mathbf{u}\dot{+}\mathbf{v}}.$$

2) Check the equation $\chi_1^{\mathbf{u}}\chi_2^{\mathbf{u}} = (\chi_1\chi_2)^{\mathbf{u}}$. Let

$$\chi_1^{\mathbf{u}} = \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k\mathbf{u}}, \; \chi_2^{\mathbf{u}} = \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{b}_k\mathbf{u}}.$$

Using Lemma 4.2 we have

$$\chi_1^{\mathbf{u}}\chi_2^{\mathbf{u}} = \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k\mathbf{u}} \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{b}_k\mathbf{u}} = \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{(\mathbf{a}_k\dot{+}\mathbf{b}_k)\mathbf{u}} = (\chi_1\chi_2)^{\mathbf{u}}.$$

3) Since the vector $\mathbf{1} = (1, 0, \ldots, 0)$ is a unity element of multiplicative group of the field $GF(p^s)$ it follows that $\chi^{\mathbf{1}} = \chi^{(1,0,\ldots,0)} = \prod\limits_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k\cdot\mathbf{1}} = \prod\limits_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k} = \chi$.

4) The equality $(\chi^{\mathbf{u}})^{\mathbf{v}} = \chi^{\mathbf{uv}}$ is true by the definition.

So, all axioms for exterior operation are fulfilled. By Lemma 4.2 all axioms for interior operation are fulfilled too. □

It follow from (4.2) that annihilator $(F_k^{(s)})^\perp$ consists of characters of the form $\chi = \mathbf{r}_{k-1}^{\mathbf{a}_{k-1}}\mathbf{r}_{k-2}^{\mathbf{a}_{k-2}}\ldots$. It is evident also that

1) Rademacher system $(\mathbf{r}_k)$ forms a basis of $(X, *, \cdot^{GF(p^s)})$,
2) any sequences of characters $\chi_k \in (F_{k+1}^{(s)})^\perp \setminus (F_k^{(s)})^\perp$ forms a basis of $(X, *, \cdot^{GF(p^s)})$.
3) $(F_k^{(s)})^\perp = \bigsqcup\limits_{\mathbf{a}_{k-1}\in GF(p^s)} (F_{k-1}^{(s)})^\perp \mathbf{r}_{k-1}^{\mathbf{a}_{k-1}}$.

The next lemma is the basic property of Rademacher functions on local field with positive characteristic.

Lemma 4.3. *Let $g_j = (\ldots, \mathbf{0}_{j-1}, (1, 0, \ldots, 0)_j, \mathbf{0}_{j+1}, \ldots) \in F^{(s)}$, $\mathbf{a}_k, \mathbf{u} \in GF(p^s)$. Then $(\mathbf{r}_k^{\mathbf{a}_k}, \mathbf{u}g_j) = 1$ for any $k \neq j$.*

Proof. Since $\mathbf{u}g_j = (\ldots, \mathbf{0}_{j-1}, (u^{(0)}, u^{(1)}, \ldots, u^{(s-1)})_j, \mathbf{0}_{j+1}, \ldots)$, it follows that

$$(\mathbf{r}_k^{\mathbf{a}_k}, \mathbf{u}g_j) = \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} a_k^{(l)} u^{(l)}} = \prod_{l=0}^{s-1} e^0 = 1. \qquad \square$$

Definition 4.4. Define a dilation operator $\mathcal{A}$ on the set of characters by the equation $(\chi\mathcal{A}, x) = (\chi, \mathcal{A}x)$.

Remark. Since additive group $F^{(s)+}$ is Vilenkin group, it follows that $g_j\mathcal{A} = g_{j+1}$, $(K_n^+)^\perp\mathcal{A} = (K_{n+1}^+)^\perp$ and $\int\limits_X f(\chi\mathcal{A})\,d\nu = \frac{1}{p^s}\int\limits_X f(\chi)\,d\nu$.

5. MRA on local fields of positive characteristic

We will use Rademacher function to construct MRA on local fields of positive characteristic. We will assume

$$g_n = (\dots, \mathbf{0}_{n-1}, (1,0,\dots,0)_n, \mathbf{0}_{n+1}, \dots).$$

Let us denote

$$H_0 = \{h \in F^{(s)} : h = \mathbf{a}_{-1}g_{-1} \dot{+} \mathbf{a}_{-2}g_{-2} \dot{+} \dots \dot{+} \mathbf{a}_{-\sigma}g_{-\sigma}, \sigma \in \mathbb{N}\},\ \mathbf{a}_j \in GF(p^s),$$
$$H_0^{(\sigma)} = \{h \in F^{(s)} : h = \mathbf{a}_{-1}g_{-1} \dot{+} \mathbf{a}_{-2}g_{-2} \dot{+} \dots \dot{+} \mathbf{a}_{-\sigma}g_{-\sigma}\}, \sigma \in \mathbb{N},\ \mathbf{a}_j \in GF(p^s).$$

The set H_0 is an analog of the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 5.1. *Let $K = F^{(s)}$ be a local field with characteristic p. Then for any $n \in \mathbb{Z}$*

1) $\int\limits_{(K_n^+)^\perp} (\chi, x)\, d\nu(\chi) = p^{sn}\mathbf{1}_{K_n^+}(x)$,
2) $\int\limits_{K_n^+} (\chi, x)\, d\mu(x) = \frac{1}{p^{sn}}\mathbf{1}_{(K_n^+)^\perp}(\chi)$.

Proof. First we prove the equation 1). Since K^+ is a zero-dimensional group, it follows

$$\int\limits_{(K_0^+)\perp} (\chi, x) d\nu(\chi) = \mathbf{1}_{K_0^+}(x), \quad \int\limits_{K_0^+} (\chi, x) d\mu(x) = \mathbf{1}_{(K_0^+)^\perp}(\chi).$$

By the definition of dilation operator

$$\int\limits_X f(\chi\mathcal{A})\, d\nu(\chi) = p^s \int\limits_X f(\chi)\, d\nu(\chi), \quad \mathbf{1}_{K_n^+}(x) = \mathbf{1}_{K_0^+}(\mathcal{A}^n x).$$

Using these equations we have

$$\int\limits_{(K_n^+)^\perp} (\chi, x)\, d\nu(\chi) = \int\limits_X \mathbf{1}_{(K_n^+)^\perp}(\chi)(\chi, x)\, d\nu(\chi) =$$
$$= p^{sn} \int\limits_X (\chi\mathcal{A}^n, x)\mathbf{1}_{(K_n^+)^\perp}(\chi\mathcal{A}^n)\, d\nu(\chi) =$$
$$= p^{sn} \int\limits_X (\chi, \mathcal{A}^n x)\mathbf{1}_{(K_0^+)^\perp}(\chi)\, d\nu(\chi) = p^{sn}\mathbf{1}_{K_0^+}(\mathcal{A}^n x) = p^{sn}\mathbf{1}_{K_n^+}(x).$$

The second equation is proved by analogy. □

Lemma 5.2. *Let $\chi_{n,l} = \mathbf{r}_n^{\mathbf{a}_n}\mathbf{r}_{n+1}^{\mathbf{a}_{n+1}} \dots \mathbf{r}_{n+l}^{\mathbf{a}_{n+l}}$ be a character does not belong to $(K_n^+)^\perp$. Then*

$$\int\limits_{(K_n^+)^\perp \chi_{n,l}} (\chi, x)\, d\nu(\chi) = p^{ns}(\chi_{n,l}, x)\mathbf{1}_{K_n^+}(x).$$

Proof. Denote $\mathfrak{G}_n := K_n^+$. By analogy with previously we have

$$\int\limits_{\mathfrak{G}_n^\perp \chi_{n,l}} (\chi, x)\, d\nu(\chi) = \int\limits_X \mathbf{1}_{\mathfrak{G}_n^\perp \chi_{n,l}}(\chi)(\chi, x)\, d\nu(\chi) = \int\limits_X \mathbf{1}_{\mathfrak{G}_n^\perp}(\chi)(\chi_{n,l}\chi, x)\, d\nu(\chi) =$$

$$= \int\limits_{\mathfrak{G}_n^\perp} (\chi_{n,l}, x)(\chi, x)\, d\nu(\chi) = p^{ns}(\chi_{n,l}, x)\mathbf{1}_{\mathfrak{G}_n}(x). \qquad \square$$

Lemma 5.3. *Let* $h_{n,l} = \mathbf{a}_{n-1}g_{n-1} \dot{+} \mathbf{a}_{n-2}g_{n-2} \dot{+} \ldots \dot{+} \mathbf{a}_{n-l}g_{n-l} \notin K_n^+$. *Then*

$$\int\limits_{K_n^+ \dot{+} h_{n,l}} (\chi, x)\, d\mu(x) = \frac{1}{p^{ns}}(\chi, h_{n,l})\mathbf{1}_{(K_n^+)^\perp}(\chi).$$

This lemma is dual to Lemma 5.2.

Definition 5.1. Let $M, N \in \mathbb{N}$. Denote by $\mathfrak{D}_M(K_{-N})$ the set of step-functions $f \in L_2(K)$ such that 1) $\operatorname{supp} f \subset K_{-N}$, and 2) f is constant on cosets $K_M \dot{+} g$. Similarly is defined $\mathfrak{D}_{-N}(K_M^\perp)$.

Lemma 5.4. *Let* $M, N \in \mathbb{N}$. $f \in \mathfrak{D}_M(K_{-N})$ *if and only if* $\hat{f} \in \mathfrak{D}_{-N}(K_M^\perp)$.

Proof. It is evident since additive group F^+ is Vilenkin group. $\square$

Lemma 5.5. *Let* $\varphi \in L_2(K)$. *The system* $(\varphi(x \dot{-} h))_{h \in H_0}$ *is orthonormal if and only if the system* $\left(p^{\frac{ns}{2}}\varphi(\mathcal{A}^n x \dot{-} h)\right)_{h \in H_0}$ *is orthonormal.*

Proof. This lemma follows from the equation

$$\int\limits_K p^{\frac{ns}{2}}\varphi(\mathcal{A}^n x \dot{-} h) p^{\frac{ns}{2}}\overline{\varphi(\mathcal{A}^n x \dot{-} g)}\, d\mu = \int\limits_K \varphi(x \dot{-} h)\overline{\varphi(x \dot{-} g)}\, d\mu. \qquad \square$$

Definition 5.2. A family of closed subspaces V_n, $n \in \mathbb{Z}$, is said to be a multiresolution analysis of $L_2(K)$ if the following axioms are satisfied:

A1) $V_n \subset V_{n+1}$;
A2) $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L_2(K)$ and $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$;
A3) $f(x) \in V_n \iff f(\mathcal{A}x) \in V_{n+1}$ ($\mathcal{A}$ is a dilation operator);
A4) $f(x) \in V_0 \implies f(x \dot{-} h) \in V_0$ for all $h \in H_0$ (H_0 is analog of $\mathbb{Z}$);
A5) there exists a function $\varphi \in L_2(K)$ such that the system $(\varphi(x \dot{-} h))_{h\in H_0}$ is an orthonormal basis for V_0.

A function φ occurring in axiom A5 is called a *scaling function*.

It is clear that the axiom A5 implies the axiom A4. Next we will follow the conventional approach. Let $\varphi(x) \in L_2(K)$, and suppose that $(\varphi(x \dot{-} h))_{h\in H_0}$ is an orthonormal system in $L_2(K)$. With the function φ and the dilation operator $\mathcal{A}$, we define the linear subspaces $L_n = (\varphi(\mathcal{A}x \dot{-} h))_{h\in H_0}$ and closed subspaces $V_n = \overline{L_n}$. It is evident that the functions $p^{\frac{ns}{2}}\varphi(\mathcal{A}x \dot{-} h)_{h\in H_0}$ form an orthonormal basis for V_n, $n \in \mathbb{Z}$. Therefore the axiom A3 is fulfilled. If subspaces V_j form a MRA, then the function φ is said to *generate* an MRA in $L_2(K)$. If a function φ generates an MRA, then we obtain from the axiom A1

$$\varphi(x) = \sum_{h\in H_0} \beta_h \varphi(\mathcal{A}x \dot{-} h) \quad \left(\sum |\beta_h|^2 < +\infty\right). \tag{5.1}$$

Therefore we will look up a function $\varphi \in L_2(K)$, which generates an MRA in $L_2(K)$, as a solution of the refinement equation (5.1). A solution of refinement equation (5.1) is called a *refinable function*.

Lemma 5.6. *Let $\varphi \in \mathfrak{D}_M(K_{-N})$ be a solution of (5.1). Then*

$$\varphi(x) = \sum_{h\in H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x \dot{-} h) \tag{5.2}$$

The proof is a repetition of the proof of Lemma 4.1 in [12].

Theorem 5.1. *Let $\varphi \in \mathfrak{D}_M(K_{-N})$ and let $(\varphi(x \dot{-} h))_{h\in H_0}$ be an orthonormal system. $V_n \subset V_{n+1}$ if and only if the function $\varphi(x)$ is a solution of refinement equation (5.2).*

The proof is a repetition of the proof of Theorem 4.2 in [12].

Theorem 5.2. *(See [3], Th. 4.1.) Let $\varphi \in \mathfrak{D}_M(K_{-N})$ be a solution of the equation (5.2), $(\varphi(x \dot{-} h))_{h\in H_0}$ an orthonormal basis in V_0 Then $\bigcap\limits_{n\in\mathbb{Z}} V_n = \{0\}$.*

Theorem 5.3. *(See [3], Th. 4.3.) Let $\varphi \in \mathfrak{D}_M(K_{-N})$ be a solution of the equation (5.2), $(\varphi(x \dot{-} h))_{h\in H_0}$ an orthonormal basis in V_0, and $\hat{\varphi}(0) \neq 0$. Then $\overline{\bigcup\limits_{n\in\mathbb{Z}} V_n} = L_2(K)$.*

The refinement equation (5.2) may be written in the form

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}), \tag{5.3}$$

where

$$m_0(\chi) = \frac{1}{p^s} \sum_{h\in H_0^{(N+1)}} \beta_h \overline{(\chi\mathcal{A}^{-1}, h)} \tag{5.4}$$

is a mask of the equation (5.3).

Lemma 5.7. *Let $\varphi \in \mathfrak{D}_M(K_{-N})$. Then the mask $m_0(\chi)$ is constant on cosets $K_{-N}^{\perp}\zeta$.*

Proof. We will prove that $(\chi, \mathcal{A}^{-1}h)$ are constant on cosets $K_{-N}^{\perp}\zeta$. Without loss of generality, we can assume that $\zeta = \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_{-N+s}^{\mathbf{a}_{-N+s}} \notin K_{-N}^{\perp}$. If

$$h = \mathbf{a}_{-1}g_{-1} \dot{+} \dots \dot{+} \mathbf{a}_{-N-1}g_{-N-1} \in H_0^{(N+1)},\ \mathbf{a}_j \in GF(p^s)$$

then

$$\mathcal{A}^{-1}h = \mathbf{a}_{-1}g_0 \dot{+} \dots \dot{+} \mathbf{a}_{-N-1}g_{-N} \in K_{-N}.$$

If $\chi \in K_{-N}^{\perp}\zeta$ then $\chi = \chi_{-N}\zeta$ where $\chi_{-N} \in K_{-N}^{\perp}$. Therefore $(\chi, \mathcal{A}^{-1}h) = (\chi_{-N}\zeta, \mathcal{A}^{-1}h) = (\zeta, \mathcal{A}^{-1}h)$. This means that $(\chi, \mathcal{A}^{-1}h)$ depends on ζ only. $\square$

Lemma 5.8. *The mask $m_0(\chi)$ is a periodic function with any period $\mathbf{r}_1^{\mathbf{a}_1}\mathbf{r}_2^{\mathbf{a}_2}\dots\mathbf{r}_l^{\mathbf{a}_l}$ ($l \in \mathbb{N}$, $\mathbf{a}_j \in GF(p^s)$, $j = \overline{1,l}$).*

Proof. Using the equation $(\mathbf{r}_k^{\mathbf{b}_k}, \mathbf{u}g_j) = 1, (k \neq j)$ we find

$$(\chi\mathbf{r}_1^{\mathbf{b}_1}\mathbf{r}_2^{\mathbf{b}_2}\dots\mathbf{r}_l^{\mathbf{b}_l}, \mathcal{A}^{-1}h) = (\chi\mathbf{r}_1^{\mathbf{b}_1}\mathbf{r}_2^{\mathbf{b}_2}\dots\mathbf{r}_l^{\mathbf{b}_l}, \mathbf{a}_{-1}g_0 \dot{+} \mathbf{a}_{-2}g_{-1} \dot{+} \dots \dot{+} \mathbf{a}_{-N-1}g_{-N}) =$$
$$= (\chi, \mathbf{a}_{-1}g_0 \dot{+} \mathbf{a}_{-2}g_{-1} \dot{+} \dots \dot{+} \mathbf{a}_{-N-1}g_{-N}) = (\chi\mathcal{A}^{-1}, h).$$

Therefore $m_0(\chi\mathbf{r}_1^{\mathbf{b}_1}\mathbf{r}_2^{\mathbf{b}_2}\dots\mathbf{r}_l^{\mathbf{b}_l}) = m_0(\chi)$ and the lemma is proved. □

Lemma 5.9. *The mask $m_0(\chi)$ is defined by its values on cosets $K_{-N}^{\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0}$ ($\mathbf{a}_j = (a_j^{(0)}, a_j^{(1)}, \dots, a_j^{(s-1)}) \in GF(p^s)$).*

Proof. Let us denote

$$k = \sum_{j=0}^{N}(a_{-j}^{(0)} + a_{-j}^{(1)}p + \dots + a_{-j}^{(s-1)}p^{s-1})p^{sj} \in [0, p^{s(N+1)} - 1],$$

$$l = \sum_{j=1}^{N+1}(\alpha_{-j}^{(0)} + \alpha_{-j}^{(1)}p + \dots + \alpha_{-j}^{(s-1)}p^{s-1})p^{s(j-1)} \in [0, p^{s(N+1)} - 1].$$

Then (5.4) can be written as the system

$$m_0(\chi_k) = \frac{1}{p^s}\sum_{l=0}^{p^{s(N+1)}-1}\beta_l\overline{(\chi_k, \mathcal{A}^{-1}h_l)},\ k = \overline{0, p^{s(N+1)} - 1} \tag{5.5}$$

in the unknowns β_l. We consider the characters χ_k on the subgroup K_{-N}^{+}. Since $\mathcal{A}^{-1}h_l \in K_{-N}^{+}$, it follows that the matrix $p^{-\frac{s(N+1)}{2}}\overline{(\chi_k, \mathcal{A}^{-1}h_l)}$ is unitary, and so the system (5.5) has a unique solution for each finite sequence $(m_0(\chi_k))_{k=0}^{p^{s(N+1)}-1}$. □

Remark. The function $m_0(\chi)$ constructed in Lemma 5.9 may not be a mask for $\varphi \in \mathfrak{D}_M(\mathfrak{G}_{-N})$. In the Section 4 we find conditions under which the function $m_0(\chi)$ will be a mask.

Lemma 5.10. *Let $\hat{f}_0(\chi) \in \mathfrak{D}_{-N}((K_1^+)^{\perp})$. Then*

$$\hat{f}_0(\chi) = \frac{1}{p^s}\sum_{h\in H_0^{(N+1)}}\beta_h\overline{(\chi, \mathcal{A}^{-1}h)}. \tag{5.6}$$

Proof. Since $\int\limits_{(K_0^+)^{\perp}}(\chi, g)\overline{(\chi, h)}\,d\nu(\chi) = \delta_{h,g}$ for $h, g \in H_0$ it follows that $\int\limits_{(K_1^+)^{\perp}}(\chi\mathcal{A}^{-1}, g)\overline{(\chi\mathcal{A}^{-1}, h)}\,d\nu(\chi) = p\delta_{h,g}$. Therefore we can consider the set $\left(\frac{\mathcal{A}^{-1}h}{\sqrt{p^s}}\right)_{h\in H_0^{(N+1)}}$ as an orthonormal system on $(K_1^+)^{\perp}$. We know (Lemma 5.7) that $(\chi, \mathcal{A}^{-1}h)$ is a constant on cosets $(K_{-N}^{+})^{\perp}\zeta$. It is evident the dimension of $\mathfrak{D}_{-N}((K_1^+)^{\perp})$ is equal to $p^{s(N+1)}$. Therefore the system $\left(\frac{\mathcal{A}^{-1}h}{\sqrt{p^s}}\right)_{h\in H_0^{(N+1)}}$ is an orthonormal basis for $\mathfrak{D}_{-N}((K_1^+)^{\perp})$ and the equation (5.6) is valid. □

6. Non-Haar wavelets

In this section we find the necessary and sufficient condition under which a step function $\varphi(x) \in \mathfrak{D}_M(K_{-N})$ generates an orthogonal MRA on the local field with positive characteristic. We will prove also that for any $n \in \mathbb{N}$ there exists a step function φ such that 1) φ generate an orthogonal MRA, 2) $\operatorname{supp}\hat{\varphi} \subset K_n^\perp$, 3) $\hat{\varphi}(K_n^\perp \setminus K_{n-1}^\perp) \not\equiv 0$.

Note that the results of Sections 6, 7 and 8, there are analogues of the corresponding results for Vilenkin groups [14]. Moreover, we use the same methods. This is possible since the basic property of Rademacher functions (Lemma 4.3) is satisfied.

First we give a test under which the system of shifts $(\varphi(x \dot- h))_{h\in H_0}$ is an orthonormal system.

Theorem 6.1. *Let $\varphi(x) \in \mathfrak{D}_M(K_{-N})$. A shift's system $(\varphi(x \dot- h))_{h\in H_0}$ will be orthonormal if and only if for any $\overline{\alpha}_{-N}, \overline{\alpha}_{-N+1}, \dots, \overline{\alpha}_{-1} \in GF(p^s)$*

$$\sum_{\overline{\alpha}_0,\overline{\alpha}_1,\dots,\overline{\alpha}_{M-1}\in GF(p^s)} |\hat{\varphi}((K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}})|^2 = 1. \tag{6.1}$$

Proof. First we prove that the system $(\varphi(x \dot- h))_{h\in H_0}$ will be orthonormal if and only if

$$\sum_{\overline{\alpha}_{-N},\dots,\overline{\alpha}_0,\dots,\overline{\alpha}_{M-1}} |\hat{\varphi}((K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}})|^2 = p^{Ns} \tag{6.2}$$

and for any vector $(\mathbf{a}_{-1}, \mathbf{a}_{-2}, \dots, \mathbf{a}_{-N}) \neq (0, 0, \dots, 0)$, $(\mathbf{a}_j \in GF(p^s))$

$$\sum_{\overline{\alpha}_{-1},\dots,\overline{\alpha}_{-N}} \exp\left(\frac{2\pi i}{p}(\mathbf{a}_{-1}^{(0)}\alpha_{-1}^{(0)} + \dots + \mathbf{a}_{-1}^{(s-1)}\alpha_{-1}^{(s-1)} + \dots + \mathbf{a}_{-N}^{(0)}\alpha_{-N}^{(0)} + \dots + \mathbf{a}_{-N}^{(s-1)}\alpha_{-N}^{(s-1)})\right) \times$$
$$\times \sum_{\overline{\alpha}_0,\overline{\alpha}_1,\dots,\overline{\alpha}_{M-1}} |\hat{\varphi}((K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}})|^2 = 0 \tag{6.3}$$

Let $(\varphi(x \dot- h))_{h\in H_0}$ be an orthonormal system. Using the Plancherel equality and Lemma 5.1 we have

$$\begin{aligned}
\delta_{h_1h_2} &= \int_{K^+} \varphi(x \dot- h_1)\overline{\varphi(x \dot- h_2)}\, d\mu(x) = \int_{(K_M^+)^\perp} |\hat{\varphi}(\chi)|^2(\chi, h_2 \dot- h_1)\, d\nu(\chi) = \\
&= \sum_{\overline{\alpha}_{-N},\dots,\overline{\alpha}_0,\dots,\overline{\alpha}_{M-1}} \int_{(K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}}} |\hat{\varphi}(\chi)|^2(\chi, h_2 \dot- h_1)\, d\nu(\chi) = \\
&= \sum_{\overline{\alpha}_{-N},\dots,\overline{\alpha}_{M-1}} |\hat{\varphi}((K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}}|^2 \int_{(K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}}} (\chi, h_2 \dot- h_1)\, d\nu(\chi) = \\
&= p^{-sN}\mathbf{1}_{K_{-N}^+}(h_2 \dot- h_1) \times \\
&\quad \times \sum_{\overline{\alpha}_{-N},\dots,\overline{\alpha}_{M-1}} |\hat{\varphi}((K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}}|^2(\mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}}, h_2 \dot- h_1).
\end{aligned}$$

If $h_2 = h_1$, we obtain the equality (6.2). If $h_2 \neq h_1$ then

$$h_2 \dot- h_1 = \mathbf{a}_{-1}g_{-1} \dot+ \dots \dot+ \mathbf{a}_{-N}g_{-N} \in K_{-N}^+ \tag{6.4}$$

or

$$h_2 \dot{-} h_1 = \mathbf{a}_{-1}g_{-1} \dot{+} \dots \dot{+} \mathbf{a}_{-N}g_{-N} \dot{+} \dots \dot{+} \mathbf{a}_{-l}g_{-l} \in K^+ \backslash K^+_{-N}. \tag{6.5}$$

If the condition (6.5) is fulfilled, then $\mathbf{1}_{(K^+_{-N})}(h_2 \dot{-} h_1) = 0$. If the condition (6.4) is fulfilled, then

$$\mathbf{1}_{K^+_{-N}}(h_2 \dot{-} h_1) = 1,$$

$$(\mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}}, h_2 \dot{-} h_1) = (\mathbf{r}_{-N}^{\mathbf{a}_{-N}}, \overline{\alpha}_{-N}g_{-N}) \dots (\mathbf{r}_{-1}^{\mathbf{a}_{-1}}, \overline{\alpha}_{-1}g_{-1}).$$

Using the equality $(\mathbf{r}_k^{\mathbf{a}_k}, \mathbf{u}g_k) = \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} u^{(l)} a_k^{(l)}}$ we obtain the equality (6.3). The converse may be proved by analogy.

Let as show now if for any vector $(\mathbf{a}_{-1}, \mathbf{a}_{-2}, \dots, \mathbf{a}_{-N}) \neq (0, 0, \dots, 0)$ the conditions (6.2), (6.3) are fulfilled, then for any $\overline{\alpha}_{-N}, \overline{\alpha}_{-N+1}, \dots, \overline{\alpha}_{-1} \in FG(p^s)$

$$\sum_{\overline{\alpha}_0, \overline{\alpha}_1, \dots, \overline{\alpha}_{M-1}} |\hat{\varphi}((K^+_{-N})^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}})|^2 = 1. \tag{6.6}$$

Let us denote

$$n = \sum_{j=1}^{N} \sum_{l=0}^{s-1} a_{-j}^{(l)} p^{s(j-1)}, \quad k = \sum_{j=1}^{N} \sum_{l=0}^{s-1} \alpha_{-j}^{(l)} p^{s(j-1)},$$

$$C_{n,k} = \exp \frac{2\pi i}{p} \left(\sum_{j=1}^{N} \sum_{l=0}^{s-1} \alpha_{-j}^{(l)} a_{-j}^{(l)} \right)$$

and write the equalities (6.2) and (6.3) as the system

$$\begin{aligned}
&C_{0,0}x_0 + C_{0,1}x_1 + \dots + C_{0,p^{sN}-1}x_{p^{sN}-1} = p^{sN} \\
&C_{1,0}x_0 + C_{1,1}x_1 + \dots + C_{1,p^{sN}-1}x_{p^{sN}-1} = 0 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&C_{p^{sN}-1,0}x_0 + C_{p^{sN}-1,1}x_1 + \dots + C_{p^{sN}-1,p^{sN}-1}x_{p^{sN}-1} = 0
\end{aligned} \tag{6.7}$$

with unknowns

$$x_k = \sum_{\overline{\alpha}_0, \overline{\alpha}_1, \dots, \overline{\alpha}_{M-1}} |\hat{\varphi}((K^+_{-N})^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}})|^2.$$

The matrix $(C_{n,k})$ is orthogonal. Indeed, if $(\mathbf{a}_{-1}, \mathbf{a}_{-2}, \dots, \mathbf{a}_{-N}) \neq (\mathbf{a}'_{-1}, \mathbf{a}'_{-2}, \dots, \mathbf{a}'_{-N})$, i.e., $n \neq n'$ we obtain

$$\sum_{k=0}^{p^N-1} C_{n,k}\overline{C_{n',k}} = \sum_{\overline{\alpha}_{-1}, \dots, \overline{\alpha}_{-N}} \exp \left(\frac{2\pi i}{p} \sum_{j=1}^{N} \sum_{l=0}^{s-1} (a_{-j}^{(l)} - a'^{(l)}_{-j}) \alpha_{-j}^{(l)} \right) = 0,$$

so at least one of differences $\mathbf{a}_{-l} - \mathbf{a}'_{-l} \neq 0$. So, the system (6.7) has unique solution. It is evident that $x_k = 1$ is a solution of this system. This means that (6.6) is fulfilled, and the necessity is proved. The sufficiency is evident. □

Now we obtain a necessary and sufficient conditions for function $m_0(\chi)$ to be a mask on the class $\mathfrak{D}_{-N}((K_M^+)^\perp)$, i.e. there exists $\hat\varphi \in \mathfrak{D}_{-N}((K_M^+)^\perp)$ for which

$$\hat\varphi(\chi) = m_0(\chi)\hat\varphi(\chi\mathcal{A}^{-1}). \tag{6.8}$$

If $m_0(\chi)$ is a mask of (6.8) then

P1) $m_0(\chi)$ is constant on cosets $(K_{-N}^+)^\perp\zeta$,
P2) $m_0(\chi)$ is periodic with any period $\mathbf{r}_1^{\overline{\alpha}_1}\mathbf{r}_2^{\overline{\alpha}_2}\dots\mathbf{r}_l^{\overline{\alpha}_l}$, $\overline{\alpha}_j \in GF(p^s)$,
P3) $m_0((K_{-N}^+)^\perp) = 1$.

Therefore we will assume that m_0 satisfies these conditions. Let

$$E_k \subset (K_k^+)^\perp \setminus (K_{k-1}^+)^\perp \quad ,(k = -N+1, -N+2, \dots, 0, 1, \dots, M, M+1)$$

be a set, on which $m_0(E_k) = 0$. Since $m_0(\chi)$ is constant on cosets $(K_{-N}^+)^\perp\zeta$, it follows that E_k is a union of such cosets or $E_k = \emptyset$.

Theorem 6.2. *$m_0(\chi)$ is a mask of some equation on the class $\mathfrak{D}_{-N}((K_M^+)^\perp)$ if and only if*

$$m_0(\chi)m_0(\chi\mathcal{A}^{-1})\dots m_0(\chi\mathcal{A}^{-M-N}) = 0 \tag{6.9}$$

on $(K_{M+1}^+)^\perp \setminus (K_M^+)^\perp$.

Proof. Indeed, if (6.9) is true we set

$$\hat\varphi(\chi) = \prod_{k=0}^{\infty} m_0(\chi\mathcal{A}^{-k}) \in \mathfrak{D}_{-N}((K_M^+)^\perp).$$

Then $\hat\varphi(\chi) = m_0(\chi)\hat\varphi(\chi\mathcal{A}^{-1})$ and

$$m_0(\chi) = \sum_{h \in H_0^{(N+1)}} \beta_h \overline{(\chi\mathcal{A}^{-1}, h)}$$

for some β_h. Therefore $m_0(\chi)$ is a mask. Conversely let $m_0(\chi)$ be a mask, i.e. $\hat\varphi(\chi) = m_0(\chi)\hat\varphi(\chi\mathcal{A}^{-1}) \in \mathfrak{D}_{-N}((K_M^+)^\perp)$. From it we find

$$\hat\varphi(\chi) = m_0(\chi)m_0(\chi\mathcal{A}^{-1})\dots m_0(\chi\mathcal{A}^{-M-N})\hat\varphi(\chi\mathcal{A}^{-M-N-1}),$$

and $\hat\varphi(\chi\mathcal{A}^{-M-N-1}) = 1$ on $(K_{M+1}^+)^\perp$. Since $\hat\varphi(\chi) = 0$ on $(K_{M+1}^+)^\perp \setminus (K_M^+)^\perp$, it follows

$$m_0(\chi)m_0(\chi\mathcal{A}^{-1})\dots m_0(\chi\mathcal{A}^{-M-N}) = 0$$

on $(K_{M+1}^+)^\perp \setminus (K_M^+)^\perp$. $\square$

Lemma 6.1. *Let $\hat\varphi \in \mathfrak{D}_{-N}((K_M^+)^\perp)$ be a solution of the refinement equation*

$$\hat\varphi(\chi) = m_0(\chi)\hat\varphi(\chi\mathcal{A}^{-1})$$

and $(\varphi(x \dot- h))_{h\in H_0}$ be an orthonormal system. Then for any $\overline{\alpha}_{-N}, \alpha_{-N+1}, \dots, \overline{\alpha}_{-1} \in GF(p^s)$

$$\sum_{\overline{\alpha}_0 \in GF(p^s)} |m_0((K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \mathbf{r}_{-N+1}^{\overline{\alpha}_{-N+1}} \dots \mathbf{r}_{-1}^{\overline{\alpha}_{-1}} \mathbf{r}_0^{\overline{\alpha}_0})|^2 = 1. \tag{6.10}$$

Proof. Since $\hat{\varphi} \in \mathfrak{D}_{-N}((K_M^+)^\perp)$, it follows that $\hat{\varphi}((K_{M+1}^+)^\perp \setminus (K_M^+)^\perp) = 0$. Using Theorem 6.1 we have

$$1 = \sum_{\overline{\alpha}_0, \overline{\alpha}_1, \dots, \overline{\alpha}_{M-1} \in GF(p^s)} |\hat{\varphi}((K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}})|^2 =$$

$$= \sum_{\overline{\alpha}_0, \dots, \overline{\alpha}_{M-1}, \overline{\alpha}_M \in GF(p^s)} |\hat{\varphi}((K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}} \mathbf{r}_M^{\overline{\alpha}_M})|^2 = \sum_{\overline{\alpha}_0=0}^{p-1} |m_0((K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_0^{\overline{\alpha}_0})|^2$$

$$\cdot \sum_{\overline{\alpha}_1, \dots, \overline{\alpha}_{M-1}, \overline{\alpha}_M \in GF(p^s)} |\hat{\varphi}((K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N+1}} \dots \mathbf{r}_{-1}^{\overline{\alpha}_0} \mathbf{r}_0^{\overline{\alpha}_1} \dots \mathbf{r}_{M-2}^{\overline{\alpha}_{M-1}} \mathbf{r}_{M-1}^{\overline{\alpha}_M})|^2 =$$

$$= \sum_{\overline{\alpha}_0 \in GF(p^s)} |m_0((K_{-N}^+)^\perp \mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \dots \mathbf{r}_0^{\overline{\alpha}_0})|^2. \qquad \square$$

Theorem 6.3. *Suppose the function $m_0(\chi)$ satisfies the conditions P1, P2, P3, (6.9), and the function*

$$\hat{\varphi}(\chi) = \prod_{n=0}^{\infty} m_0(\chi \mathcal{A}^{-n})$$

satisfies the condition (6.1). Then $\varphi \in \mathfrak{D}_M(K_{-N}^+)$ and generates an orthogonal MRA.

Proof. It is evident that $\hat{\varphi} \in \mathfrak{D}_{-N}((K_M^+)^\perp)$, $\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1})$, and $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system. From Theorems 5.1, 5.2, 5.3 we find that the function φ generates an orthogonal MRA. $\square$

7. (N, M)-elementary sets

So, to find a refinable function that generates orthogonal MRA, we need take a function $m_0(\chi)$ that satisfies conditions P1, P2, P3, (6.9), construct the function

$$\hat{\varphi}(\chi) = \prod_{k=0}^{\infty} m_0(\chi \mathcal{A}^{-k}) \in \mathfrak{D}_{-N}((K_M^+)^\perp)$$

and check that the system $\varphi(x \dot{-} h)_{h \in H_0}$ is orthonormal. We want to give a simple condition under which the system of shifts $\varphi(x \dot{-} h)_{h \in H_0}$ is orthonormal.

Definition 7.1. Let $N, M \in \mathbb{N}$. A set $E \subset X$ is called (N, M)-elementary if E is disjoint union of p^{sN} cosets

$$(K_{-N}^+)^\perp \zeta_j = (K_{-N}^+)^\perp \underbrace{\mathbf{r}_{-N}^{\overline{\alpha}_{-N}} \mathbf{r}_{-N+1}^{\overline{\alpha}_{-N+1}} \dots \mathbf{r}_{-1}^{\overline{\alpha}_{-1}}}_{\xi_j} \underbrace{\mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}}}_{\eta_j} = (K_{-N}^+)^\perp \xi_j \eta_j,$$

$j = 0, 1, \dots, p^{sN} - 1$, $j = \sum_{l=0}^{N-1} (\alpha_{-N+l}^{(0)} + \alpha_{-N+l}^{(1)} p + \dots + \alpha_{-N+l}^{(s-1)} p^{s-1}) p^{sl}$ $(\overline{\alpha}_\nu \in GF(p^s))$ such that

1) $\bigsqcup_{j=0}^{p^{sN}-1} (K_{-N}^+)^\perp \xi_j = (K_0^+)^\perp$, $(K_{-N}^+)^\perp \xi_0 = (K_{-N}^+)^\perp$,
2) for any $l = \overline{0, M+N-1}$ the intersection $((K_{-N+l+1}^+)^\perp \setminus (K_{-N+l}^+)^\perp) \bigcap E \neq \emptyset$.

Lemma 7.1. *The set $H_0 \subset K$ is an orthonormal system on any (N, M)-elementary set $E \subset X$.*

Proof. Let $h, g \in H_0$. Using the definition of (N, M)-elementary set we have

$$\int\limits_E (\chi, h)\overline{(\chi, g)}\, d\nu(\chi) = \sum_{j=0}^{p^{sN}-1} \int\limits_{(K_{-N}^{+})^{\perp}\zeta_j} (\chi, h)\overline{(\chi, g)}\, d\nu(\chi) =$$

$$= \sum_{j=0}^{p^{sN}-1} \int\limits_X \mathbf{1}_{(K_{-N}^{+})^{\perp}\zeta_j}(\chi)(\chi, h)\overline{(\chi, g)}\, d\nu(\chi) =$$

$$= \sum_{j=0}^{p^{sN}-1} \int\limits_X \mathbf{1}_{(K_{-N}^{+})^{\perp}\zeta_j}(\chi\eta_j)(\chi\eta_j, h)\overline{(\chi\eta_j, g)}\, d\nu(\chi) =$$

$$= \sum_{j=0}^{p^{sN}-1} \int\limits_X \mathbf{1}_{(K_{-N}^{+})^{\perp}\xi_j}(\chi)(\chi, h)\overline{(\chi, g)}(\eta_j, h)\overline{(\eta_j, g)}\, d\nu(\chi).$$

Since

$$(\eta_j, h) = (\mathbf{r}_0^{\overline{\alpha}_0}\mathbf{r}_1^{\overline{\alpha}_1}\dots\mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}}, \mathbf{a}_{-1}g_{-1} \dot{+} \mathbf{a}_{-2}g_{-2} \dot{+} \dots \dot{+} \mathbf{a}_{-l}g_{-l}) = 1,$$
$$(\eta_j, g) = (\mathbf{r}_0^{\overline{\alpha}_0}\mathbf{r}_1^{\overline{\alpha}_1}\dots\mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}}, \mathbf{b}_{-1}g_{-1} \dot{+} \mathbf{b}_{-2}g_{-2} \dot{+} \dots \dot{+} \mathbf{b}_{-l}g_{-l}) = 1,$$

then

$$\int\limits_E (\chi, h)\overline{(\chi, g)}\, d\nu(\chi) = \sum_{j=0}^{p^{sN}-1} \int\limits_{(K_{-N}^{+})^{\perp}\xi_j} (\chi, h)\overline{(\chi, g)}\, d\nu(\chi) =$$

$$= \int\limits_{(K_0^{+})^{\perp}} (\chi, h)\overline{(\chi, g)}\, d\nu(\chi) = \delta_{h,g}. \qquad \square$$

Theorem 7.1. *Let $K = F^{(s)}$ be a local field with positive characteristic p, $E \subset (K_M^{+})^{\perp}$ an (N, M)-elementary set. If $|\hat{\varphi}(\chi)| = \mathbf{1}_E(\chi)$ on X then the system of shifts $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system on K.*

Proof. Let $\tilde{H}_0 \subset H_0$ be a finite set. Using the Plancherel equation we have

$$\int\limits_K \varphi(x \dot{-} g)\overline{\varphi(x \dot{-} g)}\, d\mu(x) = \int\limits_X |\hat{\varphi}(\chi)|^2\overline{(\chi, g)}(\chi, h)d\nu(\chi) = \int\limits_E (\chi, h)\overline{(\chi, g)}d\nu(\chi) =$$

$$= \sum_{j=0}^{p^{sN}-1} \int\limits_{(K_{-N}^{+})^{\perp}\zeta_j} (\chi, h)\overline{(\chi, g)}\, d\nu(\chi).$$

Transform the inner integral

$$\int\limits_{(K_{-N}^{+})^{\perp}\zeta_j} (\chi, h)\overline{(\chi, g)}\, d\nu(\chi) = \int\limits_X \mathbf{1}_{(K_{-N}^{+})^{\perp}\zeta_j}(\chi)(\chi, h)\overline{(\chi, g)}\, d\nu(\chi) = \int\limits_X \mathbf{1}_{(K_{-N}^{+})^{\perp}\zeta_j}(\chi\eta_j)(\chi\eta_j, h \dot{-} g)\, d\nu(\chi) =$$

$$= \int\limits_X \mathbf{1}_{(K_{-N}^{+})^{\perp}\xi_j}(\chi)(\chi\eta_j, h \dot{-} g)\, d\nu(\chi) = \int\limits_{(K_{-N}^{+})^{\perp}\xi_j} (\chi\eta_j, h \dot{-} g)\, d\nu(\chi).$$

Repeating the arguments of Lemma 7.1 we obtain

$$\int\limits_K \varphi(x \dot{-} h)\overline{\varphi(x \dot{-} g)}\, d\mu(x) = \delta_{h,g}. \qquad \square$$

8. Trees and wavelets

Let $K = F^{(s)}$ be a local field of characteristic p. In this section we reduce the problem of construction of step refinable function on the field K to construction of some tree.

We will consider some special class of refinable functions $\varphi(\chi)$ for which $|\hat{\varphi}(\chi)|$ is a characteristic function of a set. Define this class.

Definition 8.1. A mask $m_0(\chi)$ is called N-elementary ($N \in \mathbb{N}_0$) if $m_0(\chi)$ is constant on cosets $(K_{-N}^{+})^{\perp}\chi$, its modulus $m_0(\chi)$ has two values only: 0 and 1, and $m_0((K_{-N}^{+})^{\perp}) = 1$. The refinable function $\varphi(x)$ with Fourier transform

$$\hat{\varphi}(\chi) = \prod_{n=0}^{\infty} m_0(\chi\mathcal{A}^{-n})$$

is called N-elementary too. N-elementary function φ is called (N, M)-elementary if $\hat{\varphi}(\chi) \in \mathfrak{D}_{-N}(K_M^{\perp})$. In this case we will call the Fourier transform $\hat{\varphi}(\chi)$ (N, M)-elementary, also.

Definition 8.2. Let $\tilde{E} = \bigsqcup\limits_{\overline{\alpha}_{-1},\overline{\alpha}_0} (K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0} \subset (K_1^{+})^{\perp}$ be an $(1, 1)$-elementary set. We say that the set $\tilde{E}_X$ is a periodic extension of $\tilde{E}$ if

$$\tilde{E}_X = \bigcup_{l=1}^{\infty} \bigsqcup_{\overline{\alpha}_1,\dots,\overline{\alpha}_l \in GF(p^s)} \tilde{E}\mathbf{r}_1^{\overline{\alpha}_1}\mathbf{r}_2^{\overline{\alpha}_2}\dots\mathbf{r}_l^{\overline{\alpha}_l}.$$

We say that the set $\tilde{E}$ generates an $(1, M)$ elementary set E, if $\bigcap\limits_{n=0}^{\infty} \tilde{E}_X\mathcal{A}^n = E$.

Let us write the set $GF(p^s)$ in the form

$$\{\mathbf{0}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q, \overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_{p^s-q-1}\} = V, \quad \mathbf{0} = \mathbf{u}_0,$$

where $1 \le q \le p^s - 1$. We will consider the set V as a set of vertices. By $T(\mathbf{0}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q, \overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_{p^s-q-1}) = T(V)$ we will denote a rooted tree on the set of vertices V, where $\mathbf{0}$ is a root, $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q$ are first level vertices, $\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_{p^s-q-1}$ are remaining vertices.

For example for $p = 3$, $s = 2$, $q = 2$, $\mathbf{u}_1 = (2, 1)$, $\mathbf{u}_2 = (1, 0)$ we have trees (see Fig. 1 or Fig. 2) and so on.

For any tree path

$$P_j = (\mathbf{0} \to \mathbf{u}_j \to \overline{\alpha}_{l-1} \to \overline{\alpha}_{l-2} \to \dots \to \overline{\alpha}_0 \to \overline{\alpha}_{-1})$$

we construct the set of cosets

$$(K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\mathbf{u}_j}, (K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{l-1}}\mathbf{r}_0^{\mathbf{u}_j}, (K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{l-2}}\mathbf{r}_0^{\overline{\alpha}_{l-1}}, \dots, (K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_0}\mathbf{r}_0^{\overline{\alpha}_1}, (K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}. \tag{8.1}$$

For example for the tree from Fig. 2 and the path

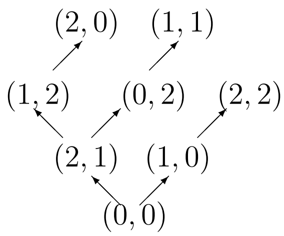

Fig. 1. Each pair is present in the tree exactly 1 times.

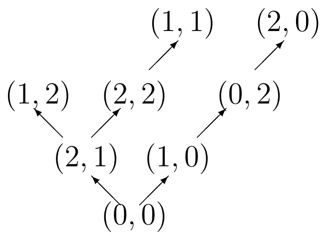

Fig. 2. The tree contains the same vertices, the root of the tree is the pair $(0, 0)$.

$$(0,0) \to (1,0) \to (0,2) \to (2,0)$$

we have 3 cosets

$$(K^+_{-1})^\perp \mathbf{r}^{(1,0)}_{-1}, (K^+_{-1})^\perp \mathbf{r}^{(0,2)}_{-1} \mathbf{r}^{(1,0)}_{0}, (K^+_{-1})^\perp \mathbf{r}^{(2,0)}_{-1} \mathbf{r}^{(0,2)}_{0},$$

for the path $(0,0), (2,1), (2,2)$ we have two cosets

$$(K^+_{-1})^\perp \mathbf{r}^{(2,1)}_{-1}, (K^+_{-1})^\perp \mathbf{r}^{(2,2)}_{-1} \mathbf{r}^{(2,1)}_{0}.$$

We will represent the tree $T(V)$ as the tree $\begin{array}{c} T_1 \cdots T_q \\ (0,0) \end{array}$ where T_j are tree branches of $T(V)$ with $\mathbf{u}_j$ as a root. By E_j denote a union of all cosets (8.1) for fixed j and set

$$\tilde{E} = \left(\bigsqcup_{j=1}^{q} E_j \right) \bigsqcup (K^+_{-1})^\perp. \tag{8.2}$$

It is clear that $\tilde{E}$ is an $(1,1)$ elementary set and $\tilde{E} \subset (K^+_1)^\perp$.

Definition 8.3. Let $\tilde{E}_X$ be a periodic extension of $\tilde{E}$. We say that the tree $T(V)$ generates a set E, if $E = \bigcap\limits_{n=0}^{\infty} \tilde{E}_X \mathcal{A}^n$.

Lemma 8.1. *Let $T(V)$ be a rooted tree with $\mathbf{0} = (0,0,\dots,0)$ as a root. Let $E \subset X$ be a set generated by the tree $T(V)$, H a height of $T(V)$. Then E is an $(1, H-2)$-elementary set.*

Proof. Let us denote

$$m(\chi) = \mathbf{1}_{\tilde{E}_X}(\chi), \quad M(\chi) = \prod_{n=0}^{\infty} m(\chi \mathcal{A}^{-n}).$$

First we note that $M(\chi) = \mathbf{1}_E(\chi)$. Indeed

$$\mathbf{1}_E(\chi)=1 \Leftrightarrow \chi\in E \Leftrightarrow \forall n,\ \chi\mathcal{A}^{-n}\in \tilde{E}_X \Leftrightarrow \forall n,\ \mathbf{1}_{\tilde{E}_X}(\chi\mathcal{A}^{-n})=1 \Leftrightarrow$$

$$\forall n,\ m(\chi\mathcal{A}^{-n})=1 \Leftrightarrow \prod_{n=0}^{\infty} m(\chi\mathcal{A}^{-n})=1 \Leftrightarrow M(\chi)=1.$$

It means that $M(\chi)=\mathbf{1}_E(\chi)$. Now we will prove, that $\mathbf{1}_E(\chi)=0$ for $\chi\in (K_{H-1}^+)^\perp\setminus (K_{H-2}^+)^\perp$. Since $\tilde{E}_X\supset (K_{-1}^+)^\perp$ it follows that $\mathbf{1}_{\tilde{E}_X}((K_{H-1}^+)^\perp\mathcal{A}^{-H})=\mathbf{1}_{\tilde{E}_X}((K_{-1}^+)^\perp)=1$. Consequently

$$\prod_{n=0}^{\infty}\mathbf{1}_{\tilde{E}_X}(\chi\mathcal{A}^{-n})=\prod_{n=0}^{H-1}\mathbf{1}_{\tilde{E}_X}(\chi\mathcal{A}^{-n})$$

for $\chi\in (K_{H-1}^+)^\perp\setminus (K_{H-2}^+)^\perp$. Let us denote $m((K_{-1}^+)^\perp\mathbf{r}_{-1}^{\mathbf{i}}\mathbf{r}_0^{\mathbf{k}})=\lambda_{\mathbf{i},\mathbf{k}}$. By the definition of cosets (8.1) $m((K_{-1}^+)^\perp\mathbf{r}_{-1}^{\mathbf{i}}\mathbf{r}_0^{\mathbf{k}})\neq 0 \Leftrightarrow$ the pair $(\mathbf{k},\mathbf{i})$ is an edge of the tree $T(V)$.

We need prove that

$$\mathbf{1}_E((K_{-1}^+)^\perp\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}\dots\mathbf{r}_{H-2}^{\overline{\alpha}_{H-2}})=0$$

for $\overline{\alpha}_{H-2}\neq 0$. Since $\tilde{E}_X$ is a periodic extension of $\tilde{E}$ it follows that the function $m(\chi)=\mathbf{1}_{\tilde{E}_X}(\chi)$ is periodic with any period $\mathbf{r}_1^{\overline{\alpha}_1}\mathbf{r}_2^{\overline{\alpha}_2}\dots\mathbf{r}_l^{\overline{\alpha}_l}$, $l\in\mathbb{N}$, i.e. $m(\chi\mathbf{r}_1^{\overline{\alpha}_1}\mathbf{r}_2^{\overline{\alpha}_2}\dots\mathbf{r}_l^{\overline{\alpha}_l})=m(\chi)$ when $\chi\in (K_1^+)^\perp$. Using this fact we can write $M(\chi)$ for $\chi\in (K_{H-1}^+)^\perp\setminus (K_{H-2}^+)^\perp$ in the form

$$\begin{aligned}M((K_{-1}^+)^\perp\zeta)&=M((K_{-1}^+)^\perp\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}\dots\mathbf{r}_{H-2}^{\overline{\alpha}_{H-2}})=\\&=m((K_{-1}^+)^\perp\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0})m((K_{-1}^+)^\perp\mathbf{r}_{-1}^{\overline{\alpha}_0}\mathbf{r}_0^{\overline{\alpha}_1})\dots\\&\quad m((K_{-1}^+)^\perp\mathbf{r}_{-1}^{\overline{\alpha}_{H-3}}\mathbf{r}_0^{\overline{\alpha}_{H-2}})m((K_{-1}^+)^\perp\mathbf{r}_{-1}^{\overline{\alpha}_{H-2}})=\\&=\lambda_{\overline{\alpha}_{-1},\overline{\alpha}_0}\lambda_{\overline{\alpha}_0,\overline{\alpha}_1}\dots\lambda_{\overline{\alpha}_{H-3},\overline{\alpha}_{H-2}}\lambda_{\overline{\alpha}_{H-2},\mathbf{0}},\ \overline{\alpha}_{H-2}\neq\mathbf{0}.\end{aligned}$$

If $\lambda_{\overline{\alpha}_{H-2},\mathbf{0}}=0$ then $M((K_{-1}^+)^\perp\zeta)=0$. Let $\lambda_{\overline{\alpha}_{H-2},\mathbf{0}}\neq 0$. It means that $\overline{\alpha}_{H-2}=\mathbf{u}_j$ for some $j=\overline{1,q}$. If $\lambda_{\overline{\alpha}_{H-3},\overline{\alpha}_{H-2}}=0$ then $M((K_{-1}^+)^\perp\zeta)=0$. Assume that $\lambda_{\overline{\alpha}_{H-3},\overline{\alpha}_{H-2}}\neq 0$. It is true iff the pair $(\overline{\alpha}_{H-2},\overline{\alpha}_{H-3})$ is an edge of $T(V)$. Repeating these arguments, we obtain a path

$$(\mathbf{0}\to\mathbf{u}_j=\overline{\alpha}_{H-2}\to\overline{\alpha}_{H-3}\to\dots\to\overline{\alpha}_l)$$

of the tree $T(V)$. Since $\mathrm{height}(T)=H$ it follows that $l\geq 0$. Consequently $(\overline{\alpha}_l,\overline{\alpha}_{l-1})$ is not edge and $\lambda_{\overline{\alpha}_{l-1},\overline{\alpha}_l}=0$, where $l\geq 0$. It means that $M((K_{-1}^+)^\perp\zeta)=0$.

Now we prove that E is $(1,H-2)$-elementary set. Indeed, any path

$$(\mathbf{0}\to\mathbf{u}_j=\overline{\alpha}_{l-1}\to\overline{\alpha}_{l-2}\to\dots\to\overline{\alpha}_0\to\overline{\alpha}_{-1})$$

defines the coset $(K_{-1}^+)^\perp\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}\dots\mathbf{r}_{l-1}^{\overline{\alpha}_{l-1}}\subset E$. But for any $\overline{\alpha}_{-1}\in GF(p^s)$ there exists unique path with endpoint $\overline{\alpha}_{-1}$ and starting point zero. It means that E is $(1,H-2)$-elementary set. □

Theorem 8.1. *Let $M,s\in\mathbb{N}$, $p^s\geq 3$. Let $E\subset (K_M^+)^\perp$ be an $(1,M)$-elementary set, $\hat{\varphi}\in\mathfrak{D}_{-1}((K_M^+)^\perp)$, $|\hat{\varphi}(\chi)|=\mathbf{1}_E(\chi)$, $\hat{\varphi}(\chi)$ the solution of the equation*

$$\hat{\varphi}(\chi)=m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}),\tag{8.3}$$

where $m_0(\chi)$ is an 1-elementary mask. Then there exists a rooted tree $T(V)$ with $\mathrm{height}(T)=M+2$ *that generates the set E.*

Proof. Since the set E is $(1,M)$-elementary set and $|\hat{\varphi}(\chi)| = \mathbf{1}_E(\chi)$, it follows from Theorem 7.1 that the system $(\varphi(x \dot{-} h))_{h\in H_0}$ is an orthonormal system in $L_2(K)$. Using the Theorem 6.1 we obtain that for any $\overline{\alpha}_{-1} \in GF(p^s)$

$$\sum_{\overline{\alpha}_0,\overline{\alpha}_1,\dots,\overline{\alpha}_{M-1}\in GF(p^s)} |\hat{\varphi}((K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}\dots\mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}})|^2 = 1. \tag{8.4}$$

Since $\hat{\varphi}$ is a solution of refinement equation (8.3) it follows from Lemma 6.1 that for $\overline{\alpha}_{-1} \in GF(p^s)$

$$\sum_{\overline{\alpha}_0\in GF(p^s)} |m_0((K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0})|^2 = 1. \tag{8.5}$$

Let as denote $\lambda_{\overline{\alpha}_{-1},\overline{\alpha}_0} := m_0((K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0})$. Then we write (8.5) in the form

$$\sum_{\overline{\alpha}_0\in GF(p^s)} |\lambda_{\overline{\alpha}_{-1},\overline{\alpha}_0}|^2 = 1. \tag{8.6}$$

Since the mask $m_0(\chi)$ is 1-elementary it follows that $|\lambda_{\overline{\alpha},\overline{\beta}}|$ take two values only: 0 or 1.

Now we will construct the tree T. Let $\mathfrak{U}$ be a family of cosets $(K_{-1}^{+})^{\perp}\zeta \subset (K_M^{+})^{\perp}$ such that $\hat{\varphi}((K_{-1}^{+})^{\perp}\zeta) \neq 0$ and $(K_{-1}^{+})^{\perp} \notin \mathfrak{U}$. We can write a coset $(K_{-1}^{+})^{\perp}\zeta \in \mathfrak{U}$ in the form

$$(K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}\dots\mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}}.$$

If $(K_{-1}^{+})^{\perp}\zeta \subset (K_l^{+})^{\perp} \setminus (K_{l-1}^{+})^{\perp}$ $(l \le M)$ then

$$(K_{-1}^{+})^{\perp}\zeta = (K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}\dots\mathbf{r}_{l-1}^{\overline{\alpha}_{l-1}},\ \overline{\alpha}_{l-1} \neq \mathbf{0}.$$

Let $\mathbf{u} \neq \mathbf{0}$. By $T_{\mathbf{u}}$ we denote the set of vectors $(\mathbf{u},\overline{\alpha}_{n-1},\dots,\overline{\alpha}_0,\overline{\alpha}_{-1})$ for which $(K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}\dots\mathbf{r}_{n-1}^{\overline{\alpha}_{n-1}}\mathbf{r}_n^{\mathbf{u}} \in \mathfrak{U}$. We will name the vector $(\mathbf{u},\overline{\alpha}_{n-1},\dots,\overline{\alpha}_0,\overline{\alpha}_{-1})$ as a path too. So $T_{\mathbf{u}}$ is the set of paths with starting point $\mathbf{u}$, for which $\hat{\varphi}((K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}\dots\mathbf{r}_{n-1}^{\overline{\alpha}_{n-1}}\mathbf{r}_n^{\mathbf{u}}) \neq 0$. Denote (it follow from (8.5)), if $\hat{\varphi}((K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}\dots\mathbf{r}_{n-1}^{\overline{\alpha}_{n-1}}\mathbf{r}_n^{\mathbf{u}}) \neq 0$ then $\hat{\varphi}((K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}\dots\mathbf{r}_{n-1}^{\overline{\alpha}_{n-1}}\mathbf{r}_n^{\mathbf{u}}\mathbf{r}_{n+1}^{\overline{\alpha}_{n+1}}) = 0$ for any $\overline{\alpha}_{n+1} \neq \mathbf{0}$. We will show that $T_{\mathbf{u}}$ is a rooted tree with $\mathbf{u}$ as a root.

1) All vertices $\overline{\alpha}_j,\mathbf{u}$ of the path $(\mathbf{u},\overline{\alpha}_{n-1},\dots,\overline{\alpha}_0,\overline{\alpha}_{-1})$ are pairwise distinct. Indeed

$$\hat{\varphi}((K_{-1}^{+})^{\perp}\mathbf{r}_{-1}^{\overline{\alpha}_{-1}}\mathbf{r}_0^{\overline{\alpha}_0}\dots\mathbf{r}_{n-1}^{\overline{\alpha}_{n-1}}\mathbf{r}_n^{\mathbf{u}}) = \lambda_{\overline{\alpha}_{-1},\overline{\alpha}_0}\lambda_{\overline{\alpha}_0,\overline{\alpha}_1}\dots\lambda_{\overline{\alpha}_{n-1},\mathbf{u}}\overline{\lambda}_{\mathbf{u},\mathbf{0}} \neq 0,\ \mathbf{u} \neq 0.$$

If $\overline{\alpha}_{n-1} = \mathbf{u}$ then $|\lambda_{\mathbf{u},\mathbf{u}}| = |\lambda_{\mathbf{u},\mathbf{0}}| = 1$ that contradicts the equation (8.5).

If $\overline{\alpha}_{n-1} = 0$ then $|\lambda_{\mathbf{0},\mathbf{u}}| = |\lambda_{\mathbf{0},\mathbf{0}}| = 1$ that contradicts the equation (8.5) too. Consequently $\overline{\alpha}_{n-1} \notin \{\mathbf{0},\mathbf{u}\}$. By analogy we obtain that $\overline{\alpha}_i \notin \{\mathbf{0},\mathbf{u},\overline{\alpha}_{n-1},\dots,\overline{\alpha}_{i+2},\overline{\alpha}_{i+1}\}$.

2) If two paths $(\mathbf{u},\overline{\alpha}_{k-1},\dots,\overline{\alpha}_0,\overline{\alpha}_{-1})$ and $(\mathbf{u},\overline{\beta}_{l-1},\dots,\overline{\beta}_0,\overline{\beta}_{-1})$ have the common subpath $(\mathbf{u},\overline{\alpha}_{k-1},\dots,\overline{\alpha}_{k-j+1},\overline{\alpha}_{k-j}) = (\mathbf{u},\overline{\beta}_{l-1},\dots,\overline{\beta}_{l-j+1},\overline{\beta}_{l-j})$ and $\overline{\alpha}_{k-j-1} \neq \overline{\beta}_{l-j-1}$ then $\{\overline{\alpha}_{-1},\overline{\alpha}_0,\dots,\overline{\alpha}_{k-j-1}\}\bigcap\{\overline{\beta}_{-1},\overline{\beta}_0,\dots,\overline{\beta}_{l-j-1}\} = \emptyset$. Indeed, let

$$\{\overline{\alpha}_{-1},\overline{\alpha}_0,\dots,\overline{\alpha}_{k-j-1}\}\bigcap\{\overline{\beta}_{-1},\overline{\beta}_0,\dots,\overline{\beta}_{l-j-1}\} \neq \emptyset.$$

Then there exists $\mathbf{v} \in \{\overline{\alpha}_{-1},\overline{\alpha}_0,\dots,\overline{\alpha}_{k-j-1}\}\bigcap\{\overline{\beta}_{-1},\overline{\beta}_0,\dots,\overline{\beta}_{l-j-1}\}$. Assume that $\mathbf{v} \neq \overline{\alpha}_{k-j-1}$. Then $\mathbf{v} = \overline{\alpha}_\nu$, $-1 \le \nu \le k-j-2$ and $\mathbf{v} = \overline{\beta}_\mu$, $-1 \le \mu \le l-j-1$. It follows that

$$(\mathbf{u} = \overline{\alpha}_k, \dots, \overline{\alpha}_{k-j}, \overline{\alpha}_{k-j-1}, \dots, \overline{\alpha}_{\nu+1}, \overline{\alpha}_\nu = \overline{\beta}_\mu, \overline{\beta}_{\mu-1}, \dots, \overline{\beta}_0, \overline{\beta}_{-1}) \in T_{\mathbf{u}}$$
$$(\mathbf{u} = \overline{\beta}_l, \dots, \overline{\beta}_{l-j} = \overline{\alpha}_{k-j}, \overline{\beta}_{l-j-1}, \dots, \overline{\beta}_{\mu+1}, \overline{\beta}_\mu, \overline{\beta}_{\mu-1}, \dots, \overline{\beta}_0, \overline{\beta}_{-1}) \in T_u.$$

So we have two different paths with the same sheet $\overline{\beta}_{-1}$. But this contradicts Theorem 6.1. This means that $T_{\mathbf{u}}$ has no cycles, consequently $T_{\mathbf{u}}$ is a graph with $\mathbf{u}$ as a root.

3) By analogy we can proof that different trees $T_{\mathbf{u}}$ an $T_{\mathbf{v}}$ has no common vertices. It follows that the graph $T = (\mathbf{0}, T_{\mathbf{u}_1}, \dots, T_{\mathbf{u}_q})$ is a tree with $\mathbf{0}$ as a root.

4) It is evident that this tree generates refinable function $\hat{\varphi}$ with a mask m_0. Show that height$(T) = M+2$. Indeed, since $\hat{\varphi} \in \mathfrak{D}_{-1}((K_M^+)^\perp)$ it follows that there exists a coset $(K_{-1}^+)^\perp r_{-1}^{\overline{\alpha}_{-1}} r_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}}$, $\overline{\alpha}_{M-1} \neq 0$ for which

$$|\hat{\varphi}((K_{-1}^+)^\perp \mathbf{r}_{-1}^{\overline{\alpha}_{-1}} \mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{M-1}^{\overline{\alpha}_{M-1}})| = 1.$$

This coset generates a path $(\mathbf{0}, \overline{\alpha}_{M-1} = \mathbf{u}, \overline{\alpha}_{M-2}, \dots, \overline{\alpha}_0, \overline{\alpha}_{-1})$ of T. This path contain $M+2$ vertex. It means that height$(T) \geq M+2$. On the other hand there isn't coset $(K_{-1}^+)^\perp \zeta \in \mathfrak{U}$ with condition $(K_{-1}^+)^\perp \zeta \subset (K_{M+1}^+)^\perp \setminus (K_M^+)^\perp$, consequently there isn't path with $L > M+2$. So height$(T) = M+2$. Since supp $\hat{\varphi}(\chi)$ is $(1, M)$-elementary set, it follows that the set of all vertices of the tree T is the set $GF(p^s)$. The theorem is proved. $\square$

Definition 8.4. Let $T(V)$ be a rooted tree with $\mathbf{0}$ as a root, H a height of $T(V)$, $V = GF(p^s)$. Using cosets (8.1) we define the mask $m_0(\chi)$ in the subgroup $(K_1^+)^\perp$ as follows: $m_0((K_{-1}^+)^\perp) = 1, m_0((K_{-1}^+)^\perp \mathbf{r}_{-1}^{\mathbf{i}} r_0^{\mathbf{j}}) = \lambda_{\mathbf{i},\mathbf{j}}$, $|\lambda_{\mathbf{i},\mathbf{j}}| = 1$ when $(K_{-1}^+)^\perp \mathbf{r}_{-1}^{\mathbf{i}} \mathbf{r}_0^{\mathbf{j}} \subset \tilde{E}$, (q.v. (8.2)), $|\lambda_{\mathbf{i},\mathbf{j}}| = 0$ when $(K_{-1}^+)^\perp \mathbf{r}_{-1}^{\mathbf{i}} \mathbf{r}_0^{\mathbf{j}} \subset (K_1^+)^\perp \setminus \tilde{E}$. Let us extend the mask $m_0(\chi)$ on the $X \setminus (K_1^+)^\perp$ periodically, i.e. $m_0(\chi \mathbf{r}_1^{\overline{\alpha}_1} \mathbf{r}_2^{\overline{\alpha}_2} \dots \mathbf{r}_l^{\overline{\alpha}_l}) = m_0(\chi)$. Then we say that the tree $T(V)$ generates the mask $m_0(\chi)$. Set $\hat{\varphi}(\chi) = \prod_{n=0}^{\infty} m_0(\chi \mathcal{A}^{-n})$. It follows from Lemma 8.1 that

1) supp $\hat{\varphi}(\chi) \subset (K_{H-2}^+)^\perp$,
2) $\hat{\varphi}(\chi)$ is $(1, H-2)$-elementary function,
3) $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system.

In this case we say that the tree $T(V)$ generates the refinable function $\varphi(x)$.

Theorem 8.2. *Let $p \geq 2$ be a prime number, $s \in \mathbb{N}, p^s \geq 3$,*

$$V = \{\mathbf{0}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{p^s-q-1}\}$$

a set of vertices, $T(V)$ a rooted tree, $\mathbf{0}$ the root, $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q$ a first level vertices. Let H be are height of $T(V)$. By $\varphi(x)$ denote the function generated by the $T(V)$. Then $\varphi(x)$ generate an orthogonal MRA on $F^{(s)}$.

Proof. Since $T(V)$ generates the function φ, it follows that 1) $\hat{\varphi} \in \mathfrak{D}_{-1}((K_1^+)^\perp)$, 2) $\hat{\varphi}(\chi)$ is $(1, H-2)$ elementary function, 3) $\hat{\varphi}(\chi)$ is a solution of refinable equation (8.3), 4) $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system. From the Theorem 6.3 it follows that $\varphi(x)$ generates an orthogonal MRA. $\square$

Remark. Now we can give a simple algorithm for constructing non-Haar refinable function $\varphi(x)$. Let $T(V)$ be a tree on the set $V = GF(p^s)$. Construct a finite sequence $(\lambda_{\mathbf{i},\mathbf{j}})_{\mathbf{i},\mathbf{j} \in GF(p^s)}$ as follows: $\lambda_{\mathbf{0},\mathbf{0}} = 1$, $|\lambda_{\mathbf{i},\mathbf{j}}| = 1$ if the pair $(\mathbf{j}, \mathbf{i})$ is an edge of $T(f)$. For any vertex $\overline{\alpha}_{-1}$ we take the path $(\mathbf{0} = \overline{\alpha}_{l+1}, \mathbf{u}_j = \overline{\alpha}_l, \overline{\alpha}_{l-1}, \dots, \overline{\alpha}_0, \overline{\alpha}_{-1})$ and suppose

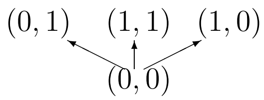

Fig. 3. All nonzero vertices have the first level.

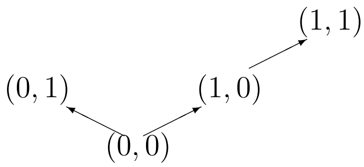

Fig. 4. Two nonzero vertices have the first level.

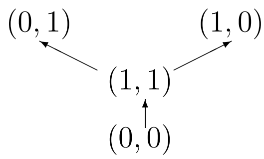

Fig. 5. One nonzero vertex has the first level.

$$\hat{\varphi}((K^+_{-1})^\perp \mathbf{r}_{-1}^{\overline{\alpha}_{-1}} \mathbf{r}_0^{\overline{\alpha}_0} \dots \mathbf{r}_{l-1}^{\overline{\alpha}_{l-1}} \mathbf{r}_l^{\overline{\alpha}_l} \mathbf{r}_{l+1}^{\mathbf{0}}) = \lambda_{\overline{\alpha}_{-1},\overline{\alpha}_0} \cdot \lambda_{\overline{\alpha}_0,\overline{\alpha}_1} \cdot \dots \cdot \lambda_{\overline{\alpha}_{l-1},\overline{\alpha}_l} \cdot \lambda_{\overline{\alpha}_l,\mathbf{0}}.$$

Otherwise we suppose $\hat{\varphi}((K^+_{-1})^\perp \zeta) = 0$. Then φ generates an orthogonal MRA on the field $GF(p^s)$.

Example. Let $p = s = 2$. For these values we have trees (see Figs. 3–5) and so on. For the tree in Fig. 5 we obtain $\hat{\varphi}(\chi)$ in the form $\hat{\varphi}(K^+_{-1}) = 1$, $\hat{\varphi}((K^+_{-1})^\perp \mathbf{r}_{-1}^{(1,1)}) = \lambda_{1,1}$, $\hat{\varphi}((K^+_{-1})^\perp \mathbf{r}_{-1}^{(0,1)} \mathbf{r}_0^{(1,1)}) = \lambda_{0,1}$, $\hat{\varphi}((K^+_{-1})^\perp \mathbf{r}_{-1}^{(1,0)} \mathbf{r}_0^{(1,1)}) = \lambda_{1,0}$. $|\lambda_{i,j}| = 1$ and $\hat{\varphi}((K^+_{-1})^\perp \zeta) = 0$ otherwise. Suppose for simplicity $\lambda_{i,j} = 1$. Then we can calculate the scaling function

$$\varphi(x) = \int\limits_X \hat{\varphi}(\chi)(\chi, x)d\,\nu(\chi) = \int\limits_{(K^+_{-1})^\perp} (\chi, x)d\,\nu(\chi) + \int\limits_{(K^+_{-1})^\perp \mathbf{r}_{-1}^{(1,1)}} (\chi, x)d\,\nu(\chi) +$$
$$+ \int\limits_{(K^+_{-1})^\perp \mathbf{r}_{-1}^{(0,1)} \mathbf{r}_0^{(1,1)}} (\chi, x)d\,\nu(\chi) + \int\limits_{(K^+_{-1})^\perp \mathbf{r}_{-1}^{(1,0)} \mathbf{r}_0^{(1,1)}} (\chi, x)d\,\nu(\chi) = 2^{-2}(\mathbf{1}_{K^+_{-1}}(x) +$$
$$+ \mathbf{r}_{-1}^{(1,1)}(x)\mathbf{1}_{K^+_{-1}}(x) + \mathbf{r}_{-1}^{(0,1)}(x)\mathbf{r}_0^{(1,1)}(x)\mathbf{1}_{K^+_{-1}}(x) + \mathbf{r}_{-1}^{(1,0)}(x)\mathbf{r}_0^{(1,1)}(x)\mathbf{1}_{K^+_{-1}}(x)) = \mathbf{1}_E(x)$$

where

$$E = K_1^+ \bigsqcup (K_1^+ \dot{+} (0,0)g_{-1}) \bigsqcup (K_1^+ \dot{+} (1,1)g_{-1}) \bigsqcup (K_1^+ \dot{+} (1,0)g_{-1} \dot{+} (1,1)g_0)$$
$$\bigsqcup (K_1^+ \dot{+} (0,1)g_{-1} \dot{+} (1,1)g_0).$$

We can consider additive group K^+ as product $\mathfrak{G} \times \mathfrak{G}$ of Cantor groups. In this case $\hat{\varphi}$ and φ may be defined on the product $\mathfrak{G}_1^\perp \times \mathfrak{G}_1^\perp$ and $\mathfrak{G}_{-1} \times \mathfrak{G}_{-1}$ respectively by Figs. 6 and 7.

Since supp$\hat{\varphi} \neq (K_0^+)^\perp$ and supp$\varphi \neq (K_0^+)$, it follows that φ generates non-Haar MRA. From this example we see that MRA on local field gives an effective method to construct multidimensional step wavelets.

0	0	1	0
0	0	0	1
0	1	0	0
1	0	0	0

$\mathfrak{G}_{-1}^{\perp}$ $\mathfrak{G}_0^{\perp}$ $\mathfrak{G}_1^{\perp}$

Fig. 6. The table of the Fourier transform $\hat{\varphi}$.

0	0	1	0
0	0	0	1
0	1	0	0
1	0	0	0

$\mathfrak{G}_{1}$ $\mathfrak{G}_0$ $\mathfrak{G}_{-1}$

Fig. 7. The table of the refinable function φ.

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