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An L_p-theory for a class of non-local elliptic equations related to nonsymmetric measurable kernels ☆

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A B S T R A C T

We study the integro-differential operators L with kernels $K(y) = a(y)J(y)$, where $J(y)$ is rotationally invariant and $J(y)dy$ is a Lévy measure on $\mathbb{R}^d$ (i.e. $\int_{\mathbb{R}^d}(1 \wedge |y|^2)J(y)dy < \infty$) and $a(y)$ is an only measurable function with positive lower and upper bounds. Under few additional conditions on $J(y)$, we prove the unique solvability of the equation $Lu - \lambda u = f$ in L_p-spaces and present some L_p-estimates of the solutions.

1. Introduction

There has been growing interest in the integro-differential equations related to pure jump processes owing to their applications in various models in physics, economics, engineering and many others involving long-range interactions. In this article we study the non-local elliptic equations with the operators

$$Lu := \int_{\mathbb{R}^d} \Big(u(x+y) - u(x) - y \cdot \nabla u(x)\chi(y) \Big) K(x,y)dy,$$

and

$$\tilde{L}u := \int_{\mathbb{R}^d} \Big(u(x+y) - u(x) - y \cdot \nabla u(x)1_{|y|<1} \Big) K(x,y)dy,$$

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where the kernel $K(x,y) = a(y)J(y) = a(y)j(|y|)$ depends only on y,

$$\chi(y) = 0 \text{ if } \sigma \in (0,1), \quad \chi(y) = 1_{|y|<1} \text{ if } \sigma = 1, \quad \chi(y) = 1 \text{ if } \sigma \in (1,2].$$

The constant σ depends on $j(|y|)$ and is defined in (2.7). In particular, if $j(|y|) = c(d,\alpha)|y|^{-d-\alpha}$ for some $\alpha \in (0,2)$ then $\sigma = \alpha$. Note that if $a(y)$ is symmetric then $\tilde{L} = L$, and in general we (formally) have

$$\tilde{L}u = Lu + b \cdot \nabla u,$$

where

$$b^i = -\int_{B_1} y^i a(y)J(y)dy \quad \text{if } \sigma \in (0,1), \qquad b^i = \int_{\mathbb{R}^d \setminus B_1} y^i a(y)J(y)dy \quad \text{if } \sigma \in (1,2].$$

The main goal of this article is to prove the unique solvability of the equations

$$Lu - \lambda u = f \quad \text{and} \quad \tilde{L}u - \lambda u = f, \quad \lambda > 0 \tag{1.1}$$

in appropriate L_p-spaces and present some L_p-estimates of the solutions. Here $p > 1$. If $p = 2$, the only condition we are assuming is that $a(y)$ has positive lower and upper bounds and $J(y)$ is rotationally invariant. If $p \neq 2$, we assume some additional conditions on $J(y)$, which are described in Section 2.

If $K(x,y) = c(d,\alpha)|y|^{-d-\alpha}$, where $\alpha \in (0,2)$ and $c(d,\alpha)$ is some normalization constant, then L becomes the fractional Laplacian operator $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. For the fractional Laplacian operator, L_p-estimates can be easily obtained by the Fourier multiplier theory (for instance, [16]). On the basis of this operator, many results have been made for kernels $K(x,y) = a(x,y)|y|^{-d-\alpha}$. In [7], the authors obtained an L_p-estimate of the equation $Lu - \lambda u = f$ with the kernel $K(x,y) = a(y)|y|^{-d-\alpha}$, where the coefficient $a(y)$ is measurable and has lower and upper bounds. For parabolic equations, in [11] the authors handle the equations with the kernel $K(x,y) = a(x,y)|y|^{-d-\alpha}$ under the condition that the coefficient $a(x,y)$ is homogeneous of order zero in y and sufficiently smooth in y, but it is allowed that a also depends on x. The homogeneity and smoothness conditions with respect to y are dropped in [12,13]. Lately, in [18] L_p-maximal regularity theory was constructed for a class of Lévy measures $\nu(dy)$ satisfying

$$\nu_1^\alpha(B) \le \nu(B) \le \nu_2^\alpha(B), \quad \forall B \in \mathcal{B}(\mathbb{R}^d). \tag{1.2}$$

Here ν_1 and ν_2 are Lévy measures of α-stable processes taking the form

$$\nu_i^{(\alpha)}(B) := \int_{\mathbb{S}^{d-1}} \left(\int_0^\infty \frac{1_B(r\theta)dr}{r^{1+\alpha}} \right) \Sigma_i(d\theta), \tag{1.3}$$

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and Σ_i, $i = 1,2$, are finite measures on $\mathbb{S}^{d-1}$. See Remark 2.13 for a comparison between [18] and our results. Briefly speaking, our Lévy measures $J(y)dy$ do not satisfy (1.3) in general and are related to the subordinate Brownian motions. We also refer to [2], where general "symmetric" kernels are handled through the theory of martingale transforms. For other works such as Harnack inequality, Hölder estimate and ABP estimate, see [4,5,8,9,15] and the references therein.

From the probabilistic point of view, the fractional Laplacian operator can be described as the infinitesimal generator of α-stable processes. That is, for any $f \in C_0^\infty$

$$\Delta^{\alpha/2} f(x) := -(-\Delta)^{\alpha/2} f(x) = \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}[f(x + X_t) - f(x)],$$

where X_t is an $\mathbb{R}^d$-valued Lévy process in a probability space (Ω, P) with the characteristic function $\mathbb{E}e^{i\lambda\cdot X_t} := \int_\Omega e^{i\lambda\cdot X_t}\,dP = e^{-t|\lambda|^\alpha}$. More generally, for any Bernstein function ϕ with $\phi(0+) = 0$ (equivalently, $\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\mu(dt)$ for some measure μ satisfying $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$), the operator $\phi(\Delta)$ is the infinitesimal generator of the process $X_t := W_{S_t}$, where S_t is a subordinator (i.e. an increasing Lévy process satisfying $S_0 = 0$) with Laplace exponent ϕ (i.e. $\mathbb{E}e^{\lambda S_t} = \exp\{t\phi(\lambda)\}$) and W_t is a d-dimensional Brownian motion independent of S_t. Such process is called the subordinate Brownian motion. Actually ϕ is a Bernstein function with $\phi(0+) = 0$ if and only if it is a Laplace exponent of a subordinator.

The operator $\phi(\Delta)$ turns out to be the following integro-differential operator

$$\phi(\Delta)f := -\phi(-\Delta)f = \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - \nabla f(x)\cdot y\chi(y)\right) J(y)\,dy,$$

where $f \in C_0^\infty$ and $J(y) = j(|y|)$ with $j : (0,\infty) \to (0,\infty)$ given by

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)}\,\mu(dt). \tag{1.4}$$

For the equations with the kernel $K(x,y) = a(y)J(y)$, an L_p-estimate is obtained in aforementioned article [2] if $a(y)$ is symmetric. However to the best of our mathematical knowledge, if the coefficient $a(y)$ is only measurable then the L_p-estimate has not been known yet because $J(y)dy$ is not of the form (1.3). For these reasons, it is quite natural to consider two equations in (1.1) with a measurable coefficient $a(y)$.

To handle a measurable coefficient $a(y)$, we estimate the sharp functions of the solutions and apply the Hardy–Littlewood theorem and the Fefferman–Stein theorem. This approach is typically used to treat the second-order PDEs with small BMO or VMO coefficients (for instance, see [10]). H. Dong and D. Kim applied this method to a non-local operator with the kernel $K(x,y) = a(y)|y|^{-d-\alpha}$ ([7]). As in [7], our sharp function estimates are based on some Hölder estimates of solutions. The original idea of obtaining Hölder estimates is from [3]. Nonetheless, since we are considering much general $J(y)$ rather then $c(d,\alpha)|y|^{-d-\alpha}$, many new difficulties arise. In particular, our operators do not have the nice scaling property which is used in [10] and [7], and this causes many difficulties in the estimates.

The article is organized as follows. In Section 2 we introduce the main results. Section 3 contains the unique solvability in the L_2-space. In Section 4 we establish some Hölder estimates of solutions. Using these estimates we obtain the sharp function and maximal function estimates in Section 5. In Section 6, the proofs of main results are given.

We finish the introduction with some notation. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, \ldots, x^d)$, $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ and $B_r := B_r(0)$. For $i = 1, \ldots, d$, multi-indices $\beta = (\beta_1, \ldots, \beta_d)$, $\beta_i \in \{0, 1, 2, \ldots\}$, and functions $u(x)$ we set

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\beta u = D_1^{\beta_1} \cdot \ldots \cdot D_d^{\beta_d} u, \quad |\beta| = \beta_1 + \ldots + \beta_d.$$

For an open set $U \subset \mathbb{R}^d$ and a nonnegative non-integer constant γ, by $C^\gamma(U)$ we denote the usual Hölder space. For a nonnegative integer n, we write $u \in C^n(U)$ if u is real-valued and n-times continuously differentiable in U. By $C_0^n(U)$ (resp. $C_0^\infty(U)$) we denote the set of all functions in $C^n(U)$ (resp. $C^\infty(U)$) with compact supports. Similarly by $C_b^n(U)$ (resp. $C_b^\infty(U)$) we denote the set of functions in $C^n(U)$ (resp. $C^\infty(U)$) with bounded derivatives. The standard L_p-space on U with Lebesgue measure is denoted by $L_p(U)$. We simply use L_p, C^n, C_b^n, C_0^n, C_b^∞, and C_0^∞ when $U = \mathbb{R}^d$. We use ":=" to denote a definition. $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. If we write $N = N(a, \ldots, z)$, this means that the constant N depends only on $a, \ldots, z$. The constant N may change from location to location, even within a line.

By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the Fourier transform and the inverse Fourier transform, respectively. That is, $\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx$ and $\mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x} f(\xi)d\xi$. For a Borel set $A \subset \mathbb{R}^d$, we use $|A|$ to denote its Lebesgue measure and by $I_A(x)$ we denote the indicator of A.

2. Setting and main results

Throughout this article, we assume that $J(y)$ is nonnegative and rotationally invariant,

$$\nu \leq a(y) \leq \Lambda \tag{2.5}$$

for some constants $\nu, \Lambda > 0$, and

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2) J(y)\, dy < \infty. \tag{2.6}$$

Let e_1 be a unit vector. Obviously, the condition that $J(y)$ is rotationally invariant can be replaced by the condition that $J(y)$ is comparable to $j(|y|) := J(|y|e_1)$, because $J(y)a(y) = j(|y|)\cdot a(y)J(y)j^{-1}(|y|) := j(|y|)\tilde{a}(y)$ and $\tilde{a}$ also has positive lower and upper bounds.

Denote

$$\sigma := \inf\{\delta > 0 : \int_{|y|\leq 1} |y|^\delta J(y)\, dy < \infty\},$$
$$\chi(y) = 0 \text{ if } \sigma \in (0,1), \quad \chi(y) = 1_{B_1} \text{ if } \sigma = 1, \quad \chi(y) = 1 \text{ if } \sigma \in (1,2]. \tag{2.7}$$

Note that if $J(y) = c(d,\alpha)|y|^{-d-\alpha}$ for some $\alpha \in (0,2)$ then we have $\sigma = \alpha$.

For $u \in C_b^2$ we introduce the non-local elliptic operators

$$\mathcal{A}u = \int_{\mathbb{R}^d} \big(u(x+y) - u(x) - y\cdot\nabla u(x)\chi(y)\big)\, J(y)\, dy,$$

$$Lu = \int_{\mathbb{R}^d} \big(u(x+y) - u(x) - y\cdot\nabla u(x)\chi(y)\big)\, a(y)J(y)\, dy,$$

$$\tilde{L}u = \int_{\mathbb{R}^d} \big(u(x+y) - u(x) - y\cdot\nabla u(x)I_{B_1}(y)\big)\, a(y)J(y)\, dy,$$

$$L^*u = \int_{\mathbb{R}^d} \big(u(x+y) - u(x) - y\cdot\nabla u(x)\chi(y)\big)\, a(-y)J(-y)\, dy,$$

and

$$\tilde{L}^*u = \int_{\mathbb{R}^d} \big(u(x+y) - u(x) - y\cdot\nabla u(x)I_{B_1}(y)\big)\, a(-y)J(-y)\, dy.$$

We start with a simple but interesting result, which will be used later in the proof of Theorem 2.22.

Lemma 2.1. *For any $p > 1$ and $\lambda > 0$,*

$$\|u\|_{L_p} \leq \frac{1}{\lambda}\|\tilde{L}u - \lambda u\|_{L_p}, \quad \forall\, u \in C_0^\infty.$$

Proof. Put

$$\Phi(\xi) := -\int_{\mathbb{R}^d} (e^{i\xi\cdot y} - 1 - i(y\cdot\xi)I_{B_1})a(-y)J(-y)\, dy$$

and

$$f := \tilde{L}u - \lambda u.$$

Since $a(-y)J(-y)$ is a Lévy measure (i.e. $\int_{\mathbb{R}^d}(1\wedge|y|^2)a(-y)J(-y)\, dy < \infty$), there exists a Lévy process whose characteristic exponent is $-t\Phi(\xi)$ (for instance, see Corollary 1.4.6 of [1]). Denoting by $p_\Phi(t,dx)$ its law at t, we have

$$\int_{\mathbb{R}^d} e^{-i\xi\cdot x}p_\Phi(t,dx) = \int_{\mathbb{R}^d} e^{i(-\xi)\cdot x}p_\Phi(t,dx) = e^{-t\Phi(-\xi)}. \tag{2.8}$$

In non-probabilistic terminology it can be rephrased that if $\int_{\mathbb{R}^d}(1\wedge|y|^2)a(-y)J(-y)\, dy < \infty$ then there exists a continuous measure-valued function $p_\Phi(t,dx)$ such that $p_\Phi(t,\mathbb{R}^d) = 1$ and (2.8) holds. Since

$$(-\Phi(-\xi) - \lambda)\mathcal{F}u = \mathcal{F}f$$

and $\mathrm{Re}\,\Phi(-\xi) \geq 0$, we have

$$\begin{aligned}
\mathcal{F}u(\xi) &= -\frac{1}{\Phi(-\xi)+\lambda}\mathcal{F}f(\xi) \\
&= -\Big(\int_0^\infty e^{-t\Phi(-\xi)-\lambda t}\, dt\ \mathcal{F}f(\xi)\Big) \\
&= -\Big(\int_0^\infty\int_{\mathbb{R}^d} e^{-i\xi\cdot x}p_\Phi(t,dx)e^{-\lambda t}\, dt\ \mathcal{F}f(\xi)\Big) \\
&= -\mathcal{F}\Big(\int_0^\infty (p_\Phi(t,\cdot) * f(x))e^{-\lambda t}\, dt\Big)(\xi).
\end{aligned}$$

Therefore,

$$u(x) = -\int_0^\infty (p_\Phi(t,\cdot) * f)e^{-\lambda t}\, dt$$

and by Young's inequality,

$$\|u\|_{L_p} \leq \int_0^\infty\int_{\mathbb{R}^d} p_\Phi(t,dx)e^{-\lambda t}\, dt\|f\|_{L_p} \leq \frac{1}{\lambda}\|f\|_{L_p}.$$

Hence the lemma is proved. □

Definition 2.2. We write $u \in \mathcal{H}_p^{\mathcal{A}}$ if and only if there exists a sequence of functions $u_n \in C_0^\infty$ such that $u_n \to u$ in L_p and $\{\mathcal{A}u_n : n = 1,2,\cdots\}$ is a cauchy sequence in L_p. By $\mathcal{A}u$ we denote the limit of $\mathcal{A}u_n$ in L_p.

Lemma 2.3. *$\mathcal{H}_p^{\mathcal{A}}$ is a Banach space equipped with the norm*

$$\|u\|_{\mathcal{H}_p^{\mathcal{A}}} := \|u\|_{L_p} + \|\mathcal{A}u\|_{L_p}.$$

Proof. It is obvious. □

Definition 2.4. We say that $u \in \mathcal{H}_p^{\mathcal{A}}$ is a solution of the equation

$$Lu - \lambda u = f \qquad \text{in } \mathbb{R}^d \tag{2.9}$$

if and only if there exists a sequence $\{u_n \in C_0^\infty\}$ such that u_n converges to u in $\mathcal{H}_p^{\mathcal{A}}$ and $Lu_n - \lambda u_n$ converges to f in L_p. Similarly, we consider the equation

$$\tilde{L}u - \lambda u = f \qquad \text{in } \mathbb{R}^d \tag{2.10}$$

in the same sense.

Lemma 2.5 *(Maximum principle). Let $\lambda > 0$, $b(x)$ be an $\mathbb{R}^d$-valued bounded function on $\mathbb{R}^d$ and u be a function in C_b^2 satisfying $u(x) \to 0$ as $|x| \to \infty$. If $Lu + b(x) \cdot \nabla u - \lambda u = 0$ in $\mathbb{R}^d$, then $u \equiv 0$. Also, the same statement is true with $\tilde{L}$ in place of L.*

Proof. Suppose that u is not identically zero. Without loss of generality, assume $\sup_{\mathbb{R}^d} u > 0$ (otherwise consider $-u$). Since u goes to zero as $|x| \to \infty$, there exists $x_0 \in \mathbb{R}^d$ such that $u(x_0) = \sup_{\mathbb{R}^d} u$. Thus $\nabla u(x_0) = 0$ and

$$Lu(x_0) = \int_{\mathbb{R}^d} (u(x_0 + y) - u(x_0) - y \cdot \nabla u(x_0)\chi(y))\, a(y)J(y)\, dy \le 0.$$

Therefore we reach the contradiction. Indeed,

$$Lu(x_0) + b(x_0) \cdot \nabla u(x_0) - \lambda u(x_0) < 0.$$

The proof for $\tilde{L}$ is almost identical. The lemma is proved. □

This maximum principle yields the denseness of $(L + b \cdot \nabla - \lambda)C_0^\infty$ and $(\tilde{L} + b \cdot \nabla - \lambda)C_0^\infty$ in L_p.

Lemma 2.6. *Let $\lambda > 0$ and $b \in \mathbb{R}^d$ be independent of x. Then $(L+b\cdot\nabla-\lambda)C_0^\infty := \{Lu+b\cdot\nabla u-\lambda u : u \in C_0^\infty\}$ is dense in L_p for any $p \in (1, \infty)$. Also, the same statement holds with $\tilde{L}$ in place of L.*

Proof. Due to the similarity we only prove the first statement. Suppose that the statement is false. Then by the Hahn–Banach theorem and Riesz's representation theorem, there exists a nonzero $v \in L_{p/(p-1)}$ such that

$$\int_{\mathbb{R}^d} (Lu(x) + b \cdot \nabla u(x) - \lambda u(x))\, v(x)\, dx = 0 \tag{2.11}$$

for all $u \in C_0^\infty$.

Fixing $y \in \mathbb{R}^d$, we apply (2.11) with $u(y - \cdot)$. Then, due to Fubini's Theorem,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} (L^* u(y-x) - b \cdot \nabla u(y-x) - \lambda u(y-x))\, v(x)\, dx \\ &= L^* u * v(y) - b \cdot (\nabla u * v(y)) - \lambda u * v(y) = (L^* - b \cdot \nabla - \lambda)(u * v)(y). \end{aligned}$$

Therefore from the previous lemma, we have $u * v = 0$ for any $u \in C_0^\infty$. Therefore, $v = 0$ (a.e.) and we have a contradiction. □

Corollary 2.7 *(Uniqueness). Let $\lambda > 0$. Suppose that there exist $u, v \in \mathcal{H}_p^{\mathcal{A}}$ satisfying*

$$Lu - \lambda u = 0, \quad \tilde{L} v - \lambda v = 0.$$

Then $u = v = 0$.

Proof. By the definition of a solution and the assumption of this corollary, there exists a sequence $\{u_n \in C_0^\infty\}$ such that for all $w \in C_0^\infty$

$$0 = \int_{\mathbb{R}^d} \lim_{n \to \infty} (Lu_n - \lambda u_n) w\, dx = \int_{\mathbb{R}^d} u(L^* w - \lambda w)\, dx.$$

Since $\{L^* w - \lambda w : w \in C_0^\infty\}$ is dense in $L_{p/(p-1)}$ owing to Lemma 2.6, we conclude $u = 0$, and by the same argument we have $v = 0$. □

Here is our L_2-theory. We emphasize that only (2.5) and (2.6) are assumed for the L_2-theory. The proof of Theorem 2.8 is given in Section 3.

Theorem 2.8. *Let $\lambda > 0$. Then for any $f \in L_2$ there exist unique solutions $u, v \in \mathcal{H}_2^{\mathcal{A}}$ of equations (2.9) and (2.10) respectively, and for these solutions we have*

$$\|\mathcal{A} u\|_{L_2} + \lambda \|u\|_{L_2} \le N(d, \nu, \Lambda) \|f\|_{L_2}, \tag{2.12}$$

$$\|\mathcal{A} v\|_{L_2} + \lambda \|v\|_{L_2} \le N(d, \nu, \Lambda) \|f\|_{L_2}. \tag{2.13}$$

The issue regarding the continuity of L (or $\tilde{L}$): $\mathcal{H}_p^{\mathcal{A}} \to L_p$ will be discussed later.

For the case $p \neq 2$, we consider the following conditions on $J(y) = j(|y|)$:

(**H1**): There exist constants $\kappa_1 > 0$ and $\alpha_0 > 0$ such that

$$j(t) \le \kappa_1 (s/t)^{d+\alpha_0} j(s), \quad \forall\, 0 < s \le t. \tag{2.14}$$

Moreover, $\alpha_0 \le 1$ if $\sigma \le 1$ and $1 < \alpha_0 < 2$ if $\sigma > 1$.

(**H2**): There exists a constant $\kappa_2 > 0$ such that for all $t > 0$,

$$\int_{|y| \le 1} |y| j(t|y|)\, dy \le \kappa_2 j(t) \qquad \text{if } \sigma \in (0, 1), \tag{2.15}$$

$$\int_{|y| \le 1} |y|^2 j(t|y|)\, dy \le \kappa_2 j(t) \qquad \text{if } \sigma \ge 1. \tag{2.16}$$

Remark 2.9. (i) By taking $t = 1$ in (2.14),

$$j(1)\kappa_1^{-1}s^{-d-\alpha_0} \le j(s), \quad \forall\, s \in (0,1). \tag{2.17}$$

An upper bound of $j(s)$ near $s = 0$ is obtained in the following lemma.

(ii) **H1** and **H2** are needed even to guarantee the continuity of the operator $L : \mathcal{H}_2^A \to L_2$ (see Lemma 3.1).

Lemma 2.10. *Suppose*

$$j(s) \ge Cj(t), \quad \forall\, s \le t, \tag{2.18}$$

and **H2** *hold. Then there exists a constant $N(d, \kappa_2, C) > 0$ such that for all $0 < s \le t$*

$$j(t) \ge N(s/t)^{d+1}j(s) \quad (\textit{if } \sigma < 1), \quad j(t) \ge N(s/t)^{d+2}j(s) \quad (\textit{if } \sigma \ge 1). \tag{2.19}$$

On the other hand, if there exists $\alpha > 0$ so that $\alpha < 1$ if $\sigma < 1$, $\alpha < 2$ if $\sigma \ge 1$, and

$$j(t) \ge N(s/t)^{d+\alpha}j(s), \quad \forall\, 0 < s \le t, \tag{2.20}$$

then **H2** *holds.*

Remark 2.11. By Lemma 2.10, both **H1** and **H2** hold if $0 < \alpha_0 \le \alpha$ and

$$N^{-1}(s/t)^{d+\alpha}j(s) \le j(t) \le N(s/t)^{d+\alpha_0}j(s), \quad \forall\ 0 < s \le t.$$

Example 2.12. Let $J(y) = j(|y|)$ be defined as in (1.4), that is for a Bernstein function $\phi(\lambda) = \int_{\mathbb{R}}(1 - e^{-\lambda t})\mu(dt)$ and $u \in C_0^2$,

$$j(r) = \int_0^\infty (4\pi t)^{-d/2}e^{-r^2/(4t)}\ \mu(dt),$$

and

$$\begin{aligned}\phi(\Delta)u &= \int_{\mathbb{R}^d} \left(u(x+y) - u(x) - \nabla u(x)\cdot yI_{|y|\le 1}\right) J(y)\ dy \\ &= -\mathcal{F}(\phi(|\xi|^2)\mathcal{F}(u)(\xi)).\end{aligned}$$

Then, **H1** and **H2** are satisfied if ϕ is given, for instance, by any one of

(1) $\phi(\lambda) = \sum_{i=1}^n \lambda^{\alpha_i}$, $0 < \alpha_i < 1$;
(2) $\phi(\lambda) = (\lambda + \lambda^\alpha)^\beta$, $\alpha, \beta \in (0,1)$;
(3) $\phi(\lambda) = \lambda^\alpha(\log(1+\lambda))^\beta$, $\alpha \in (0,1)$, $\beta \in (0, 1-\alpha)$;
(4) $\phi(\lambda) = \lambda^\alpha(\log(1+\lambda))^{-\beta}$, $\alpha \in (0,1)$, $\beta \in (0,\alpha)$;
(5) $\phi(\lambda) = (\log(\cosh(\sqrt{\lambda})))^\alpha$, $\alpha \in (0,1)$;
(6) $\phi(\lambda) = (\log(\sinh(\sqrt{\lambda})) - \log\sqrt{\lambda})^\alpha$, $\alpha \in (0,1)$.

This is because all these functions satisfy the conditions

A: $\exists\, 0 < \delta_1 \le \delta_2 < 1$,

$$N^{-1}\lambda^{\delta_1}\phi(t) \le \phi(\lambda t) \le N\lambda^{\delta_2}\phi(t), \quad \forall\, \lambda \ge 1, t \ge 1$$

B: $\exists\, 0 < \delta_3 \le \delta_4 < 1$,

$$N^{-1}\lambda^{\delta_3}\phi(t) \le \phi(\lambda t) \le N\lambda^{\delta_4}\phi(t), \quad \forall\, \lambda \le 1, t \le 1,$$

and under these condition one can prove (see [9])

$$N^{-1}\Big(\frac{R}{r}\Big)^{\delta_1\wedge\delta_3} \le \frac{\phi(R)}{\phi(r)} \le N\Big(\frac{R}{r}\Big)^{\delta_2\vee\delta_4}$$

and

$$N^{-1}\phi(|y|^{-2})|y|^{-d} \le J(y) \le N\phi(|y|^{-2})|y|^{-d}, \tag{2.21}$$

and consequently our conditions **H1** and **H2** hold. One can easily construct concrete examples of $j(r)$ using (2.21) and (1)–(6) (just replace λ by r^{-2}). See the tables at the end of [14] for more examples satisfying **A** and **B**.

Remark 2.13. As can be easily checked using (2.21), our $J(y)dy$ does not satisfy (1.3) even for the simplest example $\phi(\lambda) = \sum_{i=1}^{n}\lambda^{\alpha_i}$, $0 < \alpha_i < 1$ if $\alpha_i \neq \alpha_j$ for some i, j. On the other hand, [18] deals with Lévy measures of type $\nu(dy)$, which does not necessarily have jump density.

Remark 2.14. If $p \neq 2$, our L_p-theory does not cover the case when the jump function $J(y)$ is related to the relativistic α-stable process with mass $m > 0$ (i.e. a subordinate Brownian motion with the infinitesimal generator $\phi(\Delta) = m - (m^{2/\alpha} - \Delta)^{\alpha/2}$). This is because the related jump function decreases exponentially fast at the infinity (for instance, see [6]) and thus condition **H2** fails (see (2.19)).

Proof of Lemma 2.10. Assume (2.18) and **H2** hold. We put $B_1 = \cup_{n=0}^{\infty} B_{(n)}$, where $B_{(n)} = B_{2^{-n}} \setminus B_{2^{-(n+1)}}$. Due to (2.18) for each $n \ge 0$,

$$\begin{aligned}\kappa_2 j(t) &\ge \int\limits_{|y|\le 1} |y|^2 j(t|y|)\, dy = \sum_{n=0}^{\infty}\int\limits_{B(n)} |y|^2 j(t|y|)\, dy \\ &\ge N\sum_{n=0}^{\infty} 2^{-(n+1)(d+2)} j(t2^{-n}) \ge N2^{-(n+1)(d+2)} j(t2^{-n}).\end{aligned}$$

Put $s = t\lambda$, where $\lambda \in (0,1)$, and take an integer $m(\lambda) \ge 0$ such that $2^{-(m+1)} \le \lambda \le 2^{-m}$. Then by (2.18),

$$j(t) \ge N2^{-(m+2)(d+2)} j(2^{-(m+1)}t) \ge N\lambda^{d+2} j(\lambda t).$$

Similarly, $j(\lambda t) \le \lambda^{-d-1} j(t)$ if $\sigma < 1$.

For the other direction, put $s = t|y|$ in (2.20). If $\sigma < 1$ then

$$\begin{aligned}\int\limits_{|y|\le 1} |y| j(t|y|)\, dy &\le N j(t) \int\limits_{|y|\le 1} |y| \frac{j(t|y|)}{j(t)}\, dy \\ &\le N j(t) \int\limits_{|y|\le 1} |y|^{-d-\alpha_1+1}\, dy \le N j(t)\end{aligned}$$

and otherwise, that is, if $\sigma \geq 1$ then

$$\int\limits_{|y|\leq 1} |y|^2 j(t|y|)\, dy \leq Nj(t) \int\limits_{|y|\leq 1} |y|^2 \frac{j(t|y|)}{j(t)}\, dy$$
$$\leq Nj(t) \int\limits_{|y|\leq 1} |y|^{-d-\alpha_2+2}\, dy \leq Nj(t).$$

The lemma is proved. □

Define

$$\Psi(\xi) := -\int\limits_{\mathbb{R}^d} (e^{i\xi\cdot y} - 1 - i(y\cdot\xi)\chi(y))J(y)dy = \int\limits_{\mathbb{R}^d} (1-\cos\xi\cdot y)J(y)dy.$$

Then

$$\mathcal{A}u = \mathcal{F}^{-1}(-\Psi(\xi)\mathcal{F}u), \qquad \forall\, u \in C_0^\infty.$$

By abusing the notation, we also use $\Psi(|\xi|)$ instead of $\Psi(\xi)$ because $\Psi(\xi)$ is rotationally invariant.

The following result will be used to prove the continuity of the operator L.

Lemma 2.15. *Suppose that (2.18) holds. Then there exists a constant $N(d, C) > 0$ such that for all $\xi \in \mathbb{R}^d$*

$$j(|\xi|) \leq N|\xi|^{-d}\Psi(|\xi|^{-1}). \tag{2.22}$$

Proof. By (2.18),

$$\Psi(|\xi|^{-1}) = \int\limits_{\mathbb{R}^d} (1-\cos(y^1/|\xi|))J(y)\, dy = |\xi|^d \int\limits_{\mathbb{R}^d} (1-\cos(y^1))J(|\xi|y)\, dy$$
$$\geq |\xi|^d \int\limits_{|y|\leq 1} (1-\cos(y^1))J(|\xi|y)dy$$
$$\geq Cj(|\xi|)|\xi|^d \int\limits_{|y|\leq 1} (1-\cos(y^1))\, dy \geq Nj(|\xi|)|\xi|^d.$$

Hence the lemma is proved. □

The following condition will be considered for the case $\sigma = 1$. This condition is needed even to prove the continuity of L.

Assumption 2.16. If $\sigma = 1$ then

$$\int\limits_{\partial B_r} y^i a(y)J(y)dS_r(y) = 0, \quad \forall r \in (0,\infty),\ i = 1,\cdots,d, \tag{2.23}$$

where dS_r is the surface measure on ∂B_r.

Here is our L_p-theory for equation (2.24) below.

Theorem 2.17. *Suppose that* **H1** *and* **H2** *hold and Assumption 2.16 also holds if $\sigma = 1$. Let $\lambda > 0$ and $p > 1$. Then for any $f \in L_p$ there exists a unique solution $u \in \mathcal{H}_p^{\mathcal{A}}$ of the equation*

$$Lu - \lambda u = f, \tag{2.24}$$

and for this solution we have

$$\|\mathcal{A}u\|_{L_p} + \lambda \|u\|_{L_p} \leq N(d, \nu, \Lambda, \lambda, \kappa_1, \kappa_2, \alpha_0) \|f\|_{L_p}. \tag{2.25}$$

Moreover, L is a continuous operator from $\mathcal{H}_p^{\mathcal{A}}$ to L_p, and (2.25) holds for all $u \in \mathcal{H}_p^{\mathcal{A}}$ with $f := Lu - \lambda u$.

The proof of this theorem will be given in Section 6.

Remark 2.18. Since the constant N in (2.25) does not depend on λ, for any $u \in \mathcal{H}_p^{\mathcal{A}}$

$$\|\mathcal{A}u\|_{L_p} \leq N \|Lu\|_{L_p}.$$

To study the equations with the operator $\tilde{L}$, we consider an additional condition, which always holds when $\sigma = 1$.

Assumption 2.19 ***(H3)***. Any one of the following (i)–(iv) holds:

(i) $\mathcal{A}$ is a higher order differential operator than $I_{\sigma \neq 1} \nabla u$, that is for any $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ so that for any $u \in C_0^\infty$

$$I_{\sigma \neq 1} \|\nabla u\|_p \leq \varepsilon \|\mathcal{A}u\|_p + N(\varepsilon) \|u\|_p. \tag{2.26}$$

(ii) $\sigma < 1$ and

$$\int\limits_{r \leq |y| \leq 1} y^i \Big(a(y) - [a(y) \wedge a(-y)]\Big) J(y) dy = 0, \quad \forall r \in (0,1),\ i = 1, \cdots, d. \tag{2.27}$$

(iii) $\sigma < 1$ and there exists a constant $\kappa_3 > 0$ such that for all $0 < t < 1$,

$$\int\limits_{|z| \geq 1} |z| j(t|z|)\, dz \leq \kappa_3 j(t). \tag{2.28}$$

(iv) $\sigma > 1$ and

$$\int\limits_{1 \leq |y| \leq r} y^i \Big(a(y) - [a(y) \wedge a(-y)]\Big) J(y) dy = 0, \quad \forall r > 1\ i = 1, \cdots, d. \tag{2.29}$$

Remark 2.20. (i) Note that (2.26) is satisfied if for some $\alpha > 1$,

$$\|\Delta^{\alpha/2} u\|_p \leq N(\|u\|_p + \|\mathcal{A}u\|_p), \quad \forall u \in C_0^\infty, \tag{2.30}$$

or, equivalently $|\xi|^\alpha (1 + \Psi(\xi))^{-1}$ is an L_p-Fourier multiplier. Thus, certain differentiability of $J(y)$ is required (see Lemma 2.21 below).

(ii) It is easy to check that (2.28) holds if for a $\alpha > 1$,

$$j(\lambda t) \leq N \lambda^{-d-\alpha} j(t), \quad \forall\, \lambda \in (1, \infty),\ 0 < t < 1. \tag{2.31}$$

(iii) Obviously, (2.27) holds if $a(y) = a(-y)$ for $|y| \leq 1$, and (2.29) holds if $a(y) = a(-y)$ for $|y| \geq 1$.

Below we give a sufficient condition for (2.26).

Lemma 2.21. *(i)* **H3***(i) holds if $\mathcal{A} = \phi(\Delta)$ for some Bernstein function ϕ satisfying*

$$1 + \phi(|\xi|^2) \geq N|\xi|^\alpha, \quad \forall \xi \in \mathbb{R}^d, \tag{2.32}$$

where $\alpha > 1$ and $N > 0$.

(ii) All of **H1, H2** *and* **H3** *hold if $\sigma > 1$, $\mathcal{A} = \phi(\Delta)$ and ϕ satisfies conditions* **A** *and* **B** *described in Example 2.12.*

Proof. (i). Let $\phi(\lambda) = \int_{\mathbb{R}}(1 - e^{-\lambda t})\mu(dt)$, where $\int_{\mathbb{R}}(1 \wedge |t|)\mu(dt) < \infty$. Then from $t^n e^{-t} \leq N(n)(1 - e^{-t})$, we get

$$|\lambda|^n |D^n \phi(\lambda)| \leq N\phi(\lambda). \tag{2.33}$$

For any $u \in C_0^\infty$,

$$\mathcal{A}u = \mathcal{F}^{-1}(\phi(|\xi|^2)\mathcal{F}(u)(\xi)),$$
$$\Delta^{\alpha/2}u = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(u)(\xi)) = \mathcal{F}^{-1}(\eta(\xi)(1 + \phi(|\xi|^2)\mathcal{F}(u)(\xi)),$$

where $\eta(\xi) = |\xi|^\alpha(1 + \phi(|\xi|^2))^{-1}$. Using (2.32) and (2.33), one can easily check

$$|D^n \eta(\xi)| \leq N(n)|\xi|^{-n}, \quad \forall\, \xi,$$

and therefore η is a Fourier multiplier (see Theorem IV.3.2 of [16]) and

$$\|\Delta^{\alpha/2}u\| \leq N(\|u\|_p + \|\mathcal{A}u\|_p),$$
$$\|\nabla u\|_p \leq \varepsilon\|\Delta^{\alpha/2}u\|_p + N(\varepsilon)\|u\|_p \leq N\varepsilon\|\mathcal{A}u\|_p + N\|u\|_p.$$

(ii) If **A** and **B** hold, then as explained before both **H1, H2** hold, and we also have (see (2.21)),

$$N^{-1}\phi(|y|^{-2})|y|^{-d} \leq J(y) \leq N\phi(|y|^{-2})|y|^{-d}.$$

Thus if $|\xi| \geq 1$, then

$$\phi(|\xi|^2) \geq N|\xi|^{-d}J(|\xi|^{-1}) \geq N|\xi|^{\alpha_0},$$

where (2.17) is used for the last inequality. Hence the lemma is proved. □

Here is our L_p-theory for equation (2.34) below.

Theorem 2.22. *Suppose that* **H1**, **H2** *and* **H3** *hold and Assumption 2.16 also holds if $\sigma = 1$. Let $\lambda > 0$ and $p > 1$. Then for any $f \in L_p$ there exists a unique solution $u \in \mathcal{H}_p^{\mathcal{A}}$ of the equation*

$$\tilde{L}u - \lambda u = f, \tag{2.34}$$

and for this solution we have

$$\|\mathcal{A}u\|_{L_p} + \lambda\|u\|_{L_p} \leq N(d, \nu, \Lambda, \lambda, \alpha_0, \kappa_1, \kappa_2, \kappa_3)\|f\|_{L_p}. \tag{2.35}$$

The proof of this theorem will be given in Section 6. The dependence of the constant N can be different slightly depending on which assumption in **H3** is given. Actually the constant N in (2.35) is independent of

λ except the case when **H3**(i) is assumed. Moreover the constant N in (2.35) is independent of κ_3 except the case when **H3**(iii) is assumed.

3. L_2-theory

In this section we prove (2.12) and (2.13). These estimates and Lemma 2.6 yield the unique solvability of equations (2.9) and (2.10). The Fourier transform and Parseval's identity are used to prove these estimates.

Lemma 3.1. *Let $\lambda \geq 0$ be a constant.*

(i) For any $u \in C_0^\infty$

$$\|\mathcal{A}u\|_{L_2} + \lambda\|u\|_{L_2} \leq N(d,\nu)\|Lu - \lambda u\|_{L_2} \tag{3.36}$$

and

$$\|\mathcal{A}u\|_{L_2} + \lambda\|u\|_{L_2} \leq N(d,\nu)\|\tilde{L}u - \lambda u\|_{L_2}. \tag{3.37}$$

*(ii) Let **H1** hold and $\sigma > 1$. Then both L and $\tilde{L}$ are continuous operators from $\mathcal{H}_2^{\mathcal{A}}$ to L_2, and for any $u \in C_0^\infty$,*

$$\|Lu\|_{L_2} \leq N_1\|\mathcal{A}u\|_{L_2}, \qquad \|\tilde{L}u\|_{L_2} \leq N_2\|u\|_{\mathcal{H}_2^{\mathcal{A}}}, \tag{3.38}$$

where $N_1 = N_1(d,\Lambda,\kappa_1,\alpha_0)$ and $N_2 = N_2(d,\Lambda,\kappa_1,\alpha_0,j(1))$. Moreover, (3.36) and (3.37) hold for any $u \in \mathcal{H}_2^{\mathcal{A}}$.

*(iii) Let **H1** and **H2** hold, and Assumption 2.16 also hold if $\sigma = 1$. Then the claims of (ii) hold for L (not for $\tilde{L}$) for any $\sigma \in (0,1]$ with a constant $N(d,\Lambda,\kappa_2)$.*

Proof. (i). Let $u \in C_0^\infty$. Taking the Fourier transform, we get

$$\mathcal{F}(Lu)(\xi) = \mathcal{F}u(\xi)\int_{\mathbb{R}^d}(e^{i\xi\cdot y} - 1 - iy\cdot\xi\chi(y))a(y)J(y)dy. \tag{3.39}$$

By Parseval's identity,

$$\begin{aligned}
\int_{\mathbb{R}^d}|Lu(x)|^2dx &= (2\pi)^{-d}\int_{\mathbb{R}^d}|\mathcal{F}(Lu)(\xi)|^2d\xi \\
&\geq (2\pi)^{-d}\int_{\mathbb{R}^d}|\mathcal{F}u(\xi)|^2\left|\mathrm{Re}\int_{\mathbb{R}^d}(e^{i\xi\cdot y} - 1 - iy\cdot\xi\chi(y))a(y)J(y)dy\right|^2d\xi \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}|\mathcal{F}u(\xi)|^2\left|\int_{\mathbb{R}^d}(1-\cos(\xi\cdot y))a(y)J(y)dy\right|^2d\xi \\
&\geq (2\pi)^{-d}\nu^2\int_{\mathbb{R}^d}|\mathcal{F}u(\xi)|^2\left|\int_{\mathbb{R}^d}(1-\cos(\xi\cdot y))J(y)dy\right|^2d\xi \\
&= \nu^2\int_{\mathbb{R}^d}|\mathcal{A}u|^2dx,
\end{aligned}$$

where the facts that $1 - \cos(\xi\cdot y)$ is nonnegative and $a(y) \geq \nu$ are used above.

Similarly, since uLu is real,

$$
\begin{aligned}
-\int_{\mathbb{R}^d} uLu\ dx &= -(2\pi)^{-d}\int_{\mathbb{R}^d} \mathcal{F}(Lu)(\xi)\overline{\mathcal{F}(u)(\xi)}\ d\xi \\
&= -(2\pi)^{-d}\int_{\mathbb{R}^d} |\mathcal{F}(u)(\xi)|^2 \mathrm{Re}\int_{\mathbb{R}^d} \left(e^{i\xi\cdot y} - 1 - iy\cdot\xi\chi^{(\sigma)}(y)\right) a(y)J(y)\ dyd\xi \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d} |\mathcal{F}(u)(\xi)|^2 \int_{\mathbb{R}^d} (1-\cos(\xi\cdot y))\, a(y)J(y)\ dyd\xi \\
&\geq \frac{\nu}{2}(2\pi)^{-d}\int_{\mathbb{R}^d} |\mathcal{F}(u)(\xi)|^2 \int_{\mathbb{R}^d} (1-\cos(\xi\cdot y))\, J(y)\ dyd\xi \\
&= -\frac{\nu}{2}\int_{\mathbb{R}^d} u\mathcal{A}u\ dx.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_{\mathbb{R}^d} |Lu-\lambda u|^2\ dx &= \int_{\mathbb{R}^d} |Lu|^2\ dx - 2\lambda\int_{\mathbb{R}^d} uLu\ dx + \lambda^2\int_{\mathbb{R}^d} |u|^2\ dx \\
&\geq \nu^2\int_{\mathbb{R}^d} |\mathcal{A}u|^2\ dx - \lambda\nu\int_{\mathbb{R}^d} u\mathcal{A}u\ dx + \lambda^2\int_{\mathbb{R}^d} |u|^2\ dx \\
&\geq \nu^2\int_{\mathbb{R}^d} |\mathcal{A}u|^2\ dx - \frac{\nu^2}{2}\int_{\mathbb{R}^d} u^2\ dx - \frac{\lambda^2}{2}\int_{\mathbb{R}^d} |\mathcal{A}u|^2\ dx + \lambda^2\int_{\mathbb{R}^d} |u|^2\ dx \\
&= \frac{\nu^2}{2}\int_{\mathbb{R}^d} |\mathcal{A}u|^2\ dx + \frac{\lambda^2}{2}\int_{\mathbb{R}^d} |u|^2\ dx.
\end{aligned}
$$

Thus (3.36) holds. Also, (3.37) is proved similarly.

(ii)–(iii). Next, we prove (3.38) for any $u \in C_0^\infty$. Unlike the case $j(r) = r^{-d-\alpha}$, the proof is not completely trivial. Condition **H1** is needed if $\sigma > 1$, and **H2** is additionally needed if $\sigma \leq 1$.

By using (3.39) and Parseval's identity again,

$$
\begin{aligned}
\int_{\mathbb{R}^d} |Lu(x)|^2 dx &= (2\pi)^{-d}\int_{\mathbb{R}^d} |\mathcal{F}(Lu)(\xi)|^2 d\xi \\
&= (2\pi)^{-d}\Big[\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left|\mathrm{Re}\int_{\mathbb{R}^d} (e^{i\xi\cdot y} - 1 - iy\cdot\xi\chi(y))a(y)J(y)\ dy\right|^2 d\xi \\
&\quad + \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left|\mathrm{Im}\int_{\mathbb{R}^d} (e^{i\xi\cdot y} - 1 - iy\cdot\xi\chi(y))a(y)J(y)\ dy\right|^2 d\xi\Big] \\
&\leq (2\pi)^{-d}\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left|\int_{\mathbb{R}^d} (1-\cos(\xi\cdot y))a(y)J(y)\ dy\right|^2 d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+ (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi|\geq 1} (\sin(\xi\cdot y) - y\cdot\xi\chi(y))a(y)J(y)\ dy \right|^2 d\xi \\
&+ (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi|< 1} (\sin(\xi\cdot y) - y\cdot\xi\chi(y))a(y)J(y)\ dy \right|^2 d\xi \\
&:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}
$$

Similarly,

$$
\int_{\mathbb{R}^d} |\tilde{L}u|^2 dx = \tilde{\mathcal{I}}_1 + \tilde{\mathcal{I}}_2 + \tilde{\mathcal{I}}_3,
$$

where $\tilde{\mathcal{I}}_i$ are obtained by replacing $\chi(y)$ in $\mathcal{I}_i$ with $I_{B_1}(y)$. Here $\mathcal{I}_1$ and $\tilde{\mathcal{I}}_1$ are easily controlled by $N\|\mathcal{A}u\|^2_{L_2}$.

Due to **H1**, (2.23), the definition of χ, and the change of variables $y \to \frac{y}{|\xi|}$,

$$
\begin{aligned}
\mathcal{I}_2 &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 |\xi|^{-2d} \left| \int_{|y|\geq 1} (\sin(\frac{\xi}{|\xi|}\cdot y) - y\cdot\frac{\xi}{|\xi|}\chi(\frac{y}{|\xi|}))a(\frac{y}{|\xi|})J(\frac{y}{|\xi|})\ dy \right|^2 d\xi \\
&\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 |\xi|^{-2d} j(1/|\xi|)^2 \\
&\qquad \times \left(\int_{|y|\geq 1} \left| \sin(\frac{\xi}{|\xi|}\cdot y) - I_{\sigma\neq 1} y\cdot\frac{\xi}{|\xi|}\chi(\frac{y}{|\xi|}) \right| a(\frac{y}{|\xi|})|y|^{-d-\alpha_0}\ dy \right)^2 d\xi \\
&\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 |\xi|^{-2d} j(1/|\xi|)^2\ d\xi.
\end{aligned}
$$

Hence, by Lemma 2.15,

$$
\mathcal{I}_2 \leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 (\Psi(\xi))^2\ d\xi = N \int_{\mathbb{R}^d} |\mathcal{A}u|^2\ dx.
$$

Similarly, if $\sigma > 1$,

$$
\begin{aligned}
\tilde{\mathcal{I}}_2 &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 |\xi|^{-2d} j(1/|\xi|)^2 \\
&\qquad \times \left(\int_{|y|\geq 1} \left| \sin(\frac{\xi}{|\xi|}\cdot y) - I_{\sigma>1} y\cdot\frac{\xi}{|\xi|} I_{|y|\leq|\xi|} \right|\ a(\frac{y}{|\xi|})|y|^{-d-\alpha_0}\ dy \right)^2 d\xi \\
&\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 |\xi|^{-2d} j(1/|\xi|)^2\ d\xi \leq N \int_{\mathbb{R}^d} |\mathcal{A}u|^2\ dx.
\end{aligned}
$$

Also, using the fundamental theorem of calculus, the definition of χ and (2.23),

$$\begin{aligned}\mathcal{I}_3 &\le N\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi|<1} (\sin(\xi\cdot y) - y\cdot\xi\chi(y))a(y)J(y)\,dy\right|^2 d\xi \\
&= N\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi|<1} \int_0^1 \frac{d}{dt}(\sin(t\xi\cdot y) - ty\cdot\xi\chi(y))\,dt\,a(y)J(y)\,dy\right|^2 d\xi \\
&= N\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi|<1} (\xi\cdot y)\int_0^1 (\cos(t\xi\cdot y) - \chi(y))\,dt\,a(y)J(y)\,dy\right|^2 d\xi \\
&= I_{\sigma\le 1}N\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y|||xi|<1} (\xi\cdot y)\int_0^1 \cos(t\xi\cdot y)\,dt\,a(y)J(y)\,dy\right|^2 d\xi \\
&\quad + I_{\sigma>1}N\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi|<1} (\xi\cdot y)\int_0^1 (\cos(t\xi\cdot y) - 1)\,dt\,a(y)J(y)\,dy\right|^2 d\xi.\end{aligned}$$

Observe that by **H1**, for any $t\in(0,1)$,

$$\Psi(t|\xi|) = \int_{\mathbb{R}^d}(1-\cos(ty\cdot\xi))J(y)dy = t^{-d}\int_{\mathbb{R}^d}(1-\cos(y\cdot\xi)J(t^{-1}y)dy \le Nt^{\alpha_0}\Psi(|\xi|).$$

Thus, if $\sigma>1$,

$$\mathcal{I}_3 \le N\int_{\mathbb{R}^d} |\mathcal{F}(u)|^2\left(\int_0^1 \Psi(t|\xi|)dt\right)^2 d\xi \le N\|\mathcal{A}u\|^2_{L_2}.$$

Also, if $\sigma>1$,

$$\begin{aligned}\tilde{\mathcal{I}}_3 &\le (2\pi)^{-d}\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi|<1} (\xi\cdot y)\int_0^1 \cos(t\xi\cdot y)I_{|y|\ge 1}\,dt\,a(y)J(y)\,dy\right|^2 d\xi \\
&\quad + (2\pi)^{-d}\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi|<1} (\xi\cdot y)\int_0^1 (1-\cos(t\xi\cdot y))\,dt\,a(y)J(y)\,dy\right|^2 d\xi \\
&\le N\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left(\int_{|y|\ge 1} J(y)dy\right)^2 d\xi + N\int_{\mathbb{R}^d} |\mathcal{F}(u)|^2\left(\int_0^1 \Psi(t|\xi|)dt\right)^2 d\xi \\
&\le N\|u\|^2_{\mathcal{H}^{\mathcal{A}}_2}.\end{aligned}$$

Thus (3.38) is proved if $\sigma>1$, and (3.36) and (3.37) are obtained for general $u\in\mathcal{H}^{\mathcal{A}}_2$ owing to (3.38). Therefore (ii) is proved.

Now we assume $\sigma \leq 1$. To estimate $\mathcal{I}_3$ we use the Fubini's Theorem, the change of variable $|\xi|ty \to y$, **H1**, **H2**, and Lemma 2.15

$$\begin{aligned}
\mathcal{I}_3 &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_0^1 t^{-d-1}|\xi|^{-d} \int_{|y|<t} (\frac{\xi}{|\xi|}\cdot y)\cos(\frac{\xi}{|\xi|}\cdot y) a(\frac{y}{|\xi|t}) J(\frac{y}{|\xi|t})\, dydt \right|^2 d\xi \\
&\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| |\xi|^{-d} \int_0^1 t^{-d-1} \int_{|y|<1} |y| J(\frac{y}{|\xi|t})\, dydt \right|^2 d\xi \\
&\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| |\xi|^{-d} \int_0^1 t^{\alpha_0-1}\, dt \int_{|y|<1} |y| J(y/|\xi|)\, dy \right|^2 d\xi \\
&\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| |\xi|^{-d} j(1/|\xi|) \right|^2 d\xi \leq N\|\mathcal{A}u\|_{L_2}^2.
\end{aligned}$$

Therefore the lemma is proved. $\square$

Corollary 2.7 and Lemmas 2.6 and 3.1 easily prove Theorem 2.8.

4. Some Hölder estimates

In this section we obtain some Hölder estimates for functions $u \in \mathcal{H}_2^{\mathcal{A}} \cap C_b^\infty$. The estimates will be used later for the estimates of the mean oscillation. Throughout this section we assume Assumption 2.16 holds if $\sigma = 1$.

Lemma 4.1. *For any $\alpha \in (0,1)$, $b \in \mathbb{R}^d$, and a nonnegative measurable function $\mathcal{K}(z)$, there exist $\eta_1, \eta_2 \in (0, 1/4)$, depending only on α, such that*

$$\begin{aligned}
&\int_{\mathcal{C}} [(|b+2z|^\alpha + |b-2z|^\alpha - 2|b|^\alpha)\, \mathcal{K}(z)]\, dz \\
&\leq -2^{\alpha-3}\alpha(1-\alpha) \int_{\mathcal{C}} |b|^{\alpha-2}|z|^2 \mathcal{K}(z) dz,
\end{aligned} \tag{4.40}$$

where

$$\mathcal{C} = \{|z| < \eta_1|b| : |z\cdot b| \geq (1-\eta_2)|b||z|\}.$$

Proof. We repeat the proof of Lemma 4.2 in [7] with few minor changes. Put $\eta(t) := b + 2tz$ and $\varphi(t) := |b+2tz|^\alpha = |\eta(t)|^\alpha$ for $z \in \mathcal{C}$. Then

$$\begin{aligned}
\varphi''(t) &= \sum_{i,j=1}^d \left(\alpha(\alpha-2)(\eta_i(t))(\eta_j(t))|\eta(t)|^{\alpha-4} + I_{i=j}\alpha|\eta(t)|^{\alpha-2}\right) 4z_iz_j \\
&= 4\alpha(\alpha-2)|\eta(t)|^{\alpha-4}|\eta(t)\cdot z|^2 + 4\alpha|\eta(t)|^{\alpha-2}|z|^2 \\
&= 4\alpha|b+2tz|^{\alpha-4}[(\alpha-2)|(b+2tz)\cdot z|^2 + |b+2tz|^2|z|^2].
\end{aligned}$$

For $t \in [-1,1]$ and $z \in \mathcal{C}$, observer that,

$$|b+2tz|^2 \leq (1+2\eta_1)^2|b|^2$$

and

$$\begin{aligned}|(b+2tz)\cdot z| &= |b\cdot z + 2t|z|^2| \geq |b\cdot z| - 2|z|^2 \\ &\geq (1-\eta_2)|b||z| - 2|z|^2 \geq (1-2\eta_1-\eta_2)|z||b|.\end{aligned}$$

Thus

$$\varphi''(t) \leq 4\alpha|a+2tz|^{\alpha-4}[(\alpha-2)(1-2\eta_1-\eta_2)^2 + (1+2\eta_1)^2]|b|^2|z|^2. \tag{4.41}$$

Since $(1-2\eta_1-\eta_2)^2 \to 1$ and $(1+2\eta_1)^2 \to 1$ as $\eta_1, \eta_2 \downarrow 0$, one can choose sufficiently small $\eta_1, \eta_2 \in (0, 1/4)$, depending only on $\alpha \in (0,1)$, such that

$$(\alpha-2)(1-2\eta_1-\eta_2)^2 + (1+2\eta_1)^2 \leq (\alpha-1)/2.$$

By combining this with (4.41)

$$\varphi''(t) \leq -2\alpha(1-\alpha)|b+2tz|^{\alpha-4}|b|^2|z|^2. \tag{4.42}$$

Furthermore observe that

$$|b+2tz|^{\alpha-4} \geq (1+2\eta_1)^{\alpha-4}|b|^{\alpha-4} \geq 2^{\alpha-4}|b|^{\alpha-4}.$$

Therefore, from (4.42)

$$\varphi''(t) \leq -2^{\alpha-3}\alpha(1-\alpha)|b|^{\alpha-2}|z|^2, \quad t\in[-1,1],\ z \in \mathcal{C}.$$

In addition to this, to prove (4.40), it is enough to use the fact that there exists $t_0 \in (-1,1)$ satisfying

$$\varphi(1) + \varphi(-1) - 2\varphi(0) = \varphi''(t_0),$$

which can be shown by the mean value theorem. The lemma is proved. □

Before going further, we introduce some notation used in the following theorem. For a nonnegative function h, by $L_1(\mathbb{R}^d, h)$ we denote the classe of integrable functions with the measure $h(y)dy$, i.e. $f \in L_1(\mathbb{R}^d, h)$ iff $\int_{\mathbb{R}^d} |f(y)|h(y)dy < \infty$. For a function f and an open set U, $[f]_{C^\alpha(U)}$ denotes the Hölder semi-norm with the order α on U, i.e.

$$[f]_{C^\alpha(U)} = \sup_{x,y\in U, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

The notation $\mathrm{osc}_U f$ means the difference between the maximum of f and the minimum of f on U, that is,

$$\mathrm{osc}_U f := \max_U f - \min_U f.$$

Theorem 4.2. *Let $R > 0$, $\lambda \geq 0$ and* **H1** *hold. Suppose $f \in L_\infty(B_R)$ and $u, \tilde{u} \in C_b^2(B_R) \cap L_1(\mathbb{R}^d, w_R)$, where $w_R(x) = \frac{1}{1/j(R)+1/J(x/2)}$. Also assume*

$$Lu - \lambda u = f, \qquad \tilde{L}\tilde{u} - \lambda \tilde{u} = f \quad in \ \ B_R. \tag{4.43}$$

(i) For any $\alpha \in (0, \min\{1, \alpha_0\})$ and $0 < r < R$, it holds that

$$\begin{aligned}
[u]_{C^\alpha(B_r)} \leq{}& N r_1^{-\alpha} \|u\|_{L_\infty(B_R)} \\
&+ N \frac{\|u\|_{L_\infty(B_R)}}{j(r_1) r_1^{d+\alpha}} \Big(r_1^{-2} \int\limits_{B_{r_1}} |z|^2 J(z)\, dz + I_{\sigma<1} r_1^{-1} \int\limits_{B_{r_1}} |z| J(z)\, dz \Big) \\
&+ N \Big(\frac{1}{r_1^{d+\alpha} j(R)} \|u\|_{L_1(\mathbb{R}^d, w_R)} + \frac{1}{j(r_1) r_1^{d+\alpha}} osc_{B_R} f \Big),
\end{aligned} \tag{4.44}$$

where $r_1 = (R-r)/2$ and $N = N(d, \nu, \Lambda, \kappa_1, \alpha_0, \alpha)$.

Consequently, if **H2** *is additionally assumed, then*

$$[u]_{C^\alpha(B_r)} \leq N \left(r_1^{-\alpha} \|u\|_{L_\infty(B_R)} + \frac{1}{r_1^{d+\alpha} j(R)} \|u\|_{L_1(\mathbb{R}^d, w_R)} + \frac{osc_{B_R} f}{j(r_1) r_1^{d+\alpha}} \right), \tag{4.45}$$

$N = N(d, \nu, \Lambda, \kappa_1, \kappa_2, \alpha_0, \alpha)$.

(ii) In addition to **H1***, let one of* **H3***(ii)–***H3***(iv) hold. Then (4.44) holds for $\tilde{u}$. Consequently, if* **H2** *additionally holds, (4.45) holds for $\tilde{u}$.*

Proof. We adopt the method used in [7] (cf. [3]). Assume that u is not identically zero in B_r. Set

$$r_1 = (R-r)/2, \quad r_2 = (R+r)/2, \quad w(t,x) = I_{B_R}(x) u(t,x).$$

Then $u(x) = w(x)$ and $\nabla u(x) = \nabla w(x)$ for all $x \in B_{r_2}$. Thus

$$Lu(x) = Lw(x) + \int\limits_{|z| \geq r_1} (u(x+z) - w(x+z))\, a(z) J(z) dz.$$

So in B_{r_2}

$$Lw(x) - \lambda w = g(x) + f(x), \tag{4.46}$$

where

$$g(x) = - \int\limits_{|z| \geq r_1} (u(x+z) - w(x+z))\, a(z) J(z) dz.$$

Note that by **H1**

$$\|g\|_{L_\infty(B_R)} \leq N \frac{j(r_1)}{j(R)} \|u\|_{L_1(\mathbb{R}^d, w_R)}, \tag{4.47}$$

where $N = N(d, \Lambda, \kappa_1)$. Indeed, this comes from the fact that for all $|z| \geq r_1$ and $|x| \leq R$,

$$\frac{j(|z|)}{j(R)} + \frac{j(|z|)}{j(|x+z|/2)} \leq N \frac{j(r_1)}{j(R)}.$$

For $x_0 \in B_r$ and $\alpha \in (0, \min\{1, \alpha_0\})$, we define

$$M(x,y) := w(x) - w(y) - C|x-y|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|x-x_0|^2,$$

where C is a positive constant which will be chosen later so that it is independent of x_0 and

$$\sup_{x,y\in\mathbb{R}^d} M(x,y) \le 0. \tag{4.48}$$

For $x \in \mathbb{R}^d \setminus B_{r_1/2}(x_0)$ and $y \in \mathbb{R}^d$,

$$w(x) - w(y) \le 2\|u\|_{L_\infty(B_R)} \le 8r_1^{-2}\|u\|_{L_\infty(B_R)}|x-x_0|^2. \tag{4.49}$$

This shows

$$M(x,y) \le 0, \quad x \in \mathbb{R}^d \setminus B_{r_1/2}(x_0).$$

Assume that there exist $x, y \in \mathbb{R}^d$ such that $M(x,y) > 0$. We will get the contradiction by choosing an appropriate constant C. Due to (4.49), $x \in B_{r_1/2}(x_0)$. Moreover

$$w(x) - w(y) > C|x-y|^\alpha,$$

which implies

$$|x-y|^\alpha < \frac{2\|u\|_{L_\infty(B_R)}}{C}. \tag{4.50}$$

If we take C large enough so that $C \ge 2(r_1/2)^{-\alpha}\|u\|_{L_\infty(B_R)}$, then

$$y \in B_{r+r_1}.$$

Therefore, there exist $\bar{x}, \bar{y} \in B_{r+r_1}$ satisfying

$$\sup_{x,y\in\mathbb{R}^d} M(x,y) = M(\bar{x},\bar{y}) > 0.$$

Moreover, from (4.46)

$$\begin{aligned} -2\|g\|_{L_\infty(B_R)} - \mathrm{osc}_{B_R} f &\le (Lw(\bar{x}) - \lambda w(\bar{x})) - (Lw(\bar{y}) - \lambda w(\bar{y})) \\ &= (Lw(\bar{x}) - Lw(\bar{y})) + \lambda(w(\bar{y}) - w(\bar{x})) \\ &\le Lw(\bar{x}) - Lw(\bar{y}) := \mathcal{I}. \end{aligned} \tag{4.51}$$

Put $K(z) := a(z)J(z)$ and

$$K_1(z) := K(z) \wedge K(-z), \quad K_2(z) := K(z) - K_1(z).$$

By L_1 and L_2, respectively, we denote the operators with kernels K_1 and K_2. Then

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2,$$

where

$$\mathcal{I}_1 := L_1 w(\bar{x}) - L_1 w(\bar{y}) \quad \text{and} \quad \mathcal{I}_2 := L_2 w(\bar{x}) - L_2 w(\bar{y}).$$

Since K_1 is symmetric (i.e. $K_1(z) = K_1(-z)$),

$$\mathcal{I}_1 = \frac{1}{2} \int_{\mathbb{R}^d} \mathcal{J}(\bar{x}, \bar{y}, z) K_1(z) dz,$$

where

$$\mathcal{J}(\bar{x}, \bar{y}, z) = w(\bar{x} + z) + w(\bar{x} - z) - 2w(\bar{x}) - w(\bar{y} + z) - w(\bar{y} - z) + 2w(\bar{y}).$$

Also, since $M(x, y)$ attains its maximum at $(\bar{x}, \bar{y})$,

$$\begin{aligned} &w(\bar{x} + z) - w(\bar{y} + z) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} + z - x_0|^2 \\ &\leq w(\bar{x}) - w(\bar{y}) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} - x_0|^2 \end{aligned} \tag{4.52}$$

and

$$\begin{aligned} &w(\bar{x} - z) - w(\bar{y} - z) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} - z - x_0|^2 \\ &\leq w(\bar{x}) - w(\bar{y}) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} - x_0|^2 \end{aligned} \tag{4.53}$$

for all $z \in \mathbb{R}^d$. By combining these two inequalities,

$$\mathcal{J}(\bar{x}, \bar{y}, z) \leq 8r_1^{-2}\|u\|_{L_\infty(B_R)} \left(|\bar{x} + z - x_0|^2 + |\bar{x} - z - x_0|^2 - 2|\bar{x} - x_0|^2\right). \tag{4.54}$$

Similarly,

$$\begin{aligned} &w(\bar{x} + z) - w(\bar{y} - z) - C|\bar{x} - \bar{y} + 2z|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} + z - x_0|^2 \\ &\leq w(\bar{x}) - w(\bar{y}) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} - x_0|^2, \\ &w(\bar{x} - z) - w(\bar{y} + z) - C|\bar{x} - \bar{y} - 2z|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} - z - x_0|^2 \\ &\leq w(\bar{x}) - w(\bar{y}) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} - x_0|^2. \end{aligned}$$

It follows that, for any $z \in \mathbb{R}^d$,

$$\begin{aligned} \mathcal{J}(\bar{x}, \bar{y}, z) \leq\ & C\left(|\bar{x} - \bar{y} + 2z|^\alpha + |\bar{x} - \bar{y} - 2z|^\alpha - 2|\bar{x} - \bar{y}|^\alpha\right) \\ &+ 8r_1^{-2}\|u\|_{L_\infty(B_R)} \left(|\bar{x} + z - x_0|^2 + |\bar{x} - z - x_0|^2 - 2|\bar{x} - x_0|^2\right). \end{aligned} \tag{4.55}$$

Put $b = \bar{x} - \bar{y}$. Since $(\bar{x}, \bar{y})$ satisfy (4.50), $|b| < r_1/2$ if $C \geq 2(r_1/2)^{-\alpha}\|u\|_{L_\infty(B_R)}$. Also set for $\eta_1, \eta_2 \in (0, 1/4)$ specified in Lemma 4.1,

$$\mathcal{C} = \{|z| < \eta_1|b| : |z \cdot b| \geq (1 - \eta_2)|b||z|\}.$$

Then

$$
\begin{aligned}
2\mathcal{I}_1 = &\int\limits_{|z|\geq r_1/2} \mathcal{J}(\bar{x},\bar{y},z)K_1(z)\ dz + \int\limits_{B_{r_1/2}\setminus\mathcal{C}} \mathcal{J}(\bar{x},\bar{y},z)K_1(z)\ dz \\
&+ \int\limits_{\mathcal{C}} \mathcal{J}(\bar{x},\bar{y},z)K_1(z)\ dz := \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}.
\end{aligned} \tag{4.56}
$$

Note that by **H1**,

$$
\mathcal{I}_{11} \leq Nj(r_1/2)r_1^d\|u\|_{L_\infty(B_R)}.
$$

Indeed,

$$
\begin{aligned}
\mathcal{I}_{11} &\leq N\|u\|_{L_\infty(B_R)} \int\limits_{|z|\geq r_1/2} J(z)\ dz \\
&\leq Nr_1^d\|u\|_{L_\infty(B_R)} \int\limits_{|z|\geq 1} J(r_1 z/2)\ dz \\
&\leq Nj(r_1/2)r_1^d\|u\|_{L_\infty(B_R)} \int\limits_{|z|\geq 1} |z|^{-d-\alpha_0}\ dz.
\end{aligned}
$$

On the other hand from (4.54), it follows that

$$
\begin{aligned}
\mathcal{I}_{12} &\leq 8r_1^{-2}\|u\|_{L_\infty(B_R)} \int\limits_{B_{r_1/2}\setminus\mathcal{C}} \left(|\bar{x}+z-x_0|^2 + |\bar{x}-z-x_0|^2 - 2|\bar{x}-x_0|^2\right) K_1(z)\ dz \\
&\leq Nr_1^{-2}\|u\|_{L_\infty(B_R)} \int\limits_{B_{r_1/2}} |z|^2 J(z)\ dz.
\end{aligned}
$$

Next using (4.55) we obtain

$$
\begin{aligned}
\mathcal{I}_{13} \ \leq\ & C\int\limits_{\mathcal{C}} \left(|\bar{x}-\bar{y}+2z|^\alpha + |\bar{x}-\bar{y}-2z|^\alpha - 2|\bar{x}-\bar{y}|^\alpha\right) K_1(z)\ dz \\
&+ 8r_1^{-2}\|u\|_{L_\infty(B_R)} \int\limits_{\mathcal{C}} \left(|\bar{x}+z-x_0|^2 + |\bar{x}-z-x_0|^2 - 2|\bar{x}-x_0|^2\right) K_1(z)\ dz \\
:=\ & \mathcal{I}_{131} + \mathcal{I}_{132}.
\end{aligned}
$$

The term $\mathcal{I}_{132}$ is again bounded by

$$
Nr_1^{-2}\|u\|_{L_\infty(B_R)} \int\limits_{B_{r_1/2}} |z|^2 J(z)\ dz.
$$

Furthermore, from Lemma 4.1

$$
\mathcal{I}_{131} \leq -2^{\alpha-3}C\alpha(1-\alpha)\int\limits_{\mathcal{C}} |b|^{\alpha-2}|z|^2 K_1(z)dz.
$$

Combining all these facts above, we obtain

$$\mathcal{I}_1 \leq N\|u(\cdot)\|_{L_\infty(B_R)}\left(j(r_1/2)r_1^d + r_1^{-2}\int\limits_{B_{r_1/2}}|z|^2 J(z)\;dz\right) - 2^{\alpha-3}C\alpha(1-\alpha)\int\limits_{\mathcal{C}}|b|^{\alpha-2}|z|^2K_1(z)dz. \tag{4.57}$$

For $\mathcal{I}_2$, we first consider **the case** $\sigma < 1$. In this case,

$$\mathcal{I}_2 = \int\limits_{|z|\geq r_1/2}(w(\bar{x}+z) - w(\bar{x}) - w(\bar{y}+z) + w(\bar{y}))\,K_2(z)\;dz + \int\limits_{B_{r_1/2}}(w(\bar{x}+z) - w(\bar{x}) - w(\bar{y}+z) + w(\bar{y}))\,K_2(z)\;dz := \mathcal{I}_{21} + \mathcal{I}_{22}.$$

Analogously to $\mathcal{I}_{11}$, we bound $\mathcal{I}_{21}$ by $Nj(r_1/2)r_1^d\|u\|_{L_\infty(B_R)}$. For the other term $\mathcal{I}_{22}$, since $|\bar{x} - x_0| < r_1/2$, from (4.52)

$$\begin{aligned}\mathcal{I}_{22} &\leq Nr_1^{-2}\|u\|_{L_\infty(B_R)}\int\limits_{B_{r_1/2}}\left(|\bar{x}+z-x_0|^2 - |\bar{x}-x_0|^2\right)K_2(z)\;dz \\ &\leq Nr_1^{-2}\|u\|_{L_\infty(B_R)}\int\limits_{B_{r_1/2}}\left(|z|^2 + 2|z||\bar{x}-x_0|\right)J(z)\;dz \\ &\leq Nr_1^{-1}\|u\|_{L_\infty(B_R)}\int\limits_{B_{r_1/2}}|z|J(z)\;dz.\end{aligned}$$

So

$$\mathcal{I}_2 \leq N\|u\|_{L_\infty(B_R)}\Big(j(r_1/2)r_1^d + r_1^{-1}\int\limits_{B_{r_1/2}}|z|J(z)\;dz\Big). \tag{4.58}$$

By combining (4.47), (4.51), (4.57) and (4.58),

$$0 \leq N_1\Big(\mathrm{osc}_{B_R}f + \frac{j(r_1)}{j(R)}\|u\|_{L_1(\mathbb{R}^d,w_R)} + \|u\|_{L_\infty(B_R)}\big[j(r_1/2)r_1^d + r_1^{-1}\int\limits_{B_{r_1/2}}|z|J(z)\;dz\big]\Big) - 2^{\alpha-3}C\alpha(1-\alpha)\int\limits_{\mathcal{C}}|b|^{\alpha-2}|z|^2K_1(z)\;dz.$$

Thus, if $C \geq C_1 := 2(r_1/2)^{-\alpha}\|u\|_{L_\infty(B_R)}$ and

$$C \geq C_2 := N_1C_3\Big(\mathrm{osc}_{B_R}f + \frac{j(r_1)}{j(R)}\|u\|_{L_1(\mathbb{R}^d,w_R)} + \|u\|_{L_\infty(B_R)}\Big[j(r_1/2)r_1^d + r_1^{-1}\int\limits_{B_{r_1/2}}|z|J(z)\;dz\Big]\Big),$$

then

$$
\begin{aligned}
0 \leq\ & N_1\Big(\mathrm{osc}_{B_R} f + \frac{j(r_1)}{j(R)}\|u\|_{L_1(\mathbb{R}^d, w_R)} + \|u\|_{L_\infty(B_R)}\Big[j(r_1/2)r_1^d + r_1^{-1}\int\limits_{B_{r_1/2}} |z|J(z)\ dz\Big]\Big) \\
& \times\Big(1 - C_3 2^{\alpha-3}\alpha(1-\alpha)\int\limits_{\mathcal{C}} |b|^{\alpha-2}|z|^2 K_1(z)\ dz\Big) \\
:=\ & (1 - C_3 C_4(b)).
\end{aligned}
$$

If we take C_3 so that $C_3 = 1/C_5$ for a $C_5 = C_5(r_1, \alpha) < C_4(b)$ which does not depend on b and will be chosen below, we get the contradiction. To select C_5, observe that with **H1** and the fact $|b| \leq r_1/2$

$$
\begin{aligned}
C_4(b) &= 2^{\alpha-3}\alpha(1-\alpha)\int\limits_{\mathcal{C}} |b|^{\alpha-2}|z|^2 K_1(z) dz \\
&\geq \nu 2^{\alpha-3}\alpha(1-\alpha)\int\limits_{\mathcal{C}} |b|^{\alpha-2}|z|^2 J(z) dz \\
&\geq \kappa_1^{-1}\nu 2^{\alpha-3}\alpha(1-\alpha) j(\eta_1|b|)\int\limits_{\mathcal{C}} |b|^{\alpha-2}|z|^2 dz \\
&\geq \kappa_1^{-1}\nu 2^{\alpha-3}\alpha(1-\alpha) j(\eta_1|b|)|b|^{\alpha-2}|\eta_1 b|^{d+2}\int\limits_{\mathcal{C}_{\eta_2}} |z|^2 dz \\
&\geq \kappa_1^{-1}\nu \eta_1^{d+2} 2^{\alpha-3}\alpha(1-\alpha) j(|b|)|b|^{d+\alpha}\int\limits_{\mathcal{C}_{\eta_2}} |z|^2 dz \\
&\geq \kappa_1^{-2}\nu j(r_1/2)(r_1/2)^{d+\alpha}\eta_1^{d+2} 2^{\alpha-3}\alpha(1-\alpha)\int\limits_{\mathcal{C}_{\eta_2}} |z|^2 dz \\
&= j(r_1/2) r_1^{d+\alpha} N(\alpha, \eta_1, \eta_2) := C_5,
\end{aligned}
$$

where $\mathcal{C} = \{|z| < \eta_1|b| : |z \cdot b| \geq (1-\eta_2)|b||z|\}$ and $\mathcal{C}_{\eta_2} = \{|z| < 1 : \frac{|z\cdot b|}{|b||z|} \geq (1-\eta_2)\}$. Therefore, (4.48) holds with $C = C_1 + C_2$. Since C is independent of x_0, (4.44) is proved.

Next we consider **the case** $\sigma = 1$. Note that, because K_1 is symmetric, both K_1 and K_2 satisfy (2.23). Therefore, we can replace 1_{B_1} with $I_{B_{r_1}}$ in the definition of L_2, and get $\mathcal{I}_2 = \mathcal{I}_{21} + \mathcal{I}_{22}$, where

$$
\mathcal{I}_{21} = \int\limits_{|z|\geq r_1/2} \left(w(\bar{x}+z) - w(\bar{x}) - w(\bar{y}+z) + w(\bar{y})\right) K_2(z)\ dz,
$$

$$
\mathcal{I}_{22} = \int\limits_{B_{r_1/2}} \left(w(\bar{x}+z) - w(\bar{x}) - w(\bar{y}+z) + w(\bar{y}) - z\cdot(\nabla w(\bar{x}) - \nabla w(\bar{y}))\right) K_2(z)\ dz.
$$

$\mathcal{I}_{21}$ is already estimated in the previous case. Thus we only consider $\mathcal{I}_{22}$. Since $M(x,y)$ attains its maximum at the interior point $(\bar{x}, \bar{y})$, we have $\nabla_x M(\cdot, \bar{y})(\bar{x}) = 0$, $\nabla_y M(\bar{x}, \cdot)(\bar{y}) = 0$, and therefore

$$
\nabla w(\bar{x}) - \nabla w(\bar{y}) = 16 r_1^{-2}\|u\|_{L_\infty(B_R)}(\bar{x} - x_0). \tag{4.59}
$$

We use (4.52) and (4.59) to get

$$\begin{aligned}\mathcal{I}_{22} &\le 8r_1^{-2}\|u\|_{L_\infty(B_R)} \int\limits_{B_{r_1/2}} |z|^2 K_2(z)\,dz \\ &\le 8r_1^{-2} \int\limits_{B_{r_1/2}} |z|^2 J(z)\,dz \|u\|_{L_\infty(B_R)}.\end{aligned}$$

Therefore, (4.44) is proved following the argument in the case $\sigma < 1$.

Finally, let $\sigma > 1$. Now we have $\mathcal{I}_2 = \mathcal{I}_{21} + \mathcal{I}_{22}$, where

$$\mathcal{I}_{21} = \int\limits_{|z|\ge r_1/2} [w(\bar{x}+z) - w(\bar{x}) - w(\bar{y}+z) + w(\bar{y}) - z\cdot(\nabla w(\bar{x}) - \nabla w(\bar{y}))]K_2(z)\,dz,$$

$$\mathcal{I}_{22} = \int\limits_{B_{r_1/2}} [w(\bar{x}+z) - w(\bar{x}) - w(\bar{y}+z) + w(\bar{y}) - z\cdot(\nabla w(\bar{x}) - \nabla w(\bar{y}))]K_2(z)\,dz.$$

Since $\sigma > 1$, $|\bar{x} - x_0| < r_1/2$, by (4.59) and **H1**

$$\begin{aligned}\mathcal{I}_{21} &\le \int\limits_{|z|\ge r_1/2} [4\|u\|_{L_\infty(B_R)} + 4(r_1/2)^{-1}\|u\|_{L_\infty(B_R)}|z|]K_2(z)\,dz \\ &\le N r_1^d j(r_1/2)\|u\|_{L_\infty(B_R)}.\end{aligned}$$

For $\mathcal{I}_{22}$, we apply (4.52) and (4.59) to get

$$\mathcal{I}_{22} \le N r_1^{-2}\|u\|_{L_\infty(B_R)} \int\limits_{B_{r_1/2}} |z|^2 J(z)\,dz.$$

So we again argue as in the first case to get the contradiction. Hence (i) is proved.

The proof of (ii) is quite similar to that of (i). Denote the counter parts of w and g by $\tilde{w}$ and $\tilde{g}$, respectively. Also we introduce $\mathcal{I}_1$ and $\mathcal{I}_2$ similarly. That is $\mathcal{I}_1$ is the same as before, and $\mathcal{I}_2$ is given by

$$\begin{aligned}\mathcal{I}_2 = &\int\limits_{|z|\ge r_1/2} \Big[\tilde{w}(\bar{x}+z) - \tilde{w}(\bar{x}) - \tilde{w}(\bar{y}+z) + \tilde{w}(\bar{y}) \\ &\qquad - I_{B_1}(z) z\cdot\nabla(\tilde{w}(\bar{x}) - \tilde{w}(\bar{y}))\Big]K_2(z)\,dz \\ &+ \int\limits_{B_{r_1/2}} \Big[\tilde{w}(\bar{x}+z) - \tilde{w}(\bar{x}) - \tilde{w}(\bar{y}+z) + \tilde{w}(\bar{y}) \\ &\qquad - I_{B_1}(z) z\cdot\nabla(\tilde{w}(\bar{x}) - \tilde{w}(\bar{y}))\Big]K_2(z)\,dz \\ := {} &\mathcal{I}_{21} + \mathcal{I}_{22}.\end{aligned}$$

All of the differences are as follows. If $r_1/2 \ge 1$, then by using (4.52) and (4.59),

$$\mathcal{I}_{22} \le N r_1^{-2}\|\tilde{u}\|_{L_\infty(B_R)}\Big[\int\limits_{B_1} |z|^2 K_2(z)\,dz + \int\limits_{1\le|z|\le r_1/2} (|z|^2 + (\bar{x} - x_0)\cdot z)K_2(z)\,dz\Big]$$

$$\le N I_{\sigma<1} r_1^{-1} \|\tilde{u}\|_{L_\infty(B_R)} \int\limits_{B_{r_1/2}} |z| J(z)\, dz$$
$$+ N I_{\sigma>1} r_1^{-2} \|\tilde{u}\|_{L_\infty(B_R)} \int\limits_{B_{r_1/2}} |z|^2 J(z)\, dz.$$

In the above, we also used $\int_{1\le|z|\le r_1/2} z^i K_2(z) dz = 0$ if $\sigma > 1$ (due to **H3**(iv)).

Let $\sigma < 1$ and $r_1/2 < 1$. If **H3**(ii) hold, then by (2.14),

$$\mathcal{I}_{21} \le N\|u\|_{L_\infty(B_R)} \int\limits_{|z|\ge r_1/2} J(z)\, dz$$
$$= N r_1^d \int\limits_{|z|\ge 1} J(r_1 z/2) dz \le N j(r_1/2) r_1^d \|u\|_{L_\infty(B_R)}.$$

Also, if **H3**(iii) holds, then by using (4.59),

$$\mathcal{I}_{21} \le \|u\|_{L_\infty(B_R)} \int\limits_{|z|\ge r_1/2} [1 + 8 r_1^{-1}|z|] K_2(z)\, dz$$
$$\le N j(r_1/2) r_1^d \|u\|_{L_\infty(B_R)}.$$

This completes the proof of the theorem. □

We remove $\sup_{B_R} u$ on the right hand side of (4.45) in the following corollary. Recall $w_R(x) = \frac{1}{1/j(R)+1/J(x/2)}$.

Corollary 4.3. *Suppose that* **H1** *and* **H2** *hold. Let $\lambda \ge 0$, $f \in L_\infty(B_R)$, and $u, \tilde{u} \in C_b^2(B_R) \cap L_1(\mathbb{R}^d, w_R)$ satisfy*

$$Lu - \lambda u = f, \qquad \tilde{L}\tilde{u} - \lambda \tilde{u} = f \qquad in \quad B_R. \tag{4.60}$$

(i) For any $\alpha \in (0, \min\{1, \alpha_0\})$, it holds that

$$[u]_{C^\alpha(B_{R/2})} \le \frac{N}{j(R) R^{d+\alpha}} \left(\|u\|_{L_1(\mathbb{R}^d, w_R)} + osc_{B_R} f \right), \tag{4.61}$$

where $N = N(d, \nu, \Lambda, \kappa_1, \kappa_2, \alpha_0, \alpha)$.

(ii) If one of **H3***(ii)–(iv) is additionally assumed, then (4.61) holds for $\tilde{u}$.*

Proof. For $n = 1, 2, \ldots$, set

$$r_n := R(1 - 2^{-n}).$$

Observe that $(r_{n+1} - r_n)/2 = R 2^{-n-2} \le R$ and by **H1**

$$\frac{1}{j(r_{n+1})} \|u\|_{L_1(\mathbb{R}^d, w_{r_{n+1}})} \le \left(\int\limits_{|z|<2R} |u(z)|\, dz + \frac{1}{j(r_{n+1})} \int\limits_{|z|\ge 2R} |u(z)| j(z/2)\, dz \right)$$

$$
\begin{aligned}
&\le N\left(\int\limits_{|z|<2R} |u(z)|\,dz + \frac{1}{j(R)}\int\limits_{|z|\ge 2R} |u(z)|j(z/2)\,dz\right)\\
&\le N\frac{1}{j(R)}\int\limits_{\mathbb{R}^d} |u(z)|w_R(z)\,dz.
\end{aligned}
$$

Then by Theorem 4.2(i) and **H1**,

$$
\begin{aligned}
[u]_{C^\alpha(B_{r_n})} &\le NR^{-\alpha}2^{\alpha n}\sup_{B_{r_{n+1}}}|u|\\
&\quad + N\frac{2^{(d+\alpha)n}}{j(R2^{-n-2})R^{d+\alpha}}\left(\frac{j(R2^{-n-2})}{j(r_{n+1})}\|u\|_{L_1(\mathbb{R}^d,w_{r_{n+1}})} + \operatorname{osc}_{B_{r_{n+1}}} f\right)\\
&\le N\Big[R^{-\alpha}2^{\alpha n}\sup_{B_{r_{n+1}}}|u| + \frac{2^{(d+\alpha)n}}{j(R)R^{d+\alpha}}\Big(\|u\|_{L_1(\mathbb{R}^d,w_R)} + \operatorname{osc}_{B_R} f\Big)\Big].
\end{aligned}
\tag{4.62}
$$

In order to estimate the term $\sup_{B_{r_{n+1}}}|u|$ above, we use the following:

$$
\sup_{B_{r_{n+1}}}|u| \le (\varepsilon r_{n+1})^\alpha [u]_{C^\alpha(r_{n+1})} + N(\varepsilon r_{n+1})^{-d}\|u\|_{L_1(B_{r_{n+1}})}, \quad \varepsilon\in(0,1).
\tag{4.63}
$$

Actually this inequality can be easily obtained as follows. For all $\varepsilon\in(0,1)$, $x\in B_{r_{n+1}}$, and $y\in B_{r_{n+1}}\cap B_{\varepsilon r_{n+1}}(x)$,

$$
\begin{aligned}
&|B_{r_{n+1}}\cap B_{\varepsilon r_{n+1}}(x)|\cdot|u(x)|\\
&\le \int\limits_{B_{r_{n+1}}\cap B_{\varepsilon r_{n+1}}(x)} (|u(x)-u(y)| + |u(y)|)\,dy\\
&\le |B_{r_{n+1}}\cap B_{\varepsilon r_{n+1}}(x)|\cdot(\varepsilon r_{n+1})^\alpha[u]_{C^\alpha(B_{r_{n+1}})} + \int\limits_{B_{r_{n+1}}\cap B_{\varepsilon r_{n+1}}(x)} |u(y)|\,dy.
\end{aligned}
$$

Now it is enough to note that $|B_{r_{n+1}}\cap B_{\varepsilon r_{n+1}}(x)|\sim(\varepsilon r_{n+1})^d$ because $\varepsilon\in(0,1)$ and $x\in B_{r_{n+1}}$.

Take N from (4.62) and define ε so that

$$
\varepsilon^\alpha = N^{-1}2^{-\alpha n}2^{-3d}.
$$

Then by combining (4.62) and (4.63),

$$
\begin{aligned}
[u]_{C^\alpha(B_{r_n})} &\le 2^{-3d}[u]_{C^\alpha(B_{r_{n+1}})} + NR^{-d-\alpha}2^{2dn}\|u\|_{L_1(B_{r_{n+1}})}\\
&\quad + N\frac{2^{(d+\alpha)n}}{j(R)R^{d+\alpha}}(\|u\|_{L_1(\mathbb{R}^d,w_R)} + \operatorname{osc}_{B_R} f)\\
&\le 2^{-3d}[u]_{C^\alpha(B_{r_{n+1}})} + NR^{-d-\alpha}2^{2dn}\|u\|_{L_1(B_{r_{n+1}})}\\
&\quad + N\frac{2^{2dn}}{j(R)R^{d+\alpha}}(\|u\|_{L_1(\mathbb{R}^d,w_R)} + \operatorname{osc}_{B_R} f).
\end{aligned}
\tag{4.64}
$$

Multiply both sides of (4.64) by 2^{-3dn} and take the sum over n to get

$$
\sum_{n=1}^\infty 2^{-3dn}[u]_{C^\alpha(B_{r_n})} \le \sum_{n=1}^\infty 2^{-3d(n+1)}[u]_{C^\alpha(B_{r_{n+1}})} + N\sum_{n=1}^\infty 2^{-dn}R^{-d-\alpha}\|u\|_{L_1(B_{r_{n+1}})}
$$

$$+ N\Big(\sum_{n=1}^{\infty} 2^{-dn}\Big)\frac{1}{j(R)R^{d+\alpha}}(\|u\|_{L_1(\mathbb{R}^d,w_R)} + N\mathrm{osc}_{B_R} f).$$

Since $[u]_{C^\alpha(B_{r_n})} \leq [u]_{C^\alpha(B_R)} < \infty$ and by **H1**

$$\|u\|_{L_1(B_{r_{n+1}})} \leq \|u\|_{L_1(B_R)} = \frac{j(R)}{j(R)}\|u\|_{L_1(B_R)} \leq \frac{N}{j(R)}\|u\|_{L_1(\mathbb{R}^d,w_R)},$$

(i) is proved.

(ii) is proved similarly by following the proof of (i) with Theorem 4.2(ii). □

5. Some sharp function and maximal function estimates

For $g \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$, the maximal function and sharp function are defined as follows:

$$\mathcal{M}g(x) := \sup_{r>0} ⨍_{B_r(x)} |g(y)|\, dy := \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |g(y)|\, dy,$$

and

$$g^{\#}(x) := \sup_{r>0} ⨍_{B_r(x)} |g(y) - (g)_{B_r(x)}|\, dy := \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |g(y) - (g)_{B_r(x)}|\, dy,$$

where $(g)_{B_r(x)} = \frac{1}{|B_r(x)|}\int_{B_r(x)} g(y)\, dy$ the average of g on $B_r(x)$.

Lemma 5.1. *Suppose that* **H1** *and* **H2** *hold. Let $\lambda \geq 0$, $R > 0$, $f \in C_0^\infty$, and $f = 0$ in B_{2R}. Assume that $u, \tilde{u} \in H_2^{\mathcal{A}} \cap C_b^\infty$ satisfy*

$$Lu - \lambda u = f, \qquad \tilde{L}\tilde{u} - \lambda\tilde{u} = f. \tag{5.65}$$

(i) Then for all $\alpha \in (0, \min\{1, \alpha_0\})$,

$$[u]_{C^\alpha(B_{R/2})} \leq NR^{-\alpha}\sum_{k=1}^{\infty} 2^{-\alpha_0 k}(|u|)_{B_{2^k R}}, \tag{5.66}$$

$$[\mathcal{A}u]_{C^\alpha(B_{R/2})} \leq NR^{-\alpha}\left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k}(|\mathcal{A}u|)_{B_{2^k R}} + \mathcal{M}f(0)\right), \tag{5.67}$$

where N depends only on d, ν, Λ, κ_1, κ_2, α_0, and α.

(ii) If one of **H3***(ii)–(iv) is additionally assumed, then (5.66) and (5.67) hold for $\tilde{u}$.*

Proof. By Corollary 4.3 and the assumption that $f = 0$ in B_{2R},

$$[u]_{C^\alpha(B_{R/2})} \leq N\frac{1}{j(R)R^{d+\alpha}}\|u\|_{L_1(\mathbb{R}^d,w_R)}. \tag{5.68}$$

Set

$$B_{(0)} = B_R, \quad B_{(k)} = B_{2^k R} \setminus B_{2^{k-1} R}, \quad k \geq 1.$$

Observe that

$$
\begin{aligned}
\|u\|_{L_1(\mathbb{R}^d,w_R)} &= \int_{\mathbb{R}^d} |u(y)| \frac{1}{1/j(R)+1/j(|y|/2)}\, dy \\
&= \sum_{k=0}^{\infty} \int_{B_{(k)}} |u(y)| \frac{1}{1/j(R)+1/j(|y|/2)} dy \\
&\le 2j(R) \int_{B_{2R}} |u(y)|\, dy + N \sum_{k=2}^{\infty} j(2^{k-2}R) \int_{B_{2^k R}} |u(y)|\, dy \\
&\le N \left(j(R)R^d (|u|)_{B_{2R}} + \sum_{k=2}^{\infty} 2^{-(k-2)(d+\alpha_0)} j(R) \int_{B_{2^k R}} |u(y)|\, dy \right) \\
&\le N \left(j(R)R^d (|u|)_{B_{2R}} + \sum_{k=2}^{\infty} 2^{-(k-2)(d+\alpha_0)} 2^{kd} j(R) R^d (|u|)_{B_{2^k R}} \right) \\
&\le N j(R) R^d \left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k} (|u|)_{B_{2^k R}} \right),
\end{aligned}
$$

where the first and second inequalities come from **H1**. Therefore we get (5.66).

To prove (5.67), we apply the operator $\mathcal{A}$ to both sides of $Lu - \lambda u = f$ and obtain

$$
(L-\lambda)(\mathcal{A}u) = \mathcal{A}f.
$$

By applying Corollary 4.3 again,

$$
[\mathcal{A}u]_{C^\alpha(B_{R/2})} \le N \frac{1}{j(R)R^{d+\alpha}} \left(\|\mathcal{A}u\|_{L_1(\mathbb{R}^d,w_R)} + \sup_{B_R} |\mathcal{A}f| \right). \tag{5.69}
$$

The first term on the right hand side of (5.69) is bounded by

$$
NR^{-\alpha} \left(\sum_{k=0}^{\infty} 2^{-\alpha_0 k} (|\mathcal{A}u|)_{B_{2^k R}} \right).
$$

In order to estimate the second term, we recall the definition of $\mathcal{A}$. For $|x| < R$,

$$
\begin{aligned}
|\mathcal{A}f(x)| &= \left| \int_{\mathbb{R}^d} [f(x+y) - f(x)] J(y)\, dy \right| \\
&\le \sum_{k=1}^{\infty} \int_{B_{(k)}} |f(x+y)| j(|y|)\, dy \\
&\le N \sum_{k=1}^{\infty} j(2^{k-1}R) \int_{B_{(k)}} |f(x+y)|\, dy \\
&\le N \sum_{k=1}^{\infty} 2^{-(k-1)(d+\alpha_0)} j(R) \int_{B_{2^k R}} |f(x+y)|\, dy
\end{aligned}
$$

$$\le N \sum_{k=1}^{\infty} 2^{-(k-1)(d+\alpha_0)} j(R) \int_{B_{2^{k+1}R}} |f(y)|\, dy$$

$$\le N j(R) R^d \left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k} (|f|)_{B_{2^{k+1}R}} \right) \le N j(R) R^d \mathcal{M} f(0),$$

where the first inequality is due to the assumption $f(x) = 0$ if $|x| < 2R$ and both the second and the third inequality are owing to **H1**. Therefore (i) is proved. Also, (ii) is proved similarly with Corollary 4.3(ii). □

The above lemma easily yields the following mean oscillation estimate.

Corollary 5.2. *Suppose that* **H1** *and* **H2** *hold. Let $\lambda \ge 0$, $r > 0$, and $\kappa \ge 2$. Assume $f \in C_0^\infty$, $f = 0$ in $B_{2\kappa r}$, and $u, \tilde{u} \in H_2^{\mathcal{A}} \cap C_b^\infty$ satisfy*

$$Lu - \lambda u = f, \qquad \tilde{L}\tilde{u} - \lambda \tilde{u} = f.$$

(i) Then for all $\alpha \in (0, \min\{1, \alpha_0\})$,

$$(|u - (u)_{B_r}|)_{B_r} \le N \kappa^{-\alpha} \sum_{k=1}^{\infty} 2^{-\alpha_0 k} |u|_{B_{2^k \kappa r}}, \tag{5.70}$$

$$(|\mathcal{A}u - \mathcal{A}u)_{B_r}|)_{B_r} \le N \kappa^{-\alpha} \left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k} |\mathcal{A}u|_{B_{2^k \kappa r}} + \mathcal{M} f(0) \right), \tag{5.71}$$

where N depends only on d, ν, Λ, κ_1, κ_2, α_0, and α.

(ii) If one of **H3***(ii)–(iv) is additionally assumed, then (5.70) and (5.71) hold for $\tilde{u}$.*

Proof. It is enough to use the following inequality

$$(|u - (u)_{B_r}|)_{B_r} \le 2^\alpha r^\alpha [u]_{C^\alpha(r)} \le 2^\alpha r^\alpha [u]_{C^\alpha(\kappa r/2)}$$

and apply Lemma 5.1 with $R = \kappa r$. □

Next we show that the mean oscillation of u is controlled by the maximal functions of u and $Lu - \lambda u$.

Lemma 5.3. *Suppose that* **H1** *and* **H2** *hold. Let $\lambda > 0$, $\kappa \ge 2$, $r > 0$, and $f \in C_0^\infty$. Assume $u, \tilde{u} \in H_2^{\mathcal{A}} \cap C_b^\infty$ satisfy*

$$Lu - \lambda u = f, \qquad \tilde{L}u - \lambda u = f. \tag{5.72}$$

(i) Then for all $\alpha \in (0, \min\{1, \alpha_0\})$,

$$\begin{aligned} &\lambda(|u - (u)_{B_r}|)_{B_r} + (|\mathcal{A}u - (\mathcal{A}u)_{B_r}|)_{B_r} \\ &\le N\kappa^{-\alpha} \left(\lambda \mathcal{M}u(0) + \mathcal{M}(\mathcal{A}u)(0)\right) + N\kappa^{d/2} (\mathcal{M}(f^2)(0))^{1/2}, \end{aligned} \tag{5.73}$$

where N depends only on d, ν, Λ, κ_1, κ_2, α_0 and α.

(ii) If one of **H3***(ii)–(iv) is additionally assumed, then (5.73) holds for $\tilde{u}$.*

Proof. Due to the similarity of the proof, we only prove the assertion (i).

Take a cut-off function $\eta \in C_0^\infty(B_{4\kappa r})$ satisfying $\eta = 1$ in $B_{2\kappa r}$. By Theorem 2.8, there exists a unique solution u in $H_2^{\mathcal{A}}$ satisfying

$$Lw - \lambda w = \eta f \tag{5.74}$$

and

$$\lambda \|w\|_{L_2} + \|\mathcal{A}w\|_{L_2} \le N\|\eta f\|_{L_2}. \tag{5.75}$$

From (5.75), Jensen's inequality, and the fact ηf has its support within $B_{4\kappa r}$, for any $R > 0$,

$$\begin{aligned} \lambda(|w|)_{B_R} + (|\mathcal{A}w|)_{B_R} &\le NR^{-d/2}\left(\lambda\|w\|_{L_2} + \|\mathcal{A}w\|_{L_2}\right) \\ &\le NR^{-d/2}\|\eta f\|_{L_2} \\ &\le NR^{-d/2}(\kappa r)^{d/2}(\mathcal{M}(f^2)(0))^{1/2}. \end{aligned} \tag{5.76}$$

Furthermore, for any $\gamma > 0$, taking $(1-\Delta)^\gamma$ to both sides of (5.74) and using the fact $(1-\Delta)^\gamma Lw = L(1-\Delta)^\gamma w$, we can easily check that $w \in C_b^\infty$ by Sobolev's inequality. By setting $v := u - w$, from (5.74) and (5.72)

$$Lv - \lambda v = (1-\eta)f, \quad v \in C_b^\infty \cap H_2^{\mathcal{A}}.$$

By applying Corollary 5.2 to v,

$$\begin{aligned} &(\lambda|v - (v)_{B_r}|)_{B_r} + (|\mathcal{A}v - (\mathcal{A}v)_{B_r}|)_{B_r} \\ &\le N\kappa^{-\alpha}\left(\sum_{k=1}^\infty 2^{-\alpha_0 k}[\lambda(|v|)_{B_{2^k\kappa r}} + (|\mathcal{A}v|)_{B_{2^k\kappa r}}] + \mathcal{M}f(0)\right) \\ &\le N\kappa^{-\alpha}\left(\sum_{k=1}^\infty 2^{-\alpha_0 k}[\lambda(|u|)_{B_{2^k\kappa r}} + (|\mathcal{A}u|)_{B_{2^k\kappa r}}]\right) \\ &\quad + N\kappa^{-\alpha}\left(\sum_{k=0}^\infty 2^{-\alpha_0 k}[\lambda(|w|)_{B_{2^k\kappa r}} + (|\mathcal{A}w|)_{B_{2^k\kappa r}}] + \mathcal{M}f(0)\right) \\ &\le N\kappa^{-\alpha}\left(\sum_{k=1}^\infty 2^{-\alpha_0 k}[\lambda(|u|)_{B_{2^k\kappa r}} + (|\mathcal{A}u|)_{B_{2^k\kappa r}}]\right) \\ &\quad + N\kappa^{-\alpha}\left(\sum_{k=1}^\infty 2^{-\alpha_0 k}[2^{-dk/2}(\mathcal{M}(f^2)(0))^{1/2}] + \mathcal{M}f(0)\right) \\ &\le N\kappa^{-\alpha}\left(\lambda\mathcal{M}u(0) + \mathcal{M}(\mathcal{A}u)(0) + (\mathcal{M}(f^2)(0))^{1/2}\right), \end{aligned} \tag{5.77}$$

where (5.76) is used for the third inequality with $R = 2^k\kappa r$, and for the last inequality we use $\mathcal{M}f(0) \le (\mathcal{M}(f^2)(0))^{1/2}$. By combining (5.76) and (5.77),

$$\begin{aligned} &\lambda(|u - (u)_{B_r}|)_{B_r} + (|\mathcal{A}u - (\mathcal{A}u)_{B_r}|)_{B_r} \\ &\le N\left(\lambda(|v - (v)_{B_r}|)_{B_r} + (|\mathcal{A}v - (\mathcal{A}v)_{B_r}|)_{B_r} + \lambda(|w|)_{B_r} + (|\mathcal{A}w|)_{B_r}\right) \\ &\le N\kappa^{-\alpha}\left(\lambda\mathcal{M}u(0) + \mathcal{M}(\mathcal{A}u)(0)\right) + N(\mathcal{M}(f^2)(0))^{1/2}. \end{aligned}$$

Therefore, the lemma is proved. $\square$

We make full use of Lemma 5.1 to get the mean oscillation of Lu.

Lemma 5.4. *Suppose that* **H1** *and* **H2** *hold. Let $\lambda > 0$, $\kappa \geq 2$, $r > 0$, and $f \in C_0^\infty$. Assume $u \in H_2^{\mathcal{A}} \cap C_b^\infty$ satisfy*

$$\mathcal{A}u - \lambda u = f.$$

Then for all $\alpha \in (0, \min\{1, \alpha_0\})$,

$$\begin{aligned}
&\lambda(|u - (u)_{B_r}|)_{B_r} + (|Lu - (Lu)_{B_r}|)_{B_r} \\
&\leq N\kappa^{-\alpha}\left(\lambda \mathcal{M}u(0) + \mathcal{M}(Lu)(0)\right) + N\kappa^{d/2}(\mathcal{M}(f^2)(0))^{1/2},
\end{aligned}$$

where N depends only on d, ν, Λ, κ_1, κ_2, α_0, and α.

Proof. Exchanging the roles of $\mathcal{A}$ and L in the proof of Lemma 5.1, we easily get

$$[Lu]_{C^\alpha(B_{R/2})} \leq NR^{-\alpha}\left(\sum_{k=0}^{\infty} 2^{-\alpha_0 k}(Lu)_{B_{2^k R}} + \mathcal{M}f(0)\right).$$

Therefore, the lemma is proved as we follow the proof of Lemma 5.3. □

6. Proof of Theorems 2.17 and 2.22

Proof of Theorem 2.17. The case $p = 2$ was already proved in Theorem 2.8. Due to Corollary 2.7 and Lemma 2.6, it is sufficient to prove

$$\|\mathcal{A}u\|_{L_p} + \lambda\|u\|_{L_p} \leq N\|Lu - \lambda u\|_{L_p}, \quad \forall\, u \in C_0^\infty, \tag{6.78}$$

where $N = N(d, \nu, \Lambda, \kappa_1, \kappa_2, \alpha_0)$.

First, assume $p > 2$. Put $f := Lu - \lambda u$. From Lemma 5.3, for all $\alpha \in (0, \min\{1, \alpha_0\})$

$$\begin{aligned}
&\lambda(|u - (u)_{B_r}|)_{B_r} + (|\mathcal{A}u - (\mathcal{A}u)_{B_r}|)_{B_r} \\
&\leq N\kappa^{-\alpha}\left(\lambda \mathcal{M}u(0) + \mathcal{M}(\mathcal{A}u)(0)\right) + N\kappa^{d/2}(\mathcal{M}(f^2)(0))^{1/2}.
\end{aligned}$$

By translation, it is easy to check that the above inequality holds for all $B_r(x)$ with $x \in \mathbb{R}^d$ and $r > 0$. By the arbitrariness of r,

$$\begin{aligned}
&\lambda u^{\#}(x) + (\mathcal{A}u)^{\#}(x) \\
&\leq N\kappa^{-\alpha}\left(\lambda \mathcal{M}u(x) + \mathcal{M}(\mathcal{A}u)(x)\right) + N\kappa^{d/2}(\mathcal{M}(f^2)(x))^{1/2}.
\end{aligned}$$

Therefore, by the Fefferman–Stein theorem and Hardy–Littlewood maximal theorem (see, for instance, chapter 1 of [17]), we get

$$\lambda\|u\|_{L_p} + \|\mathcal{A}u\|_{L_p} \leq N\kappa^{-\alpha}\left(\lambda\|u\|_{L_p} + \|\mathcal{A}u\|_{L_p}\right) + N\kappa^{d/2}\|f\|_{L_p}.$$

By choosing $\kappa > 2$ large enough so that $N\kappa^{-\alpha} < 1/2$,

$$\lambda\|u\|_{L_p} + \|\mathcal{A}u\|_{L_p} \leq N\|f\|_{L_p}.$$

We use the duality argument for $p \in (1,2)$. Put $q := p/(p-1)$. Then since $q \in (2,\infty)$, for any $g \in C_0^\infty$ there is a unique $v_g \in H_q^{\mathcal{A}}$ satisfying

$$L^* v_g - \lambda v_g = g \quad \text{in } \mathbb{R}^d.$$

Therefore, by applying (6.78) with $q \in (2,\infty)$, for any $u \in C_0^\infty$,

$$\begin{aligned}
\|\mathcal{A}u\|_{L_p} &\le \sup_{\|g\|_{L_q}=1,\ g\in C_0^\infty} \int_{\mathbb{R}^d} |g\mathcal{A}u|\, dx \\
&= \sup_{\|g\|_{L_q}=1,\ g\in C_0^\infty} \int_{\mathbb{R}^d} |(L^* v_g - \lambda v_g)\mathcal{A}u|\, dx \\
&= \sup_{\|g\|_{L_q}=1,\ g\in C_0^\infty} \int_{\mathbb{R}^d} |\mathcal{A}v_g(Lu - \lambda u)|\, dx \\
&\le \sup_{\|g\|_{L_q}=1,\ g\in C_0^\infty} \|\mathcal{A}v_g\|_{L_q}\|Lu-\lambda u\|_{L_p} \\
&\le \sup_{\|g\|_{L_q}=1,\ g\in C_0^\infty} N\|g\|_{L_q}\|Lu-\lambda u\|_{L_p} = N\|Lu-\lambda u\|_{L_p}.
\end{aligned}$$

Similarly,

$$\lambda\|u\|_{L_p} \le N\|Lu-\lambda u\|_{L_p}.$$

Finally, we prove the continuity of the operator L by showing

$$\|Lu\|_{L_p} \le N\|\mathcal{A}u\|_p, \quad \forall\, u \in C_0^\infty. \tag{6.79}$$

Recall that we proved (6.78) based on Lemma 5.3. Similarly, using Lemma 5.4, one can prove

$$\|Lu\|_{L_p} \le N\|\mathcal{A}u - \lambda u\|_{L_p} \quad \forall u \in C_0^\infty, \quad \forall \lambda > 0.$$

Since N is independent of λ, this leads to (6.79). The theorem is proved. □

Proof of Theorem 2.22. The proof is identical to that of Theorem 2.17 if one of **H3**(ii)–(iv) holds. So it only remains to prove

$$\|\mathcal{A}u\|_{L_p} + \lambda\|u\|_{L_p} \le N\|\tilde{L}u - \lambda u\|_{L_p}, \quad \forall\, u \in C_0^\infty$$

under the condition **H3**(i). Define

$$b^i = -\int_{B_1} y^i a(y)J(y)dy \quad \text{if } \sigma \in (0,1), \qquad b^i = \int_{\mathbb{R}^d\setminus B_1} y^i a(y)J(y)dy \quad \text{if } \sigma \in (1,2).$$

Then under **H1** and **H2**, $|b| < \infty$ and for any $u \in C_0^\infty$, we have

$$\tilde{L}u = Lu + b\cdot\nabla u,$$

and therefore

$$\|u\|_{L_p} + \|\mathcal{A}u\|_{L_p} \le N\|Lu-\lambda u\|_{L_p} \le N\big(\|\tilde{L}u - \lambda u\|_{L_p} + \|\nabla u\|_{L_p}\big).$$

Take $\varepsilon = 1/(2N)$ in **H3**(i) and apply Lemma 2.1. Then, the theorem is proved. □

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