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Uniform factorization for p-compact sets of p-compact linear operators

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ABSTRACT

Obtaining a factorization of p-compact linear operators via universal Banach spaces, and using the lifting property of quotient maps for p-compact sets we prove a factorization result for relatively r-compact subsets of p-compact operators, where $r \geq 2$, $1 \leq p \leq r < \infty$. To apply our results to homogeneous polynomials, in particular, we show that relatively p-compact subsets of a Banach space of p-compact operators are collectively p-compact.

1. Introduction

W.B. Johnson [20] proved that an operator in the closure of finite rank operators can be factorized through a universal Banach space. Following this, T. Figiel [15] proved that compact operators can be factorized through a closed subspace of Johnson's universal Banach space. D.J. Randtke [29], T. Terzioğlu [34], and J. Dazord [9] factorized compact operators defined on some certain Banach spaces, such as $\mathcal{L}_1$, $\mathcal{L}_\infty$. Then W.H. Graves and W.M. Ruess [18], extended these works to simultaneous factorization of operators belonging to compact subsets of compact operators. But the (uniform) factorization of compact subsets of compact operators on arbitrary Banach spaces was studied by R. Aron, M. Lindström, W.M. Ruess, R. Ryan in [3]. Showing that the universal Banach space established by W.B. Johnson [20] and T. Figiel [15] also serves as a uniform factorization space for factorization of operators belonging to the space of compact weak*–weak

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continuous operators, they obtain a factorization of relative compact subsets of compact operators defined on an arbitrary Banach space.

As a stronger form of compactness D.P. Sinha and A.K. Karn [31] introduced p-compactness notion, which was motivated by the well-known characterization of compact sets due to A. Grothendieck [19]. Then plenty of papers appeared related to the p-compactness notion in different directions. We mention [4,6,8,11–13,22,26,28,31,32]. As to the factorization of p-compact operators, some factorization results are obtained by some researchers. We mention the works done by D.P. Sinha and A.K. Karn [31], Y.S. Choi and J.M. Kim [8], D. Galicier, S. Lassalle and P. Turco [17]. But the uniform factorization for (p-compact) subsets of p-compact operators, as to our knowledge, has not been considered so far. The purpose of this paper is to study simultaneous factorization of operators belonging to a p-compact subset of p-compact operators, basing on the paper [3] by R. Aron, M. Lindström, W.M. Ruess, R. Ryan. In this paper we firstly consider factorization of p-compact operators via universal Banach spaces, then as a main goal we study factorization of relatively r-compact subsets of the Banach space of all p-compact operators.

After giving preliminary material, as a slight improvement of [17, Proposition 2.9] we get a factorization of p-compact operators through a universal Banach space in Section 2. In section 3 we study uniform factorization of relatively r-compact subsets of p-compact operators. Our approach is based on the use of characterization of relatively p-compact sets in the projective tensor product of Banach spaces. For this, we strengthen a result given in [2], and then making a careful modification with quantitative strengthening of a method given in [3] we show that every p-compact operator in certain relatively r-compact subsets of the Banach space of p-compact operators with $r \geq 2$ and $1 \leq p \leq r < \infty$ can be factorized simultaneously through a universal Banach space. It should be pointed out that we do not use any selection principal in our proof, rather we use the lifting property of quotient maps for p-compact sets. Finally, we prove partial p-compact versions of a result of E. Toma [35] for homogeneous polynomials. For this aim, in particular we show that for any $p \geq 1$ every relatively p-compact subset of a Banach space of p-compact operators is collectively p-compact.

2. Notation and preliminaries

The letters X and Y will always represent complex Banach spaces. The symbol B_X represents the closed unit ball of X, S_X represents the unit sphere of X. For any topology τ on X, $\overline{M}^{\tau}$ will denote the τ-closure of a set M in X. The space of all bounded linear operators from X to Y will be denoted by $L(X, Y)$. The subspace of all compact (respectively, finite rank) operators of $L(X, Y)$ is denoted by $K(X, Y)$ (respectively, $F(X, Y)$). If X is a Banach space, and $1 \leq p \leq \infty$ with the conjugate index p^* given by $\frac{1}{p} + \frac{1}{p^*} = 1$ (where $p^* = 1$ if $p = \infty$), we let $\ell_p(X)$ $(1 \leq p < \infty)$ (resp., $\ell_\infty(X)$) denote the set of all sequences $(x_n)_{n=1}^\infty$ in X such that $\sum_{n=1}^\infty \|x_n\|^p < \infty$ (resp., $(x_n)_{n=1}^\infty$ is bounded), and we let $c_0(X)$ denote the set of all sequences $(x_n)_{n=1}^\infty$ in X such that $x_n \longrightarrow 0$ in X. A set $K \subset X$ is said to be relatively p-compact if there is a sequence $(x_n)_{n=1}^\infty$ in $\ell_p(X)$ ($(x_n)_{n=1}^\infty$ in $c_0(X) \subset \ell_\infty(X)$ if $p = \infty$) such that $K \subset \{\sum_{n=1}^\infty a_n x_n : (a_n)_{n=1}^\infty \in B_{\ell_{p^*}}\}$ (see [31]). A relatively p-compact and closed set will be called p-compact. We denote this last set by $p\text{-}co\{(x_n)_{n=1}^\infty\}$ and we will call it a fundamental p-compact set since these sets form a fundamental system of p-compact sets of X. From Grothendieck's characterization of compact sets (see [19] or [23, Proposition 1.e.2]), a subset K of a Banach space X is relatively compact if and only if there is a sequence $(x_n)_{n=1}^\infty$ in $c_0(X)$ such that $K \subset \{\sum_{n=1}^\infty a_n x_n : (a_n)_{n=1}^\infty \in B_{\ell_1}\}$. Thus, by the above definition one can consider compact sets as ∞-compact. Also, note that p-compact sets are q-compact if $1 \leq p < q \leq \infty$. For $1 \leq p \leq \infty$ an operator $T \in L(X, Y)$ is said to be p-compact if $T(B_X)$ is a relatively p-compact set in Y (see [31]). The subspace of all p-compact operators of $L(X, Y)$ will be denoted by $K_p(X, Y)$. If T belongs to $K_p(X, Y)$, we define

$$k_p(T) = \inf\{\|(y_n)_{n=1}^\infty\|_p : (y_n)_{n=1}^\infty \in \ell_p(Y) \text{ and } T(B_X) \subset p\text{-}co\{(y_n)_{n=1}^\infty\}\}$$

where k_∞ coincides with the supremum norm (see [13]). It is easy to see that k_p is a norm on $K_p(X, Y)$, and that (K_p, k_p) is a Banach ideal (see [31,12]).

We recall that a mapping $P : X \to Y$ is an n-homogeneous polynomial if there is an n-linear mapping $A : X \times \ldots \times X \to Y$ such that $P(x) = A(x, \ldots, x)$ for all $x \in X$. Let $P(^nX, Y)$ denote the space of all continuous n-homogeneous polynomials from X to Y, endowed with the usual sup norm. Given a polynomial $P \in P(^nX, Y)$, there is a unique symmetric continuous n-linear mapping $\overset{\vee}{P} \in L(^nX, Y)$ such that $P(x) = \overset{\vee}{P}(\underbrace{x, \ldots, x}_{n\ times})$. It is well known that the correspondence $\overset{\vee}{P} \longleftrightarrow P$ is a topological isomorphism between $L^s(^nX, Y)$, the space of all symmetric continuous n-linear mappings from X to Y, and $P(^nX, Y)$ (see, for example, [24, Theorem 2.2]). The space of n-homogeneous polynomials that are weakly uniformly continuous on bounded subsets of X is denoted $P_{wu}(^nX, Y)$ and the corresponding space of symmetric n-linear mappings is denoted by $L^s_{wu}(^nX, Y)$. When $Y = \mathbb{C}$, instead of $P_{wu}(^nX, Y)$, $L^s(^nX, Y)$ and $L^s_{wu}(^nX, Y)$ we will shortly write $P_{wu}(^nX)$, $L^s(^nX)$ and $L^s_{wu}(^nX)$, respectively. For each n-homogeneous polynomial P there is a linear operator $T_P : X \to L^s(^{n-1}X)$, defined by $T_P(x_1)(x_2, \ldots, x_n) = A(x_1, x_2, \ldots, x_n)$. It is known that P belongs to $P_{wu}(^nX)$ if and only if the operator T_P is compact (see [5]).

Following R.M. Aron, M. Maestre and P. Rueda [4] we say that an n-homogeneous polynomial is p-compact if for each $x \in X$ there is a neighborhood V_x of x such that $P(V_x)$ is relatively p-compact in Y. We denote by $P_{K_p}(^nX, Y)$ the space of p-compact n-homogeneous polynomials. By [4, Proposition 1] an n-homogeneous polynomial P is p-compact if and only if $P(B_X)$ is relatively p-compact in Y. On $P_{K_p}(^nX, Y)$ we define

$$k_p(P) = \inf\{\|(x_n)_{n=1}^\infty\|_p : (x_n)_{n=1}^\infty \in \ell_p(Y),\ P(B_X) \subset p\text{-}co\{(x_n)_{n=1}^\infty\}\}$$

which is a norm satisfying that $\|P\| \le k_p(P)$ for any p-compact homogeneous polynomial P. Also, $(P_{K_p}(^nX, Y), k_p)$ is a Banach space (see [22]).

$X \otimes_\pi Y$ denotes the tensor product of X and Y endowed with the projective norm π, which is defined as $\pi(u) = \inf\{\sum_{n=1}^{n} \|x_n\|\,\|y_n\| :\ n \in \mathbb{N},\ u = \sum_{n=1}^{n} x_n \otimes y_n\}$ for $x \in X \otimes_\pi Y$ (see [30]). $\hat{\otimes}^{n,s}_{\pi_s} X$ will denote the completed n-fold symmetric tensor product of X endowed with the projective s-tensor norm π_s, which is defined as (see [16, p. 164]) $\pi_s(z) = \inf\{\sum_{j=1}^{\infty} |\lambda_j| \parallel x_j \parallel^n : z = \sum_{j=1}^{\infty} \lambda_j \otimes^n x_j\}$ for $z \in \hat{\otimes}^{n,s}_{\pi_s} X$, where $\otimes^n x := \underbrace{x \otimes \ldots \otimes x}_{n\text{-}times}$. We refer to [10,16,30] for tensor products of Banach spaces.

Finally, throughout the paper $\ell_{p^*} = c_0$ if $p = 1$ and the ℓ_p-sum (of Banach spaces), as usual, will stand for the c_0-sum if $p = \infty$.

D. Galicier, S. Lassalle and P. Turco in [17, Proposition 2.9] showed that a linear operator is p-compact if and only if it is quotiented in ℓ_{p^*}. To be more precise, their proof can be sketched as follows: Given $T \in K_p(X, Y)$ there is a $z = (z_n)_{n=1}^\infty \in \ell_p(Y)$ such that $T(B_X) \subset \{\sum_{n=1}^\infty \alpha_n z_n : (\alpha_n)_{n=1}^\infty \in L\}$, where L is a compact set in $B_{\ell_{p^*}}$. Then, define two linear mappings $\theta_z : \ell_{p^*} \to Y$ by $\theta_z(\alpha) = \sum_{n=1}^\infty \alpha_n z_n$, $\alpha = (\alpha_n)_{n=1}^\infty \in \ell_{p^*}$, and $\hat{\theta}_z : \ell_{p^*}/ker\theta_z \to Y$ by $\hat{\theta}_z([\alpha]) = \theta_z(\alpha)$. And define a linear operator $R : X \to \ell_{p^*}/ker\theta_z$ by $R(x) = [(\alpha_n)_{n=1}^\infty]$, where $(\alpha_n)_{n=1}^\infty \in L$ is a sequence satisfying that $T(x) = \sum_{n=1}^\infty \alpha_n z_n$. Then one can easily see that $T = \hat{\theta}_z o R$. Here, we notice that $\hat{\theta}_z$ is p-compact and R is compact. Now, with these notations and facts in mind we get the following factorization of p-compact operators through a universal Banach space.

Theorem 2.1. *Let $1 \le p < \infty$, let X and Y be Banach spaces. Then there is a universal Banach space $Z^{(p)}$ such that every operator $T \in K_p(X, Y)$ can be factored as $T = B \circ A$, where $A \in K(X, Z^{(p)})$ and $B \in K_p(Z^{(p)}, Y)$. In particular, for $1 \le q \le \infty$ $Z^{(p)}$ can be chosen as $Z^{(p)} = (\sum_{W^{(p)}} W^{(p)})_q$, for a fixed q, where $W^{(p)}$ runs through the quotient spaces of ℓ_{p^*} which are Banach spaces.*

Proof. Given $T \in K_p(X,Y)$, by [17, Proposition 2.9] there exist $z = (z_n)_{n=1}^\infty \in \ell_p(Y)$, $R \in K(X, \ell_{p^*}/ker\theta_z)$ and $\tilde{\theta}_z \in K_p(\ell_{p^*}/ker\theta_z, Y)$ such that $T = \tilde{\theta}_z \circ R$. Let $I_{\ell_{p^*}/ker\theta_z} : \ell_{p^*}/ker\theta_z \to Z^{(p)}$ denote the natural norm one embedding and let $P_{\ell_{p^*}/ker\theta_z} : Z^{(p)} \to \ell_{p^*}/ker\theta_z$ denote the natural norm one projection. If we define the mappings $A := I_{\ell_{p^*}/ker\theta_z} \circ R$ and $B := \tilde{\theta}_z \circ P_{\ell_{p^*}/ker\theta_z}$, then $A \in K(X, Z^{(p)})$, $B \in K_p(Z^{(p)}, Y)$ and $T = B \circ A$. $\square$

On the other hand, we know by results of T. Figiel [15] and W.B. Johnson [20], combined with a result of S. Banach and S. Mazur [7] (see also [21, p. 280]), that compact operators between Banach spaces can be factored compactly through a quotient space of ℓ_1. We note that by a slight modification of the proof of [17, Proposition 2.9] we recover this result easily as follows, which we include here for the sake of completeness.

Proposition 2.2. *(See [15,20,17].) Let X and Y be Banach spaces and let $T \in K(X,Y)$. Then there exist $(z_n)_{n=1}^\infty \in c_0(Y)$, $R \in K(X, \ell_1/ker\theta_z)$ and $\tilde{\theta}_z \in K(\ell_1/ker\theta_z, Y)$ such that $T = \tilde{\theta}_z \circ R$.*

Proof. Let $(z_n)_{n=1}^\infty \in c_0(Y)$ be such that $T(B_X) \subset \{\sum_{n=1}^\infty \alpha_n z_n : (\alpha_n)_{n=1}^\infty \in B_{\ell_1}\}$. Choosing a sequence $(\lambda_n)_{n=1}^\infty$ with, $\lambda_n \geq 1$ and $\lambda_n \to \infty$ such that $(\lambda_n z_n)_{n=1}^\infty \in c_0(Y)$, and defining $(y_n)_{n=1}^\infty := (\lambda_n z_n)_{n=1}^\infty$, we get $T(B_X) \subset \{\sum_{n=1}^\infty \beta_n y_n : (\beta_n)_{n=1}^\infty \in L\}$, where L is a relatively compact set in B_{ℓ_1}. Now following the lines of the proof of [17, Proposition 2.9] one can get the required factorization. $\square$

As a consequence of Proposition 2.2, we obtain the following $p = \infty$ case of Theorem 2.1, which is nothing more than a factorization of compact operators through a universal Banach space, and is well known as we already mentioned above.

Theorem 2.3. *(See [15,20].) Let X and Y be Banach spaces. Then there is a universal Banach space $Z^{(\infty)}$ such that a compact operator $T \in K(X,Y)$ can be factored as $T = B \circ A$, where $A \in K(X, Z^{(\infty)})$ and $B \in K(Z^{(\infty)}, Y)$. In particular, for $1 \leq q \leq \infty$ $Z^{(\infty)}$ can be chosen as $Z^{(\infty)} = (\sum_W W)_q$ for a fixed q, where W runs through the quotient spaces of ℓ_1 which are Banach spaces.*

The above factorization results will be used in the next section. It should be pointed out that factorization results for operators are quite useful when working with approximation properties of Banach spaces, since in many cases they have a crucial role for determining whether or not certain (classes of) Banach spaces have certain type of approximation properties. For instance, we consider the approximation and the k_p-approximation properties. We recall that a Banach space X is said to have the approximation property (AP for short) if for every compact set K in X and every $\varepsilon > 0$, there exists a $T \in F(X;X)$ such that $\sup_{x \in K} \|Tx - x\| \leq \varepsilon$ (see [19]), and a Banach space X is said to have the k_p-approximation property (k_p-AP for short) if for every Banach space Y, $\overline{F(Y,X)}^{k_p} = K_p(Y,X)$ (see [13]). It is known that there are quotient spaces of ℓ_q for $1 < q < 2$, which does not have the AP (see [33]). But using the factorization for p-compact operators given in [17, Proposition 2.9] one gets the same result at once without any effort. Moreover, by using standard methods along with [8, Theorem 3.1], we get easily another known result asserting that if $1 \leq p < \infty$, $p \neq 2$, then there are quotient spaces of ℓ_1 which does not have the k_p-AP (see, e.g., [8] and [26]).

3. The results

R. Aron, M. Lindström, W. Ruess and R. Ryan in [3] obtained (uniform) factorizations of compact subsets of compact operators between Banach spaces. Here by a suitable and careful modification of their method we obtain, speaking roughly, (uniform) factorizations of r-compact subsets of p-compact operators between Banach spaces. In order to obtain this result (Theorem 3.5) we need some preparation. For this aim we will start with a sequence of lemmas which will be needed to achieve our goal.

Lemma 3.1. *(The lifting property of quotient maps for p-compact sets.) Let X be Banach space and let $1 \leq p < \infty$. Let N be a closed subspace of X and let $Q_N^X : X \to X/N$ be the quotient mapping. If K is a relatively p-compact subset of X/N, then there is a relatively p-compact subset C of X such that $K \subset Q_N^X(C)$.*

Proof. If K is a relatively p-compact subset of X/N, there exists $(X_n)_n^\infty \in \ell_p(X/N)$ such that $K \subset \{\sum_{n=1}^\infty \alpha_n X_n : (\alpha_n)_{n=1}^\infty \in B_{\ell_{p^*}}\}$. For each $n \in \mathbb{N}$, choose $x_n \in X_n$ such that $\|x_n\| < \|X_n\| + \frac{1}{n^2}$, so that $(x_n)_{n=1}^\infty \in \ell_p(X)$. Taking $C := p\text{-}co\{(x_n)_{n=1}^\infty)\}$ ends up the proof. $\square$

The following lemma says that any p-compact subset in the range of a surjective continuous linear operator is always contained in the image of a p-compact set by the operator. More precisely, we have

Lemma 3.2. *Let X and Y be Banach spaces, let $1 \leq p < \infty$ and let $T \in L(X, Y)$ be onto. If H is a relatively p-compact subset of Y, then there exists a relatively p-compact subset A of X such that $H \subset T(A)$.*

Proof. If $T \in L(X, Y)$ is onto, then it admits a factorization $T = T_0 \circ Q$, where $T_0 : X/N(T) \to Y$ is an isomorphism and $Q : X \to X/N(T)$ is the quotient map (see, e.g., [27, B.3.7 Proposition]). Letting $C := T_0^{-1}(H)$ and applying Lemma 3.1 we get the conclusion. $\square$

A result similar to the above lemma, replacing quotient maps by continuous surjective linear maps, can be stated for fundamental p-compact sets as follows.

Lemma 3.3. *Let X and Y be Banach spaces and let $T \in L(X, Y)$ be onto.*

- a) *If $p = 1$, $\alpha > 1$ and $H \subset p\text{-}co\{(a_k)_{k=1}^\infty\}$ with $(k^\alpha a_k)_{k=1}^\infty \in \ell_p(Y)$, then there exists a sequence $(\tau_k)_{k=1}^\infty \subset X$ with $(k^\alpha \tau_k)_{k=1}^\infty \in \ell_p(X)$ such that for $L := p\text{-}co\{(\tau_k)_{k=1}^\infty\}$ we have $H \subset T(L)$.*
- b) *If $1 < p < \infty$ and $H \subset p\text{-}co\{(a_k)_{k=1}^\infty\}$ with $(ka_k)_{k=1}^\infty \in \ell_p(Y)$, then there exists a sequence $(\tau_k)_{k=1}^\infty \subset X$ with $(k\tau_k)_{k=1}^\infty \in \ell_p(X)$ such that for $L := p\text{-}co\{(\tau_k)_{k=1}^\infty\}$ we have $H \subset T(L)$.*

Proof. We give a proof for the case $p = 1$ since the proof for the case $1 < p < \infty$ is similar. Since $T \in L(X, Y)$ is onto, as in the proof of Lemma 3.2, we can write $T = T_0 \circ Q$, where $T_0 : X/N(T) \to Y$ is an isomorphism and $Q : X \to X/N(T)$ is the quotient map. If $y \in H$, then there exists $(\alpha_k)_{k=1}^\infty \in B_{\ell_{p^*}}$ such that $y = \sum_{k=1}^\infty \alpha_k a_k$. For each $k \in \mathbb{N}$, there is a $\tau_k \in T_0^{-1}(a_k) := [\tau_k] \in X/N(T)$ such that $\|\tau_k\| < \|[\tau_k]\| + \frac{1}{k^{2\alpha}}$. Therefore, since $\sum_{k=1}^\infty \|k^\alpha \tau_k\| < \|T_0^{-1}\| \sum_{k=1}^\infty \|k^\alpha a_k\| + \sum_{k=1}^\infty \frac{1}{k^\alpha} < \infty$, letting $L := p\text{-}co\{(\tau_k)_{k=1}^\infty\}$, we get that $H \subset T(L)$. $\square$

By strengthening a result in [2] we obtain the following lemma, which relies on a result of A. Grothendieck characterizing tensor products [19]. This lemma will be the key result since the main result (Theorem 3.5) will be based on the tensor representation provided therein.

Lemma 3.4. *Let X and Y be Banach spaces.*

- a) *Let $2 \leq p < \infty$. If $L \subset p\text{-}co\{(\tau_k)_{k=1}^\infty\}$ with $(k\tau_k)_{k=1}^\infty \in \ell_p(X \hat{\otimes}_\pi Y)$, then there exist sequences $(r_k)_{k=1}^\infty \in c_0(X)$, $(s_k)_{k=1}^\infty \in \ell_p(Y)$ and a subset K of $B_{\ell_{p^*}}$ with $K \subset p^*\text{-}co\{(t_k)_{k=1}^\infty\}$, $(t_k)_{k=1}^\infty \in \ell_{p^*}(B_{\ell_{p^*}})$, such that $L \subset \{\sum_{k=1}^\infty \lambda_k r_k \otimes s_k : (\lambda_k)_{k=1}^\infty \in K\}$.*
- b) *Let $1 < p < \infty$. If $L \subset p\text{-}co\{(\tau_k)_{k=1}^\infty\}$ with $(\tau_k)_{k=1}^\infty \in \ell_p(X \hat{\otimes}_\pi Y)$, then there exist sequences $(r_k)_{k=1}^\infty \in c_0(X)$, $(s_k)_{k=1}^\infty \in \ell_p(Y)$ and a compact subset K of $B_{\ell_{p^*}}$, such that $L \subset \{\sum_{k=1}^\infty \lambda_k r_k \otimes s_k : (\lambda_k)_{k=1}^\infty \in K\}$.*

Proof. a) If $L \subset p\text{-}co\{(\tau_k)_{k=1}^{\infty}\}$ with $(k\tau_k)_{k=1}^{\infty} \in \ell_p(X \hat{\otimes}_\pi Y)$, then for any $\tau \in L$, there exists $(\lambda_k^\tau)_{k=1}^{\infty} \in B_{\ell_{p^*}}$ such that $\tau = \sum_{k=1}^{\infty} \lambda_k^\tau \tau_k$. Since $\tau_k \in X \hat{\otimes}_\pi Y$, it follows from [30, Proposition 2.8, and the formula given in p. 22] (see also [16, p. 165]) that $\tau_k = \sum_{i=1}^{\infty} \lambda_{k,i} r_{k,i} \otimes s_{k,i}$, where for every $i \in \mathbb{N}$, $r_{k,i} \in S_X$, $s_{k,i} \in S_Y$ and $\sum_{i=1}^{\infty} |\lambda_{k,i}| < \infty$ with $\mu_k := \sum_{i=1}^{\infty} |\lambda_{k,i}| < \pi(\tau_k) + \frac{1}{2^k k}$. Thus we get that

$$\tau = \sum_{k=1}^{\infty} \lambda_k^\tau \tau_k = \sum_{k=1}^{\infty} \lambda_k^\tau \sum_{i=1}^{\infty} \lambda_{k,i} r_{k,i} \otimes s_{k,i} = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \frac{1}{k} \lambda_k^\tau (\frac{\lambda_{k,i}}{\mu_k})^{1/p^*} r_{k,i} \otimes k(\frac{\lambda_{k,i}}{\mu_k})^{1/p} \mu_k s_{k,i}.$$

Since the series $\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \frac{1}{k}\lambda_k^\tau \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p^*} r_{k,i} \otimes k \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p} \mu_k s_{k,i}$ converges absolutely in $X \hat{\otimes}_\pi Y$, and since $\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} |\frac{1}{k}\lambda_k^\tau (\frac{\lambda_{k,i}}{\mu_k})^{1/p^*}|^{p^*} = \sum_{k=1}^{\infty} \frac{1}{k^{p^*}} |\lambda_k^\tau|^{p^*} \leq 1$ and

$$\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \| k(\frac{\lambda_{k,i}}{\mu_k})^{1/p} \mu_k s_{k,i} \|^p \leq 2^p (\sum_{k=1}^{\infty} (\pi(k\tau_k))^p + \frac{1}{2^{pk}}) < \infty,$$

by choosing a specific order one can write

$$\begin{aligned}(\gamma_l^\tau)_{l=1}^{\infty} &:= (\frac{1}{k}\lambda_k^\tau (\frac{\lambda_{k,i}}{\mu_k})^{1/p^*})_{(k,i)\in\mathbb{N}\times\mathbb{N}} \in B_{\ell_{p^*}},\\ (x_l)_{l=1}^{\infty} &:= (r_{k,i})_{(k,i)\in\mathbb{N}\times\mathbb{N}} \in \ell_\infty(X),\\ (y_l)_{l=1}^{\infty} &:= (k(\frac{\lambda_{k,i}}{\mu_k})^{1/p}\mu_k s_{k,i})_{(k,i)\in\mathbb{N}\times\mathbb{N}} \in \ell_p(Y),\end{aligned}$$

so that we obtain a representation $\tau = \sum_{l=1}^{\infty} \gamma_l^\tau x_l \otimes y_l$. Moreover, since $\sum_{l=1}^{\infty} \|y_l\|^p < \infty$, we may choose a positive increasing sequence $(c_l)_{l=1}^{\infty}$, diverging to infinity, such that $\sum_{l=1}^{\infty} \|y_l\|^p c_l < \infty$. Thus, writing $\tau = \sum_{l=1}^{\infty} \gamma_l^\tau x_l \frac{1}{c_l^{1/p}} \otimes c_l^{1/p} y_l$, and letting $r_l := x_l \frac{1}{c_l^{1/p}}$ and $s_l := c_l^{1/p} y_l$ for each l, we get that $\tau = \sum_{l=1}^{\infty} \gamma_l^\tau r_l \otimes s_l$, where $(r_l)_{l=1}^{\infty} \in c_0(X)$ and $(s_l)_{l=1}^{\infty} \in \ell_p(Y)$.

Now, we claim that the sequences $(\gamma_l^\tau)_{l=1}^{\infty}$, $(\tau \in L)$, range over a relatively p^*-compact subset K of $B_{\ell_{p^*}}$. Indeed, according to the order chosen above, we can write

$$(\gamma_l^\tau)_{l=1}^{\infty} = (\frac{1}{k}\lambda_k^\tau (\frac{\lambda_{k,i}}{\mu_k})^{1/p^*})_{(k,i)\in\mathbb{N}\times\mathbb{N}} = \sum_{k=1}^{\infty} \lambda_k^\tau \sum_{i=1}^{\infty} \frac{1}{k} (\frac{\lambda_{k,i}}{\mu_k})^{1/p^*} e_i^k$$

where the vectors e_i^k appeared in the double indexed set $(e_i^k)_{(k,i)\in\mathbb{N}\times\mathbb{N}}$ are the standard basis vectors e_l of ℓ_{p^*} ordered as above. Next, for each $k \in \mathbb{N}$, we define $t_k := \frac{1}{k}\sum_{i=1}^{\infty} \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p^*} e_i^k$. Hence, for every $k \in \mathbb{N}$ we have that $t_k \in B_{\ell_{p^*}}$, and, since $1 < p^* \leq 2$, we have $\sum_{k=1}^{\infty} \|t_k\|_{p^*}^{p^*} = \sum_{k=1}^{\infty} \frac{1}{k^{p^*}} \sum_{i=1}^{\infty} \left|\frac{\lambda_{k,i}}{\mu_k}\right| \|e_i^k\|^{p^*} < \infty$. Thus, $(t_k)_{k=1}^{\infty} \in \ell_{p^*}(B_{\ell_{p^*}}) \subset \ell_{p^*}(\ell_{p^*})$. On the other hand since $p^* \leq p$, we have $(\lambda_k^\tau)_{k=1}^{\infty} \in B_{\ell_p}$. Therefore, since $(\gamma_l^\tau)_{l=1}^{\infty} = \sum_{k=1}^{\infty} \lambda_k^\tau t_k$ with $(\lambda_k^\tau)_{k=1}^{\infty} \in B_{\ell_p}$, if we take K as the set $\{(\gamma_l^\tau)_{l=1}^{\infty} : \tau \in L\}$, then $K \subset p^*\text{-}co\{(t_k)_{k=1}^{\infty}\}$, and the proof of part a) is complete.

b) If $L \subset p\text{-}co\{(\tau_k)_{k=1}^{\infty}\}$ with $(\tau_k)_{k=1}^{\infty} \in \ell_p(X \hat{\otimes}_\pi Y)$, then by a similar argument as in (a) any $\tau \in L$ can be written as $\tau = \sum_{i=1}^{\infty} \lambda_i^\tau r_i \otimes t_i$ with $(\lambda_i^\tau)_{i=1}^{\infty} \in B_{\ell_{p^*}}$ where $(r_i)_{i=1}^{\infty} \in c_0(X)$ and $(t_i)_{i=1}^{\infty} \in \ell_p(Y)$. Since $(t_i)_{i=1}^{\infty} \in \ell_p(Y)$, we may choose $\beta = (\beta_i)_{i=1}^{\infty} \in B_{c_0}$ such that $\left(\frac{t_i}{\beta_i}\right)_{i=1}^{\infty} \in \ell_p(Y)$. Accordingly we write $\tau = \sum_{i=1}^{\infty} \lambda_i^\tau r_i \otimes t_i = \sum_{i=1}^{\infty} \beta_i \lambda_i^\tau r_i \otimes \frac{t_i}{\beta_i}$, where $(\lambda_i^\tau)_{i=1}^{\infty} \in B_{\ell_{p^*}}$. If for every $i \in \mathbb{N}$ we let $\theta_i^\tau := \beta_i \lambda_i^\tau$ and $s_i := \dfrac{t_i}{\beta_i}$, then,

$$\tau = \sum_{i=1}^{\infty} \theta_i^\tau r_i \otimes s_i, \quad (\theta_i^\tau)_{i=1}^{\infty} \in B_{\ell_{p^*}}$$

with $(r_i)_{i=1}^{\infty} \in c_0(X)$, $(s_i)_{i=1}^{\infty} \in \ell_p(Y)$. To see that the sequences $(\theta_i^T)_{i=1}^{\infty}$ range over a compact subset of $B_{\ell_{p^*}}$, note that the set $K := \{(\beta_i\gamma i)_{i=1}^{\infty} : (\gamma i)_{i=1}^{\infty} \in B_{\ell_{p^*}}\}$ is a compact subset of $B_{\ell_{p^*}}$, so the proof of the claim is complete. □

As a final step towards our main result, let $1 \le p < \infty$, and let $Z^{(p)}$ be the universal Banach space given in Theorem 2.1. Given Banach spaces X and Y, according to Theorem 2.1, the continuous bilinear map

$$\tau : K(X, Z^{(p)}) \times (K_p(Z^{(p)}, Y), k_p) \to (K_p(X, Y), k_p), \quad \tau(u, v) = v \circ u,$$

is onto. The linearization of τ, $\hat{\tau} : K(X, Z^{(p)}) \hat{\otimes}_\pi (K_p(Z^{(p)}, Y), k_p) \to (K_p(X, Y), k_p)$, defined by $\hat{\tau}(u \otimes v) = \tau(u, v) = v \circ u$, is a continuous linear map which is onto.

Now we are ready to give the main result of the paper. For the proof we will carefully modify a method given in [3], namely, the first method in the proof of [3, Theorem 1]. It should be emphasized that in our proof we do not use any selection principal as it is done in the first method given in the proof of [3, Theorem 1], instead we use the lifting of p-compact sets (given by Lemma 3.1) which is already pointed out at the end of the same paper for the compact case. Therefore the proof for the p-compact case given below seems to be more direct proof than the ones given in [3] for the compact case.

Theorem 3.5. *Let X and Y be Banach spaces, let $r \ge 2$ and let $1 \le p \le r < \infty$. For every (balanced and convex) relatively r-compact subset H of $(K_p(X, Y), k_p)$ such that $H \subset r\text{-}co\{(a_k)_{k=1}^{\infty}\}$ with $(ka_k)_{k=1}^{\infty} \in \ell_r(K_p(X, Y), k_p)$, there exist an operator $u \in K(X, Z_{FJ})$, a (resp. balanced and convex) relatively r^*-compact subset $\{B_T : T \in H\}$ of $K(Z_{FJ}, Z^{(r)})$ and an operator $v \in K_r(Z^{(r)}, Y)$ such that $T = v \circ B_T \circ u$ for all $T \in H$, where Z_{FJ} denotes a universal factorization space of Figiel [15] and Johnson [20], and $Z^{(r)}$ is the universal Banach space given in Theorem 2.1.*

Proof. Since $\hat{\tau}$ is a continuous linear onto map and $H \subset r\text{-}co\{(a_k)_{k=1}^{\infty}\}$ with $(ka_k)_{k=1}^{\infty} \in \ell_r(K_p(X, Y), k_p)$, by Lemma 3.3 b), there exists $(\tau_k)_{k=1}^{\infty}$ in $K(X, Z^{(p)}) \hat{\otimes}_\pi (K_p(Z^{(p)}, Y), k_p)$ with $(k\tau_k)_{k=1}^{\infty} \in \ell_r(K(X, Z^{(p)}) \hat{\otimes}_\pi (K_p(Z^{(p)}, Y), k_p))$ such that for $L := r\text{-}co\{(\tau_k)_{k=1}^{\infty}\}$ we have $H \subset \hat{\tau}(L)$. Thus, for every $T \in H$ there exists $\tau_T \in L$ such that $T = \hat{\tau}(\tau_T)$. By Lemma 3.4 a) we have a representation $\tau_T = \sum_{i=1}^{\infty} \lambda_i^{\tau_T} r_i \otimes s_i$ with $(\lambda_i^{\tau_T})_{i=1}^{\infty} \in K$, where $(r_i)_{i=1}^{\infty} \in c_0(K(X, Z^{(p)}))$, $(s_i)_{i=1}^{\infty} \in \ell_r(K_p(Z^{(p)}), Y), k_p)$ and $K \subset \ell_{r^*}$ is a relatively r^*-compact subset. Now, define $r : X \to c_0(Z^{(p)})$ by $r(x) := (r_i(x))_{i=1}^{\infty}$. Then $r \in K(X, c_0(Z^{(p)}))$ (see [3, Theorem 1]). Next, for each $T \in H$ define $A_T : c_0(Z^{(p)}) \to \ell_{r^*}(Z^{(p)})$ by $A_T(z) = (\lambda_i^{\tau_T} z_i)_{i=1}^{\infty}$, $z = (z_i)_{i=1}^{\infty} \in c_0(Z^{(p)})$. Since

$$\sum_{i=1}^{\infty} \|\lambda_i^{\tau_T} z_i\|^{r^*} \le \sum_{i=1}^{\infty} |\lambda_i^{\tau_T}|^{r^*} \|z_i\|^{r^*} \le \left(\sup_{i\in\mathbb{N}} \|z_i\|\right)^{r^*} \sum_{i=1}^{\infty} |\lambda_i^{\tau_T}|^{r^*} < \infty,$$

A_T is well defined and that $A_T \in L(c_0(Z^{(p)}), \ell_{r^*}(Z^{(p)}))$. Now we consider the continuous linear map $A : \ell_{r^*} \to L(c_0(Z^{(p)}), \ell_{r^*}(Z^{(p)})$ defined by $A(\lambda)z := (\lambda_i z_i)_{i=1}^{\infty}$, $\lambda = (\lambda_i)_{i=1}^{\infty}$, $z = (z_i)_{i=1}^{\infty}$. Since $\{A_T : T \in H\} \subset A(K)$ and K is a relatively r^*-compact subset in ℓ_{r^*}, it follows that the subset $\{A_T : T \in H\}$ of $L(c_0(Z^{(p)}), \ell_{r^*}(Z^{(p)}))$ is relatively r^*-compact. Finally we define $s : \ell_{r^*}(Z^{(p)}) \to Y$ by $s(w) := \sum_{i=1}^{\infty} s_i(w_i)$, $w = (w_i)_{i=1}^{\infty} \in \ell_{r^*}(Z^{(p)})$. Since

$$\sum_{i=1}^{\infty} \|s_i(w_i)\| \le \sum_{i=1}^{\infty} k_p(s_i)\|w_i\| \le \left(\sum_{i=1}^{\infty} (k_p(s_i))^r\right)^{1/r} \left(\sum_{i=1}^{\infty} \|w_i\|^{r^*}\right)^{1/r*} < \infty,$$

s is well defined, and since $\|s\| \le \|(k_p(s_i))_{i=1}^{\infty}\|_r$, s is a continuous operator. Now we show that s is, in fact, r-compact. For every $i \in \mathbb{N}$, since $s_i \in K_p(Z^{(p)}, Y)$ and $p \le r$, then $s_i \in K_r(Z^{(p)}, Y)$ and $k_r(s_i) \le k_p(s_i)$ (see [31]). Hence, since $(s_i)_{i=1}^{\infty} \in \ell_r(K_p(Z^{(p)}, Y), k_p))$, then $\sum_{i=1}^{\infty}(k_r(s_i))^r < \infty$. Now, for every $i \in \mathbb{N}$, choose a sequence $(c_n^i)_{n=1}^{\infty} \in \ell_r(Y)$ such that $\|(c_n^i)_{n=1}^{\infty}\|_r < k_r(s_i) + \frac{1}{2^i}$ with $s_i(B_{Z^{(p)}}) \subset r\text{-}co\{(c_n^i)_{n=1}^{\infty}\}$. Since $\sum_{i=1}^{\infty}\|(c_n^i)_{n=1}^{\infty}\|_r^r < 2^r\left(\sum_{i=1}^{\infty}(k_r(s_i))^r + \sum_{i=1}^{\infty}\frac{1}{2^{ir}}\right) < \infty$, we have $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\|c_n^i\|^r = \sum_{i=1}^{\infty}\|(c_n^i)_{n=1}^{\infty}\|_r^r < \infty$. Next, let $w = (w_i)_{i=1}^{\infty} \in B_{\ell_{r^*}(Z^{(p)})}$ (without loss of generality we can assume that $w_i \ne 0$ for each $i \in \mathbb{N}$). Now, one can write $s(w) = \sum_{i=1}^{\infty}\|w_i\|\sum_{n=1}^{\infty}\alpha_n^{w_i}c_n^i$ with $(\alpha_n^{w_i})_{n=1}^{\infty} \in B_{\ell_{r^*}}$. Note that

$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\|w_i\|\alpha_n^{w_i}c_n^i \le \sum_{i=1}^{\infty}[(\sum_{n=1}^{\infty}(\|w_i\||\alpha_n^{w_i}|)^{r^*})^{1/r^*}(\sum_{n=1}^{\infty}\|c_n^i\|^r)^{1/r}]$$

$$\le (\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\|w_i\|^{r^*}|\alpha_n^{w_i}|^{r^*})^{1/r^*}(\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\|c_n^i\|^r)^{1/r} < \infty,$$

and since $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\|c_n^i\|^r < \infty$ and $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(\|w_i\||\alpha_n^{w_i}|)^{r^*} \le 1$, choosing a specific order for these double series and writing $(\lambda_l)_{l\in\mathbb{N}} := (\|w_i\|\alpha_n^{w_i})_{(i,n)\in\mathbb{N}\times\mathbb{N}} \in B_{\ell_{r^*}}$ and $(z_l)_{l\in\mathbb{N}} := \left(c_n^i\right)_{(i,n)\in\mathbb{N}\times\mathbb{N}} \in \ell_r(Y)$, we obtain a representation $s(w) = \sum_{l=1}^{\infty}\lambda_l z_l$, which shows that $s \in K_r(\ell_{r^*}(Z^{(p)}), Y)$.

Now for $T \in H$, where $T = \hat{\tau}(\tau_T) = \sum_{i=1}^{\infty}\lambda_i^{\tau_T}s_i \circ r_i$, we have $T = s \circ A_T \circ r$. Finally we factor r and s through Z_{FJ} and $Z^{(r)}$, respectively, (see [3] and Theorem 2.1). That is, there exist operators $u \in K(X, Z_{FJ})$, $\alpha \in K(Z_{FJ}, c_0(Z^{(p)}))$, $\beta \in K(\ell_{r^*}(Z^{(p)}), Z^{(r)})$ and $v \in K_r(Z^{(r)}, Y)$ such that $r = \alpha \circ u$ and $s = v \circ \beta$. For each $T \in H$, let $B_T := \beta \circ A_T \circ \alpha$. Then it can be easily seen that $\{B_T : T \in H\}$ is a relatively r^*-compact subset of $K(Z_{FJ}, Z^{(r)})$ and $T = v \circ B_T \circ u$ for every $T \in H$.

In addition if we assume that H is convex and balanced, then one can readily see that $\{B_T : T \in H\}$ is also convex and balanced, with which the proof is complete. □

Remark 3.6. a) In the hypothesis of Lemma 3.4 a) and Theorem 3.5, instead of each factor "k" if we take more generally any number $\xi_k \ge 1$ such that, for every $\gamma > 1$, $\sum_{k=1}^{\infty}\frac{1}{\xi_k^{\gamma}} < \infty$, then these results continue to be true.

b) Let X and Y be Banach spaces, let $r \ge 2$ and let $1 \le p \le r < \infty$. If H is a relatively r-compact subset of $(K_p(X, Y), k_p)$ such that $H \subset r\text{-}co\{(a_k)_{k=1}^{\infty}\}$ with $(ka_k)_{k=1}^{\infty} \in \ell_r(K_p(X, Y), k_p)$, then as a consequence of Theorem 3.5 observe that H is, in fact, a relatively r^*-compact subset of $(K_r(X, Y), k_r)$.

If we relax the hypothesis of the previous theorem by removing the factor "k" in the sequence $(ka_k)_{k=1}^{\infty}$, as compared to Theorem 3.5, we obtain the following weaker result.

Proposition 3.7. *Let X and Y be Banach spaces, let $1 \le p \le r < \infty$ with $r > 1$. For every (balanced and convex) relatively r-compact subset H of $(K_p(X, Y), k_p)$, there exist an operator $u \in K(X, Z_{FJ})$, a (resp. balanced and convex) relatively compact subset $\{B_T : T \in H\}$ of $K(Z_{FJ}, Z^{(r)})$ and an operator $v \in K_r(Z^{(r)}, Y)$ such that $T = v \circ B_T \circ u$ for all $T \in H$.*

Proof. Let H be a relatively r-compact subset of $(K_p(X, Y), k_p)$. By Lemma 3.2 there exists a relatively r-compact subset L of $K(X, Z^{(p)}) \hat{\otimes}_{\pi} (K_p(Z^{(p)}, Y), k_p)$ such that $H \subset \hat{\tau}(L)$. Now by Lemma 3.4 b) any $\tau_T \in L$ has a representation $\tau_T = \sum_{i=1}^{\infty}\theta_i^{\tau_T}r_i \otimes s_i$ with $(\theta_i^{\tau_T})_{i=1}^{\infty} \in K$, where $(r_i)_{i=1}^{\infty} \in c_0(K(X, Z^{(p)}))$, $(s_i)_{i=1}^{\infty} \in \ell_r(K_p(Z^{(p)}, Y), k_p)$, and K is a compact subset of $B_{\ell_{r^*}}$. Now the set $\{A_T : T \in H\}$ obtained in Theorem 3.5 is a relatively compact subset of $L(c_0(Z^{(p)}), \ell_{r^*}(Z^{(p)})$ and so is the corresponding set $\{B_T : T \in H\}$. Finally if H is balanced and convex then, one can see that the set $\{B_T : T \in H\}$ has the same properties. Thus, we have the proof. □

We can improve Theorem 3.5 and Proposition 3.7 a little bit more, since in the factorizations given in these results one of the spaces through which the p-compact operators factorize depends on the number $r \geq 2$.

Let $1 \leq p, q \leq \infty$ and let $Z = (\sum_{1 \leq p \leq \infty} Z^{(p)})_q$ for a fixed q, where $Z^{(p)}$ is the universal Banach space given in Theorems 2.1 and 2.3. Thus by Theorem 2.1 and Theorem 2.3, it can be easily seen that Z is a universal Banach space for the factorization of all p-compact operators between arbitrary Banach spaces, which is independent of p. That is, given Banach spaces X and Y, and any $1 \leq p \leq \infty$ and any $T \in K_p(X, Y)$, we can write $T = vou$, $u \in K(X, Z)$, $v \in K_p(Z, Y)$. As a consequence we obtain the following strengthening of Theorem 3.5 and Proposition 3.7, respectively, in which the corresponding factorizations are obtained through a universal Banach space which does not depend on the number $r \geq 2$.

Corollary 3.8. *Let X and Y be Banach spaces, let $r \geq 2$ and let $1 \leq p \leq r < \infty$. For every (balanced and convex) relatively r-compact subset H of $(K_p(X, Y), k_p)$ such that $H \subset r\text{-}co\{(a_k)_{k=1}^\infty\}$ with $(ka_k)_{k=1}^\infty \in \ell_r(K_p(X, Y), k_p)$, there exist an operator $u \in K(X, Z)$, a (resp. balanced and convex) relatively r^*-compact subset $\{B_T : T \in H\}$ of $K(Z, Z)$ and an operator $v \in K_r(Z, Y)$ such that $T = v \circ B_T \circ u$ for all $T \in H$.*

Corollary 3.9. *Let X and Y be Banach spaces, let $r > 1$ and $1 \leq p \leq r < \infty$. For every (balanced and convex) relatively r-compact subset H of $(K_p(X, Y), k_p)$ there exist an operator $u \in K(X, Z)$, a (resp. balanced and convex) relatively compact subset $\{B_T : T \in H\}$ of $K(Z, Z)$ and an operator $v \in K_r(Z, Y)$ such that $T = v \circ B_T \circ u$ for all $T \in H$.*

We now look at the use of uniform factorization result given in Corollary 3.8. The motivation is that whether or not compact sets can be replaced by p-compact sets in a result of E. Toma [35] given in [3, Corollary 6] which gives a characterization of scalar-valued homogeneous polynomials that are weakly uniformly continuous on the unit ball. It is worth saying that in the p-compact case the situation is quite complicated due to the nature of p-compact sets.

We will begin by defining collectively p-compact set, which is the natural extension of notion of collectively compactness.

Definition 3.10. Let X and Y be Banach spaces, let $p \geq 1$. A subset A of $L(X, Y)$ is said to be collectively p-compact if $A(B_X) = \{Tx : T \in A, x \in B_X\}$ is a relatively p-compact set in Y.

Now, similarly to the collectively compactness case (see [1, Theorem 2.4]), as a p-compact counterpart, we obtain the following result, which is of independent interest, and will be needed in the proof of the next theorem.

Proposition 3.11. *Let $1 \leq p < \infty$. Every relatively p-compact subset K of $(K_p(X, Y), k_p)$ is collectively p-compact.*

Proof. We give a proof for the case $1 < p < \infty$ since the proof for the case $p = 1$ is similar. Let K be a relatively p-compact subset of $(K_p(X, Y), k_p)$. Thus, for a given $T \in K$ there exist $(\alpha_n^T)_{n=1}^\infty \in B_{\ell_{p^*}}$ and $(T_n)_{n=1}^\infty \in \ell_p(K_p(X, Y), k_p))$ such that $T = \sum_{n=1}^\infty \alpha_n^T T_n$. For every $n \in N$ we choose a sequence $(z_k^n)_{k=1}^\infty \in \ell_p(Y)$ such that $\|(z_k^n)_{k=1}^\infty\|_p < k_p(T_n) + \frac{1}{2^n}$ and $T_n(B_X) \subset p\text{-}co\{(z_k^n)_{k=1}^\infty\}$. Hence, for $T \in K$ and $x \in B_X$ we have that $T(x) = \sum_{n=1}^\infty \alpha_n^T T_n(x) = \sum_{n=1}^\infty \sum_{k=1}^\infty \alpha_n^T \lambda_k^{n,x} z_k^n$, where $(\alpha_n^T)_{n=1}^\infty$, $(\lambda_k^{n,x})_{k=1}^\infty \in B_{\ell_{p^*}}$. Since $\sum_{n=1}^\infty \sum_{k=1}^\infty |\alpha_n^T \lambda_k^{n,x}|^{p^*} \leq \sum_{n=1}^\infty |\alpha_n^T|^{p^*} \leq 1$ and

$$\sum_{n=1}^\infty \sum_{k=1}^\infty \| z_k^n \|^p = \sum_{n=1}^\infty \| (z_k^n)_{k=1}^\infty \|_p^p < 2^p (\sum_{n=1}^\infty k_p^p(T_n) + \frac{1}{2^{np}}) < \infty,$$

by choosing a specific order one can write $(\gamma_l^{T,x})_{l=1}^{\infty} := (\alpha_n^T \lambda_k^{n,x})_{(n,k)\in\mathbb{N}\times\mathbb{N}} \in B_{\ell_{p^*}}$ and $(s_l)_{l=1}^{\infty} := (z_k^n)_{(n,k)\in\mathbb{N}\times\mathbb{N}} \in \ell_p(Y)$, so that we obtain $T(x) = \sum_{l=1}^{\infty} \gamma_l^{T,x} s_l$. Thus, $K(B_X) = \{Tx : T \in K, x \in B_X\} \subset p\text{-}co\{(s_l)_{l=1}^{\infty}\}$. □

The following result is a partial p-compact version of [3, Proposition 5], which we obtain for the case $n = 2$.

Theorem 3.12. *Let X be a Banach space with X' having the AP, and let $r \geq 2$, $1 \leq p \leq r < \infty$. Let H be a relatively r-compact subset of $(K_p(X, X'), k_p)$ such that $H \subset r\text{-}co\{(a_k)_{k=1}^{\infty}\}$ with $(ka_k)_{k=1}^{\infty} \in \ell_r(K_p(X, X'), k_p)$. Then for every $\varepsilon > 0$ there exists an r-compact subset K'_ε of X' such that for every $T \in H$ and $x \in X$*

$$|T(x)(x)| \leq \varepsilon \parallel x \parallel \sup_{k'\in K'_\varepsilon} |k'(x)| + \sup_{k'\in K'_\varepsilon} |k'(x)|^2.$$

Proof. By Corollary 3.8, there are a Banach space Z, a relatively r^*-compact subset $\{L_T : T \in H\}$ of $K(X, Z)$, and an operator $v \in K_r(Z, X')$ such that $T = voL_T$ for all $T \in H$. Thus, for each $x \in X \subset X''$ and for each $T \in H$, we have $|T(x)(x)| = |v \circ L_T(x)(x)| \leq \|v'(x)\|\|L_T(x)\|$, where v' is the adjoint of v. Note that $\|v'(x)\| = \sup_{z\in B_Z} |v(z)(x)| \leq \sup_{k'\in K'_1} |k'(x)|$, where $K'_1 := \overline{v(B_Z)} \subset X'$, which is an r-compact set. Furthermore,

$$\|L_T(x)\| = \sup_{z'\in B_{Z'}} |z'(L_T(x))| = \sup_{z'\in B_{Z'}} |(L'_T z')(x)|. \tag{1}$$

Let $K := \{L_T : T \in H\}$ and let $K^* := \{L'_T : T \in H\}$. Since K^* is relatively r^*-compact subset of $K(Z', X')$, there exists $(S'_n)_{n=1}^{\infty} \in \ell_{r^*}(K(Z', X'))$ such that $K^* \subset r^*\text{-}co\{(S'_n)_{n=1}^{\infty}\}$. Hence, for any $\varepsilon > 0$ there is $N = N(\varepsilon) \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \|S'_n\|^{r^*} \leq \left(\frac{\varepsilon}{2}\right)^{r^*}$. Since X' has the AP, by [23, Theorem 1.e.4], for every $n \in \mathbb{N}$ there is an $S_n^F \in F(Z', X')$ such that $\left\|S'_n - S_n^F\right\| < \frac{\varepsilon}{2n^2}(\sum_{n=1}^{\infty} \frac{1}{n^2})^{-1}$. So, if we define a sequence $(S_n^*)_{n=1}^{\infty}$ in $F(Z', X')$ by $S_n^* := S_n^F$ for $n = 1, 2, \dots, N$, and $S_n^* := 0$ for $n > N$, and consequently a set by $K^*_{F,\varepsilon} := \{\sum_{n=1}^{\infty} \alpha_n S_n^* : (\alpha_n)_{n=1}^{\infty} \in B_{\ell_r}$ with $\sum_{n=1}^{\infty} \alpha_n S'_n \in K^*\}$ then, $K^*_{F,\varepsilon}$ is a relatively r^*-compact subset of $(F(Z', X'), k_{r^*})$. Now, given any $L'_T = \sum_{n=1}^{\infty} \alpha_n^T S'_n \in K^*$, let $L_T^* := \sum_{n=1}^{\infty} \alpha_n^T S_n^*$. Thus, we have that

$$\parallel L'_T - L_T^* \parallel \leq \sum_{n=1}^{N} \parallel S'_n - S_n^F \parallel + (\sum_{n=N+1}^{\infty} \mid \alpha_n^T \mid^r)^{1/r} \sum_{n=N+1}^{\infty} \parallel S'_n \parallel^{r^*})^{1/r^*} < \varepsilon.$$

Hence, we have shown that for any $L_T \in K$ there is $L_T^* \in K^*_{F,\varepsilon}$ such that $\|L'_T - L_T^*\| < \varepsilon$. Therefore, by (1), for every $x \in X$ we get that

$$\|L_T(x)\| \leq \|L'_T - L_T^*\| \sup_{z'\in B_{Z'}} \|z'\| \|x\| + \sup_{z'\in B_{Z'}} |L_T^* z'(x)| < \varepsilon \|x\| + \sup_{z'\in B_{Z'}} |L_T^* z'(x)|.$$

Since $K^*_{F,\varepsilon}$ is a relatively r^*-compact subset of $(K_{r^*}(Z', X'), k_{r^*})$, thus by Proposition 3.11 the set $K^*_{F,\varepsilon}$ is collectively r^*-compact in $L(Z', X')$, so that the set $K'_2 := \overline{\{L_T^*(z') : L_T^* \in K^*_{F,\varepsilon},\ z' \in B_{Z'}\}}$ is an r^*-compact, hence r-compact, subset of X'. Therefore, $\parallel L_T(x) \parallel < \varepsilon \parallel x \parallel + \sup_{z'\in B_{Z'}} |L_T^* z'(x)| \leq \varepsilon \parallel x \parallel + \sup_{k'\in K'_2} |k'(x)|$. Finally, letting $K'_\varepsilon := K'_1 \cup K'_2$, which is also r-compact, for all $T \in H$ and $x \in X$ we obtain $|T(x)(x)| \leq \varepsilon \parallel x \parallel \sup_{k'\in K'_\varepsilon} |k'(x)| + \sup_{k'\in K'_\varepsilon} |k'(x)|^2$. □

Now as an application of Theorem 3.12 we get the following partial p-compact version of a result of E. Toma [35] for 2-homogeneous polynomials.

Corollary 3.13. *Let X be a Banach space with X' having the AP and let $2 \le r < \infty$. Then for a $P \in P_{wu}(^2X)$ with T_P being r-compact, given any $\varepsilon > 0$ there exists an r-compact subset K'_ε of X' such that $|P(x)| \le \varepsilon \parallel x \parallel \sup_{k' \in K'_\varepsilon} |k'(x)| + \sup_{k' \in K'_\varepsilon} |k'(x)|^2$ for all $x \in X$.*

Proof. Let $P \in P_{wu}(^2X)$ with T_P being r-compact, $2 \le r < \infty$. Then taking $H := \{T_P\}$ and applying Theorem 3.12 we obtain the desired inequality. $\square$

Recall that a polynomial $P \in P(^nX, Y)$ is of finite type if it can be written as a linear combination of functions $\phi^n \otimes y$ $(n \in \mathbb{N}, \phi \in X', y \in Y)$, where $\phi^n \otimes y(x) = \phi^n(x)y$ for each $x \in X$. Note that if a polynomial P is of finite type then the corresponding operator is also of finite type, hence, is r-compact for any $r \ge 2$.

We do not know if the reverse implication in Corollary 3.13 is true. If that would be the case, Corollary 3.13 would be an improvement of [3, Corollary 6] for the case $n = 2$, since the compact sets are replaced by r-compact sets.

Motivated by [3, Corollary 6] and Corollary 3.13, a result for vector-valued p-compact n-homogeneous polynomials can be stated in a similar fashion. Therefore, as a consequence of Theorem 3.12 we prove the following interesting result concerning p-compact polynomials with values in $(\hat{\otimes}^{n,s}_{\pi_s} X)'$.

Corollary 3.14. *Let X be a Banach space such that $(\hat{\otimes}^{n,s}_{\pi_s} X)'$ has the AP. Let $r \ge 2$, $1 \le p \le r < \infty$, and let H_n be a relatively r-compact subset of $(P_{k_p}(^nX, (\hat{\otimes}^{n,s}_{\pi_s} X)'), k_p)$ such that $H_n \subset r\text{-}co\{(a^n_k)^\infty_{k=1}\}$ with $(ka^n_k)^\infty_{k=1} \in \ell_r\left((P_{k_p}(^nX, (\hat{\otimes}^{n,s}_{\pi_s} X)'), k_p)\right)$. Then for every $\varepsilon > 0$ there exists an r-compact subset K'_ε of $(\hat{\otimes}^{n,s}_{\pi_s} X)'$ such that for all $P \in H_n$ and all $x \in X$, $|P(x)(\otimes^n x)| \le \sup_{k' \in K'_\varepsilon} |k'(\otimes^n x)|(\varepsilon \|x\|^n + \sup_{k' \in K'_\varepsilon} |k'(\otimes^n x)|)$.*

Proof. Since by [6, Theorem 3.1] (see [16, Proposition, p. 163]) $(P_{k_p}(^nX, (\hat{\otimes}^{n,s}_{\pi_s} X)'), k_p)$ and $(K_p(\hat{\otimes}^{n,s}_{\pi_s} X, (\hat{\otimes}^{n,s}_{\pi_s} X)'), k_p)$ are isometrically isomorphic, there is a sequence $(T^n_k)^\infty_{k=1} \subset K_p(\hat{\otimes}^{n,s}_{\pi_s} X, (\hat{\otimes}^{n,s}_{\pi_s} X)')$ such that $(kT^n_k)^\infty_{k=1} \in \ell_r(K_p(\hat{\otimes}^{n,s}_{\pi_s} X, (\hat{\otimes}^{n,s}_{\pi_s} X)'), k_p)$ and $C_n := \left\{P^L : P \in H_n\right\} \subset r\text{-}co\{(T^n_k)^\infty_{k=1}\}$, where the mapping $P^L : \hat{\otimes}^{n,s}_{\pi_s} X \to Y$, defined by $P^L(\otimes^n x) = P(x)$ is the linearization of P. Now since $(\hat{\otimes}^{n,s}_{\pi_s} X)'$ has the AP hence, by Theorem 3.12, given any $\varepsilon > 0$, there exits an r-compact subset K'_ε of $(\hat{\otimes}^{n,s}_{\pi_s} X)'$ such that for all $P^L \in C_n$ and for all $x \in X$, we have

$$|P^L(\otimes^n x)(\otimes^n x)| \le \sup_{k' \in K'_\varepsilon} |k'(\otimes^n x)|(\varepsilon \parallel \otimes^n x \parallel + \sup_{k' \in K'_\varepsilon} |k'(\otimes^n x)|),$$

from which we get the conclusion. $\square$

Note that in Corollary 3.14 taking $n = 1$ one gets exactly Theorem 3.12. In this sense it is a generalization of Theorem 3.12.

Remark 3.15. Since $P(^nX) = \left(\hat{\otimes}^{n,s}_{\pi_s} X\right)'$ (see, e.g., [16, p. 165]) the hypothesis of Corollary 3.14 concerning the AP is satisfied for the spaces c_0, ℓ_1 and T (Tsirelson space), and for ℓ_q whenever $n < q < \infty$ (see [14] and [25]).

We end the paper with some natural questions that arise from our results.

Question 3.16.

a) In Lemma 3.4 a), can the factor "k" be removed to get the same conclusion for general r-compact sets?
b) Is it possible to extend Lemma 3.4 a) to the case $1 \le p$, and accordingly, Theorem 3.5 to the case $1 \le r < p$?
c) In Theorem 3.5, what conditions should be imposed (e.g., on the set H) to get the set $\{B_T : T \in H\}$ being p-compact, for any $1 \le p < \infty$?

Question 3.17. Is it true that the result of E. Toma [35], stated in [3, Corollary 6], continue to be true if one replaces compact sets by p-compact sets, for any $1 \leq p < \infty$?

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