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The Bishop–Phelps–Bollobás property for operators from c_0 into some Banach spaces [☆]

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This paper is dedicated to our dear colleague Richard Aron

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ABSTRACT

We exhibit a new class of Banach spaces Y such that the pair (c_0, Y) has the Bishop–Phelps–Bollobás property for operators. This class contains uniformly convex Banach spaces and spaces with the property β of Lindenstrauss. We also provide new examples of spaces in this class.

1. Introduction

The well-known Bishop–Phelps Theorem [8] states that the set of norm attaining (continuous and linear) functionals on a Banach space is dense in its topological dual. After this result was proved, a lot of attention was devoted to extend it to operators (see [1,11,17], for instance).

In 1970, Bollobás showed the following "quantitative version" which is now called Bishop–Phelps–Bollobás Theorem [9]. As usual, for a normed space X, we denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. By X^* we stand the topological dual of X.

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Theorem 1.1 *(Bishop–Phelps–Bollobás Theorem). (See [10, Theorem 16.1].) Let X be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \frac{\varepsilon^2}{4}$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.*

For a refinement of the above result see [13, Corollary 2.4(a)]. In 2008 Acosta, Aron, García and Maestre initiated the study of parallel versions of this result for operators [3]. For two normed spaces X and Y over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $\mathcal{L}(X, Y)$ denotes the space of (bounded and linear) operators from X into Y, endowed with the usual operator norm. For an operator $T \in \mathcal{L}(X, Y)$, T^t denotes the transpose of T.

Definition 1.2. (See [3].) Let X and Y be both either real or complex Banach spaces. It is said that the pair (X, Y) *has the Bishop–Phelps–Bollobás property for operators* (*BPBp*), if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that for any $T \in S_{\mathcal{L}(X,Y)}$, if $x \in S_X$ is such that $\|Tx\| > 1 - \eta(\varepsilon)$, then there exist an element u in S_X and an operator S in $S_{\mathcal{L}(X,Y)}$ satisfying the following conditions

$$\|Su\| = 1, \quad \|u - x\| < \varepsilon \quad \text{and } \|S - T\| < \varepsilon.$$

It was shown that the pair (X, Y) has the *BPBp* whenever X and Y are finite dimensional spaces [3, Proposition 2.4]. It was also proved that (X, Y) has the *BPBp* for every Banach space X, whenever Y has the property β of Lindenstrauss [3, Theorem 2.2]. A characterization of the Banach spaces Y such that the pair (ℓ_1, Y) has the *BPBp* was also provided [3, Theorem 4.1].

However, up to now there is no characterization of the spaces Y such that (c_0, Y) has the *BPBp*. First let us notice that the previous property is not trivially satisfied for every Banach space Y (see [7, Example 4.1]). Now we are going to mention some results known in the real case. Acosta et al. showed that (ℓ_∞^n, Y) has the *BPBp* for every nonnegative integer n whenever Y is uniformly convex [3]. Kim proved that the pair (c_0, Y) has the *BPBp* if Y is a uniformly convex Banach space [15]. There is a characterization of the Banach spaces Y such that (ℓ_∞^3, Y) has the *BPBp* for operators [5]. Moreover, Kim, Lee and Lin proved that the pair (L_∞, Y) has the *BPBp* for operators, whenever Y is uniformly convex [16].

Recently, Acosta showed that for the complex case the pair $(\mathcal{C}_0(L), Y)$ has the *BPBp* for every complex uniformly convex space Y and any locally compact Hausdorff topological space L [2].

On the other hand, for real or complex spaces, Aron, Cascales and Kozhushkina showed that the pair $(X, \mathcal{C}_0(L))$ has the *BPBp* for any locally compact Hausdorff space L in case that X is Asplund [6]. Later, Cascales, Guirao and Kadets extended this result to uniform algebras [12]. From here it follows that $(c_0, \mathcal{C}_0(L))$ has the *BPBp*.

In the real case, it was shown that the pair $(\mathcal{C}_0(L), \mathcal{C}_0(S))$ has the *BPBp* for any locally compact Hausdorff spaces L and S [4]. It is not known whether or not the parallel result holds in the complex case.

In this paper, we provide a new class of Banach spaces Y, containing uniformly convex spaces and spaces with the property β of Lindenstrauss, and such that the pair (c_0, Y) satisfies the Bishop–Phelps–Bollobás property for operators. Hence, spaces in this class can be very different from $\mathcal{C}_0(L)$. Moreover, elements in that class are not necessarily neither uniformly convex spaces nor spaces with the property β of Lindenstrauss.

2. The main result

The Banach spaces Y for which (ℓ_1, Y) has the *BPBp* for operators have been characterized in [3]. However nowadays it is considered as a main question in this subject to characterize the Banach spaces Y such that (c_0, Y) has the *BPBp* for operators.

As we already mentioned, our goal is to provide a new class of Banach spaces Y such that the pair (c_0, Y) has the Bishop–Phelps–Bollobás property for operators. To this purpose the following notion will be useful.

Definition 2.1. Let Y be a (real or complex) Banach space, $E \subset S_Y$ and $F : E \longrightarrow S_{Y^*}$. We recall that the family E is *uniformly strongly exposed* by F if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$y \in B_Y, \quad e \in E, \quad \operatorname{Re} F(e)(y) > 1 - \delta \;\Rightarrow\; \|y - e\| < \varepsilon.$$

This concept was used by Lindenstrauss related to norm attaining operators (see [17]). Indeed, if there is a set $E \subset S_Y$ satisfying Definition 2.1 such that B_Y is the closure of the absolutely convex hull of E, then the set of norm attaining operators from Y into Z is dense in the space $\mathcal{L}(Y, Z)$, for every Banach space Z [17, Proposition 1].

We recall the following elementary result.

Lemma 2.2. *(See [3, Lemma 3.3].) Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \le 1$ for every n, and let $\eta > 0$ be such that there is some sequence $\{\alpha_n\}$ in $\mathbb{R}_0^+$ satisfying $\sum_{n=1}^{\infty} \alpha_n \le 1$ and $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N} : \operatorname{Re} c_i > r\}$, satisfies the estimate*

$$\sum_{i \in A} \alpha_i > 1 - \frac{\eta}{1 - r}$$

For a subset $A \subset \mathbb{N}$, $P_A : c_0 \longrightarrow c_0$ is defined by $P_A(x) = \sum_{n \in A} x(n) e_n$, where $\{e_n\}$ is the usual Schauder basis of c_0. The following result extends [3, Lemma 5.1]:

Lemma 2.3. *Let Y be a (real or complex) Banach space and assume that $E \subset S_Y$, $F : E \longrightarrow S_{Y^*}$ and E is uniformly strongly exposed by F. Assume that for some $\varepsilon > 0$, δ satisfies the condition stated in Definition 2.1. If $T \in S_{L(c_0, Y)}$ and $x \in B_{c_0}$ satisfy $\operatorname{Re} F(e)\big(T(x)\big) > 1 - \delta$ for some $e \in E$ and $A = \operatorname{supp} x$ then $\|T(I - P_A)\| \le 2\varepsilon$.*

Proof. Let $u \in B_{c_0}$ and so $x \pm (I - P_A)(u) \in B_{c_0}$, hence $T(x \pm (I - P_A)(u)) \in B_Y$. By assumption we have $\operatorname{Re} F(e)\big(T(x)\big) > 1 - \delta$ for some $e \in E$. Hence for some choice of sign s we have that $\operatorname{Re} F(e)\big(T(x + s(I - P_A)(u))\big) > 1 - \delta$. As a consequence $\|T(x) - e\| < \varepsilon$ and $\|T(x + s(I - P_A)(u)) - e\| < \varepsilon$. So we obtain that

$$\|T(I - P_A)(u)\| = \|sT(I - P_A)(u)\| \le \|T(x + sT(I - P_A)(u)) - e\| + \|e - T(x)\| < 2\varepsilon.$$

Since u is any element in B_{c_0}, we proved that $\|T(I - P_A)\| \le 2\varepsilon$. $\square$

The main result of the paper reads as follows.

Theorem 2.4. *Assume that Y is a (real or complex) Banach space such that there are a set I, $\{y_i : i \in I\} \subset S_Y$, $\{y_i^* : i \in I\} \subset S_{Y^*}$, a subset $E \subset S_Y$, a mapping $F : E \longrightarrow S_{Y^*}$ and $0 \le \rho < 1$ satisfying that*

i) $y_i^*(y_i) = 1$, $\forall i \in I$.
ii) $|y_i^*(y_j)| \le \rho$, $\forall i, j \in I$, $i \ne j$.
iii) *E is uniformly strongly exposed by F.*
iv) $|F(e)(y_i)| \le \rho$, $\forall e \in E$, $i \in I$.
v) *The set $F(E) \cup \{y_i^* : i \in I\}$ is a 1-norming set for Y, that is, for any $y \in Y$ it is satisfied that* $\|y\| = \max\big\{\sup\{|y_i^*(y)| : i \in I\}, \sup\{|F(e)(y)| : e \in E\}\big\}$.

Then the pair (c_0, Y) has the BPBp *for operators.*

Proof. Fix $0 < \varepsilon < 1$. By assumption iii) there are positive numbers $\nu < \frac{\varepsilon}{12}$ and $\delta < 1$ satisfying

$$e \in E, \quad y \in B_Y, \quad \operatorname{Re} F(e)(y) > 1 - \nu \;\Rightarrow\; \|y - e\| < \frac{\varepsilon}{8} \tag{2.1}$$

and

$$e \in E, \quad y \in B_Y, \quad \operatorname{Re} F(e)(y) > 1 - \delta \;\Rightarrow\; \|y - e\| < \frac{\nu^3}{8}. \tag{2.2}$$

We write $\varepsilon_1 = \dfrac{\delta\nu^3}{16}$ and choose positive real numbers η and s satisfying

$$\eta < \frac{4}{5}\min\Big\{\frac{\varepsilon_1(1-\rho)}{1+\rho}, \frac{\varepsilon}{3}, 2\delta\Big\}, \quad s < \min\Big\{\frac{\varepsilon^2\varepsilon_1(\delta - 2\varepsilon_1)}{24(1+2\varepsilon_1)}, \frac{\eta^2\varepsilon_1\varepsilon^2}{3\cdot 2^5}\Big\}.$$

Assume that $T \in S_{\mathcal{L}(c_0,Y)}$ and $x_0 \in S_{c_0}$ satisfy that

$$\|T(x_0)\| > 1 - s.$$

Our goal is to find an operator $S \in S_{\mathcal{L}(c_0,Y)}$ and $z \in S_{c_0}$ such that

$$\|S(z)\| = 1, \qquad \|S - T\| < \varepsilon \qquad \text{and} \qquad \|z - x_0\| < \varepsilon.$$

In order to define the appropriate operator S we will consider several cases.

Case 1. Assume that there is $i \in I$ such that $|y_i^*(T(x_0))| > 1 - \dfrac{\eta^2}{4}$.

By multiplying x_0 by an appropriate scalar of modulus 1, we may assume that $y_i^*(T(x_0)) = |y_i^*(T(x_0))| > 1 - \dfrac{\eta^2}{4}$.

By the Bishop–Phelps–Bollobás Theorem, we can choose $z^* \in S_{c_0^*}$ and $z \in S_{c_0}$ such that

$$z^*(z) = 1, \; \|z - x_0\| < \eta \;\text{ and }\; \Big\| z^* - \frac{T^t(y_i^*)}{\|T^t(y_i^*)\|}\Big\| < \eta. \tag{2.3}$$

Clearly we get that

$$\big\|z^* - T^t(y_i^*)\big\| < \eta + \frac{\eta^2}{4}. \tag{2.4}$$

We define the operator

$$\tilde{S}(x) = T(x) + \big[(1+\varepsilon_1)z^*(x) - T^t(y_i^*)(x)\big]y_i \qquad (x \in c_0).$$

It follows that

$$\tilde{S}^t(y^*) = T^t(y^*) + \big[(1+\varepsilon_1)z^* - T^t(y_i^*)\big]y^*(y_i), \qquad \forall y^* \in Y^*.$$

We will estimate the norm of $\tilde{S}$. We clearly have that

$$\|\tilde{S}\| = \|\tilde{S}^t\| \geq \|\tilde{S}^t(y_i^*)\| = \|(1+\varepsilon_1)z^*\| = 1 + \varepsilon_1. \tag{2.5}$$

Let us fix any element $e \in E$. By condition iv) of assumptions it is satisfied that $|F(e)(y_i)| \leq \rho$ and so

$$\begin{aligned}\|\tilde{S}^t(F(e))\| &\leq \|T^t\| + \|(1+\varepsilon_1)z^* - T^t(y_i^*)\|\,\rho \\ &\leq 1 + \rho\Big(\eta + \frac{\eta^2}{4}\Big) + \varepsilon_1\rho < 1 + \varepsilon_1 \quad \text{by (2.4)}. \end{aligned} \tag{2.6}$$

By using ii) instead of iv) and the previous argument again we also obtain that

$$\|\tilde{S}^t(y_j^*)\| < 1 + \varepsilon_1 \qquad \forall j \in I\backslash\{i\}. \tag{2.7}$$

By v), in view of (2.5), (2.6) and (2.7) we get that

$$\begin{aligned}\|\tilde{S}\| = \|\tilde{S}^t\| &= \max\Big\{\sup\{\|\tilde{S}^t(y_j^*)\| : j \in I\}, \sup\{\|\tilde{S}^t(F(e))\| : e \in E\}\Big\} \\ &= \|\tilde{S}^t(y_i^*)\| = 1 + \varepsilon_1.\end{aligned}$$

Since $\tilde{S}^t(y_i^*) = (1+\varepsilon_1)z^*$ and z^* attains its norm at z, $S = \dfrac{\tilde{S}}{\|\tilde{S}\|}$ attains its norm at z and it is also satisfied that

$$\begin{aligned}\|S - T\| &\leq \|S - \tilde{S}\| + \|\tilde{S} - T\| \\ &\leq \big|1 - \|\tilde{S}\|\big| + \varepsilon_1 + \big\|z^* - T^t(y_i^*)\big\| \\ &\leq \big|1 - \|\tilde{S}\|\big| + \varepsilon_1 + \eta + \frac{\eta^2}{4} \quad \text{by (2.4)} \\ &\leq 2\varepsilon_1 + \eta + \frac{\eta^2}{4} < \varepsilon.\end{aligned}$$

Also we have that $\|z - x_0\| < \eta < \varepsilon$ by (2.3) and the proof is finished in this case.

Case 2. Assume that $\sup\{|y_i^*(T(x_0))| : i \in I\} \leq 1 - \dfrac{\eta^2}{4}$.

Since $\|T(x_0)\| > 1 - s > 1 - \dfrac{\eta^2}{4}$, in view of assumption v) we can choose $e_0 \in E$ such that the element $z_1^* = F(e_0)$ satisfies $|z_1^*(T(x_0))| > 1 - s$. By rotating x_0, if needed, we can assume that $z_1^*(T(x_0)) = |z_1^*(T(x_0))|$.

For every positive integer n, let λ_n be a scalar satisfying $|\lambda_n| = 1$ and $(T^t(z_1^*))(e_n) = \lambda_n|(T^t(z_1^*))(e_n)|$.

We consider the set A given by

$$A = \Big\{n \in \mathbb{N} \ : \ (T^t(z_1^*))(e_n) \neq 0 \ \text{ and } \ \mathrm{Re}\,\big(x_0(n)\lambda_n\big) > 1 - \frac{\varepsilon^2}{8}\Big\}.$$

Clearly A is a finite set. We define the element $a \in S_{c_0}$ by

$$a(k) = \overline{\lambda_k} \ \ \forall k \in A, \qquad a(k) = x_0(k) \ \ \forall k \in \mathbb{N}\backslash A.$$

Let $a_1 = P_A(a)$. Now we apply Lemma 2.2 for the sequences $\{\alpha_n\} = \{|T^t(z_1^*)(e_n)|\}$ and $\{c_n\} = \{x_0(n)\lambda_n\}$. Since $z_1^*(T(x_0)) = \sum_{n=1}^{\infty} \alpha_n x_0(n)\lambda_n$ we obtain that

$$1 - \delta < 1 - \frac{8s}{\varepsilon^2} < \sum_{k\in A} \alpha_k = \sum_{k\in A} |T^t(z_1^*)(e_k)| = \mathrm{Re}\, z_1^*(T(a_1)) = |z_1^*(T(a_1))| \leq \|T(a_1)\|. \tag{2.8}$$

Let us notice

$$a, b \in \mathbb{R}, \quad |a + ib| \leq 1 \quad \Rightarrow \quad |1 - (a + ib)| \leq \sqrt{2(1 - a)}. \tag{2.9}$$

As a consequence

$$\|a - x_0\| = \max_{k \in A} |\overline{\lambda_k} - x_0(k)| < \frac{\varepsilon}{2}. \tag{2.10}$$

Since $z_1^* = F(e_0)$, from (2.2) and (2.8) we have that

$$\|T(a_1) - e_0\| < \frac{\nu^3}{8} \tag{2.11}$$

In view of (2.8) and (2.2) we can apply Lemma 2.3 and obtain that

$$\|T - TP_A\| \leq \frac{\nu^3}{4}. \tag{2.12}$$

Now we consider two subcases.

Subcase 2.1. Assume that for some $i \in I$ it is satisfied that $\left| y_i^* \left(\frac{T(P_A(a))}{\|T(P_A(a))\|} \right) \right| > 1 - \frac{\eta^2}{4}$. By the same argument used in Case 1, we can find $u \in S_{c_0}$ and $S \in S_{\mathcal{L}(c_0, Y)}$ such that

$$\left\| S - \frac{TP_A}{\|TP_A\|} \right\| \leq 2\varepsilon_1 + \eta + \frac{\eta^2}{4}, \qquad \|u - a\| < \eta \qquad \text{and} \qquad \|S(u)\| = 1.$$

Since $\|T\| = 1$, in view of the previous estimate and from (2.12), we deduce that

$$\begin{aligned} \|S - T\| &\leq \left\| S - \frac{TP_A}{\|TP_A\|} \right\| + \left\| \frac{TP_A}{\|TP_A\|} - TP_A \right\| + \|TP_A - T\| \\ &< 2\varepsilon_1 + \eta + \frac{\eta^2}{4} + \big|1 - \|TP_A\|\big| + \|TP_A - T\| \\ &\leq 2\varepsilon_1 + \eta + \frac{\eta^2}{4} + 2\|T - TP_A\| \\ &< 2\varepsilon_1 + \eta + \frac{\eta^2}{4} + \frac{\nu^3}{2} < \varepsilon, \end{aligned}$$

and in view of (2.10) we also have

$$\begin{aligned} \|u - x_0\| &\leq \|u - a\| + \|a - x_0\| \\ &< \eta + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This completes the proof.

Subcase 2.2. Assume that $\sup_{i \in I} \left| y_i^* \left(\frac{T(P_A(a))}{\|T(P_A(a))\|} \right) \right| \leq 1 - \frac{\eta^2}{4}$. We define the operator $\tilde{S}$ in $\mathcal{L}(c_0, Y)$ by

$$\tilde{S}(x) = T\big(P_A(x)\big) + \varepsilon_1 z_1^* \big(T(P_A(x))\big)\, T(a_1) \qquad (x \in c_0).$$

It is clear that $\tilde{S} \neq 0$ and $\tilde{S} = \tilde{S}P_A$. We write $S = \dfrac{\tilde{S}}{\|\tilde{S}\|}$. Since A is finite, $P_A(B_{c_0})$ is a compact set. Hence there is an element $a_2 \in B_{c_0} \cap \text{Ext}\ (P_A(B_{c_0}))$ such that

$$\|S(a_2)\| = 1 \quad \text{and} \quad z_1^*(T(a_2)) = |z_1^*(T(a_2))|. \tag{2.13}$$

Thus

$$P_A(a_2) = a_2 \quad \text{and} \quad |a_2(k)| = 1 \ \text{ for every } \ k \in A. \tag{2.14}$$

Let $\mathbb{T}$ be the unit sphere of the scalar field. Since $\|S(a_2)\| = 1$, by condition v) and Banach–Alaoglu Theorem, there exists z_2^* in the w^*-closure of the set $\mathbb{T}F(E) \cup \{\mathbb{T}y_i^* : i \in I\}$ such that

$$z_2^*(S(a_2)) = |z_2^*(S(a_2))| = \|S(a_2)\| = 1. \tag{2.15}$$

In view of (2.8) we have that

$$\begin{aligned}\|\tilde{S}\| &\geq |z_1^*(\tilde{S}(a_1))| \\ &= \left|z_1^*(T(a_1)) + \varepsilon_1 \big(z_1^*(T(a_1))\big)^2\right| \\ &\geq 1 - \frac{8s}{\varepsilon^2} + \varepsilon_1 \Big(1 - \frac{8s}{\varepsilon^2}\Big)^2 > 1.\end{aligned} \tag{2.16}$$

As a consequence

$$1 < \|\tilde{S}\| \leq 1 + \varepsilon_1,$$

and so $\big|1 - \|\tilde{S}\|\big| \leq \varepsilon_1$. We obtain that

$$\begin{aligned}\|S - TP_A\| &\leq \|S - \tilde{S}\| + \|\tilde{S} - TP_A\| \\ &\leq \big|1 - \|\tilde{S}\|\big| + \varepsilon_1 \leq 2\varepsilon_1.\end{aligned} \tag{2.17}$$

By (2.12) we deduce that

$$\|S - T\| \leq \|S - TP_A\| + \|TP_A - T\| \leq 2\varepsilon_1 + \frac{\nu^3}{4}. \tag{2.18}$$

On the other hand, we have

$$\begin{aligned}\|\tilde{S}\| &= \operatorname{Re} z_2^*\big(\tilde{S}(a_2)\big) \qquad \text{(by (2.15))} \\ &= \operatorname{Re} z_2^*\big(T(a_2)\big) + \varepsilon_1 \operatorname{Re}\big(z_1^*\big(T(a_2)\big) z_2^*\big(T(a_1)\big)\big) \\ &\leq 1 + \varepsilon_1 \operatorname{Re}\big(z_1^*\big(T(a_2)\big) z_2^*\big(T(a_1)\big)\big) \\ &= 1 + \varepsilon_1 |z_1^*\big(T(a_2)\big)| \operatorname{Re} z_2^*\big(T(a_1)\big) \qquad \text{(by (2.13))} \\ &\leq 1 + \varepsilon_1 \operatorname{Re} z_1^*\big(T(a_2)\big) \qquad \text{(by (2.13))}.\end{aligned} \tag{2.19}$$

Linking inequalities (2.16) and (2.19) we obtain that

$$|z_1^*(T(a_2))| = \operatorname{Re} z_1^*(T(a_2)) \geq \Big(1 - \frac{8s}{\varepsilon^2}\Big)^2 - \frac{8s}{\varepsilon_1 \varepsilon^2} > 1 - \frac{\eta^2}{4} > 0. \tag{2.20}$$

Bearing in mind (2.19) and repeating the same argument we also deduce that

$$|z_2^*(T(a_1))| > 1 - \frac{\eta^2}{4}.$$

Since we assume that $|y_i^*(T(a_1))| \leq 1 - \frac{\eta^2}{4}$ for every $i \in I$, z_2^* belongs to the w^*-closure of $\mathbb{T}F(E)$.

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We clearly have that

$$\begin{aligned}\text{Re } z_1^*(S(a_2)) &\geq \text{Re } z_1^*(T(a_2)) - \|S - TP_A\| \\ &\geq 1 - \frac{\eta^2}{4} - 2\varepsilon_1 \qquad \text{(by (2.20) and (2.17))} \\ &> 1 - \delta.\end{aligned}$$

Since $z_1^* = F(e_0)$, by (2.8) and the previous inequality, in view of (2.2) we obtain that $\|T(a_1) - S(a_2)\| < \frac{\nu^3}{4}$. As a consequence

$$\begin{aligned}\text{Re } z_2^*(S((a_1))) &= \text{Re } \Big(z_2^*(S(a_2)) + z_2^*\big(S(a_1) - S(a_2)\big)\Big) \\ &\geq 1 - \|S(a_1) - S(a_2)\| \qquad \text{(by (2.15))} \\ &\geq 1 - \|S(a_1) - T(a_1)\| - \|T(a_1) - S(a_2)\| \\ &> 1 - 2\varepsilon_1 - \frac{\nu^3}{4} \qquad \text{(by (2.17))} \\ &> 1 - \frac{\nu^3}{8} - \frac{\nu^3}{4} = 1 - \frac{3\nu^3}{8}.\end{aligned} \tag{2.21}$$

We identify the element $S^t\big(z_2^*\big) \in B_{c_0^*}$ with the sequence $\big\{\alpha_k s_k\big\} \in B_{\ell_1}$, where $\alpha_k \geq 0$ and $s_k \in \mathbb{T}$ for every k. By (2.15) and (2.14) we know that $z_2^*(S(a_2)) = 1$ and $|a_2(k)| = 1$ for every $k \in A$. Hence

$$k \in A, \quad \alpha_k \neq 0 \Rightarrow s_k a_2(k) = 1 \quad \text{(i.e.} \quad s_k = \overline{a_2(k)}). \tag{2.22}$$

We write

$$B = \Big\{k \in A : \alpha_k > 0, \quad \text{Re } (s_k a_1(k)) > 1 - \frac{\nu^2}{2}\Big\}.$$

Our next goal is to prove that

$$\text{Re } z_2^*\big(S(P_B(a_1))\big) = \text{Re} \sum_{j \in B} \alpha_j s_j a_1(j) > 1 - \nu. \tag{2.23}$$

From (2.21) the inequality Re $\sum_{k \in A} \alpha_k s_k a_1(k) = \text{Re } z_2^*\big(S(a_1)\big) > 1 - \frac{3\nu^3}{8}$ is satisfied. By applying Lemma 2.2 we get that

$$\sum_{j \in B} \alpha_j > 1 - \frac{3\nu}{4}.$$

Thus

$$\begin{aligned}\text{Re} \sum_{j \in B} \alpha_j s_j a_1(j) &> \Big(1 - \frac{\nu^2}{2}\Big) \sum_{j \in B} \alpha_j \\ &> \Big(1 - \frac{\nu^2}{2}\Big)\Big(1 - \frac{3\nu}{4}\Big) \\ &> (1 - \nu^2)\Big(1 - \frac{\nu}{1+\nu}\Big) \\ &= 1 - \nu.\end{aligned}$$

Since z_2^* belongs to the w^*-closure of $\mathbb{T}F(E)$, condition (2.23) implies that there is some element $y^* = \lambda F(e) \in \mathbb{T}F(E)$ satisfying

$$\text{Re } F(e)(S(\lambda P_B(a_1))) = \text{Re } \lambda F(e)(S(P_B(a_1))) = \text{Re } y^*(S(P_B(a_1))) > 1 - \nu.$$

Taking into account also (2.1) we use Lemma 2.3 to obtain

$$\|S(I - P_B)\| \leq \frac{\varepsilon}{4}. \tag{2.24}$$

Finally we define the operator $U = SM$, where $M : c_0 \longrightarrow c_0$ is the operator given by

$$M(x) = \big(a_2(n)\overline{a_1(n)}x(n)\big) \qquad (x \in c_0).$$

Clearly $\|U\| \leq \|S\| = 1$ and by (2.15) we get that

$$z_2^*\big(U(P_A(a_1))\big) = z_2^*\big(S(a_2)\big) = 1.$$

Therefore, $\|U\| = 1$ and U attains its norm at $P_A(a_1) = a_1$. We also have that

$$\begin{aligned}
\|SP_B(M - I)\| &\leq \|P_B(M - I)\| \\
&= \max\{|(a_2(k)\overline{a_1(k)}x(k) - x(k)| : \ k \in B, x \in B_{c_0}\} \\
&= \max\big\{|(a_2(k)\overline{a_1(k)}x(k) - x(k)| : \ k \in B, x \in \text{Ext } B_{P_B(c_0)}\big\} \\
&= \max_{k\in B} |a_2(k)\overline{a_1(k)} - 1| = \max_{k\in B} |\overline{a_2(k)}a_1(k) - 1| \\
&= \max_{k\in B} |s_k a_1(k) - 1| \qquad (\text{by } \ (2.22)) \\
&\leq \sqrt{2}\sqrt{1 - \min_{k\in B} \text{Re } s_k a_1(k)} \qquad (\text{by } \ (2.9)) \\
&< \sqrt{2}\sqrt{\frac{\nu^2}{2}} = \nu.
\end{aligned}$$

From here we have

$$\begin{aligned}
\|U - T\| &\leq \|U - S\| + \|S - T\| \\
&= \|S(M - I)\| + \|S - T\| \\
&\leq \|SP_B(M - I)\| + \|S(I - P_B)(M - I)\| + 2\varepsilon_1 + \frac{\nu}{2} \qquad (\text{by } (2.18)) \\
&\leq \nu + 2\|S(I - P_B)\| + 2\varepsilon_1 + \frac{\nu}{2} \\
&\leq \frac{\varepsilon}{2} + 2\varepsilon_1 + \frac{3\nu}{2} < \varepsilon \qquad (\text{by } (2.24)).
\end{aligned}$$

We know that U attains its norm at $P_A(a_1)$ and $U = UP_A$. Since $P_A(a) = a_1$ the operator U also attains its norm at a. By (2.10) we know that $\|a - x_0\| < \frac{\varepsilon}{2} < \varepsilon$. Now the proof is completed. □

Definition 2.5. A Banach space Y is said to have the *property* β (of Lindenstrauss) if there are two sets $\{y_\alpha : \alpha \in \Lambda\} \subset S_Y$, $\{y_\alpha^* : \alpha \in \Lambda\} \subset S_{Y^*}$ and $0 \leq \rho < 1$ such that the following conditions hold

1) $y_\alpha^*(y_\alpha) = 1$, for every $\alpha \in \Lambda$.
2) $|y_\alpha^*(y_\beta)| \le \rho < 1$ for any $\alpha, \beta \in \Lambda$ such that $\alpha \ne \beta$.
3) $\|y\| = \sup\{|y_\alpha^*(y)| : \alpha \in \Lambda\}$, for all $y \in Y$.

Clearly, $c_0(\Lambda)$ and $\ell_\infty(\Lambda)$ satisfy the above property for $\{y_\alpha : \alpha \in \Lambda\} = \{e_\alpha : \alpha \in \Lambda\}$, $\{y_\alpha^* : \alpha \in \Lambda\}$ equals the set of coordinate functionals, and $\rho = 0$.

We recall that a Banach space X is *uniformly convex* if for every $\varepsilon > 0$ there is $0 < \delta < 1$ such that

$$u, v \in B_X, \quad \frac{\|u+v\|}{2} > 1 - \delta \;\Rightarrow\; \|u - v\| < \varepsilon \,.$$

In this case, the modulus of convexity is given by

$$\delta_X(\varepsilon) := \inf\Big\{1 - \frac{\|u+v\|}{2} : u, v \in B_X, \|u - v\| \ge \varepsilon\Big\} \,.$$

Let Y be a uniformly convex space. For every $y \in S_Y$, we choose a functional $y^* \in S_{Y^*}$ such that $y^*(y) = 1$. If we define $F(y) = y^*$, it is immediate that S_Y is uniformly strongly exposed by F. As a consequence of Theorem 2.4 we obtain the following known result.

Corollary 2.6. *Let Y be either a uniformly convex space or a Banach space satisfying the property β of Lindenstrauss. Then the pair (c_0, Y) has the* BPBp *for operators.*

The next results show that there are more classes of spaces where the main result can be applied.

It is convenient now to recall that a *biorthogonal system* in a Banach space X is a subset $\{(x_i, x_i^*) : i \in I\} \subset X \times X^*$ such that $x_i^*(x_j) = \delta_j^i$ for every $i, j \in I$. The biorthogonal system is said to be *bounded* if $\sup\{\|x_i\|\|x_i^*\| : i \in I\} < +\infty$. The biorthogonal system is *total* whenever the subset $\{x_i^* : i \in I\}$ separates the elements in X.

Proposition 2.7. *Let $(X, \|\ \|)$ be a Banach space, I a nonempty set and $\{(x_i, x_i^*) : i \in I\}$ a bounded biorthogonal system in X. Assume that $\|x_i\| = 1$, for every $i \in I$.*

If $K = \sup\{\|x_i^*\| : i \in I\}$ and $M > K$, let B be the set given by

$$B = MB_X \cap \{x \in X : \ |x_i^*(x)| \le 1, \quad \forall i \in I\}.$$

Then the following assertions hold

(a) *The set B is the closed unit ball of a new norm $|||\ |||$ equivalent to the original norm of X that satisfies*

$$\frac{1}{K}|||x||| \le \|x\| \le M|||x|||, \quad \forall \;\; x \in X.$$

(b) *If X is uniformly convex, then $Y = (X, |||\ |||)$ satisfies the assumptions of Theorem 2.4.*
(c) *If $\dim X > 1$ then Y is not strictly convex. In the complex case Y is not $\mathbb{C}$-rotund.*
(d) *If the biorthogonal system is not total and X is uniformly convex, then $Y = (X, |||\ |||)$ does not have the property β of Lindenstrauss.*

Proof. Let us notice that $K \ge 1$ since we are assuming that $I \ne \varnothing$, and $1 = \|x_i\| = x_i^*(x_i)$ for every $i \in I$.

(a) Clearly B is convex, balanced, bounded and closed. It is also satisfied that

$$\frac{1}{K} B_X \subset B \subset MB_X, \tag{2.25}$$

so B is the closed unit ball of a norm on X, that we will denote by $|||\ |||$. As a consequence of (2.25) we obtain

$$\frac{1}{K}|||x||| \leq \|x\| \leq M|||x|||, \quad \text{for all } \ x \in X.$$

In particular, both norms are equivalent.

(b) Assume that X is uniformly convex. Hence the following condition is satisfied for every $\varepsilon > 0$

$$y \in B_X,\ x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1,\ \text{Re}\ x^*(y) > 1 - 2\delta_X(\varepsilon) \implies \|x - y\| < \varepsilon. \tag{2.26}$$

That is, if we choose for every $x \in S_X$ any functional $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and define $G(x) = x^*$ then S_X is uniformly strongly exposed by G.

The following description of the unit sphere of Y is easily obtained

$$\begin{aligned} S_Y \ = \ & \Big(MS_X \cap \{x \in X : |x_i^*(x)| \leq 1,\ \forall\ i \in I\}\Big) \\ & \bigcup \Big(MB_X \cap \{x \in X : \sup\{|x_i^*(x)| :\ i \in I\} = 1\}\Big). \end{aligned}$$

Let $E = MS_X \cap \{x \in X : |x_i^*(x)| \leq 1,\ \forall\ i \in I\}$. Hence for every $x \in E$ there is $x^* \in S_{X^*}$ such that $x^*(x) = M$. We choose such a functional x^* and define

$$F(x) = \frac{x^*}{M} \quad \text{for every} \quad x \in E.$$

Clearly $F(x) \in S_{Y^*}$ for every $x \in E$.

On the other hand, for every $i \in I$, it is clear that $x_i \in B$ and $|||x_i^*||| \leq 1$. Since $x_i^*(x_i) = 1$, then $1 = |||x_i||| = |||x_i^*|||$, for every $i \in I$. From the description of S_Y we obtain that the set $F(E) \cup \{x_i^* : i \in I\}$ is a 1-norming set for Y, so condition v) in the assumptions of Theorem 2.4 is satisfied. Conditions i) and ii) are also trivially satisfied for $\rho = \dfrac{1}{M}$. Now we check that E is uniformly strongly exposed by F, which is condition iii) in Theorem 2.4.

Let $\varepsilon > 0$. If we assume that $e \in E$, $y \in B_Y$ and $\text{Re}\ F(e)(y) > 1 - 2\delta_X(\varepsilon)$, since $MF(e) \in S_{X^*}$, $\frac{y}{M} \in B_X$ and $MF(e)(\frac{y}{M}) = F(e)(y)$, in view of condition (2.26) we obtain that $\|\frac{y}{M} - \frac{e}{M}\| < \varepsilon$, so $\|y - e\| < \varepsilon M$. Since both norms $\|\ \|$ and $|||\ |||$ are equivalent, we checked that E is uniformly strongly exposed by F.

Finally we need only to check condition iv) in Theorem 2.4. For every $e \in E$ and $i \in I$ we know that $|F(e)(x_i)| \leq \|F(e)\|\|x_i\| = \|F(e)\| = \frac{1}{M} < 1$.

(c) Since $I \neq \varnothing$, we can choose $i_0 \in I$. We are also assuming that $\dim X > 1$, so there is $y \in \text{Ker}\ x_{i_0}^* \cap S_X$. Since $\|x_{i_0} + ty\| \leq 1 + |t|$, then for any scalar t satisfying $|t| < \min\{M - 1, \frac{1}{K}\}$ we have $x_{i_0} + ty \in B$ and $x_{i_0}^*(x_{i_0} + ty) = 1$, so $x_{i_0} + ty \in S_Y$. Hence Y is not strictly convex. In the complex case we also proved that Y is not even $\mathbb{C}$-rotund.

(d) If the biorthogonal system is not total, then there is $e \in S_Y$ such that $x_i^*(e) = 0$ for every $i \in I$. We checked in b) that $F(E) \cup \{x_i^* : i \in I\}$ is a 1-norming set for Y, so by the description of S_Y we deduce that $e \in E$. Since we already proved that e is strongly exposed by $F(e)$ whenever X is uniformly convex, then Y contains LUR points. In view of [18, Proposition 3.3] Y does not satisfy the property β of Lindenstrauss. □

A result due to Auerbach states that every (non-trivial) finite-dimensional normed space X contains a biorthogonal system $\{(x_j, x_j^*) : j \in J\} \subset S_X \times S_{X^*}$ and such that $\{x_j : j \in J\}$ is a Hamel basis of X (see for instance [14, Proposition 20.12]). Hence for every finite-dimensional strictly convex space, the previous result can be applied by using any proper subset $I \subset J$. Notice that in this case we have $K = 1$.

In the case that X is infinite-dimensional and separable, a well-known result due to Pełczyński states that for every $\varepsilon > 0$, X admits a fundamental total biorthogonal system $\{(x_n, x_n^*) : n \in \mathbb{N}\}$ such that

$\sup\{\|x_n\|\|x_n^*\| : n \in \mathbb{N}\} < 1 + \varepsilon$ [19, Theorem 1]. In the non-separable case we can simply apply the previous result to an infinite-dimensional separable subspace of X and use Hahn–Banach extensions of the functionals. After removing one of the functionals of the biorthogonal system, if needed, we can apply the procedure in Proposition 2.7.

So we obtain the following result.

Corollary 2.8. *Let X be a uniformly convex Banach space with $\dim X \geq 2$. There is an equivalent norm $|||\ |||$ on X arbitrarily close to the original norm of X and also satisfying the following two conditions*

(1) *The assumptions of Theorem 2.4.*
(2) *The space $(X, |||\ |||)$ is neither uniformly convex nor satisfies the property β of Lindenstrauss.*

In the argument used to deduce the previous result, we consider a countable set I to apply Proposition 2.7. Of course, the set I can be uncountable. Indeed, if I is any infinite set and $X = \ell_p(I)$ for $1 < p < \infty$, as usual, we denote by $e_i(j) = \delta_i^j$ for every $i, j \in I$. Let $\{e_j^* : j \in I\}$ be the set of coordinate functionals in $\big(\ell_p(I)\big)^*$. For that space we can apply the procedure described in Proposition 2.7 for the biorthogonal system $\{(e_i, e_i^*) : i \in I\}$, even if this system is total and all the statements of Proposition 2.7 hold true for $(\ell_p(I), |||\ |||)$. So this space satisfies all the assumptions of Theorem 2.4 and it is neither uniformly convex nor has the property β of Lindenstrauss.

Proceeding as in Proposition 2.7, we obtain a quite simple example of a Banach space Y satisfying conditions in Corollary 2.8. Indeed, take a uniformly convex space X with $\dim X \geq 2$, $x^* \in S_{X^*}$ and $0 < \alpha < 1$. Then the set $B = B_X \cap \{x \in X : |x^*(x)| \leq \alpha\}$ is the closed unit ball of an equivalent norm $|||\ |||$ on X that satisfies the assumptions of Theorem 2.4. The above mentioned example is $(Y, |||\ |||)$.

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