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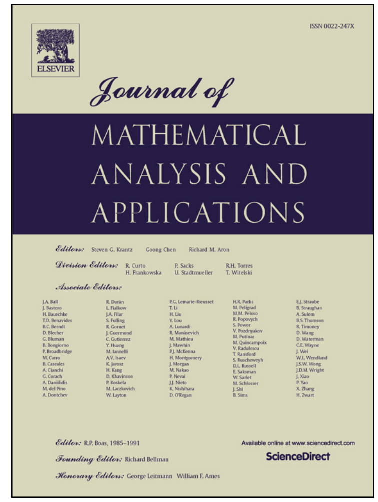


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# Ground state sign-changing solutions for a Schrödinger-Poisson system with a 3-linear growth nonlinearity

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**Abstract**: In this paper, we investigate the existence and asymptotic behavior of ground state sign-changing solutions to a class of Schrödinger-Poisson systems

$$\begin{cases} -\triangle u + V(x)u + \mu\phi u = \lambda f(x)u + |u|^2 u, & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $V$ is a smooth function, $f$ is nonnegative, $\mu > 0$, $\lambda < \lambda_1$ and $\lambda_1$ is the first eigenvalue of the problem $-\triangle u + V(x)u = \lambda f(x)u$ in $H$. With the help of the sign-changing Nehari manifold, we obtain that the Schrödinger-Poisson system possesses at least one ground state sign-changing solution $u_\mu$ for all $\mu > 0$ and each $\lambda < \lambda_1$. Moreover, we prove that its energy is strictly larger than twice that of ground state solutions. Besides, we give a convergence property of $u_\mu$ as $\mu \searrow 0$. This paper can be regarded as the complementary work of Shuai and Wang [23], Wang and Zhou [24].

**Key words:** Schrödinger-Poisson system; nonlocal term; sign-changing solution; ground state

## 1 Introduction and main results

In this paper, we are concerned the existence and asymptotic behavior of ground state sign-changing solutions of the following Schrödinger-Poisson system:

$$\begin{cases} -\triangle u + V(x)u + \mu\phi u = \lambda f(x)u + |u|^2 u, & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $V$ is a smooth function, $\mu > 0$ and $f$ is nonnegative.

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System (1) stems from quantum mechanics models and semiconductor theory, and it has been studied extensively. From a physical standpoint, Schrödinger-Poisson systems describe systems of identical charged particles interacting each other if magnetic effects could be ignored and their solutions are standing waves. The nonlinearity models the interaction between the particles. System (1) is coupled with a Poisson equation, which implies that the potential is determined by the charge of the wave function. The term $\mu\phi u$ considers the interaction with the electric field. For more details about the mathematical and physical background of system (1), please refer to the papers [2, 3, 4, 5] and the references therein.

When $\mu = 0$ in system (1), it reduces to the classic semilinear elliptic problem. Bartsch, Weth and Willem [8] have obtained a ground state sign-changing solution. After that many authors are devoted to the investigations for a variety of elliptic equations on a bounded domain or the whole space. Remarkably, system (1) is nonlocal because of the presence of the term $\mu\phi u$, which causes that the energy functional has totally different properties from the case $\mu = 0$. This phenomenon provokes some mathematical difficulties, which make the study of system (1) particularly interesting.

Schrödinger-Poisson systems have been paid much attention to various authors, especially on the existence of positive solutions, multiple solutions, ground state solutions, semiclassical states and the concentration behavior of positive solutions, see for example, [10, 12, 17, 21, 22, 27] and the references therein. However, regarding the existence of sign-changing solutions for the following Schrödinger-Poisson system

$$\begin{cases} -\triangle u + V(x)u + \mu\phi u = h(u), & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{2}$$

to the best of our knowledge, there are a few results, such as [1, 11, 13, 14, 15, 16, 18, 19, 20, 23, 24, 26]. Especially, when the nonlinearity $h$ satisfies 3-superlinear growth condition, for example, if $h(u) = |u|^{p-1}u$ and $p \in (3, 5)$, $\mu = 1$, Liu, Wang and Zhang [19] proved that system (2) has infinitely many sign-changing solutions by using the method of invariant sets of descending flow. Wang and Zhou [24] obtained a sign-changing solution by means of a constraint variational method combining the Brouwer degree theory if the potential function $V$ satisfies the following condition:

$(V)$ $V \in C(\mathbb{R}^3, \mathbb{R}^+)$ such that $H \subset H^1(\mathbb{R}^3)$ and for all $s \in (2, 6)$, the continuous embedding $H \hookrightarrow L^s(\mathbb{R}^3)$ is compact, where

$$H := \left\{ u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty \right\}. \tag{3}$$

Noting that the method in [24] strongly depends on the fact that the nonlinearity $h$ is homogeneous, Shuai and Wang [23] used constraint variational methods and quantitative deformation lemma, and studied the existence and asymptotic behavior of ground state sign-changing solutions for system (2), if $\mu$ is a positive parameter, $V$ satisfies condition $(V)$ and $h \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following conditions:

$(h_1)$ $h(s) = o(|s|)$ as $s \to 0$;
$(h_2)$ $\lim_{s\to+\infty} \frac{h(s)}{s^5} = 0$;
$(h_3)$ $\lim_{s\to+\infty} \frac{H(s)}{s^4} = +\infty$, where $H(s) = \int_0^s h(t)dt$;
$(h_4)$ $\frac{h(s)}{|s|^3}$ is an increasing function of $s \in \mathbb{R}\backslash\{0\}$.

Recently, Chen and Tang [11] obtained the similar results with Shuai and Wang [23] if Schrödinger-Poisson system (2) involves the non-autonomous nonlinearity $K(x)h(u)$ and the vanishing potential function by replacing condition $(h_4)$ with the weaker condition

$(h_5)$ there exists a $\theta_0 \in (0, 1)$ such that for all $t > 0$, $x \in \mathbb{R}^3$ and $\tau \in \mathbb{R}\backslash\{0\}$,

$$K(x)\left[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3}\right]\operatorname{sign}(1-t) + \theta_0 V(x)\frac{|1-t^2|}{(t\tau)^2} \geq 0.$$

Here, we must also point out that Chen and Tang [11] and Shuai and Wang [23] investigated the existence and asymptotic behavior of ground state sign-changing solutions for system (2) when the nonlinearity satisfies 3-superlinear growth condition at infinity and superlinear growth at zero. So, a natural question is whether these conditions can be relaxed to obtain the same results. Motivated by the previously mentioned works, in the present paper, we shall consider the case the nonlinearity satisfies 3-linear growth condition at infinity and linear growth at zero, in other words, we will investigate the existence and asymptotic behavior of ground state sign-changing solutions to system (1).

In order to avoid involving too much details for checking the compactness, we also assume that the potential function $V$ also satisfies condition $(V)$ and the weight function $f$ satisfies

$(f)$ $f \in L^{\frac{3}{2}}(\mathbb{R}^3)\backslash\{0\}$ is nonnegative.

When dealing with system (1), we delicately analyze the behaviors of the term $\mu\phi u$ and the term $|u|^2u$, and find that both $\mu\int_{\mathbb{R}^3}\phi u^2dx$ and $\int_{\mathbb{R}^3}u^4dx$ are 4-order, and neither of them can control each other. This observation indicates that system (1) may have sign-changing solutions for all $\mu > 0$. On the other hand, this observation also indicates that the methods used in above papers cannot be used here directly.

Next, we give some notations. Throughout this paper, let $H$ be the Sobolev spaces defined by (3) and equipped with the norms

$$\|u\| = \left(\int_{\mathbb{R}^3}|\nabla u|^2 + V(x)u^2dx\right)^{\frac{1}{2}},$$

$\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^{\frac{1}{2}}$ be the norms of $D^{1,2}(\mathbb{R}^3)$, $|\cdot|_s$ be the usual Lebesgue space $L^s(\mathbb{R}^3)$ norm, $S$ be the best Sobolev constant for the embedding of $H$ in $L^4(\mathbb{R}^3)$. In particular,

$$S = \inf_{u\in H\backslash\{0\}}\frac{\|u\|^2}{|u|_4^2}, \qquad |u|_4 \leq S^{-\frac{1}{2}}\|u\|. \tag{4}$$

For $u \in H^1(\mathbb{R}^3)$, Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that for all $v \in D^{1,2}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \nabla \phi_u \cdot \nabla v = \int_{\mathbb{R}^3} u^2 v dx,$$

that is, $\phi_u$ is the weak solution of $-\triangle \phi = u^2$. Furthermore,

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy,$$

$$L_{\phi_u}(u) = \int_{\mathbb{R}^3} \phi_u u^2 dx = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dx dy. \tag{5}$$

Define an energy functional $J_\mu$ on the space $H$ by

$$J_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\lambda}{2} \int_{\mathbb{R}^3} f(x) u^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} u^4 dx, \quad \forall u \in H.$$

Then $J_\mu$ is well defined on $H$ and is of $C^1$, and for each $u,\ v \in H$, we have

$$\langle J'_\mu(u), v \rangle \;\; = \;\; \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x) uv + \mu \phi_u uv)\, dx - \int_{\mathbb{R}^3} (\lambda f(x) uv dx + |u|^2 uv) dx,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality. It is standard to verify that the weak solutions of system (1) correspond to the critical points of the functional $J_\mu$. Furthermore, if $u \in H$ is a critical point of $J_\mu$, $(u, \phi_u)$ is a solution of system (1). Since $\phi_u$ is nonnegative, $(u, \phi_u)$ is a sign-changing solution of system (1) if and only if $u$ is a critical point of $J_\mu$ and $u^\pm \neq 0$, where

$$u^+(x) := \max\{u(x), 0\} \quad \text{and} \quad u^-(x) := \min\{u(x), 0\}.$$

Here a solution is called a ground state (or least energy) sign-changing one if it possesses the least energy among all sign-changing solutions. It follows from (5) and Fubini theorem that

$$L_{\phi_{u^+}}(u^-) = \int_{\mathbb{R}^3} \phi_{u^+} |u^-|^2 dx = \int_{\mathbb{R}^3} \phi_{u^-} |u^+|^2 dx = L_{\phi_{u^-}}(u^+).$$

By a simple calculation, we can obtain that

$$J_\mu(u) = J_\mu(u^+) + J_\mu(u^-) + \frac{\mu}{2} L_{\phi_{u^+}}(u^-), \tag{6}$$

$$\langle J'_\mu(u), u^+ \rangle = \langle J'_\mu(u^+), u^+ \rangle + \mu L_{\phi_{u^+}}(u^-), \tag{7}$$

$$\langle J'_\mu(u), u^- \rangle = \langle J'_\mu(u^-), u^- \rangle + \mu L_{\phi_{u^+}}(u^-). \tag{8}$$

When $\mu = 0$, system (1) does not depend on the nonlocal term $\mu\phi_u u$ any more, i.e., it becomes

$$-\triangle u + V(x)u = \lambda f(x)u + |u|^2 u, \quad x \in \mathbb{R}^3, \tag{9}$$

which corresponds to the energy functional $J_0 : H \to \mathbb{R}$ by

$$J_0(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{\mathbb{R}^3} f(x)u^2 dx - \frac{1}{4}\int_{\mathbb{R}^3} u^4 dx, \quad \forall u \in H.$$

Similarly, $J_0$ is well defined on $H$ and is of $C^1$, and for any $u,\ v \in H$,

$$\langle J_0'(u), v\rangle \ = \ \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)dx - \lambda \int_{\mathbb{R}^3} f(x)uvdx - \int_{\mathbb{R}^3} |u|^2 uvdx.$$

From (6), (7), (8), it is easy to see that there are some essential differences in studying the sign-changing solutions for system (1) between $\mu > 0$ and $\mu = 0$. Therefore, the methods of seeking sign-changing solutions for problems as (9) seem to be not applicable to system (1).

In order to obtain the existence results of sign-changing solutions, we will consider the following minimization problems:

$$m_0 := \inf\{J_0(u) : u \in \mathbf{M}_0\},\ m_\mu := \inf\{J_\mu(u) : u \in \mathbf{M}_\mu\},$$

$$\mathbf{M}_\mu = \{u \in H : u^\pm \neq 0,\ \langle J_\mu'(u), u^+\rangle = \langle J_\mu'(u), u^-\rangle = 0\},$$

$$\mathbf{M}_0 = \{u \in H : u^\pm \neq 0,\ \langle J_0'(u), u^+\rangle = \langle J_0'(u), u^-\rangle = 0\}.$$

It is easy to see that if $(u, \phi_u)$ is a sign-changing solution of system (1), one can get $u \in \mathbf{M}_\mu$. Clearly, $\mathbf{M}_\mu$ is a much smaller set than $H$ and so it is easier to study $J_\mu$ on $\mathbf{M}_\mu$. The minimizers of $\mathbf{M}_\mu$ and $\mathbf{M}_0$ are corresponding to the sign-changing solutions for system (1) and problem (9), respectively.

Another aim of the paper is to show the energy of any sign-changing solutions of system (1) is strictly larger than twice that of the ground state solutions of system (1), and establish the convergence of the ground state sign-changing solution as $\mu \searrow 0$. As usual, we seek the ground state solutions of system (1) and problem (9) as minimizers of corresponding energy functionals $J_\mu$ and $J_0$ on the following Nehari manifolds:

$$\mathbf{N}_\mu = \{u \in H\backslash\{0\} : \langle J_\mu'(u), u\rangle = 0\},$$

$$\mathbf{N}_0 = \{u \in H\backslash\{0\} : \langle J_0'(u), u\rangle = 0\},$$

respectively. Similarly, let

$$c_0 := \inf\{J_0(u) : u \in \mathbf{N}_0\},\ c_\mu := \inf\{J_\mu(u) : u \in \mathbf{N}_\mu\}.$$

Defined by $\lambda_1$ the first eigenvalue of the problem $-\triangle u + V(x)u = \lambda f(x)u$ in $H$ under hypothesis $(f)$, our main results can be stated as follows.

**Theorem 1.** *If hypotheses $(f)$ and $(V)$ hold, $\mu > 0$ and $\lambda < \lambda_1$, system (1) has at least one ground state sign-changing solution which changes sign only once, and $m_b > 2c_b$.*

**Theorem 2.** *If hypotheses $(f)$ and $(V)$ hold, $\mu > 0$ and $\lambda < \lambda_1$, for each sequence $\{\mu_n\}$ with $\mu_n \searrow 0$ as $n \to \infty$, there exists a subsequence, still denoted by $\{\mu_n\}$, such that $u_{\mu_n}$ convergent to $u_0$ strongly in $H$, where $u_0$ is a ground state sign-changing solution of problem (9) which changes sign only once.*

**Remark 3.** Our results make good explanation for the existence and asymptotic behavior of ground state sign-changing solutions to system (1). However, if $N = 4$, system (1) involves the critical nonlinearity $|u|^2u$ because $2^* = 4$. Whether system (1) admits any sign-changing solution for $\lambda < \lambda_1$ and $\mu > 0$ or not remains incognito. So, we propose an open question whether system (1) has sign-changing solutions if $N = 4$.

**Remark 4.** Comparing with [11, 18, 23, 24], we investigate the existence of ground state sign-changing solutions to system (1) and give a convergence property of ground state sign-changing solutions as $\mu \searrow 0$ when the nonlinearity satisfies 3-linear growth condition at infinity and linear growth at zero. However, Chen and Tang [11], Liang, Xu and Zhu [18], Shuai and Wang [23], Wang and Zhou [24] considered the case the nonlinearity satisfies 3-superlinear growth condition at infinity and superlinear growth at zero. Since both $\mu \int_{\mathbb{R}^3} \phi u^2 dx$ and $\int_{\mathbb{R}^3} u^4 dx$ are 4-order, we introduce some new and more intuitive ideas to prove that $\mathbf{M}_\mu \neq \emptyset$. Furthermore, our results can be regarded as the supplementary work of [23] and [24].

We organize this paper as follows. In Section 2 we present some notations and prove some useful preliminary lemmas which pave the way for getting one ground state sign-changing solution. Then Section 3 is devoted to proving Theorem 1 and Theorem 2.

## 2 Some preliminary lemmas

In this section, we give some preliminary lemmas which are crucial and pave the way for proving our results. We begin this section by introducing some lemma familiar with us.

**Lemma 5.** *Assume that hypotheses $(f)$ and $(V)$ hold. Then the functional $F : u \in H \mapsto \int_{\mathbb{R}^3} f(x)u^2 dx$ is weakly continuous. For each $v \in H$, $G : u \in H \mapsto \int_{\mathbb{R}^3} f(x)uvdx$ is also weakly continuous.*

**Lemma 6.** *Assume that condition $(V)$ holds. Then for any $u \in H$, the following statements are valid.*
*(i) There exists $C > 0$ such that $\int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx \leq C\|u\|^4$ and*

$$L_{\phi_u}(u) \leq C|u|_\alpha^4 \leq C\|u\|^4, \quad \text{where } \alpha = \frac{12}{5}.$$

*(ii) If $u_n \rightharpoonup u$ in $H$, we have $\phi_{u_n} \rightharpoonup \phi_u$ in $H$ and*

$$\lim_{n\to\infty} L_{\phi_{u_n}}(u_n) = L_{\phi_u}(u).$$

Now we check that the set $\mathbf{M}_\mu \neq \emptyset$ if there exists $u \in H$ with some conditions.

**Lemma 7.** *If $\mu > 0$, $\lambda < \lambda_1$, $u \in H$ satisfies $u^\pm \neq 0$ and*

$$\begin{cases} \mu L_{\phi_{u^+}}(u^+) + \mu L_{\phi_{u^+}}(u^-) < \int_{\mathbb{R}^3} |u^+|^4 dx, \\ \mu L_{\phi_{u^-}}(u^-) + \mu L_{\phi_{u^+}}(u^-) < \int_{\mathbb{R}^3} |u^-|^4 dx, \end{cases} \tag{10}$$

*there is a unique pair $(s_u,\ t_u)$ of positive numbers such that $s_u u^+ + t_u u^- \in \mathbf{M}_\mu$ and*

$$J_\mu(s_u u^+ + t_u u^-) = \max_{s,t\geq 0} J_\mu(su^+ + tu^-).$$

**Proof.** Let $\lambda < \lambda_1$, $u \in H$ with $u^\pm \neq 0$ and (10), then $su^+ + tu^-$ is contained in $\mathbf{M}_\mu$ if and only if

$$\begin{cases} s^2\|u^+\|^2 + \mu s^4 L_{\phi_{u^+}}(u^+) + \mu s^2 t^2 L_{\phi_{u^+}}(u^-) = \lambda s^2 \int_{\mathbb{R}^3} f(x)|u^+|^2 dx + s^4 \int_{\mathbb{R}^3} |u^+|^4 dx, \\ t^2\|u^-\|^2 + \mu t^4 L_{\phi_{u^-}}(u^-) + \mu s^2 t^2 L_{\phi_{u^+}}(u^-) = \lambda t^2 \int_{\mathbb{R}^3} f(x)|u^-|^2 dx + t^4 \int_{\mathbb{R}^3} |u^-|^4 dx. \end{cases}$$

Hence, we only need to show that there is only one positive solution $(S, T)$ to the following system

$$\begin{cases} S\left(\int_{\mathbb{R}^3} |u^+|^4 dx - \mu L_{\phi_{u^+}}(u^+)\right) - \mu T L_{\phi_{u^+}}(u^-) = \|u^+\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^+|^2 dx, \\ T\left(\int_{\mathbb{R}^3} |u^-|^4 dx - \mu L_{\phi_{u^-}}(u^-)\right) - \mu S L_{\phi_{u^+}}(u^-) = \|u^-\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^-|^2 dx. \end{cases} \tag{11}$$

It is easy to see from (10) that

$$\begin{cases} \mu L_{\phi_{u^+}}(u^-) < \int_{\mathbb{R}^3} |u^+|^4 dx - \mu L_{\phi_{u^+}}(u^+), \\ \mu L_{\phi_{u^+}}(u^-) < \int_{\mathbb{R}^3} |u^-|^4 dx - \mu L_{\phi_{u^-}}(u^-). \end{cases}$$

Consequently,

$$D = \begin{vmatrix} \int_{\mathbb{R}^3} |u^+|^4 dx - \mu L_{\phi_{u^+}}(u^+) & -\mu L_{\phi_{u^+}}(u^-) \\ -\mu L_{\phi_{u^+}}(u^-) & \int_{\mathbb{R}^3} |u^-|^4 dx - \mu L_{\phi_{u^-}}(u^-) \end{vmatrix} > 0.$$

Together with $\lambda < \lambda_1$, we have $\|u^\pm\|^2 > \lambda \int_{\mathbb{R}^3} f(x)|u^\pm|^2 dx$ and

$$D_S = \begin{vmatrix} \|u^+\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^+|^2 dx & -\mu L_{\phi_{u^+}}(u^-) \\ \|u^-\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^-|^2 dx & \int_{\mathbb{R}^3} |u^-|^4 dx - \mu L_{\phi_{u^-}}(u^-) \end{vmatrix} > 0,$$

$$D_T = \begin{vmatrix} \int_{\mathbb{R}^3} |u^+|^4 dx - \mu L_{\phi_{u^+}}(u^+) & \|u^+\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^+|^2 dx \\ -\mu L_{\phi_{u^+}}(u^-) & \|u^-\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^-|^2 dx \end{vmatrix} > 0.$$

Let $S = \frac{D_S}{D}$ and $T = \frac{D_T}{D}$, then $(S,T) \in (0,+\infty) \times (0,+\infty)$ is the unique solution to system (11). Choosing $s_u = \sqrt{S}$ and $t_u = \sqrt{T}$, we can obtain that $(s_u,\ t_u)$ is the unique pair of positive numbers such that $s_u u^+ + t_u u^- \in \mathbf{M}_\mu$.

Furthermore, since

$$\begin{aligned} J_\mu(su^+ + tu^-) =& \frac{s^2}{2}\|u^+\|^2 + \frac{\mu s^4}{4} L_{\phi_{u^+}}(u^+) + \frac{\mu s^2 t^2}{2} L_{\phi_{u^+}}(u^-) + \frac{t^2}{2}\|u^-\|^2 \\ & + \frac{\mu t^4}{4} L_{\phi_{u^-}}(u^-) - \frac{\lambda s^2}{2}\int_{\mathbb{R}^3} f(x)|u^+|^2dx - \frac{s^4}{4}\int_{\mathbb{R}^3} |u^+|^4 dx \\ & - \frac{\lambda t^2}{2}\int_{\mathbb{R}^3} f(x)|u^-|^2dx - \frac{t^4}{4}\int_{\mathbb{R}^3} |u^-|^4 dx, \end{aligned}$$

it is not difficult to verify that

$$\begin{aligned} \frac{\partial^2 J_\mu(su^+ + tu^-)}{\partial s^2} &= \left(\|u^+\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^+|^2dx\right) + \mu t^2 L_{\phi_{u^+}}(u^-) \\ & \quad + 3s^2\left(\mu L_{\phi_{u^+}}(u^+) - \int_{\mathbb{R}^3} |u^+|^4dx\right), \\ \frac{\partial^2 J_\mu(su^+ + tu^-)}{\partial t^2} &= \left(\|u^-\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^-|^2dx\right) + \mu s^2 L_{\phi_{u^+}}(u^-) \\ & \quad + 3t^2\left(\mu L_{\phi_{u^-}}(u^-) - \int_{\mathbb{R}^3} |u^-|^4dx\right). \end{aligned}$$

From the fact that $(s_u^2,\ t_u^2)$ is the solution of system (11), we have

$$\frac{\partial^2 J_\mu(su^+ + tu^-)}{\partial s^2}\Big|_{(s_u,\ t_u)} = -2s_u^2\left(\int_{\mathbb{R}^3} |u^+|^4dx - \mu L_{\phi_{u^+}}(u^+)\right) < 0, \tag{12}$$

$$\frac{\partial^2 J_\mu(su^+ + tu^-)}{\partial t^2}\Big|_{(s_u,\ t_u)} = -2t_u^2\left(\int_{\mathbb{R}^3} |u^-|^4dx - \mu L_{\phi_{u^-}}(u^-)\right) < 0, \tag{13}$$

$$\frac{\partial^2 J_\mu(su^+ + tu^-)}{\partial s \partial t}\Big|_{(s_u,\ t_u)} = 2\mu s_u t_u L_{\phi_{u^+}}(u^-) > 0. \tag{14}$$

We consider the Hessian matrix of $J_\mu(su^+ + tu^-)$, i.e.

$$H(s_u, t_u) = \begin{pmatrix} \frac{\partial^2 J_\mu(su^+ + tu^-)}{\partial s^2} & \frac{\partial^2 J_\mu(su^+ + tu^-)}{\partial s \partial t} \\ \frac{\partial^2 J_\mu(su^+ + tu^-)}{\partial s \partial t} & \frac{\partial^2 J_\mu(su^+ + tu^-)}{\partial t^2} \end{pmatrix}_{(s_u, t_u)}.$$

Combining with (10), one can obtain that

$$\begin{aligned} \det H(s_u, t_u) =& 4s_u^2 t_u^2 \left(\int_{\mathbb{R}^3} |u^+|^4dx - \mu L_{\phi_{u^+}}(u^+)\right)\left(\int_{\mathbb{R}^3} |u^-|^4dx - \mu L_{\phi_{u^-}}(u^-)\right) \\ & - 4\mu^2 s_u^2 t_u^2 (L_{\phi_{u^+}}(u^-))^2 \\ =& 4s_u^2 t_u^2 \Bigg[\left(\int_{\mathbb{R}^3} |u^+|^4dx - \mu L_{\phi_{u^+}}(u^+)\right)\left(\int_{\mathbb{R}^3} |u^-|^4dx - \mu L_{\phi_{u^-}}(u^-)\right) \\ & - \left(\mu L_{\phi_{u^+}}(u^-)\right)^2\Bigg] \\ >& 0. \end{aligned} \tag{15}$$

It is easy to see that the maximum point can not be achieved on the boundary of $\mathbb{R}^2_+$. Therefore, it follows from (12),(13),(14),(15) that

$$J_\mu(s_u u^+ + t_u u^-) = \max_{s,t\geq 0} J_\mu(su^+ + tu^-),$$

and we complete the proof. □

**Lemma 8.** *Assume that $\lambda < \lambda_1$ and $u \in \mathbf{M}_\mu$, then (10) holds.*

**Proof.** Let $u \in \mathbf{M}_\mu$, we have from the definition of $\mathbf{M}_\mu$ that $u^\pm \neq 0$ and

$$\begin{cases} \|u^+\|^2 + \mu L_{\phi_{u^+}}(u^+) + \mu L_{\phi_{u^+}}(u^-) = \lambda \int_{\mathbb{R}^3} f(x)|u^+|^2 dx + \int_{\mathbb{R}^3} |u^+|^4 dx, \\ \|u^-\|^2 + \mu L_{\phi_{u^-}}(u^-) + \mu L_{\phi_{u^+}}(u^-) = \lambda \int_{\mathbb{R}^3} f(x)|u^-|^2 dx + \int_{\mathbb{R}^3} |u^-|^4 dx. \end{cases} \tag{16}$$

Since $\lambda < \lambda_1$ and $\lambda_1$ is the first eigenvalue of the problem $-\triangle u + V(x)u = \lambda f(x)u$ in $H$, we can obtain that

$$\|u^+\|^2 > \lambda \int_\Omega f(x)|u^+|^2 dx, \quad \|u^-\|^2 > \lambda \int_\Omega f(x)|u^-|^2 dx,$$

which implies from (16) that

$$\begin{cases} \mu L_{\phi_{u^+}}(u^+) + \mu L_{\phi_{u^+}}(u^-) < \int_{\mathbb{R}^3} |u^+|^4 dx, \\ \mu L_{\phi_{u^-}}(u^-) + \mu L_{\phi_{u^+}}(u^-) < \int_{\mathbb{R}^3} |u^-|^4 dx, \end{cases}$$

Then we have completed the proof. □

**Lemma 9.** *Assume that $\mu > 0$, $\lambda < \lambda_1$, $u \in H$ with $u^\pm \neq 0$, we have*
*(i) if $\langle J'_\mu(u), u^\pm \rangle \leq 0$, there is a unique pair $(s_u, t_u) \in (0,1] \times (0,1]$ such that*

$$s_u u^+ + t_u u^- \in \mathbf{M}_\mu;$$

*(ii) if (10) holds and $\langle J'_\mu(u), u^\pm \rangle \geq 0$, there is a unique pair $(s_u, t_u) \in [1,+\infty) \times [1,+\infty)$ such that*

$$s_u u^+ + t_u u^- \in \mathbf{M}_\mu.$$

**Proof.** (i) If $u \in H$ with $u^\pm \neq 0$ and $\langle J'_\mu(u), u^\pm \rangle \leq 0$, we have

$$\begin{cases} \|u^+\|^2 + \mu L_{\phi_{u^+}}(u^+) + \mu L_{\phi_{u^+}}(u^-) \leq \lambda \int_{\mathbb{R}^3} f(x)|u^+|^2 dx + \int_{\mathbb{R}^3} |u^+|^4 dx, \\ \|u^-\|^2 + \mu L_{\phi_{u^-}}(u^-) + \mu L_{\phi_{u^+}}(u^-) \leq \lambda \int_{\mathbb{R}^3} f(x)|u^-|^2 dx + \int_{\mathbb{R}^3} |u^-|^4 dx, \end{cases}$$

then

$$\begin{cases} \|u^+\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^+|^2 dx \leq \int_{\mathbb{R}^3} |u^+|^4 dx - \mu L_{\phi_{u^+}}(u^+) - \mu L_{\phi_{u^+}}(u^-), \\ \|u^-\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^-|^2 dx \leq \int_{\mathbb{R}^3} |u^-|^4 dx - \mu L_{\phi_{u^-}}(u^-) - \mu L_{\phi_{u^+}}(u^-). \end{cases} \tag{17}$$

Since $\lambda < \lambda_1$, it is clear that $\|u^{\pm}\|^2 > \lambda \int_{\mathbb{R}^3} f(x)|u^{\pm}|^2 dx$, then

$$\begin{cases} \mu L_{\phi_{u^+}}(u^+) + \mu L_{\phi_{u^+}}(u^-) < \int_{\mathbb{R}^3} |u^+|^4 dx, \\ \mu L_{\phi_{u^-}}(u^-) + \mu L_{\phi_{u^+}}(u^-) < \int_{\mathbb{R}^3} |u^-|^4 dx. \end{cases}$$

By Lemma 7, there is a unique pair $(s_u,\ t_u)$ of positive numbers such that

$$s_u u^+ + t_u u^- \in \mathbf{M}_\mu.$$

It means that $(s_u^2,\ t_u^2)$ is the solution of system (11). Similar to the argument of Lemma 7, we have from (17) that

$$\begin{aligned} D_{s_u^2} &= \left(\|u^+\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^+|^2 dx\right)\left(\int_{\mathbb{R}^3} |u^-|^4 dx - \mu L_{\phi_{u^-}}(u^-)\right) \\ &\quad + \mu L_{\phi_{u^+}}(u^-)\left(\|u^-\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^-|^2 dx\right) \\ &\le \left(\int_{\mathbb{R}^3} |u^+|^4 dx - \mu L_{\phi_{u^+}}(u^+) - \mu L_{\phi_{u^+}}(u^-)\right)\left(\int_{\mathbb{R}^3} |u^-|^4 dx - \mu L_{\phi_{u^-}}(u^-)\right) \\ &\quad + \mu L_{\phi_{u^+}}(u^-)\left(\int_{\mathbb{R}^3} |u^-|^4 dx - \mu L_{\phi_{u^-}}(u^-) - \mu L_{\phi_{u^+}}(u^-)\right). \\ &= \left(\int_{\mathbb{R}^3} |u^+|^4 dx - \mu L_{\phi_{u^+}}(u^+)\right)\left(\int_{\mathbb{R}^3} |u^-|^4 dx - \mu L_{\phi_{u^-}}(u^-)\right) - \mu^2 (L_{\phi_{u^+}}(u^-))^2 \\ &= \ D. \end{aligned}$$

Therefore, $s_u^2 = \frac{D_{s_u^2}}{D} \le 1$. Similarly, $t_u^2 = \frac{D_{t_u^2}}{D} \le 1$. Then there is a unique pair $(s_u, t_u) \in (0,1] \times (0,1]$ such that $s_u u^+ + t_u u^- \in \mathbf{M}_\mu$.

(ii) If (10) holds, by Lemma 7, there is a unique pair $(s_u,\ t_u)$ of positive numbers such that $s_u u^+ + t_u u^- \in \mathbf{M}_\mu$. Similar to the proof of (i), we can obtain from $\langle J_\mu'(u), u^{\pm}\rangle \ge 0$ that (ii) holds. $\square$

**Lemma 10.** *If $\lambda < \lambda_1$, for any $u \in H$ with $\mu L_{\phi_u}(u) < \int_{\mathbb{R}^3} u^4 dx$, there exists a unique $\bar{s}_u > 0$ such that $\bar{s}_u u \in \mathbf{N}_\mu$. Moreover, $J_\mu(\bar{s}_u u) > J_\mu(su)$ for all $s \ge 0$ and $s \neq \bar{s}_u$.*

**Proof.** If $\lambda < \lambda_1$ and $u \in H$ satisfies $\mu L_{\phi_u}(u) < \int_{\mathbb{R}^3} u^4 dx$. Clearly, $su \in \mathbf{N}_\mu$ if and only if

$$s^2\|u\|^2 + s^4 \mu L_{\phi_u}(u) = \lambda s^2 \int_{\mathbb{R}^3} f(x) u^2 dx + s^4 \int_{\mathbb{R}^3} u^4 dx,$$

it is easy to see that there exists a unique $\bar{s}_u = \left(\frac{\|u\|^2 - \lambda \int_{\mathbb{R}^3} f(x)u^2 dx}{\int_{\mathbb{R}^3} u^4 dx - \mu L_{\phi_u}(u)}\right)^{\frac{1}{2}}$ such that $\bar{s}_u u \in \mathbf{N}_\mu$. Furthermore, since

$$\frac{\partial J_\mu(su)}{\partial s} = s\left[\left(\|u\|^2 - \lambda \int_{\mathbb{R}^3} f(x)u^2 dx\right) - s^2\left(\int_{\mathbb{R}^3} u^4 dx - \mu L_{\phi_u}(u)\right)\right],$$

we have $J_\mu(\bar{s}_u u) > J_\mu(su)$ for all $s \ge 0$ and $s \neq \bar{s}_u$. $\square$

**Lemma 11.** *Assume that hypotheses* $(f)$, $(V)$ *hold,* $\mu > 0$ *and* $\lambda < \lambda_1$, *we have*
*(i)* $c_\mu > 0$ *is attained by some* $v_\mu \in \mathbf{N}_\mu$ *and* $v_\mu$ *is a constant sign critical point of* $J_\mu$;
*(ii)* $m_\mu > 0$ *is attained by some* $u_\mu \in \mathbf{M}_\mu$ *and* $u_\mu$ *is a sign-changing critical point of* $J_\mu$.

**Proof.** (i) Firstly, we will show that for all $\mu > 0$, there exists $u \in H$ such that $\mu L_{\phi_u}(u) < \int_{\mathbb{R}^3} u^4 dx$, which implies from Lemma 10 that $\mathbf{N}_\mu \neq \emptyset$. Otherwise, there exists $\mu_0 > 0$ such that for all $u \in H$, $\int_{\mathbb{R}^3} u^4 dx \le \mu_0 L_{\phi_u}(u) \le C\mu_0 |u|_\alpha^4$ by Lemma 6. Choosing $u_0 \neq 0$, $u_0 \in H$ and $u_0^\rho(x) = u_0(\frac{x}{\rho})$ for all $\rho > 0$ and $x \in \mathbb{R}^3$, we have that $u_0^\rho \in H$ and

$$\frac{|u_0^\rho|_\alpha^4}{\int_{\mathbb{R}^3} |u_0^\rho|^4 dx} = \frac{\left(\int_{\mathbb{R}^3} |u_0(\frac{x}{\rho})|^\alpha dx\right)^{\frac{4}{\alpha}}}{\int_{\mathbb{R}^3} |u_0(\frac{x}{\rho})|^4 dx} = \frac{\left(\int_{\mathbb{R}^3} |u_0(x)|^\alpha \rho^3 dx\right)^{\frac{4}{\alpha}}}{\int_{\mathbb{R}^3} |u_0(x)|^4 \rho^3 dx} = \frac{\rho^2 |u_0|_\alpha^4}{\int_{\mathbb{R}^3} |u_0|^4 dx}.$$

Therefore, $\frac{|u_0^\rho|_\alpha^4}{\int_{\mathbb{R}^3} |u_0^\rho|^4 dx} < \frac{1}{C\mu_0}$ for small enough $\rho > 0$, which contradicts our assumption.

Secondly, for each $u \in \mathbf{N}_\mu$, it follows from $\lambda < \lambda_1$ and Sobolev inequality (4) that

$$\|u\|^2 + \mu L_{\phi_u}(u) = \lambda \int_{\mathbb{R}^3} f(x) u^2 dx + \int_{\mathbb{R}^3} u^4 dx \le \frac{\lambda}{\lambda_1} \|u\|^2 + \frac{1}{S^2} \|u\|^4.$$

Then

$$\|u\| \ge S \left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{1}{2}} > 0,$$

and

$$J_\mu(u) = \frac{1}{4} \left(\|u\|^2 - \lambda \int_{\mathbb{R}^3} f(x) u^2 dx\right) \ge \frac{1}{4} \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^2.$$

Therefore,

$$c_\mu = \inf_{u \in \mathbf{N}_\mu} J_\mu(u) \ge \frac{1}{4} S^2 \left(1 - \frac{\lambda}{\lambda_1}\right)^2 > 0,$$

and $J_\mu$ is coercive and bounded below on $\mathbf{N}_\mu$ for all $\mu > 0$ and $\lambda < \lambda_1$.

Let $\{v_n\} \subset \mathbf{N}_\mu$ is a minimizing sequence for $J_\mu$. Obviously, $J_\mu(v_n) = J_\mu(|v_n|)$ and $|v_n| \in \mathbf{N}_\mu$ and therefore we can assume from the beginning that $v_n(x) \ge 0$ a.e. in $\mathbb{R}^3$ and for all $n$. It follows from the fact $J_\mu$ is coercive on $\mathbf{N}_\mu$ that the sequence $\{v_n\}$ is bounded in $H$, so that, up to subsequences, $v_n \rightharpoonup v_\mu$ in $H$ and $v_\mu(x) \ge 0$. We now prove that $v_n \to v_\mu$ strongly in $H$. Supposing the contrary, then $\|v_\mu\| < \liminf_{n\to\infty} \|v_n\|$, we get from Lemmas 5, 6 that

$$\|v_\mu\|^2 + \mu L_{\phi_{v_\mu}}(v_\mu) < \lambda \int_{\mathbb{R}^3} f(x) v_\mu^2 dx + \int_{\mathbb{R}^3} v_\mu^4 dx,$$

which means that $v_\mu(x) \not\equiv 0$ in $\mathbb{R}^3$ and $\mu L_{\phi_{v_\mu}}(v_\mu) < \int_{\mathbb{R}^3} v_\mu^4 dx$ by $\lambda < \lambda_1$. By Lemma 10, there exists a unique $\bar{s}_v > 0$ such that $\bar{s}_v v_\mu \in \mathbf{N}_\mu$. Moreover, $J_\mu(\bar{s}_v v_n) \le J_\mu(v_n)$ for all

$v_n \in \mathbf{N}_\mu$. Therefore, we obtain

$$
\begin{aligned}
c_\mu \leq & J_\mu(\bar{s}_v v_\mu) \\
= & \frac{1}{2}\|\bar{s}_v v_\mu\|^2 + \frac{\mu}{4} L_{\phi_{\bar{s}_v v_\mu}}(\bar{s}_v v_\mu) - \frac{\lambda}{2}\int_{\mathbb{R}^3} f(x)|\bar{s}_v v_\mu|^2 dx - \frac{1}{4}\int_{\mathbb{R}^3} |\bar{s}_v v_\mu|^4 dx \\
< & \liminf_{n\to\infty}\left[\frac{1}{2}\|\bar{s}_v v_n\|^2 + \frac{\mu}{4} L_{\phi_{\bar{s}_v v_n}}(\bar{s}_v v_n) - \frac{\lambda}{2}\int_{\mathbb{R}^3} f(x)|\bar{s}_v v_n|^2 dx - \frac{1}{4}\int_{\mathbb{R}^3} |\bar{s}_v v_n|^4 dx\right] \\
= & \liminf_{n\to\infty} J_\mu(\bar{s}_v v_n) \\
\leq & \liminf_{n\to\infty} J_\mu(v_n) = c_\mu,
\end{aligned}
$$

which leads to a contradiction. Thus $v_n \to v_\mu$ strongly in $H$, $v_\mu \in \mathbf{N}_\mu$ and $J_\mu(v_\mu) = c_\mu$. Similar to the argument in Brown and Zhang [9], we can conclude $v_\mu$ is a constant sign critical point of $J_\mu$.

(ii) Let us define $B_r(y) = \{x \in \mathbb{R}^3 : |x-y| < r\}$. For each fixed $\mu > 0$, we can pick up $w \in H$ with supp$w \subset B_1(x_0 + \frac{x_0}{\rho})$ and

$$\int_{\mathbb{R}^3} |w|^4 dx = \int_{B_1(x_0+\frac{x_0}{\rho})} |w|^4 dx \neq 0,$$

where $x_0 = (1,0,0)$ and $\rho = \min\left\{\frac{1}{2(C\mu)^{\frac{1}{2}}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}},\ 1\right\}$. Let $u_1(x) = w(x_0 + \frac{x}{\rho})$, then supp$u_1 \subset B_\rho(x_0)$, and we can obtain from Hölder's inequality that

$$
\begin{aligned}
\frac{\int_{\mathbb{R}^3} u_1^4 dx}{2|u_1|_\alpha^4} = & \frac{\int_{B_\rho(x_0)} |w(x_0+\frac{x}{\rho})|^4 dx}{2\left(\int_{B_\rho(x_0)} |w(x_0+\frac{x}{\rho})|^\alpha dx\right)^{\frac{4}{\alpha}}} = \frac{\int_{B_1(x_0+\frac{x_0}{\rho})} |w(x)|^4 dx}{2\rho^2\left(\int_{B_1(x_0+\frac{x_0}{\rho})} |w(x)|^\alpha dx\right)^{\frac{4}{\alpha}}} \\
\geq & \frac{\int_{B_1(x_0+\frac{x_0}{\rho})} |w(x)|^4 dx}{2\rho^2\left(\int_{B_1(x_0+\frac{x_0}{\rho})} |w(x)|^4 dx\right)\left(\int_{B_1(x_0+\frac{x_0}{\rho})} 1 dx\right)^{\frac{2}{3}}} \\
= & \frac{1}{2\rho^2(\frac{4\pi}{3})^{\frac{2}{3}}} \geq 2C\mu > C\mu.
\end{aligned}
\tag{18}
$$

Obviously $|u_1| \in H$ also satisfies (18) and therefore we can assume from the beginning that $u_1(x) \geq 0$ a.e. in $B_\rho(x_0)$. Similarly, we can pick up $u_2(x) = -u_1(-x)$ for all $x \in B_\rho(-x_0)$. Then $u_2 \in H$ such that $u_2(x) \leq 0$ for $x \in B_\rho(-x_0)$, supp$u_1 \cap$ supp$u_2 = \emptyset$ and

$$\frac{\int_{\mathbb{R}^3} u_2^4 dx}{2|u_2|_\alpha^4} = \frac{\int_{\mathbb{R}^3} u_1^4 dx}{2|u_1|_\alpha^4} > C\mu. \tag{19}$$

Let $u = u_1 + u_2$, we can obtain that $u \in H$, $u^+ = u_1$, $u^- = u_2$. It follows from Lemma 6 that

$$
\begin{cases}
\mu L_{\phi_{u^+}}(u^+) + \mu L_{\phi_{u^+}}(u^-) \leq C\mu|u_1|_\alpha^4 + C\mu|u_1|_\alpha^2|u_2|_\alpha^2, \\
\mu L_{\phi_{u^-}}(u^-) + \mu L_{\phi_{u^+}}(u^-) \leq C\mu|u_1|_\alpha^4 + C\mu|u_1|_\alpha^2|u_2|_\alpha^2.
\end{cases}
\tag{20}
$$

Furthermore, we can get from the definition of $u$, (18) and (19) that

$$\begin{cases} C\mu|u_1|_\alpha^4 + C\mu|u_1|_\alpha^2|u_2|_\alpha^2 = 2C\mu|u_1|_\alpha^4 < \int_{\mathbb{R}^3} u_1^4 dx, \\ C\mu|u_2|_\alpha^4 + C\mu|u_1|_\alpha^2|u_2|_\alpha^2 = 2C\mu|u_2|_\alpha^4 < \int_{\mathbb{R}^3} u_2^4 dx. \end{cases} \tag{21}$$

Combining (20) and (21) gives

$$\begin{cases} \mu L_{\phi_{u^+}}(u^+) + \mu L_{\phi_{u^+}}(u^-) < \int_{\mathbb{R}^3} u_1^4 dx = \int_{\mathbb{R}^3} |u^+|^4 dx, \\ \mu L_{\phi_{u^-}}(u^-) + \mu L_{\phi_{u^+}}(u^-) < \int_{\mathbb{R}^3} u_2^4 dx = \int_{\mathbb{R}^3} |u^-|^4 dx. \end{cases}$$

It implies from Lemma 7 that $\mathbf{M}_\mu \neq \emptyset$ for all $\mu > 0$. Clearly, it follows from $u \in \mathbf{M}_\mu \subset \mathbf{N}_\mu$ that $m_\mu \geq c_\mu > 0$.

Assume that $\{u_n\} \subset \mathbf{M}_\mu$ is a minimizing sequence for $J_\mu$, namely such that $J_\mu(u_n) \to m_\mu$. We have already observed that $J_\mu$ is coercive on $\mathbf{N}_\mu$, this implies that the sequence $\{u_n\}$ is bounded in $H$, going if necessary to a subsequence, still denoted by $\{u_n\}$, we can assume from condition (V) that there exists a $u_\mu \in H$ such that, for $n$ sufficiently large,

$$u_n^\pm \rightharpoonup u_\mu^\pm \ \ weakly \ \ in \ \ H,$$

$$u_n(x) \to u_\mu(x) \ \ almost \ \ everywhere \ \ on \ \ \mathbb{R}^3,$$

$$u_n^\pm \to u_\mu^\pm \ \ strongly \ \ in \ \ L^s(\mathbb{R}^3) \ \ for \ \ 2 < s < 6.$$

Next since $\{u_n\} \subset \mathbf{M}_\mu \subset \mathbf{N}_\mu$, we have $\langle J'_\mu(u_n), u_n^\pm \rangle = 0$, that is

$$\begin{cases} \|u_n^+\|^2 + \mu L_{\phi_{u_n^+}}(u_n^+) + \mu L_{\phi_{u_n^+}}(u_n^-) = \lambda \int_{\mathbb{R}^3} f(x)|u_n^+|^2 dx + \int_{\mathbb{R}^3} |u_n^+|^4 dx, \\ \|u_n^-\|^2 + \mu L_{\phi_{u_n^-}}(u_n^-) + \mu L_{\phi_{u_n^+}}(u_n^-) = \lambda \int_{\mathbb{R}^3} f(x)|u_n^-|^2 dx + \int_{\mathbb{R}^3} |u_n^-|^4 dx. \end{cases}$$

Then

$$\|u_n^\pm\|^2 \leq \lambda \int_{\mathbb{R}^3} f(x)|u_n^\pm|^2 dx + \int_{\mathbb{R}^3} |u_n^\pm|^4 dx,$$

similarly, we can obtain that

$$\|u_n^\pm\|^2 \geq S^2 \left(1 - \frac{\lambda}{\lambda_1}\right) > 0. \tag{22}$$

Combining with $\lambda < \lambda_1$, we also have

$$\left(1 - \frac{\lambda}{\lambda_1}\right) \|u_n^\pm\|^2 \leq \int_{\mathbb{R}^3} |u_n^\pm|^4 dx.$$

Passing to the limit, we obtain from (22) and Lemmas 5, 6 that

$$0 < S^2\left(1-\frac{\lambda}{\lambda_1}\right)^2 \leq \liminf_{n\to\infty}\left(1-\frac{\lambda}{\lambda_1}\right)\|u_n^{\pm}\|^2 \leq \int_{\mathbb{R}^3}|u_\mu^{\pm}|^4dx,$$

which implies that $u_\mu^{\pm}\neq 0$ and

$$\begin{cases} \|u_\mu^+\|^2+\mu L_{\phi_{u_\mu^+}}(u_\mu^+)+\mu L_{\phi_{u_\mu^+}}(u_\mu^-)\leq \lambda\int_{\mathbb{R}^3}f(x)|u_\mu^+|^2dx+\int_{\mathbb{R}^3}|u_\mu^+|^4dx,\\ \|u_\mu^-\|^2+\mu L_{\phi_{u_\mu^-}}(u_\mu^-)+\mu L_{\phi_{u_\mu^+}}(u_\mu^-)\leq \lambda\int_{\mathbb{R}^3}f(x)|u_\mu^-|^2dx+\int_{\mathbb{R}^3}|u_\mu^-|^4dx. \end{cases}$$

Then by Lemma 9, there is a unique pair $(s_u,t_u)\in(0,1]\times(0,1]$ such that

$$s_uu_\mu^+ + t_uu_\mu^- \in \mathbf{M}_\mu.$$

And thus

$$J_\mu(s_uu_\mu^+ + t_uu_\mu^-)\geq m_\mu.$$

Furthermore, it follows from $\lambda<\lambda_1$ that

$$\begin{aligned}
&J_\mu(s_uu_\mu^+ + t_uu_\mu^-)\\
=&J_\mu(s_uu_\mu^+ + t_uu_\mu^-)-\frac{1}{4}\langle J_\mu'(s_uu_\mu^+ + t_uu_\mu^-), s_uu_\mu^+ + t_uu_\mu^-\rangle\\
=&\frac{1}{4}\left(\|s_uu_\mu^+ + t_uu_\mu^-\|^2-\lambda\int_{\mathbb{R}^3}f(x)|s_uu_\mu^+ + t_uu_\mu^-|^2dx\right)\\
=&\frac{1}{4}\left[s_u^2\left(\|u_\mu^+\|^2-\lambda\int_{\mathbb{R}^3}f(x)|u_\mu^+|^2dx\right)+t_u^2\left(\|u_\mu^-\|^2-\lambda\int_{\mathbb{R}^3}f(x)|u_\mu^-|^2dx\right)\right]\\
\leq&\frac{1}{4}\left[\left(\|u_\mu^+\|^2-\lambda\int_{\mathbb{R}^3}f(x)|u_\mu^+|^2dx\right)+\left(\|u_\mu^-\|^2-\lambda\int_{\mathbb{R}^3}f(x)|u_\mu^-|^2dx\right)\right]\\
=&\frac{1}{4}\left(\|u_\mu\|^2-\lambda\int_{\mathbb{R}^3}f(x)|u_\mu|^2dx\right)\\
\leq&\liminf_{n\to\infty}\left[J_\mu(u_n)-\frac{1}{4}\langle J_\mu'(u_n),u_n\rangle\right]\\
=&m_\mu,
\end{aligned}$$

which implies that $s_u=t_u=1$, $u_\mu\in\mathbf{M}_\mu$ and $J_\mu(u_\mu)=m_\mu$, then $u_\mu$ is the required minimizer.

Thirdly, if $J_\mu'(u_\mu)\neq 0$, there exist $\delta>0$ and $\alpha>0$ such that

$$u\in H,\quad \|J_\mu'(u_\mu)\|\geq\alpha,\quad \|u-u_\mu\|\leq 3\delta.$$

Let $D=(1-\sigma,1+\sigma)\times(1-\sigma,1+\sigma)$, $0<\sigma<1$ and $\psi(s,t)=su^+ + tu^-$, $(s,t)\in D$. It follows from Lemma 7 that

$$m:=\max_{\partial D}J_\mu\circ\psi<m_\mu.$$

Let $\varepsilon = \min\{\frac{m_\mu - m}{2}, \frac{\alpha\delta}{8}\}$ and $S_\delta = \{u \in H : \|u - u_\mu\| \leq \delta\}$, there exists a deformation $\eta \in C([0,1] \times H, H)$ such that

(a) $\eta(1,u) = u$ if $u \notin J_\mu^{-1}([m_\mu - 2\varepsilon, m_\mu + 2\varepsilon]) \cap S_{2\delta}$;
(b) $\eta(1, J_\mu^{m_\mu+\varepsilon} \cap S_\delta) \subset J_\mu^{m_\mu-\varepsilon}$;
(c) $J_\mu(\eta(1,u)) \leq J_\mu(u), \forall u \in H$.

Similar to the proof of Theorem 1 in [23], we can obtain that $u_b$ is a sign-changing critical point of $J_\mu$ by the degree theory, and we complete the proof. □

## 3 Proof of the main results

In this section, we will prove the main results. To begin with, we can show that the ground state sign-changing solution $u_\mu$ of system (1) changes sign only once, and its energy is strictly larger than twice that of the ground state energy, which means that the energy of any sign-changing solutions of system (1) is strictly larger than twice that of the ground state solutions.

**Proof of Theorem 1.** In view of Lemma 11, there exists a $u_\mu \in \mathbf{M}_\mu$ such that $m_\mu = J_\mu(u_\mu)$ and $J'_\mu(u_\mu) = 0$. In other words, $u_\mu$ is a ground state sign-changing solution to system (1). Then by Lemma 8, we have that

$$\begin{cases} \mu L_{\phi_{u_\mu^+}}(u_\mu^+) + \mu L_{\phi_{u_\mu^+}}(u_\mu^-) < \displaystyle\int_{\mathbb{R}^3} |u_\mu^+|^4 dx, \\ \mu L_{\phi_{u_\mu^-}}(u_\mu^-) + \mu L_{\phi_{u_\mu^+}}(u_\mu^-) < \displaystyle\int_{\mathbb{R}^3} |u_\mu^-|^4 dx, \end{cases}$$

it follows from Lemma 10 that there exist $s_1$, $t_1 > 0$ such that $s_1 u_\mu^+$, $t_1 u_\mu^- \in \mathbf{N}_\mu$. Then

$$\begin{aligned} m_\mu &= J_\mu(u_\mu) \geq J_\mu(s_1 u_\mu^+ + t_1 u_\mu^-) \\ &= J_\mu(s_1 u_\mu^+) + J_\mu(t_1 u_\mu^-) + \frac{\mu s_1^2 t_1^2}{2} L_{\phi_{u_\mu^+}}(u_\mu^-) \\ &> J_\mu(s_1 u_\mu^+) + J_\mu(t_1 u_\mu^-) \geq 2c_\mu. \end{aligned}$$

Now, we show that $u_\mu$ changes sign only once. We assume by contradiction that $u_\mu = u_1 + u_2 + u_3$ with

$$u_i \neq 0, \quad u_1 \geq 0, \quad u_2 \leq 0, \quad u_3 \geq 0,$$

$$\mathrm{supp}(u_i) \cap \mathrm{supp}(u_j) = \emptyset, \quad i \neq j \ (i,j = 1,2,3).$$

Moreover, using the fact that $J'_\mu(u_\mu) = 0$, we get

$$\begin{cases} \langle J'_\mu(u_1 + u_2), u_1 \rangle = \langle J'_\mu(u_\mu), u_1 \rangle - \mu L_{\phi_{u_3}}(u_1) < 0, \\ \langle J'_\mu(u_1 + u_2), u_2 \rangle = \langle J'_\mu(u_\mu), u_2 \rangle - \mu L_{\phi_{u_3}}(u_2) < 0. \end{cases}$$

Consequently, by Lemma 9, there exist $(\bar{s}, \bar{t}) \in (0,1] \times (0,1]$ such that

$$\bar{s}u_1 + \bar{t}u_2 \in \mathbf{M}_\mu, \quad J_\mu(\bar{s}u_1 + \bar{t}u_2) \geq m_\mu.$$

Noting that $\lambda < \lambda_1$, $\langle J'_\mu(u_\mu), u_\mu \rangle = 0$ and $\langle J'_\mu(\bar{s}u_1 + \bar{t}u_2), \bar{s}u_1 + \bar{t}u_2 \rangle = 0$, we have

$$\begin{aligned}
m_\mu =& J_\mu(u_\mu) - \frac{1}{4}\langle J'_\mu(u_\mu), u_\mu \rangle = \frac{1}{4}\left( \|u_\mu\|^2 - \lambda \int_{\mathbb{R}^3} f(x) u_\mu^2 dx \right) \\
=& \frac{1}{4}\left[ \left( \|u_1\|^2 - \lambda \int_{\mathbb{R}^3} f(x) u_1^2 dx \right) + \left( \|u_2\|^2 - \lambda \int_{\mathbb{R}^3} f(x) u_2^2 dx \right) \right. \\
& \left. + \left( \|u_3\|^2 - \lambda \int_{\mathbb{R}^3} f(x) u_3^2 dx \right) \right] \\
>& \frac{1}{4}\left[ \left( \|u_1\|^2 - \lambda \int_{\mathbb{R}^3} f(x) u_1^2 dx \right) + \left( \|u_2\|^2 - \lambda \int_{\mathbb{R}^3} f(x) u_2^2 dx \right) \right] \\
\geq& \frac{1}{4}\left[ \bar{s}^2 \left( \|u_1\|^2 - \lambda \int_{\mathbb{R}^3} f(x) u_1^2 dx \right) + \bar{t}^2 \left( \|u_2\|^2 - \lambda \int_{\mathbb{R}^3} f(x) u_2^2 dx \right) \right] \\
=& J_\mu(\bar{s}u_1 + \bar{t}u_2) - \frac{1}{4}\langle J'_\mu(\bar{s}u_1 + \bar{t}u_2), \bar{s}u_1 + \bar{t}u_2 \rangle \\
=& J_\mu(\bar{s}u_1 + \bar{t}u_2) \\
\geq& \, m_\mu,
\end{aligned}$$

which leads to a contradiction, and thus the minimizer $u_\mu$ changes sign only once. $\square$

Now, we are in a situation to prove Theorem 2. In the following, we regard $\mu > 0$ as a parameter in system (1). We shall analyze the convergence property of $u_\mu$ as $\mu \searrow 0$.

For any $\mu \searrow 0$, let $u_\mu \in \mathbf{M}_\mu$ be the ground state sign-changing solution to system (1), which changes sign only once.

**Proof of Theorem 2.** Firstly, choose a nonzero function $w_0 \in C_0^\infty(\mathbb{R}^3)$ and $\beta > 0$ such that $w_0^\pm \neq 0$ and

$$\begin{cases}
\|w_0^+\|^2 + \beta L_{\phi_{w_0^+}}(w_0^+) + \beta L_{\phi_{w_0^+}}(w_0^-) \leq \lambda \displaystyle\int_{\mathbb{R}^3} f(x)|w_0^+|^2 dx + \int_{\mathbb{R}^3} |w_0^+|^4 dx, \\
\|w_0^-\|^2 + \beta L_{\phi_{w_0^-}}(w_0^-) + \beta L_{\phi_{w_0^+}}(w_0^-) \leq \lambda \displaystyle\int_{\mathbb{R}^3} f(x)|w_0^-|^2 dx + \int_{\mathbb{R}^3} |w_0^-|^4 dx.
\end{cases}$$

Thus, for any $\mu \in [0, \beta]$, $\langle J'_\mu(w_0), w_0^\pm \rangle \leq 0$. It follows from Lemma 9 that for any $\mu \in [0, \beta]$, there is a unique pair $(s_\mu, t_\mu) \in (0,1] \times (0,1]$ such that

$$s_\mu w_0^+ + t_\mu w_0^- \in \mathbf{M}_\mu.$$

Thus, for any $\mu \in [0, \beta]$, we have

$$J_\mu(s_\mu w_0^+ + t_\mu w_0^-) = J_\mu(s_\mu w_0^+ + t_\mu w_0^-) - \frac{1}{4}\langle J'_\mu(s_\mu w_0^+ + t_\mu w_0^-), s_\mu w_0^+ + t_\mu w_0^- \rangle$$

$$
\begin{aligned}
&=\frac{1}{4}\left(\|s_\mu w_0^+ + t_\mu w_0^-\|^2 - \lambda\int_{\mathbb{R}^3} f(x)|s_\mu w_0^+ + t_\mu w_0^-|^2 dx\right)\\
&<\frac{1}{4}\|s_\mu w_0^+ + t_\mu w_0^-\|^2\\
&\leq\frac{1}{4}\|w_0\|^2 = \theta.
\end{aligned}
$$

For any sequence $\{\mu_n\}$ with $\mu_n \searrow 0$ as $n \to \infty$, one can obtain from Theorem 1 that for large $n$, there exists $u_{\mu_n} \in \mathbf{M}_{\mu_n}$ is a ground state sign-changing critical point of $J_{\mu_n}$, then

$$
\theta + 1 \quad \geq \quad J_{\mu_n}(u_{\mu_n}) - \frac{1}{4}\langle J'_{\mu_n}(u_{\mu_n}), u_{\mu_n}\rangle \geq \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u_{\mu_n}\|^2.
$$

This shows that $\{u_{\mu_n}\}$ is bounded in $H$, then there exists a subsequence of $\{\mu_n\}$, still denoted by $\{\mu_n\}$, such that $u_{\mu_n} \rightharpoonup u_0$ weakly in $H$. By the compactness of the embedding $H \hookrightarrow L^s(\mathbb{R}^3)$ for $2 < s < 6$, using a standard argument, we can prove that $u^\pm_{\mu_n} \to u^\pm_0$ strongly in $H$, and $u_0^\pm \neq 0$. Furthermore, we deduce that for all $u \in H$,

$$
\begin{aligned}
0 &= \lim_{n\to\infty}\langle J'_{\mu_n}(u_{\mu_n}), u\rangle\\
&= \lim_{n\to\infty}\left[\int_{\mathbb{R}^3}(\nabla u_{\mu_n}\cdot\nabla u + V(x)u_{\mu_n}u)\,dx + \mu_n\int_{\mathbb{R}^3}\phi_{u_{\mu_n}}u_{\mu_n}u dx\right.\\
&\quad\left. -\lambda\int_{\mathbb{R}^3} f(x)u_{\mu_n}u dx - \int_{\mathbb{R}^3}|u_{\mu_n}|^2 u_{\mu_n} u dx\right]\\
&= \int_{\mathbb{R}^3}(\nabla u_0\cdot\nabla u + V(x)u_0 u)\,dx - \lambda\int_{\mathbb{R}^3} f(x)u_0 u dx - \int_{\mathbb{R}^3}|u_0|^2 u_0 u dx\\
&=\langle J'_0(u_0), u\rangle,
\end{aligned}
$$

which implies that

$$
J'_0(u_0) = 0, \quad u_0 \in \mathbf{M}_0, \quad J_0(u_0) \geq m_0. \tag{23}
$$

Secondly, in the proof of Theorem 1, $\mu = 0$ is allowed. Then there exists a $v_0 \in \mathbf{M}_0$ such that

$$
J_0(u_0) = m_0 = \inf_{u\in\mathbf{M}_0} J_0(u),
$$

and $v_0$ is a sign-changing solution to system (1) which changes sign only once. Similarly, we can pick up $\epsilon > 0$ which is independent on $\mu_n$ such that

$$
\begin{cases}
\epsilon L_{\phi_{v_0^+}}(v_0^+) + \epsilon L_{\phi_{v_0^+}}(v_0^-) < \int_{\mathbb{R}^3}|v_0^+|^4 dx,\\
\epsilon L_{\phi_{v_0^-}}(v_0^-) + \epsilon L_{\phi_{v_0^+}}(v_0^-) < \int_{\mathbb{R}^3}|v_0^-|^4 dx.
\end{cases}
$$

According to Lemma 7, there is a unique pair $(s_0,\ t_0)$ of positive numbers such that

$$
s_0 v_0^+ + t_0 v_0^- \in \mathbf{M}_\epsilon.
$$

Let $\mu_n \in [0, \epsilon]$, we can know that

$$\begin{aligned}
\langle J'_{\mu_n}(s_0v_0^+ + t_0v_0^-), s_0v_0^+\rangle =& \|s_0v_0^+\|^2 + \mu_n L_{\phi_{s_0v_0^+}}(s_0v_0^+) + \mu_n L_{\phi_{s_0v_0^+}}(t_0v_0^-) \\
& - \lambda \int_{\mathbb{R}^3} f(x)|s_0v_0^+|^2 dx - \int_{\mathbb{R}^3} |s_0v_0^+|^4 dx \\
\leq & \|s_0v_0^+\|^2 + \epsilon L_{\phi_{s_0v_0^+}}(s_0v_0^+) + \epsilon L_{\phi_{s_0v_0^+}}(t_0v_0^-) \\
& - \lambda \int_{\mathbb{R}^3} f(x)|s_0v_0^+|^2 dx - \int_{\mathbb{R}^3} |s_0v_0^+|^4 dx \\
=& \langle J'_{\epsilon}(s_0v_0^+ + t_0v_0^-), s_0v_0^+\rangle = 0.
\end{aligned}$$

In the same way, we can obtain that

$$\langle J'_{\mu_n}(s_0v_0^+ + t_0v_0^-), t_0v_0^-\rangle \leq \langle J'_{\epsilon}(s_0v_0^+ + t_0v_0^-), t_0v_0^-\rangle = 0.$$

It follows from Lemma 9 that for all $\mu_n \in [0, \epsilon]$, there is a unique pair $(s_n, t_n) \in (0, s_0] \times (0, t_0]$ such that

$$s_nv_0^+ + t_nv_0^- \in \mathbf{M}_{\mu_n}. \tag{24}$$

Then for any sequence $\{\mu_n\}$ with $\mu_n \searrow 0$ as $n \to \infty$, we have as $n \to \infty$,

$$\mu_n s_n^4 L_{\phi_{v_0^+}}(v_0^+) \to 0, \quad \mu_n s_n^2 t_n^2 L_{\phi_{v_0^+}}(v_0^-) \to 0, \quad \mu_n t_n^4 L_{\phi_{v_0^-}}(v_0^-) \to 0,$$

together with $\langle J'_{\mu_n}(s_nv_0^+ + t_nv_0^-), s_nv_0^+\rangle = \langle J'_{\mu_n}(s_nv_0^+ + t_nv_0^-), t_nv_0^-\rangle = 0$, we can get

$$\begin{cases} \|v_0^+\|^2 + o(1) = \lambda \int_{\mathbb{R}^3} f(x)|v_0^+|^2 dx + s_n^2 \int_{\mathbb{R}^3} |v_0^+|^4 dx, \\ \|v_0^-\|^2 + o(1) = \lambda \int_{\mathbb{R}^3} f(x)|v_0^-|^2 dx + t_n^2 \int_{\mathbb{R}^3} |v_0^-|^4 dx, \end{cases} \tag{25}$$

and by $\langle J'_0(v_0), v_0^{\pm}\rangle = 0$, we have

$$\begin{cases} \|v_0^+\|^2 = \lambda \int_{\mathbb{R}^3} f(x)|v_0^+|^2 dx + \int_{\mathbb{R}^3} |v_0^+|^4 dx, \\ \|v_0^-\|^2 = \lambda \int_{\mathbb{R}^3} f(x)|v_0^-|^2 dx + \int_{\mathbb{R}^3} |v_0^-|^4 dx, \end{cases} \tag{26}$$

Combining with (25) and (26), one has that as $n \to \infty$,

$$s_n \to 1, t_n \to 1.$$

Lastly, we only need to show $J_0(u_0) = J_0(v_0)$, then by (23), $u_0$ is a ground state sign-changing solution of problem (9) which changes sign only once. In fact, it follows from (24) that

$$\begin{aligned}
J_0(v_0) \leq J_0(u_0) &= \lim_{n\to\infty} J_{\mu_n}(u_{\mu_n}) \\
&\leq \lim_{n\to\infty} J_{\mu_n}(s_nv_0^+ + t_nv_0^-) = J_0(v_0^+ + v_0^-) = J_0(v_0).
\end{aligned}$$

This completes the proof of Theorem 2. □

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## References

[1] C.O. Alves and M.A.S. Souto, Existence of least energy nodal solution for a Schrödinger-Poisson system in bounded domain, Z. Angew. Math. Phys. 65 (2014) 1153–1166.

[2] A. Ambrosetti, On Schrödinger-Poisson systems. Milan J. Math. 76 (2008) 257–274.

[3] A. Ambrosetti, D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, Commun. Contemp. Math. 10 (2008) 391–404.

[4] A. Azzollini, P. D'Avenia, A. Pomponio, On the Schrödinger-Maxwell equations under the effect of a general nonlinear term. Ann. Inst. H. Poincaré Anal. Non-linéaire 27 (2010) 779–791.

[5] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Schrödinger-Maxwell equations. J. Math. Anal. Appl. 345 (2008) 90–108.

[6] T. Bartsch, Z.L. Liu, T. Weth, Sign changing solutions of superlinear Schrödinger equations, Comm. Partial Differential Equations 29 (2004) 25–42.

[7] T. Bartsch, T. Weth, Three nodal solutions of singular perturbed elliptic equations on domains without topology, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005) 259–281.

[8] T. Bartsch, T. Weth, M. Willem, Partial symmetry of least energy nodal solutions to some variational problems, J. Anal. Math. 96 (2005) 1–18.

[9] K.J. Brown, Y. Zhang, The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function, J. Differential Equations 193 (2003) 481–499.

[10] G. Cerami, G. Vaira, Positive solutions for some non-autonomous Schrödinger-Poisson systems, J. Differential Equations 248 (2010) 521–543.

[11] S.T. Chen and X.H. Tang, Ground state sign-changing solutions for a class of Schrödinger-Poisson type problems in $\mathbb{R}^3$, Z. Angew. Math. Phys. (2016) 67–102.

[12] P. D'Avenia, A. Pomponio, G. Vaira, Infinitely many positive solutions for a Schrödinger-Poisson system, Nonlinear Anal. 74 (2011) 5705–5721.

[13] H. Guo, Nonexistence of least energy nodal solutions for Schrödinger-Poisson equation, Applied Mathematics Letters (2017) In Press.

[14] I. Ianni, Sign-changing radial solutions for the Schrödinger-Poisson-Slater problem, Topol. Methods Nonlinear Anal. 41 (2013) 365–385.

[15] I. Ianni, G. Vaira, Non-radial sign-changing solutions for the Schrödinger-Poisson problem in semiclassic limit, Nonlinear Differential Equations and Applications 22 (2012) 741–776.

[16] S. Kim, J. Seok, On nodal solutions of the nonlinear Schrödinger-Poisson equations, Commun. Contemp. Math. 14 (2012) 1–16.

[17] G.B. Li, S.J. Peng, S.S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger-Poisson system, Commun. Contemp. Math. 12 (2010) 1069–1092.

[18] Z.P. Liang, J. Xu, X.L. Zhu, Revisit to sign-changing solutions for the nonlinear Schrödinger-Poisson system in $\mathbb{R}^3$, J. Math. Anal. Appl. 435(2016):783–799.

[19] Z.L. Liu, Z.Q. Wang, J.J. Zhang, Infnitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system, Annali di Matematica 195 (2016) 775–794.

[20] A.M. Mao, S.X. Luan, Sign-changing solutions of a class of nonlocal quasilinear elliptic boundary value problems, J. Math. Anal. Appl. 383 (1) (2011) 239–243.

[21] D. Mugnai, The Schrödinger-Poisson system with positive potential, Comm. Partial Differential Equations 36 (2011) 1099–1117.

[22] Z.P. Shen, Z.Q. Han, Multiple solutions for a class of Schrödinger-Poisson system with indefinite nonlinearity, J. Math. Anal. Appl. 426 (2015) 839–854.

[23] W. Shuai, Q.F. Wang, Existence and asymptotic behavior of sign-changing solutions for the nonlinear Schrödinger-Poisson system in $\mathbb{R}^3$, Z. Angew. Math. Phys. 66 (2015) 3267–3282.

[24] Z.P. Wang, H.S. Zhou, Sign-changing solutions for the nonlinear Schrödinger-Poisson system in $\mathbb{R}^3$. Calc. Var. Partial. Differ. Equ. 52(2015) 927–943.

[25] M. Willem, Minimax Theorems, Progress in nonlinear differential equations and their applications, vol. 24, Birkhäuser Boston, inc., Boston, MA, 1996.

[26] J. Zhang, On ground state and nodal solutions of Schrödinger-Poisson equations with critical growth, J. Math. Anal. Appl. 428 (2015) 387–402.

[27] L. Zhao, H. Liu, F. Zhao, Existence and concentration of solutions for the Schrödinger-Poisson equations with steep well potential, J. Differential Equations 255 (2013) 1–23.

[28] W.M. Zou, Sign-Changing Critical Point Theory, Spring, New York, 2008.