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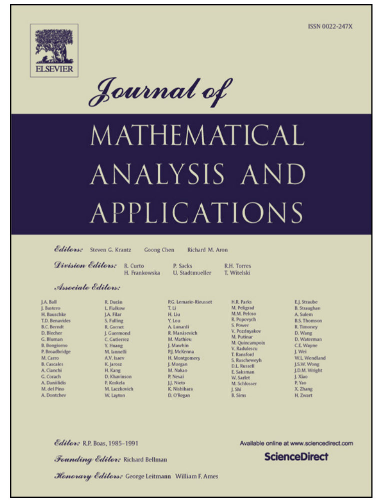

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Bounded solutions to a singular parabolic system

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Abstract

In this paper, we are concerned with the singular parabolic system $u_t = \Delta u + f(x)v^{-p}, v_t = \Delta v + g(x)u^{-q}$ in a smooth bounded domain $\Omega \subset \mathbf{R}^N$ subject to zero Dirichlet conditions, with initial conditions $u_0(x), v_0(x) > 0$. This problem is of interest as it is related to some problems in biology and physics. Under suitable assumptions on p, q and $f(x), g(x)$, some existence results of weak and classical solutions are obtained using a functional method. This method is motivated by such results found in [4] and [5] when dealing with singular parabolic systems and the related references within.

Keywords: bounded solutions, singular parabolic systems, Dirichlet problems

1. Introduction

In this paper, we consider the following parabolic system:

$$\begin{cases} u_t = \Delta u + \dfrac{f(x)}{v^p}, & \\ v_t = \Delta v + \dfrac{g(x)}{u^q}, & t > 0, x \in \Omega, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

in a smooth and bounded domain $\Omega \subset \mathbf{R}^N$ with $N \geq 1$. This system is a natural extension of various related problems concerned with a single equation. Such problems arise in relation to the study of enzyme kinetics, as well as in relations to some problems in physics when considering the steady state solutions. Enzyme kinetics is a form of reaction-diffusion processes, specifically chemical reactions catalyzed by enzymes. Enzymes are biological molecules which help complex reactions to occur virtually everywhere in life. Understanding how these chemical reactions occur is crucial to our understanding of metabolic processes and how they occur

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within cells, [15]. Readers are directed to [1] and [2] for an in-depth exploration of enzyme kinetics, as well as the references found within [7]. In addition, steady state solutions of equations taking the form of (1.1) and other variations relate to Lane-Emden type equations with negative exponents. Such equations are used in describing the gravitational potential Newtonian self-gravitating, spherically symmetric, polytropic fluid. Readers are directed to [10, 11, 12, 19] and the references within.

In 2004, Davila and Montenegro [7] investigated existence of solutions to the problem:

$$\begin{cases} u_t = \Delta u - \left[\dfrac{1}{u^\beta} - f(u)\right] \chi_{\{u>0\}}, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & x \in \partial\Omega, \\ u(x,0) = u_0(x)\ , & x \in \Omega, \end{cases} \tag{1.2}$$

where $\chi_{\{u>0\}}$ is the characteristic function for $u > 0$, $\beta \in (0,1)$, $f(u) \geq 0$ is C^2 in $[0,\infty)$ and $f(u) \leq C(1+u)$. These results were obtained using the method of sub- and super-solutions. In 2007, Winkler [17] considered the similar problem with $f(u) \equiv 0$ and Dirichlet boundary condition $B = B(x)$ for $x \in \partial\Omega$. He proved the existence of nonnegative weak solutions with quenching property as well as some nonuniqueness results of (1.2). In 2009, Xia and Yao [18] derived some existence and uniqueness results for the following problem:

$$\begin{cases} v_t = \Delta v - \mu \dfrac{|\nabla v|^2}{v} + f(x,t), & \text{in } \Omega \times (0,T], \\ v(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \\ v(x,0) = v_0(x), & \text{in } \bar{\Omega}, \end{cases} \tag{1.3}$$

which can be transformed into

$$\begin{cases} u_t = \Delta u + \dfrac{f(x,t)}{u^\gamma}\ , & \text{in } \Omega \times (0,T], \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = \left[(1+\gamma)\, v_0\right]^{1/(1+\gamma)}\ , & \text{in } \bar{\Omega}, \end{cases} \tag{1.4}$$

if we set $\mu = \gamma/(\gamma+1), \gamma > 0$ and $u = \left[(1+\gamma)\, v\right]^{1/(1+\gamma)}$. Recently, Boccardo, Escobedo and Porzio [3] discussed the existence of some problems whose prototype takes the form:

$$\begin{cases} u_t = \Delta u + \dfrac{\lambda}{u^\gamma} + \mu u^p & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.5}$$

where $\lambda, \gamma > 0$, and $\mu \geq 0$. They first constructed approximate solutions and then used the method of sub- and super-solutions to estimated lower and upper bounds.

The results in this paper are further motivated by similar elliptic (or steady-state) systems, taking the form:

$$\begin{cases} 0 = \Delta u + \dfrac{1}{u^p} + \dfrac{1}{v^q}, \; u > 0 \text{ in } \Omega, \\ 0 = \Delta v + \dfrac{1}{u^r} + \dfrac{1}{v^s}, \; v > 0 \text{ in } \Omega, \\ u(x) = v(x) = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

or

$$\begin{cases} 0 = \Delta u + \dfrac{1}{u^p v^q}, \; u > 0 \text{ in } \Omega, \\ 0 = \Delta v + \dfrac{1}{u^r v^s}, \; v > 0 \text{ in } \Omega, \\ u(x) = v(x) = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where conditions are put upon $p, q, r, s > 0$, see [10, 19] and the references within. If $p, r < 0$, then the solutions of (1.7) are steady states of a generalized Gierer-Meinhardt or activator-inhibitor system, see [5, 13]. As far as we know, no results have yet been published investigating the existence or boundedness of solutions to the singular parabolic system taking the form of (1.1).

We use a functional method to obtain lower bounds of solutions for a perturbed system. That is, we consider the following integral

$$\int_\Omega \frac{\phi^{n+2}(x)}{[u_\varepsilon(x,t)+\varepsilon]^{\alpha n}[v_\varepsilon(x,t)+\varepsilon]^{\beta n}} dx, \tag{1.8}$$

where $(u_\varepsilon, v_\varepsilon)$ is a solution to the perturbed system of (1.1) (see (3.3 in Section 3 for details), $n > 1$, $\alpha, \beta > 0$, $\alpha + \beta \leq 1$, and $\phi(x)$ is the first eigenfunction of

$$\Delta\phi(x) + \lambda\phi(x) = 0, \quad \phi(x)|_{\partial\Omega} = 0. \tag{1.9}$$

Then, taking derivatives, substituting the perturbed system and integrating by parts, we can show that (1.8) is bounded and independent of ε. We can use the similar method to obtain the upper bound for $(u_\varepsilon, v_\varepsilon)$. The functional method is a very powerful method to obtain a priori estimates for elliptic and parabolic equations (see [5, 6]) and is completely different than the traditional methods of sub and super solutions.

This paper will be organized as follows. In Section 2, we establish some important inequalities relating a linear system and the eigenfunctions by using the functional method. In Section 3, we use these inequalities to obtain a uniform bound to a related perturbation problem, and then, use Sobolev embedding theorem to get the existence of positive solutions to (1.1). Finally, for the convenience of readers, we put the proof of an inequality in Appendix because the proof is already given in [6].

2. Some Useful Inequalities

In this section, we collect some useful estimates for solutions to some linear systems. We first start with a generalized Young's inequality which will be used many times.

Lemma 2.1. *For any functions $u(x), v(x), f(x), g(x) > 0$, any indices $p_1, p_2, q_1, q_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1$, where $\theta_1 < p_1 < \alpha_1$ (not necessarily positive) and any constant $c > 0$, we have that*

$$\frac{u^{p_1} f^{p_2}}{v^{q_1} g^{q_2}} \leq c \frac{u^{\alpha_1} f^{\alpha_2}}{v^{\beta_1} g^{\beta_2}} + c^{-(p_1-\theta_1)/(\alpha_1-p_1)} \frac{u^{\theta_1} f^{\theta_2}}{v^{\eta_1} g^{\eta_2}},$$

where

$$\theta_2 = [p_2(\alpha_1 - \theta_1) - \alpha_2(p_1 - \theta_1)](\alpha_1 - p_1)^{-1},$$
$$\eta_1 = [q_1(\alpha_1 - \theta_1) - \beta_1(p_1 - \theta_1)](\alpha_1 - p_1)^{-1},$$
$$\eta_2 = [q_2(\alpha_1 - \theta_1) - \beta_2(p_1 - \theta_1)](\alpha_1 - p_1)^{-1}.$$

proof. See the proof of Lemma 1 in [5].

Let $\psi(x)$ to be the solution of:

$$\begin{cases} \Delta\psi(x) + \psi^{-\sigma}(x) = 0, \\ \psi(x) = 0, \quad x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $\sigma \in (-1, 1)$. By [14], we have that:

$$\delta_0\phi(x) \leq \psi(x) \leq \delta_1\phi(x), \ \delta_1, \delta_0 > 0,$$
$$\partial\phi/\partial\mathbf{n} < 0, \ x \in \partial\Omega. \tag{2.2}$$

That is, $\phi(x)$ and $\psi(x)$ are separated by some constant, which will be used in Section 3.

Suppose that (u, v) is a solution of the following linear system:

$$\begin{cases} u_t = \Delta u + A(x,t), \\ v_t = \Delta v + B(x,t), \qquad t > 0, x \in \Omega, \\ u, v > 0, \qquad x \in \Omega, \end{cases} \tag{2.3}$$

where $A(x,t), B(x,t)$ are continuous functions.

Lemma 2.2. *For any $n > 2$, we have that*

$$\frac{d}{dt}\int_\Omega \frac{\psi^{n+2}}{u^n} dx \leq n \int_\Omega \frac{\psi^{n+1-\sigma}}{u^n} dx - n \int_\Omega \frac{\psi^{n+2}}{u^{n+1}} A(x,t) dx, \tag{2.4}$$

where u and ψ are solutions of (2.1) and (2.3).

proof. Differentiating with respect to t and integrating by parts, we have that:

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{\psi^{n+2}}{u^n}dx &= -n\int_\Omega \frac{\psi^{n+2}}{u^{n+1}}\Delta u dx - n\int_\Omega \frac{\psi^{n+2}}{u^{n+1}}A(x,t)dx \\ &= n\int_\Omega \nabla\left(\frac{\psi^{n+2}}{u^{n+1}}\right)\nabla u dx - n\int_\Omega \frac{\psi^{n+2}}{u^{n+1}}A(x,t)dx \\ &= -n(n+1)\int_\Omega \frac{\psi^{n+2}}{u^{n+2}}\left|\nabla u\right|^2 dx + n(n+2)\int_\Omega \frac{\psi^{n+1}}{u^{n+1}}\nabla u\nabla\psi dx \\ &\quad - n\int_\Omega \frac{\psi^{n+2}}{u^{n+1}}A(x,t)dx. \end{aligned} \tag{2.5}$$

Using the following identity:

$$\psi^2\left|\nabla u\right|^2 = \left|\psi\nabla u - u\nabla\psi\right|^2 + 2u\psi\nabla u\nabla\psi - u^2\left|\nabla\psi\right|^2,$$

we then see that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{\psi^{n+2}}{u^n}dx =& -n(n+1)\int_\Omega \frac{\psi^{n+2}}{u^{n+2}}\left|\psi\nabla u - u\nabla\psi\right|^2 dx - dn^2\int_\Omega \frac{\psi^{n+1}}{u^{n+1}}\nabla u\nabla\psi dx \\ &+ n(n+1)\int_\Omega \frac{\psi^n}{u^n}\left|\nabla\psi\right|^2 dx - n\int_\Omega \frac{\psi^{n+2}}{u^{n+1}}A(x,t)dx. \end{aligned} \tag{2.6}$$

Integrating by parts for the second term of (2.6), we find

$$\begin{aligned}-n^2\int_\Omega \frac{\psi^{n+1}}{u^{n+1}}\nabla u\nabla\psi dx &= n\int_\Omega \psi^{n+1}\nabla\psi\nabla\left(\frac{1}{u^n}\right)dx \\ &= -n\int_\Omega \nabla\left(\psi^{n+1}\nabla\psi\right)\frac{1}{u^n}dx \\ &= -n(n+1)\int_\Omega \frac{\psi^n}{u^n}\left|\nabla\psi\right|^2 dx - n\int_\Omega \frac{\psi^{n+1}}{u^n}\Delta\psi dx. \end{aligned}$$

Combining this with the rest of (2.6) gives us that:

$$\frac{d}{dt}\int_\Omega \frac{\psi^{n+2}}{u^n}dx \le -n\int_\Omega \frac{\psi^{n+1}}{u^n}\Delta\psi dx - n\int_\Omega \frac{\psi^{n+2}}{u^{n+1}}A(x,t)dx. \tag{2.7}$$

Lastly, we use (2.1) to arrive at

$$\frac{d}{dt}\int_\Omega \frac{\psi^{n+2}}{u^n}dx \le n\int_\Omega \frac{\psi^{n+1-\sigma}}{u^n}dx - n\int_\Omega \frac{\psi^{n+2}}{u^{n+1}}A(x,t)dx. \tag{2.8}$$

This completes the proof.

Lemma 2.3. *For $\alpha,\beta>0$ and $\alpha+\beta\le 1$, we have that*

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n}} \le& -\alpha n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1}v^{\beta n}}A(x,t)dx - \beta n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n+1}}B(x,t)dx \\ &+ \lambda_1(n+2)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n}}dx, \end{aligned} \tag{2.9}$$

where ϕ is the first normalized eigenfunction of (1.9), (u, v) is a solution of (2.3). **proof.** We first set

$$z_n(t) = \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n}} dx.$$

Consider the following after taking the derivative with respect to time:

$$\begin{aligned} z_n'(t) =& -\alpha n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n}} u_t dx - \beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+1}} v_t dx \\ =& -\alpha n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n}} \Delta u dx - \beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+1}} \Delta v dx \\ & -\alpha n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n}} A(x,t) dx - \beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+1}} B(x,t) dx. \end{aligned} \tag{2.10}$$

Looking at only the Laplace terms and integrating by parts gives us that:

$$\begin{aligned} & \alpha n \int_\Omega \nabla \left(\frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n}} \right) \nabla u dx + \beta n \int_\Omega \nabla \left(\frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+1}} \right) \nabla v dx \\ =& \, \alpha n(n+2) \int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1} v^{\beta n}} \nabla \phi \nabla u dx + \beta n(n+2) \int_\Omega \frac{\phi^{n+1}}{u^{\alpha n} v^{\beta n+1}} \nabla \phi \nabla v dx \\ & - \alpha n(\alpha n+1) \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2} v^{\beta n}} |\nabla u|^2 dx - \beta n(\beta n+1) \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+2}} |\nabla v|^2 dx \\ & - 2\alpha\beta n^2 \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n+1}} \nabla u \nabla v dx \\ \triangleq & \sum_{k=1}^{5} I_k. \end{aligned} \tag{2.11}$$

Consider the following term and integrate by parts:

$$\begin{aligned} J \triangleq & -\alpha n(n+2) \int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1} v^{\beta n}} \nabla \phi \nabla u dx \\ =& \, (n+2) \int_\Omega \frac{\phi^{n+1} \nabla \phi}{v^{\beta n}} \nabla \left(\frac{1}{u^{\alpha n}} \right) dx \\ =& -(n+2) \int_\Omega \nabla \left(\frac{\phi^{n+1} \nabla \phi}{v^{\beta n}} \right) \frac{1}{u^{\alpha n}} dx \\ =& -(n+2)(n+1) \int_\Omega \frac{\phi^n}{u^{\alpha n} v^{\beta n}} |\nabla \phi|^2 dx - (n+2) \int_\Omega \frac{\phi^{n+1}}{u^{\alpha n} v^{\beta n}} \Delta \phi dx \\ & + \beta n(n+2) \int_\Omega \frac{\phi^{n+1}}{u^{\alpha n} \phi^{\beta n+1}} \nabla \phi \nabla v dx. \end{aligned} \tag{2.12}$$

We notice some similar terms, giving us the following:

$$\begin{aligned} (I_1 - J) \; + \; (J + I_2) =& \, 2\alpha n(n+2) \int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1} v^{\beta n+1}} \nabla \phi \left(v \nabla u + \frac{\beta}{\alpha} u \nabla v \right) dx \\ & - (n+2)(n+1) \int_\Omega \frac{\phi^n}{u^{\alpha n} v^{\beta n}} |\nabla \phi|^2 dx - (n+2) \int_\Omega \frac{\phi^{n+1}}{u^{\alpha n} v^{\alpha n}} \Delta \phi dx \; . \end{aligned} \tag{2.13}$$

This gives us that:

$$
\begin{aligned}
\sum_{k=1}^{5} I_k = {} & 2\alpha n(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1}v^{\beta n+1}}\nabla\phi\left(v\nabla u+\frac{\beta}{\alpha}u\nabla v\right)dx \\
& -\alpha n(\alpha n+1)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\beta n}}\left|\nabla u\right|^2 dx-\beta n(\beta n+1)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n+2}}\left|\nabla v\right|^2 dx \\
& -2\alpha\beta n^2\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1}v^{\beta n+1}}\nabla u\nabla v dx-(n+2)(n+1)\int_\Omega \frac{\phi^{n}}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx \\
& -(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\alpha n}}\Delta\phi dx. \qquad (2.14)
\end{aligned}
$$

Then, we complete the square in a specific way:

$$
\begin{aligned}
\sum_{k=1}^{5} I_k = {} & 2\alpha n(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1}v^{\beta n+1}}\nabla\phi\left(v\nabla u+\frac{\beta}{\alpha}u\nabla v\right)dx \\
& -\alpha^2 n(n+1)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\beta n+2}}\left|v\nabla u+\frac{\beta}{\alpha}u\nabla v\right|^2 dx \\
& -\alpha(1-\alpha)n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\alpha n}}\left|\nabla u\right|^2 dx-\beta(1-\beta)n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n+2}}\left|\nabla v\right|^2 dx \\
& +2\alpha\beta n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1}v^{\beta n+1}}\nabla u\nabla v dx-(n+2)(n+1)\int_\Omega \frac{\phi^{n}}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx \\
& -(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\alpha n}}\Delta\phi dx. \qquad (2.15)
\end{aligned}
$$

If we recall that $\alpha+\beta\leq 1$, we combine terms $3,4$ and 5 of (2.15) as follows:

$$
\begin{aligned}
& -\alpha(1-\alpha)n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\alpha n}}\left|\nabla u\right|^2 dx-\beta(1-\beta)n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n+2}}\left|\nabla v\right|^2 dx \\
& +2\alpha\beta n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1}v^{\beta n+1}}\nabla u\nabla v dx \\
\leq {} & -\alpha\beta n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\alpha n}}\left|\nabla u\right|^2 dx-\alpha\beta n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n+2}}\left|\nabla v\right|^2 dx \\
& +2\alpha\beta n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1}v^{\beta n+1}}\nabla u\nabla v dx \\
= {} & -\alpha\beta n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\beta n+2}}\left|v\nabla u-u\nabla v\right|^2 dx \\
\leq {} & 0. \qquad (2.16)
\end{aligned}
$$

Thus, we have that

$$\begin{aligned}
\sum_{k=1}^{5} I_k \leq\, & 2\alpha n(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1}v^{\beta n+1}}\nabla\phi\left(v\nabla u+\frac{\beta}{\alpha}u\nabla v\right)dx \\
& -\alpha^2 n(n+1)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\beta n+2}}\left|v\nabla u+\frac{\beta}{\alpha}u\nabla v\right|^2 dx \\
& -(n+2)(n+1)\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx-(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\alpha n}}\Delta\phi dx. \qquad (2.17)
\end{aligned}$$

Applying Young's Inequality to the second term of (2.17), we have that:

$$\begin{aligned}
& \left|2\alpha n(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1}v^{\beta n+1}}\nabla\phi\left(v\nabla u+\frac{\beta}{\alpha}u\nabla v\right)dx\right| \\
\leq\, & \frac{1}{2}\left(2\alpha^2 n(n+1)\right)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\beta n+2}}\left|v\nabla u+\frac{\beta}{\alpha}u\nabla v\right|^2 dx \\
& +\frac{1}{2}\left(\frac{2\alpha^2 n^2(n+2)^2}{\alpha^2 n(n+1)}\right)\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx \\
=\, & \alpha^2 n(n+1)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\beta n+2}}\left|v\nabla u+\frac{\beta}{\alpha}u\nabla v\right|^2 dx \\
& +\frac{n(n+2)^2}{n+1}\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx. \qquad (2.18)
\end{aligned}$$

Combining this with our remaining terms leaves us with the following:

$$\begin{aligned}
\sum_{k=1}^{5} I_k \leq & -\left[(n+2)(n+1)-\frac{n(n+2)^2}{n+1}\right]\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx \\
& -(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n}}\Delta\phi dx \\
= & -\frac{(n+2)}{(n+1)}\left[(n+1)^2-n(n+2)\right]\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx \\
& -(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n}}\Delta\phi dx \\
= & -\frac{(n+2)}{(n+1)}\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx-(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n}}\Delta\phi dx. \qquad (2.19)
\end{aligned}$$

We are thus left with:

$$\begin{aligned}
z_n'(t) \leq & -\alpha n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1}v^{\beta n}}A(x,t)dx-\beta n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n+1}}B(x,t)dx \\
& +\lambda_1(n+2)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n}}dx. \qquad (2.20)
\end{aligned}$$

This completes the proof.

Lemma 2.4. *Suppose that u, v is a solution of (2.3) with $u|_{\partial\Omega} = 0$ and ψ is a solution of (2.2). For any $\alpha, \beta \in (0,1)$ such that $\alpha + \beta \le 1$, we have*

$$\frac{d}{dt}\int_\Omega \frac{u^n}{v^{\alpha n}\psi^{\beta n-2}}dx \le n\int_\Omega \frac{u^{n-1}}{v^{\alpha n}\psi^{\beta n-2}}A(x,t)dx - \alpha n\int_\Omega \frac{u^n}{v^{\alpha n+1}\psi^{\beta n-2}}B(x,t)dx$$
$$- (\beta n - 2)\int_\Omega \frac{u^n}{v^{\alpha n}\psi^{\beta n-1+\sigma}}dx. \tag{2.21}$$

proof. See the proof of Lemma 2.5 in [6] (The proof is given in Appendix).

3. Main Results

In this section, we state the main result and prove it. Consider the following singular parabolic system:

$$\begin{cases} u_t = \Delta u + \dfrac{f(x)}{v^p}, & \\ v_t = \Delta v + \dfrac{g(x)}{u^q}, & t > 0, x \in \Omega, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

We say that (u, v) is a weak solution to (3.1) if $u, v \in L^2(0,T; W_0^{1,2}(\Omega)) \bigcap L^\infty(\overline{\Omega} \times (0,T))$,

$$\frac{f(x)}{v^p}, \frac{g(x)}{u^q} \in L^1(\Omega \times (0,T)),$$

and

$$\int_\Omega u_0\xi dx + \int_0^T\int_\Omega \left(u\xi_t - \nabla u\nabla\xi + \frac{f(x)}{v^p}\xi\right)dxdt = 0$$
$$\int_\Omega v_0\xi dx + \int_0^T\int_\Omega \left(v\xi_t - \nabla v\nabla\xi + \frac{g(x)}{u^q}\xi\right)dxdt = 0,$$

for all $\xi \in C^\infty(\overline{\Omega} \times (0,T))$, with $\xi(x) = 0$ on $\partial\Omega \times (0,T)$ and $\xi(T) = 0$ in Ω. We assume that the initial conditions

$$u_0(x), v_0(x) \in C_0^1(\Omega) \quad \text{and} \quad u_0(x), v_0(x) \ge \varepsilon_0\phi(x), \tag{3.2}$$

for some $\varepsilon_0 > 0$, where $\phi(x)$ is a normalized eigenfunction defined in (1.9).

Theorem 3.1. *Suppose that $p, q \in (0,1)$, and that $\delta_0\phi^\theta(x) \le f(x) \le c_0\phi^\eta(x)$, $\delta_0\phi^\tau(x) \le g(x) \le c_0\phi^\mu(x)$, where $\delta_0, c_0 > 0$, $0 \le \eta \le \theta \le 1$ and $0 \le \mu \le \tau \le 1$. Then system (3.1) has a pair of positive, weak solution (u, v). Furthermore, if $N(\eta - p) > -1$, then u is a classical solution; if $N(\mu - q) > -1$, then v is a classical solution.*

proof. To start, we consider the related perturbation problem:

$$\begin{cases} u_t = \Delta u + \dfrac{f(x)}{(v+\varepsilon)^p}, & \\ v_t = \Delta v + \dfrac{g(x)}{(u+\varepsilon)^q}, & t>0, x\in\Omega, \\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\Omega, \\ u(x,t)=v(x,t)=0, & x\in\partial\Omega. \end{cases} \tag{3.3}$$

Denote the solution of (3.3) by $(u_\varepsilon, v_\varepsilon)$ and set $w_\varepsilon = u_\varepsilon + \varepsilon$, $z_\varepsilon = v_\varepsilon + \varepsilon$. For any $\alpha, \beta > 0$ and $\alpha + \beta \le 1$, Lemma 2.3 gives us the following:

$$\begin{aligned} \frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx &\le \lambda_1(n+2)\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx - \alpha n\int_\Omega \frac{\phi^{n+2}f(x)}{w_\varepsilon^{\alpha n+1} z_\varepsilon^{\beta n+p}}dx \\ &\quad - \beta n\int_\Omega \frac{\phi^{n+2}g(x)}{w_\varepsilon^{\alpha n+q} z_\varepsilon^{\beta n+1}}dx \\ &\le \lambda_1(n+2)\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx - \delta_0\alpha n\int_\Omega \frac{\phi^{n+2+\theta}}{w_\varepsilon^{\alpha n+1} z_\varepsilon^{\beta n+p}}dx \\ &\quad - \delta_0\beta n\int_\Omega \frac{\phi^{n+2+\tau}}{w_\varepsilon^{\alpha n+q} z_\varepsilon^{\beta n+1}}dx. \end{aligned} \tag{3.4}$$

For any $\delta > 0$, Lemma 2.1 with $p_1 = \alpha n$, $\alpha_1 = \alpha n + 1$ and $\theta_1 = \alpha n - \delta$ gives us that

$$\begin{aligned} \lambda_1(n+2)\frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} &= \lambda_1(n+2)\frac{(w_\varepsilon^{-1})^{\alpha n}\phi^{n+2}}{z_\varepsilon^{\beta n}} \\ &\le \delta_0\alpha n\frac{(w_\varepsilon^{-1})^{\alpha n+1}\phi^{n+2+\theta}}{z_\varepsilon^{\beta n+p}} + c_1(n)\frac{(w_\varepsilon^{-1})^{\alpha n-\delta}\phi^{\theta_2}}{z_\varepsilon^{\eta_1}}, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \eta_1 &= \beta n(1+\delta) - (\beta n + p)\delta = \beta n - p\delta, \\ \theta_2 &= (n+2)(1+\delta) - (n+2+\theta)\delta = n+2-\theta\delta, \\ c_1(n) &= \lambda_1(n+2)\left(\frac{\delta_0\alpha n}{\lambda_1(n+2)}\right)^{-\delta} \le \lambda_1(n+2)\left(\frac{2\lambda_1}{\alpha\delta_0}\right)^{\delta}. \end{aligned} \tag{3.6}$$

(3.4) then becomes

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx \le c_1(n)\int_\Omega \frac{\phi^{n+2-\theta\delta}}{w_\varepsilon^{\alpha n-\delta} z_\varepsilon^{\beta n-\delta p}}dx - \delta_0\beta n\int_\Omega \frac{\phi^{n+2+\tau}}{w_\varepsilon^{\alpha n+q} z_\varepsilon^{\beta n+1}}dx. \tag{3.7}$$

Using Lemma 2.1 again with $p_1 = \alpha n - \delta$, $\alpha_1 = \alpha n + q$ and $\theta_1 = 0$ yields

$$c_1(n)\frac{(w_\varepsilon^{-1})^{\alpha n-\delta}\phi^{n+2-\theta\delta}}{z_\varepsilon^{\beta n-p\delta}} \le \delta_0\beta n\frac{(w_\varepsilon^{-1})^{\alpha n+q}\phi^{n+2+\tau}}{z_\varepsilon^{\beta n+1}} + c_2(n)\frac{\phi^{\theta_3}}{z_\varepsilon^{\eta_2}}, \tag{3.8}$$

where

$$\begin{aligned}
\eta_2 &= [(\beta n - \delta p)(\alpha n + q) - (\beta n + 1)(\alpha n - \delta)] / (\delta + q) \\
&= [\beta q n - \delta p \alpha n - q \delta p - \alpha n + \delta \beta n + \delta] / (\delta + q) \ , \\
\theta_3 &= [(n + 2 - \theta\delta)(\alpha n + q) - (n + 2 + \tau)(\alpha n - \delta)] / (q + \delta) \\
&= [(q - \alpha\tau + \delta - \theta\alpha\delta)n + 2q - \theta\delta q + 2\delta + \tau\delta] / (q + \delta) \ , \\
c_2(n) &= c_1(n) \left(\frac{\beta\delta_0 n}{c_1(n)} \right)^{-(\alpha n - \delta)/(q+\delta)} \le c_1(n) \left[\left(\frac{2\lambda_1}{\alpha\beta\delta_0} \right)^{1+\delta} \right]^{(\alpha n - \delta)/(q+\delta)} .
\end{aligned} \tag{3.9}$$

If we then set $\eta_2 = 0$ and solve for δ, we find that

$$\delta = \frac{(\alpha - \beta q)n}{(\beta - \alpha p)n + 1 - pq} = \frac{\alpha - \beta q}{\beta - \alpha p} \left(1 - \frac{1 - pq}{(\beta - \alpha p)n + 1 - pq} \right). \tag{3.10}$$

Choose

$$\alpha = 1/(1 + p) - \gamma \tag{3.11}$$

and $\beta = 1 - \alpha$ for some small $\gamma > 0$. Then,

$$\delta = \left(\frac{1 - pq - \gamma(1 + q)(1 + p)}{\gamma(1 + p)^2} \right) \left(1 - \frac{1 - pq}{\gamma(1 + p)n + 1 - pq} \right) > 0. \tag{3.12}$$

Since $1 - \alpha\theta > 0$, $1 - pq > 0$ and $q - \alpha\tau + \delta(1 - \alpha\theta) > 0$ for γ sufficiently small, we have that $\theta_3 > 0$ for sufficiently large n. Thus, (3.7) becomes

$$\frac{d}{dt} \int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} dx \le c_2(n) \int_\Omega \phi^{\theta_3} dx, \tag{3.13}$$

or

$$\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} dx \le \int_\Omega \frac{\phi^{n+2}}{(u_0 + \varepsilon)^{\alpha n} (v_0 + \varepsilon)^{\beta n}} dx + c_2(n) t \int_\Omega \phi^{\theta_3} dx. \tag{3.14}$$

Extracting n^{th} roots and letting $n \to \infty$, we obtain

$$\begin{aligned}
\max_{\bar\Omega} \frac{\phi(x)}{w_\varepsilon^{\alpha} z_\varepsilon^{1-\alpha}} &\le \max \left[\max_{\bar\Omega} \frac{\phi(x)}{(u_0 + \varepsilon)^{\alpha} (v_0 + \varepsilon)^{1-\alpha}}, c_3 \max_{\bar\Omega} \phi^{\sigma_1}(x) \right] \\
&\le \max \left[\max_{\bar\Omega} \frac{\phi(x)}{u_0^{\alpha} v_0^{1-\alpha}}, c_3 \max_{\bar\Omega} \phi^{\sigma_1}(x) \right] \\
&\triangleq M_1,
\end{aligned} \tag{3.15}$$

where M_1 is a constant independent of ε, and

$$\begin{aligned}
c_3 &= \left(\frac{2\lambda_1}{\alpha\beta\delta_0} \right)^{\frac{\alpha(\delta+1)}{q+\delta}} , \\
\sigma_1 &= [q - \alpha\tau + \delta - \delta\alpha\theta] / (q + \delta).
\end{aligned} \tag{3.16}$$

Next, define w_ε and ϕ as before with $\sigma \in (0,1)$. For any $\beta \in (0,1)$, Lemma 2.4 with α replaced by β and β replaced by $1-\beta$ gives us the following:

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\beta n}\phi^{(1-\beta)n-2}}dx \le n\int_\Omega \frac{u_\varepsilon^{n-1}f(x)}{w_\varepsilon^{\beta n}z_\varepsilon^p\phi^{(1-\beta)n-2}}dx - \beta n\int_\Omega \frac{u_\varepsilon^n f(x)}{w_\varepsilon^{\beta n+1}z_\varepsilon^p\phi^{(1-\beta)n-2}}dx\\ - [(1-\beta)n-2]\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\beta n}\phi^{(1-\beta)n-1+\sigma}}dx.\end{aligned}\tag{3.17}$$

For any $\delta_1 > 1$, Lemma 2.1 gives us

$$\frac{u_\varepsilon^{n-1}}{w_\varepsilon^{\beta n}} \le \beta\frac{u_\varepsilon^n}{w_\varepsilon^{\beta n+1}} + \beta^{1-\delta_1}\frac{u_\varepsilon^{n-\delta_1}}{w_\varepsilon^{\eta_1}},\tag{3.18}$$

where

$$\eta_1 = \beta\delta_1 n - (\beta n+1)(\delta_1-1) = \beta n - \delta_1 + 1.\tag{3.19}$$

(3.17) then becomes

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\beta n}\phi^{(1-\beta)n-2}}dx \le n\beta^{1-\delta_1}\int_\Omega \frac{u_\varepsilon^{n-\delta_1}f(x)}{w_\varepsilon^{\eta_1}z_\varepsilon^p\phi^{(1-\beta)n-2}}dx\\ - [(1-\beta)n-2]\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\beta n}\phi^{(1-\beta)n-1+\sigma}}dx.\end{aligned}\tag{3.20}$$

Note that

$$\begin{aligned}\frac{u_\varepsilon^{n-\delta_1}f(x)}{w_\varepsilon^{\eta_1}z_\varepsilon^p\phi^{(1-\beta)n-2}} &\le c_0\frac{u_\varepsilon^{n-\delta_1}\psi^\eta}{w_\varepsilon^{\eta_1-\alpha p/(1-\alpha)}}\left(\frac{\psi}{w_\varepsilon^\alpha z_\varepsilon^{1-\alpha}}\right)^{\frac{p}{1-\alpha}}\left(\frac{\phi}{\psi}\right)^{\frac{p}{1-\alpha}}\frac{1}{\phi^{(1-\beta)n-2+p/(1-\alpha)}}\\ &\le M_2\frac{u_\varepsilon^{n-\delta_1}}{w_\varepsilon^{\eta_1-\alpha p/(1-\alpha)}\phi^{(1-\beta)n-2+p/(1-\alpha)-\eta}},\end{aligned}\tag{3.21}$$

where α is defined as (3.11). Using Lemma 2.1 again paired with (3.21), we see that

$$n\beta^{1-\delta_1}\frac{u_\varepsilon^{n-\delta_1}f(x)}{w_\varepsilon^{\eta_1}z_\varepsilon^p\phi^{(1-\beta)n-2}} \le [(1-\beta)n-2]\frac{u_\varepsilon^n}{w_\varepsilon^{\beta n}\phi^{(1-\beta)n-1+\sigma}} + c_4(n)\frac{1}{w_\varepsilon^{\eta_2}\phi^{\eta_3}},\tag{3.22}$$

where

$$\begin{aligned}\eta_2 &= \left[(\beta n - \delta_1 + 1 - \alpha p/(1-\alpha))n - \beta n(n-\delta_1)\right]/\delta_1\\ &= \left[\beta\delta_1 + 1 - \delta_1 - \alpha p/(1-\alpha)\right]n/\delta_1,\\ \eta_3 &= \left[((1-\beta)n - 2 + p/(1-\alpha) - \eta)n - ((1-\beta)n - 1 + \sigma)(n-\delta_1)\right]/\delta_1\\ &= -\left[(1+\sigma+\eta-p/(1-\alpha) - \delta_1(1-\beta)\right]n/\delta_1 - 1 + \sigma.\end{aligned}\tag{3.23}$$

If we set $\eta_2 = 0$ and solve for δ_1, we find:

$$\delta_1 = \frac{1-\alpha p/(1-\alpha)}{1-\beta} > 1,\tag{3.24}$$

since

$$\frac{\alpha p}{1-\alpha} = \frac{p - p(1+p)\gamma}{p + \gamma(1+p)} < 1. \tag{3.25}$$

Additionally, since

$$\begin{aligned} 1+\sigma+\eta-p/(1-\alpha)-\delta_1(1-\beta) &= 1+\sigma+\eta-p/(1-\alpha)-1+\alpha p/(1-\alpha) \\ &= \sigma+\eta-p \\ &> 0, \end{aligned} \tag{3.26}$$

we thus have that $\eta_3 < 0$. (3.20) then becomes

$$\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\beta n}\phi^{(1-\beta)n-2}}dx \le c_5(n)\int_\Omega \psi^{-\eta_3}dx, \tag{3.27}$$

which implies that $u_\varepsilon \le M_3 w_\varepsilon^\beta \phi^{1-\beta}$, and so u_ε is uniformly bounded. Similarly, we also find that v_ε is uniformly bounded.

Let $\phi_1(x)$ be a solution of (2.1) with $\sigma = -\theta$. By Lemma 2.2,

$$\frac{d}{dt}\int_\Omega \frac{\phi_1^{n+2}}{w_\varepsilon^n}dx \le (n+2)\int_\Omega \frac{\phi_1^{n+1+\theta}}{w_\varepsilon^n}dx - \delta_0 n\int_\Omega \frac{\phi_1^{n+2+\theta}}{w_\varepsilon^{n+1}z_\varepsilon^p}dx. \tag{3.28}$$

By Lemma 2.1,

$$(n+2)\frac{\phi_1^{n+1+\theta}}{w_\varepsilon^n} \le \delta_0 n\frac{\phi_1^{n+2+\theta}}{w_\varepsilon^{n+1}z_\varepsilon^p} + c_6(n)\frac{1}{w_\varepsilon^{\eta_1}z_\varepsilon^{\eta_2}}, \tag{3.29}$$

where

$$\begin{aligned} \eta_1 &= [n(n+2+\theta)-(n+1)(n+1+\theta)] = -(1+\theta) < 0, \\ \eta_2 &= -p(n+1+\theta) < 0, \\ c_6(n) &= (n+2)\left(\frac{\delta_0 n}{n+2}\right)^{-(n+1+\theta)}. \end{aligned} \tag{3.30}$$

Thus, the last term in (3.29) is bounded and $\phi_1 \le M_4 w_\varepsilon$. Similarly, we find that $\phi_1 \le M_5 z_\varepsilon$.

Finally, for any $r \in (1, \frac{1}{p})$,

$$\left\|\frac{f(x)}{z_\varepsilon^p}\right\|_{L^r}^r \le c_0^r\int_\Omega \frac{\psi^{r\eta}(x)}{z_\varepsilon^{rp}}dx \le c_0^r M_5^{rp}\int_\Omega \frac{\psi^{r\eta}(x)}{\phi_1^{rp}(x)}dx \le M_6. \tag{3.31}$$

A similar ineqaulity can be found for v_ε. Hence, by L^p Theory (see Theorem 6 in [16]), $(u_\varepsilon, v_\varepsilon)$ are uniformly bounded in $\left[W_r^{2,1}(\Omega\times(0,T))\right]^2$, and so a subsequence $(u_{\varepsilon_k}, v_{\varepsilon_k})$ can be found which converges to a weak solution of (3.1). If $N(\eta-p) > -1$, we can choose $r > N$ and $r(\eta-p) > -1$ so that (3.31) is true. Then, by the Sobolev embedding theorem, $u_\varepsilon \to u$ in $C^{1,\kappa}(\bar{\Omega})$ for some $\kappa \in (0,1)$, and so u is a classical solution of (3.1). Similar results can be found for v. This completes the proof.

As we can see, this functional method proves to be very powerful when dealing with singular parabolic systems. Future works may include relaxing the conditions put on $f(x)$ and $g(x)$, as well as consideration of the different diffusion coefficients.

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4. Appendix

Here we give the proof of Lemma 2.4 (for details, please see [6]). Similar to the proof of Lemma 2.3, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{u^n}{v^{\alpha n}\phi^{\beta n-2}}dx = \ & n\int_\Omega \frac{u^{n-1}}{v^{\alpha n}\phi^{\beta n-2}}[\Delta u + A(x,t)]dx \\
& - \alpha n \int_\Omega \frac{u^n}{v^{\alpha n+1}\phi^{\beta n-2}}[\Delta v + B(x,t)]dx \\
= & - n(n-1)\int_\Omega \frac{u^{n-2}}{v^{\alpha n}\phi^{\beta n-2}}\left|\nabla u\right|^2 dx + 2\alpha n^2 \int_\Omega \frac{u^{n-1}}{v^{\alpha n+1}\phi^{\beta n-2}}\nabla u \nabla v dx \\
& + n(\beta n-2)\int_\Omega \frac{u^{n-1}}{v^{\alpha n}\phi^{\beta n-1}}\nabla u\nabla\phi dx + n\int_\Omega \frac{u^{n-1}}{v^{\alpha n}\phi^{\beta n-2}}A(x,t)dx \\
& - \alpha n(\alpha n+1)\int_\Omega \frac{u^n}{v^{\alpha n+2}\phi^{\beta n-2}}\left|\nabla v\right|^2 dx \\
& - \alpha n(\beta n-2)\int_\Omega \frac{u^n}{v^{\alpha n+1}\phi^{\beta n-1}}\nabla v\nabla\phi dx \\
& - \alpha n\int_\Omega \frac{u^n}{v^{\alpha n+1}\phi^{\beta n-2}}B(x,t)dx \\
\triangleq & \sum_{k=1}^{7} I_k. \qquad (4.1)
\end{aligned}$$

Letting $k = 1/(1-\alpha)$ and completing the square, we have

$$\begin{aligned}
I_1 + I_2 + I_5 =& -n(n-k)\int_\Omega \frac{u^{n-2}}{v^{\alpha n+2}\phi^{\beta n-2}}\left|v\nabla u - \alpha u\nabla v\right|^2 dx \\
&- n(k-1)\int_\Omega \frac{u^{n-2}}{v^{\alpha n}\phi^{\beta n-2}}\left|\nabla u\right|^2 dx \\
&+ 2k\alpha n\int_\Omega \frac{u^{n-1}}{v^{\alpha n+1}\phi^{\beta n-2}}\nabla u\nabla v dx - \alpha n(1+k\alpha)\int_\Omega \frac{u^n}{v^{\alpha n+2}\phi^{\beta n-2}}\left|\nabla v\right|^2 dx \\
=& -n(n-k)\int_\Omega \frac{u^{n-2}}{v^{\alpha n+2}\phi^{\beta n-2}}\left|v\nabla u - \alpha u\nabla v\right|^2 dx \\
&- \frac{\alpha n}{1-\alpha}\int_\Omega \frac{u^{n-2}}{v^{\alpha n+2}\phi^{\beta n-2}}\left|v\nabla u - u\nabla v\right|^2 dx. \qquad (4.2)
\end{aligned}$$

Consider the following term and integrate by parts,

$$\begin{aligned}
J \triangleq& -n(\beta n-2)\int_\Omega \frac{u^{n-1}}{v^{\alpha n}\phi^{\beta n-1}}\nabla u\nabla\phi dx = -(\beta n-2)\int_\Omega \frac{1}{v^{\alpha n}\phi^{\beta n-1}}\nabla(u)^n\nabla\phi dx \\
=& -(\beta n-2)\alpha n\int_\Omega \frac{u^n}{v^{\alpha n+1}\phi^{\beta n-1}}\nabla v\nabla\phi - (\beta n-2)(\beta n-1)\int_\Omega \frac{u^n}{v^{\alpha n}\psi^{\beta n}}\left|\nabla\phi\right|^2 dx \\
&+ (\beta n-2)\int_\Omega \frac{u^n}{v^{\alpha n}\phi^{\beta n-1}}\Delta\psi dx.
\end{aligned}$$

Then,

$$\begin{aligned}
I_3 + I_6 =& (I_3 - J) + (I_6 + J) \\
=& 2n(\beta n-2)\int_\Omega \frac{u^{n-1}}{v^{\alpha n}\phi^{\beta n-1}}\nabla u\nabla\phi dx - 2\alpha n(\beta n-2)\int_\Omega \frac{u^n}{v^{\alpha n+1}\phi^{\beta n-1}}\nabla v\nabla\phi \\
&- (\beta n-2)(\beta n-1)\int_\Omega \frac{u^n}{v^{\alpha n}\phi^{\beta n}}\left|\nabla\phi\right|^2 dx - (\beta n-2)\int_\Omega \frac{u^n}{v^{\alpha n}\phi^{\beta n-1+\sigma}}dx \\
=& 2n(\beta n-2)\int_\Omega \frac{u^{n-1}}{v^{\alpha n+1}\phi^{\beta n-1}}(v\nabla u - \alpha u\nabla v)\nabla\phi dx \\
&- (\beta n-2)(\beta n-1)\int_\Omega \frac{u^n}{v^{\alpha n}\phi^{\beta n}}\left|\nabla\phi\right|^2 dx - (\beta n-2)\int_\Omega \frac{u^n}{v^{\alpha n}\phi^{\beta n-1+\sigma}}dx. \qquad (4.3)
\end{aligned}$$

Using Cauchy's inequality, we obtain

$$\begin{aligned}
&\left|2n(\beta n-2)\int_\Omega \frac{u^{n-1}}{v^{\alpha n+1}\phi^{\beta n-1}}(v\nabla u - \alpha u\nabla v)\nabla\phi dx\right| \\
&\quad\le n(n-k)\int_\Omega \frac{u^{n-2}}{v^{\alpha n+2}\phi^{\beta n-2}}\left|v\nabla u - \alpha u\nabla v\right|^2 dx \\
&\quad+ \frac{n(\beta n-2)^2}{n-k}\int_\Omega \frac{u^n}{v^{\alpha n}\phi^{\beta n}}\left|\nabla\phi\right|^2 dx. \qquad (4.4)
\end{aligned}$$

Substituting (4.2)-(4.4) into (4.1), we find

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{u^n}{v^{\alpha n}\phi^{\beta n-2}}dx \le& I_4 + I_7 - (\beta n-2)\int_\Omega \frac{u^n}{v^{\alpha n}\phi^{\beta n-1+\sigma}}dx \\
&- \left[(\beta n-2)(\beta n-1) - \frac{n(\beta n-2)^2}{n-k}\right]\int_\Omega \frac{u^n}{v^{\alpha n}\phi^{\beta n}}\left|\nabla\phi\right|^2 dx. \qquad (4.5)
\end{aligned}$$

Since

$$(\beta n - 1)(n - k) - n(\beta n - 2) = n(1 - k\beta) + k = \frac{n(1 - \alpha - \beta)}{1 - \alpha} + k > 0,$$

the last term of (4.5) is negative. Hence, (2.18) is true. This completes the proof.

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