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Effect of the protection zone on coexistence of the species for a ratio-dependent predator-prey model [☆]

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ABSTRACT

This paper deals with the effect of the protection zone Ω_0 on coexistence of the species for a ratio-dependent predator-prey model. We obtain a critical value $\lambda_1^N(b\delta(x), \Omega)$ which is less than the well-known critical value $\lambda_1^D(\Omega_0)$ obtained in the previous literatures. Furthermore, we show that if $\lambda > \lambda_1^N(b\delta(x), \Omega)$, then the prey persists regardless of the growth rates of the predator; while if $\lambda \leq \lambda_1^N(b\delta(x), \Omega)$, then there exists a real number μ^* , such that the prey is ultimately extinct when $\mu > \mu^*$. As for $\lambda < \lambda_1^N(b\delta(x), \Omega)$ and $\mu < \mu^*$, the curve of the positive steady state solutions of the model emanating from $(\lambda, 0; -c)$ ends at a singular point $(0, 0; \mu_2)$. Meantime, by using the Lyapunov–Schmidt reduction method, we obtain a fine profile of its bifurcation diagrams, and the uniqueness or multiplicity of its positive steady state solutions. In addition, as generally expected, the chances of survival of the prey will increase with the size of the protection zone.

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1. Introduction

In the ecology and the mathematical ecology, the dynamical relationships between the predators and their preys always are one of the dominant themes due to its universal existence and importance. Although these problems can appear to be some simple mathematical models, they often are very challenging and complicated. The classical predator-prey model, due independently to Lotka and Volterra in the 1920s, only reflects the population changes due to the predation in a situation where the densities of the prey and the predator are not spatially dependent. When the spatial distributions of the two species are considered, the passive dispersal of the two species can be modelled by a diffusion operator. Thus, a natural mathe-

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mathematical model, which the predator and the prey are interacting and migrating in the same habitat Ω , is a reaction–diffusion system as follows

$$\begin{cases} u_t - d_1 \Delta u = \lambda u - u^2 - b\phi(u, v)v, & x \in \Omega, \quad t > 0; \\ v_t - d_2 \Delta v = \mu v - v^2 + c\phi(u, v)v, & x \in \Omega, \quad t > 0; \end{cases} \quad (1.1)$$

where, Ω is a bounded domain in \mathbf{R}^N with the smooth boundary $\partial\Omega$, $N \geq 1$; u and v are the densities of the prey and the predator respectively, and both the predator and the prey have the logistic growth rates; The terms $b\phi(u, v)$ and $c\phi(u, v)$ respectively account for the functional response of the predator and the conversion rates of the prey captured by the predator. In the classical Lotka–Volterra predator–prey model, it is assumed that $\phi(u, v) = u$. If the handling time of each prey is considered, then the Holling type II response $b\phi(u, v) = \frac{bu}{1+mu}$ usually is chosen [22]. However, the classical predator–prey models with $\phi(u, v) = u$ or $\frac{u}{1+mu}$ exhibit the “paradox of enrichment” and the so-called “biological control paradox” [2,3,21]. Thus, a ratio dependent functional response $b\phi(u, v) = \frac{bu/v}{1+mu/v} = \frac{bu}{mu+v}$ is introduced in [23,24,37], and the corresponding model is called as a ratio-dependent predator–prey model as follows

$$\begin{cases} u_t - d_1 \Delta u = \lambda u - u^2 - \frac{buv}{mu+v}, & x \in \Omega, \quad t > 0; \\ v_t - d_2 \Delta v = \mu v - v^2 + \frac{cuv}{mu+v}, & x \in \Omega, \quad t > 0. \end{cases} \quad (1.2)$$

The researches [23,24,37] showed that the ratio-dependent predator–prey model does not produce those paradox phenomena in [2,3,21]. Therefore, the ratio-dependent predator–prey model should be a more reasonable model in the prey-dependent predator–prey models.

To our knowledge, the ratio-dependent predator–prey model possesses many peculiar characters. For example, $\frac{uv}{mu+v}$ is singular and non-differential at $(u, v) = (0, 0)$, and the curve of its positive solutions can be emanated from the singular point $(0, 0)$ [49], which are different from those of some classical bio-mathematical models such as the Lotka–Volterra predator–prey models [26,27,46], the Holling type II predator–prey model [19] and the competition models [15,29,31,32,35]. In addition, we may re-define $\frac{buv}{u+mv} := 0$ at $(0, 0)$.

For some endangered species, it is necessary that the nature reserves are established to protect these endangered species and their habitats. If a protection zone Ω_0 to the prey is introduced, then the corresponding model becomes as follows

$$\begin{cases} u_t - d_1 \Delta u = \lambda u - u^2 - b\delta(x)\phi(u, v)v, & x \in \Omega, \quad t > 0; \\ v_t - d_2 \Delta v = \mu v - v^2 + c\phi(u, v)v, & x \in \Omega_1, \quad t > 0, \end{cases} \quad (1.3)$$

where Ω_0 is a subdomain of Ω with the smooth boundary $\partial\Omega_0$. The larger region Ω is the habitat of the prey, while the predator only lives in $\Omega_1 := \Omega \setminus \overline{\Omega_0}$. Thus, Ω_0 is called as the protection zone of the prey. $\delta(x)$ satisfies that $\delta(x) \equiv 0$ in $\overline{\Omega_0}$ and $\delta(x) = 1$ in Ω_1 , which implies that the prey enjoys the predation-free growth in Ω_0 . To our knowledge, Du and his coauthors [15,17,18] investigated some biomathematics models with the diffusion and the protect zone Ω_0 , and obtained a critical path $\lambda_1^D(\Omega_0)$ of the persistence/extinction of the prey. Oeda [38] and Wang and Li [43,44] studied effects of the cross-diffusion on some predator–prey models with the protect zones. Cui and his coauthors [10] studied strong Allee effect of a predator–prey model with a protection zone. The pioneer Professor Lopez-Gomez and his coauthors investigated some biomathematics models with the crowing terms and the protection zones, and obtained many prominent results [20,30,31].

In (1.2), if $\lambda < b$, then the prey is ultimately extinct for any sufficient large μ . To the persistence of the prey, the nature reserves must be introduced into (1.2). Thus, in this paper, we will investigate the following predator–prey model with a protection zone

$$\begin{cases} u_t - d_1 \Delta u = \lambda u - u^2 - \frac{b\delta(x)uv}{mu+v}, & x \in \Omega, \quad t > 0; \\ v_t - d_2 \Delta v = \mu v - v^2 + \frac{cuv}{mu+v}, & x \in \Omega_1, \quad t > 0; \\ \partial_\nu u = 0, & x \in \partial\Omega, \quad t > 0; \\ \partial_\nu v = 0, & x \in \partial\Omega_1, \quad t > 0; \\ u(x, 0) = u_0(x), & x \in \bar{\Omega}; \\ v(x, 0) = v_0(x), & x \in \bar{\Omega}_1, \end{cases} \quad (1.4)$$

where $\lambda, \mu, b, c, m, d_1$ and d_2 are constants, and all of these constants are positive except μ which may take negative values. Ω_0 is an open and connected subset of Ω , and $\partial\Omega_0 \in C^2$. On $\partial\Omega$, the homogeneous Neumann boundary conditions indicate that (1.4) is self-contained with zero population flux across the boundary $\partial\Omega$. The boundary $\partial\Omega_0$ does not affect the dispersal of the prey, but it works as a barrier to block the predator from entering Ω_0 ; thus a no-flux boundary condition should be imposed on $\partial\Omega_0$ for the predator. For technical reasons, we assume further that $\Omega_0 \subset \Omega$. Therefore, we call Ω_0 an interior protection zone. For the meanings of other terms and coefficients, one can refer to the references [39,40,47,50].

In (1.4), since the protect zone exists, the coefficient $b\delta(x)$ of the functional response is a function of the space variable x . As far as we know, in the spatial population models, on account of the effect of the environment, some coefficients such as the growth rates, the capacity of the species and the population interaction coefficients, are usually replaced by the functions of the space variable x . The spatially heterogeneous models are very meaningful and valuable in the control to the alien species and in the protection to the endangered species *etc* [1,5–7,14,15,17–20,25,28–36,38,43,44,48,49,51]. It has been observed that in general, the behaviors of the solutions of the class of population models are very sensitive to the change of certain coefficient functions in part of the underlying spatial region. Therefore, the results of these population models [1,5–7,14,15,17–20,25,28–36,38,43,44,48,49,51] are different from those of the corresponding systems in the spatially homogeneous environment [16,26,27,39,40,42,46,47,50]. One can see [1,5–7,14,15,17–20,25,28–36,38,43,44,48,49,51] and references therein for the more discussions on the models in a spatially heterogeneous environment.

In this paper, we study the steady state problem of (1.4). To keep things simple, we assume that $d_1 = d_2 = m = 1$. By employing the bifurcation theories, we will obtain that for $\lambda < b$, there exists a critical value $\lambda_1^N(b\delta(x), \Omega)$, such that when $\lambda > \lambda_1^N(b\delta(x), \Omega)$, (1.4) possesses at least a positive steady state solution for any $\mu > -c$; while when $\lambda \leq \lambda_1^N(b\delta(x), \Omega)$, there exists a real number μ^* , such that (1.4) has no positive steady state solution for any $\mu > \mu^*$. As for $\lambda < \lambda_1^N(b\delta(x), \Omega)$ and $\mu < \mu^*$, (1.4) possesses at least a positive steady state solution, and the branch of the positive steady state solutions of (1.4) emanating from $(\lambda, 0; -c)$ ends at a singular point $(0, 0; \mu_2)$, where $-c < \mu_2 \leq \mu^*$. Furthermore, by employing the Lyapunov–Schmidt reduction method, we will obtain the fine profile of the curve of the positive steady state solutions of (1.4) and the more information of this curve at the singular bifurcation point $(0, 0; \mu_2)$. Our results have three novelties as follows: (i) the critical value $\lambda_1^N(b\delta(x), \Omega)$ in our paper is different from the well-known critical value $\lambda_1^D(\Omega_0)$ obtained in the previous literatures [10,15,17,18,20,30,31,38,43,44]; (ii) we solve the bifurcation of the positive steady state solutions of (1.4) at $(0, 0; \mu_2(\varepsilon))$, but the local bifurcation theory of Crandall–Rabinowitz [8] does not apply to the singular point $(0, 0; \mu_2(\varepsilon))$; (iii) if the protection zone is large enough such that $\lambda > \lambda_1^N(b\delta(x), \Omega)$, then our results show that the prey species of (1.4) may persist in Ω_1 except Ω_0 regardless of the growth rates of the predator. In addition, we also discuss incidentally the case $\lambda > b$. As for the stability of the positive steady state solutions of (1.4) and the dynamical behaviors of the positive solutions of (1.4), we will discuss in another paper.

This paper is organized as follows: in Section 2, we give some basic setups and preliminaries; in Section 3, by using the bifurcation theories [8,31], we discuss the existence of the positive steady state solutions of (1.4); in Section 4, by using the Lyapunov–Schmidt reduction method, we discuss the uniqueness or multiplicity of the positive steady state solutions of (1.4), and obtain some fine bifurcation diagrams.

2. Preliminaries

To keep things simple, we let $d_1 = d_2 = m = 1$, then (1.4) becomes

$$\begin{cases} u_t - \Delta u = \lambda u - u^2 - \frac{b\delta(x)uv}{u+v}, & x \in \Omega, \quad t > 0; \\ v_t - \Delta v = \mu v - v^2 + \frac{cuv}{u+v}, & x \in \Omega_1, \quad t > 0; \\ \partial_\nu u = 0, & x \in \partial\Omega, \quad t > 0; \\ \partial_\nu v = 0, & x \in \partial\Omega_1, \quad t > 0; \\ u(x, 0) = u_0(x), & x \in \overline{\Omega}; \\ v(x, 0) = v_0(x), & x \in \overline{\Omega}_1, \end{cases} \quad (2.1)$$

and the corresponding steady state problem is as follows

$$\begin{cases} -\Delta u = \lambda u - u^2 - \frac{b\delta(x)uv}{u+v}, & x \in \Omega, \\ -\Delta v = \mu v - v^2 + \frac{cuv}{u+v}, & x \in \Omega_1, \\ \partial_\nu u = 0, & x \in \partial\Omega, \\ \partial_\nu v = 0, & x \in \partial\Omega_1. \end{cases} \quad (2.2)$$

In the following, we give some basic setups and preliminaries. Let $\lambda_1^D(\phi, O)$ and $\lambda_1^N(\phi, O)$ be the principal eigenvalues of $-\Delta + \phi$ over the region O subject to Dirichlet or Neumann boundary conditions respectively. If the region O is omitted in the notation, then we understand that $O = \Omega$. If the potential function ϕ is omitted, then $\phi = 0$. It is well known (see [4,5,20,30,36,45]) that the principal eigenvalues $\lambda_1^D(\phi, O)$ and $\lambda_1^N(\phi, O)$ have the following properties: (i) $\lambda_1^D(\phi, O) > \lambda_1^N(\phi, O)$; (ii) $\lambda_1^B(\phi_1, O) > \lambda_1^B(\phi_2, O)$ if $\phi_1 \geq \phi_2$ and $\phi_1 \not\equiv \phi_2$ in \overline{O} for $B = D$ or N ; (iii) $\lambda_1^D(\phi, O_1) \geq \lambda_1^D(\phi, O_2)$ if $O_1 \subset O_2$.

To discuss Problems (2.1) and (2.2), it is necessary that we collect some results of the following problem

$$\begin{cases} u_t - \Delta u + b\delta(x)u = \lambda u - u^2, & x \in \Omega, \quad t > 0; \\ \partial_\nu u = 0, & x \in \partial\Omega, \quad t > 0; \\ u(x, 0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \quad (2.3)$$

and the corresponding steady state problem as follows

$$-\Delta u + b\delta(x)u = \lambda u - u^2, x \in \Omega; \quad \partial_\nu u = 0, x \in \partial\Omega. \quad (2.4)$$

First, we investigate the following eigenvalue problem

$$-\Delta u + b\delta(x)u = ku, x \in \Omega; \quad \partial_\nu u = 0, x \in \partial\Omega. \quad (2.5)$$

By the properties of the principal eigenvalue of $-\Delta + \phi$ and the relationships between the existence of the positive solution of the logistic equation with the potential function $b\delta(x)$ and the principal eigenvalue of (2.5) (see [4,5,20,29,30,36,45]), it is easily obtain the following Lemmas 2.1 and 2.2 when b is finite [20]. As for the case as $b \rightarrow +\infty$, Lemmas 2.1 and 2.2 go back to [5,6,29,30] where the general elliptic operators with Dirichlet boundary condition or a general mixed boundary condition have been discussed. Thus, $\lim_{b \rightarrow +\infty} \lambda_1^N(b\delta(x), \Omega) = \lambda_1^D(\Omega_0)$ is a special case of (11.1) in [5] and Lemma 2.2 (iii) is a special case of Theorem 1.3 in [6]. On the other hand, one also sees the arguments of Lemmas 2.1 and 2.2 in [15].

Lemma 2.1. *Let $\lambda_1^N(b\delta(x), \Omega)$ be the principal eigenvalue of (2.5), then $\lambda_1^N(b\delta(x), \Omega) < \lambda_1^D(\Omega_0)$, and $\lim_{b \rightarrow +\infty} \lambda_1^N(b\delta(x), \Omega) = \lambda_1^D(\Omega_0)$.*

Lemma 2.2. (i) If $\lambda \leq \lambda_1^N(b\delta(x), \Omega)$, then (2.4) only possesses the trivial solution $u(x) = 0$, and $u(x) = 0$ is globally asymptotically stable for (2.3).

(ii) If $\lambda > \lambda_1^N(b\delta(x), \Omega)$, then (2.4) possesses a unique positive solution $u_{\lambda, b\delta(x)}(x)$, and $u_{\lambda, b\delta(x)}(x)$ is globally asymptotically stable for (2.3).

(iii) For $\lambda > \lambda_1^D(\Omega_0)$, $u_{\lambda, b\delta(x)}(x) \rightarrow 0$ in $\overline{\Omega}_1$ and $u_{\lambda, b\delta(x)}(x) \rightarrow \tilde{u}_\lambda$ in Ω_0 as $b \rightarrow +\infty$, where \tilde{u}_λ is the unique positive solution of the following classical logistic equation

$$-\Delta u = \lambda u - u^2, x \in \Omega; \quad u = 0, x \in \partial\Omega_0. \quad (2.6)$$

3. Existence of the positive solutions of (2.2)

In this section, by using the local and global bifurcation theories [8,31], we discuss the existence of the positive solutions of (2.2), and obtain the rough bifurcation diagrams of the curve of the positive solutions of (2.2). First, we investigate the boundedness of the positive solutions of (2.2) and the parameter μ .

Lemma 3.1. Suppose that λ, b and c are fixed. If (2.2) possesses a positive solution (u_μ, v_μ) , then the following results hold:

- (i) $0 < u_\mu(x) < \lambda$ in $\overline{\Omega}$ and $\max\{\mu, 0\} < v_\mu(x) < \mu + c$ in $\overline{\Omega}_1$;
- (ii) if $\lambda > b$, then $\lambda - b < u_\mu(x) < \lambda$ in $\overline{\Omega}$ for any $\mu \in (-c, +\infty)$;
- (iii) if $\lambda_1^N(b\delta(x), \Omega) < \lambda < b$, then $u_{\lambda, b\delta(x)}(x) < u_\mu(x) < \lambda$ in $\overline{\Omega}$ for any $\mu \in (-c, +\infty)$;
- (iv) if $\lambda \leq \lambda_1^N(b\delta(x), \Omega)$, then there exists a positive number μ^* , such that $0 < u_\mu(x) < u_{\lambda, \frac{b\delta(x)\mu}{\lambda+\mu}}(x)$ in $\overline{\Omega}$ for any $\mu \in (0, \mu^*)$ and $0 < u_\mu(x) < \lambda$ in $\overline{\Omega}$ for any $\mu \in (-c, 0]$, while (2.2) has no positive solution for any $\mu \in (\mu^*, +\infty)$, where $u_{\lambda, q(x)}(x)$ is the unique positive solution of the following logistic equation with $\lambda > \lambda_1^N(q(x), \Omega)$,

$$-\Delta u + q(x)u = \lambda u - u^2, x \in \Omega; \quad \partial_\nu u = 0, x \in \partial\Omega. \quad (3.1)$$

Proof. By the standard comparison arguments, Part (i) is easily obtained. As for Parts (ii), (iii) and (iv), from Part (i) and the second equation of (2.2), we have

$$\mu = \lambda_1^N\left(v - \frac{cu}{u+v}\right) > \lambda_1^N\left(-\frac{c\lambda}{\lambda}\right) = -c. \quad (3.2)$$

From the first equation of (2.2), we have

$$-\Delta u \geq \lambda u - u^2 - \frac{b\delta(x)uv}{0+v} = (\lambda - b\delta(x))u - u^2, x \in \Omega; \quad \partial_\nu u = 0, x \in \partial\Omega. \quad (3.3)$$

Thus, if $\lambda > b$, then by using the comparison arguments and the maximum principle, we have $u_\mu(x) > \lambda - b$ in $\overline{\Omega}$ for any $\mu \in (-c, +\infty)$. For $\lambda < b$, if $\lambda > \lambda_1^N(b\delta(x), \Omega)$, then $u_\mu(x) > u_{\lambda, b\delta(x)}(x)$ in $\overline{\Omega}$ for any $\mu \in (-c, +\infty)$.

Next, we discuss the case $\lambda \leq \lambda_1^N(b\delta(x), \Omega)$. From the first equation of (2.2), we have

$$-\Delta u \leq \lambda u - u^2 - \frac{b\delta(x)u\mu}{\lambda+\mu}, x \in \Omega; \quad \partial_\nu u = 0, x \in \partial\Omega \quad (3.4)$$

provided $\mu > 0$. If $\lambda < \lambda_1^N(b\delta(x), \Omega)$, then there exists a real number $\mu^* > 0$, such that $\lambda \leq \lambda_1^N\left(\frac{b\delta(x)\mu}{\lambda+\mu}, \Omega\right) < \lambda_1^N(b\delta(x), \Omega)$ for any $\mu \in (\mu^*, +\infty)$. Thus, $u_\mu = 0$ in $\overline{\Omega}$ for any $\mu \in (\mu^*, +\infty)$. Therefore, for $\lambda < \lambda_1^N(b\delta(x), \Omega)$, if (2.2) possesses a positive solution, then $\mu < \mu^*$ and $u_\mu(x) < u_{\lambda, \frac{b\delta(x)\mu}{\lambda+\mu}}(x)$ in $\overline{\Omega}$ for $\mu \in (0, \mu^*)$. As for $\mu \leq 0$, we have $u_\mu(x) < \lambda$ in $\overline{\Omega}$.

For $\lambda = \lambda_1^N(b\delta(x), \Omega)$, we assume on the contrary that (2.2) possesses a positive solution (u_μ, v_μ) for any $\mu \in (-c, +\infty)$. From (3.4), we have $\lambda_1^N(\frac{b\delta(x)\mu}{\lambda+\mu}, \Omega) < \lambda_1^N(b\delta(x), \Omega) = \lambda$ for any $\mu \in (0, +\infty)$. Thus, $u_\mu(x) < u_{\lambda, \frac{b\delta(x)\mu}{\lambda+\mu}}(x)$ in $\overline{\Omega}$ for any $\mu \in (0, +\infty)$. Since $u_{\lambda, \frac{b\delta(x)\mu}{\lambda+\mu}}(x) \rightarrow 0$ in $\overline{\Omega}$ as $\mu \rightarrow +\infty$, we have $u_\mu(x) \rightarrow u_\infty(x) = 0$ in $\overline{\Omega}$ as $\mu \rightarrow +\infty$. Let $\overline{u}_\mu = \frac{u_\mu}{\max_{x \in \overline{\Omega}} u_\mu}$, then, by dividing the first equation of (2.2) by $\max_{x \in \overline{\Omega}} u_\mu$, we have

$$-\Delta \overline{u}_\mu = \lambda \overline{u}_\mu - u_\mu \overline{u}_\mu - \frac{b\delta(x)\overline{u}_\mu v_\mu}{u_\mu + v_\mu}, \quad x \in \Omega; \quad \partial_\nu \overline{u}_\mu = 0, \quad x \in \partial\Omega. \quad (3.5)$$

By using the estimates for the elliptic equations and the Sobolev imbedding theorem, we obtain that $\overline{u}_\mu \rightarrow \overline{u}_\infty(x)$ in $C^1(\Omega)$ as $\mu \rightarrow +\infty$, and $\overline{u}_\infty(x)$ satisfies that

$$-\Delta \overline{u}_\infty = \lambda \overline{u}_\infty - b\delta(x)\overline{u}_\infty, \quad x \in \Omega; \quad \partial_\nu \overline{u}_\infty = 0, \quad x \in \partial\Omega. \quad (3.6)$$

Since $\max_{x \in \overline{\Omega}} \overline{u}_\mu = 1$, we have $\overline{u}_\infty = \phi_b > 0$ in $\overline{\Omega}$, where ϕ_b is the corresponding eigenfunction of $\lambda_1^N(b\delta(x), \Omega)$. By multiplying the first equation of (2.2) by ϕ_b and integrating over Ω by parts, we have

$$\int_{\Omega} -\Delta \phi_b u_\mu dx = \int_{\Omega} [\lambda u_\mu - u_\mu^2 - \frac{b\delta(x)u_\mu v_\mu}{u_\mu + v_\mu}] \phi_b dx. \quad (3.7)$$

Since $\lambda = \lambda_1^N(b\delta(x), \Omega)$, we have

$$\int_{\Omega} \frac{b\delta(x)u_\mu^2}{u_\mu + v_\mu} \phi_b dx = \int_{\Omega} u_\mu^2 \phi_b dx. \quad (3.8)$$

Thus for $\mu > 0$, we have

$$\frac{b}{\mu} \int_{\Omega_1} u_\mu^2 \phi_b dx \geq \int_{\Omega} u_\mu^2 \phi_b dx. \quad (3.9)$$

It is easily checked that $u_\mu(x) = 0$ in $\overline{\Omega}$ provided $\mu > b$. Therefore, if (2.2) possesses a positive solution, then there exists μ^* such that $-c < \mu < \mu^*$. \square

3.1. Local bifurcation analysis

In this subsection, let λ, b and c be fixed and assume $\lambda \neq \lambda_1^N(b\delta(x), \Omega)$. We will use the predator growth μ as the bifurcation parameter and consider the local structure of the curve of the positive solutions of (2.2).

For $\mu \leq 0$, (2.2) possesses the trivial solution $(0, 0)$ and a semi-trivial solution $(\lambda, 0)$; while for $\mu > 0$, (2.2) possesses the trivial solution $(0, 0)$ and two semi-trivial solutions $(\lambda, 0)$ and $(0, \mu)$. Therefore, (2.2) possesses three curves of the solutions in the space of $(u, v; \mu)$:

$$\Gamma_u := \{(\lambda, 0; \mu) : \mu \in \mathbf{R}\}, \quad \Gamma_v := \{(0, \mu; \mu) : \mu > 0\}, \quad \Gamma_0 := \{(0, 0; \mu) : \mu \in \mathbf{R}\}.$$

By the strong maximum principle, any nonnegative solution (u, v) of (2.2) is either $(0, 0)$, or a semi-trivial solution, or a positive solution.

For $p > 1$, let $X_1 := \{u \in W^{2,p}(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}$ and $Y_1 := L^p(\Omega)$, and $X_2 := \{v \in W^{2,p}(\Omega_1) : \partial_\nu v = 0 \text{ on } \partial\Omega_1\}$ and $Y_2 := L^p(\Omega_1)$. We now carry through the local bifurcation analysis of the positive solutions to (2.2) along the curves Γ_u, Γ_v and Γ_0 respectively.

(i) We first discuss the bifurcation along Γ_u . We change the variables $w = \lambda - u$ and $v = v$, and let $F : X_1 \times X_2 \times \mathbf{R} \rightarrow Y_1 \times Y_2$ by

$$F(w, v; \mu) := \begin{pmatrix} \Delta w - \lambda w + w^2 + \frac{b\delta(x)(\lambda-w)v}{\lambda-w+v} \\ \Delta v + \mu v - v^2 + \frac{c(\lambda-w)v}{\lambda-w+v} \end{pmatrix},$$

then

$$F_{(w,v)}(0, 0; \mu)(\varphi, \psi) = \begin{pmatrix} \Delta\varphi - \lambda\varphi + b\delta(x)\psi \\ \Delta\psi + \mu\psi + c\psi \end{pmatrix},$$

$$F_\mu(0, 0; \mu) = \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, F_{\mu(w,v)}(0, 0; \mu)(\varphi, \psi) = \begin{pmatrix} 0 \\ \psi \end{pmatrix}.$$

We find that only when $\mu = -c$, $F_{(w,v)}(0, 0; \mu)(\varphi, \psi) = \mathbf{0}$ possesses a solution with $\psi_1 := \psi(x) = 1 > 0$ in $\overline{\Omega}_1$ and then $\varphi_1 := (-\Delta + \lambda)^{-1}(b\delta(x)\psi) > 0$ in $\overline{\Omega}$. Thus, $\mu_1 := -c$ can be a bifurcation point along Γ_u where the positive solutions of (2.2) are emanated. Furthermore,

$$\text{Ker}(F_{(w,v)}(0, 0; \mu_1)) = \text{span}\{(\varphi_1, \psi_1)\}$$

which is of dimension one; The range of the operator $F_{(w,v)}(0, 0; \mu_1)$ is given by

$$R(F_{(w,v)}(0, 0; \mu_1)) = \{(f, g) \in Y^2 : \int_{\Omega_1} g(x)\psi_1 dx = 0\}$$

which is of co-dimension one, and $F_{\mu(w,v)}(0, 0; \mu_1)(\varphi_1, \psi_1) = (0, 1) \notin R(F_{(w,v)}(0, 0; \mu_1))$ since $\int_{\Omega_1} 1^2 dx > 0$. Hence, we may apply the local bifurcation theory of Crandall–Rabinowitz [8] to conclude that a unique branch of the set of the positive solutions to (2.2) near to $(\lambda, 0; \mu_1)$ is a smooth curve as follows

$$\Gamma_1 := \{(\lambda - u_1(s), v_1(s); \mu_1(s)) : s \in (0, \delta)\}, \quad (3.10)$$

where $\mu_1(0) = -c$, $u_1(s) = s\varphi_1(x) + o(s)$ and $v_1(s) = s + o(s)$, $0 < \delta \ll 1$.

By integrating the second equation of (2.2) over Ω_1 , we have

$$(\mu_1(s) + c) \int_{\Omega_1} v dx = \int_{\Omega_1} v^2 dx + \int_{\Omega_1} \frac{cv^2}{u+v} dx. \quad (3.11)$$

By dividing (3.11) by s^2 , we have

$$\frac{\mu_1(s) + c}{s} \int_{\Omega_1} \frac{v}{s} dx = \int_{\Omega_1} \left(\frac{v}{s}\right)^2 dx + \int_{\Omega_1} \frac{cv^2}{s^2(u+v)} dx. \quad (3.12)$$

Since $\lim_{s \rightarrow 0^+} \frac{v(s)}{s} = 1$, $\lim_{s \rightarrow 0^+} u(s) = \lambda$, $\lim_{s \rightarrow 0^+} v(s) = 0$ and $\mu_1(0) = -c$, we have

$$\mu'_1(0) = 1 + \frac{c}{\lambda} > 0. \quad (3.13)$$

Therefore, the curve Γ_1 at $(\lambda, 0; \mu_1)$ turns right.

(ii) For $\mu > 0$, we claim that there has no positive solution emanating from any point on Γ_v . In fact, let $v = \mu + w$ and $u = u$, and let $G : X_1 \times X_2 \times \mathbf{R} \rightarrow Y_1 \times Y_2$ by

$$G(u, w; \mu) := \begin{pmatrix} \Delta u + \lambda u - u^2 - \frac{b\delta(x)u(\mu+w)}{u+\mu+w} \\ \Delta w - \mu w - w^2 + \frac{cu(\mu+w)}{u+\mu+w} \end{pmatrix},$$

then

$$G_{(u,w)}(0, 0; \mu)(\varphi, \psi) = \begin{pmatrix} \Delta \varphi + (\lambda - b(x))\varphi \\ \Delta \psi - \mu \psi + c\varphi \end{pmatrix}.$$

For any given λ and Ω_0 , since $\lambda \neq \lambda_1^N(b\delta(x), \Omega)$ and $\varphi \geq 0$ in $\overline{\Omega}$, by $G_{(u,w)}(0, 0; \mu)(\varphi, \psi) = \mathbf{0}$, we obtain that $\varphi(x) \equiv 0$ in $\overline{\Omega}$ and then $\psi(x) \equiv 0$ in $\overline{\Omega}_1$. Therefore, any point $(0, \mu; \mu)$ on Γ_v is not bifurcation point of the curve of the positive solutions of (2.2) for $\mu > 0$.

Since $uv/(u+v)$ is singular and non-differential at $(0, 0)$, the bifurcation along Γ_0 can not be discussed by the local bifurcation theory [8]. In the following subsection, we may discuss the bifurcation along Γ_0 by the global bifurcation arguments.

3.2. Global bifurcation analysis

In this subsection, we consider the global structure of the curve of the positive solutions of (2.2) emanating from $(\lambda, 0; \mu_1)$. According to the global bifurcation theory [11–13, 31, 41], the union set of the nontrivial solutions of (2.2) emanating from the point $(\lambda, 0; \mu_1)$ plus all bifurcation points form two closed continuous components Γ^+ and Γ^- which meet at $(\lambda, 0; \mu_1)$ (see Theorem 7.2.2 in [31]), and Γ^+ satisfies the following alternatives:

- (a) Γ^+ is unbounded in $X_1 \times X_2 \times \mathbf{R}$;
- (b) there exists a real number $\mu_2 \neq \mu_1$ such that $(\lambda, 0; \mu_2) \in \Gamma^+$;
- (c) there exists $(u, v; \mu_3) \in \Gamma_0$ or Γ_v such that $(u, v; \mu_3) \in \Gamma^+$.

For $\lambda > b$ or $\lambda_1^N(b\delta(x), \Omega) < \lambda < b$, by Lemma 3.1 and the arguments of Parts (i) and (ii) of Subsection 3.1, we know that the alternatives (b) and (c) don't occur, and then the alternative (a) holds. Furthermore, by Lemma 3.1, since all positive solutions of (2.2) are bounded for any fixed μ , (2.2) possesses at least a positive solution for any $\mu \in (-c, +\infty)$.

For $\lambda < \lambda_1^N(b\delta(x), \Omega)$, by Lemma 3.1 and the arguments of Part (i) of Subsection 3.1, we know that $-c < \mu < \mu^*$, and the alternatives (a) and (b) don't occur. Thus, the alternative (c) holds. Furthermore, by $-c < \mu < \mu^*$ and the Part (ii) of Subsection 3.1, we obtain that Γ^+ ends at some point $(0, 0; \mu_2)$ on Γ_0 for some $\mu_2 \leq 0$, where μ_2 is a non-positive real number to be determined. Therefore, $(0, 0; \mu_2)$ may be considered as another bifurcation point of the branch of the set of positive solutions of (2.2).

We now determine μ_2 . Let $\Gamma_2 := \{(u(s), v(s); \mu_2(s)) : 0 < s \ll 1\} \subset \Gamma^+$ be a continuous component of the set of the positive solutions of (2.2) emanating from $(0, 0; \mu_2)$, where $\mu_2(0) = \mu_2$, $u(0) = 0$ and $v(0) = 0$. We claim that $(u(s), v(s); \mu_2(s))$ on Γ_2 satisfies that $\frac{u(s)}{v(s)} = O(1)$ in $\overline{\Omega}_1$ for $0 < s \ll 1$. In fact, we assume on the contrary that if $\frac{u(s)}{v(s)} = o(1)$ in $\overline{\Omega}_1$ for $0 < s \ll 1$, then for any small $\varepsilon > 0$, we have $u(s) \leq \varepsilon v(s)$ in $\overline{\Omega}_1$ for $0 < s \ll 1$. Thus, by the first equation of (2.2), we have

$$-\Delta u \leq \lambda u - u^2 - \frac{b\delta(x)u}{1+\varepsilon}, \quad x \in \Omega; \quad \partial_\nu u = 0, \quad x \in \partial\Omega. \quad (3.14)$$

Since $\lambda < b$ and $\lambda < \lambda_1^N(b\delta(x), \Omega)$, we may take $\varepsilon > 0$ small enough such that $\lambda < \lambda_1^N[\frac{b\delta(x)}{1+\varepsilon}, \Omega]$. Thus $u(s) \leq 0$ in $\overline{\Omega}$, which contradicts $u(s) > 0$ in $\overline{\Omega}$. While if $\frac{v(s)}{u(s)} = o(1)$ in $\overline{\Omega}_1$ for $0 < s \ll 1$, then for any small $\varepsilon > 0$, we have $v(s) \leq \varepsilon u(s)$ in $\overline{\Omega}_1$ for $0 < s \ll 1$. Thus, by the first equation of (2.2), we have

$$-\Delta u \geq \lambda u - u^2 - \frac{b\delta(x)\varepsilon u}{1+\varepsilon}, \quad x \in \Omega; \quad \partial_\nu u = 0, \quad x \in \partial\Omega. \quad (3.15)$$

By taking $\varepsilon > 0$ small enough and by using the comparison principle, we have $u(s) > u_{\lambda, \frac{b\delta(x)\varepsilon}{1+\varepsilon}}(x) > u_{\frac{\lambda}{2}, 0} > 0$ in $\overline{\Omega}$, which contradicts $u(s) \rightarrow 0$ in $\overline{\Omega}$ as $s \rightarrow 0$.

For simplicity, let $u(s) = \varphi_2 s + o(s)$ and $v(s) = \psi_2 s + o(s)$ for $0 < s \ll 1$, then $\lim_{s \rightarrow 0^+} \frac{u(s)}{s} = \varphi_2$, $\lim_{s \rightarrow 0^+} \frac{v(s)}{s} = \psi_2$, where φ_2 and ψ_2 are the positive functions in $\overline{\Omega}$ and $\overline{\Omega}_1$ respectively. By dividing (2.2) by s and letting $s \rightarrow 0^+$, we have

$$\begin{cases} -\Delta \varphi_2 = \lambda \varphi_2 - \frac{b\delta(x)\varphi_2\psi_2}{\varphi_2 + \psi_2}, & x \in \Omega; \\ -\Delta \psi_2 = \mu_2 \psi_2 + \frac{c\varphi_2\psi_2}{\varphi_2 + \psi_2}, & x \in \Omega_1; \\ \partial_\nu \varphi_2 = 0, & x \in \partial\Omega; \\ \partial_\nu \psi_2 = 0, & x \in \partial\Omega_1. \end{cases} \quad (3.16)$$

Since both φ_2 and ψ_2 are the positive functions, we obtain that $\mu_2 \neq 0$. Thus, $\mu_2 < 0$. Furthermore, by Lemma 3.1, we know that $-c = \mu_1 < \mu_2 < 0$. By integrating two equations of (3.16) over $\overline{\Omega}$ and $\overline{\Omega}_1$ respectively and by computing simply, we obtain that

$$c\lambda \int_{\Omega} \varphi_2 dx + b\mu_2 \int_{\Omega_1} \psi_2 dx = 0. \quad (3.17)$$

Thus, $\mu_2 = -\frac{c\lambda \int_{\Omega} \varphi_2 dx}{b \int_{\Omega_1} \psi_2 dx}$.

Next, we consider the turn direction of the component Γ_2 at $(0, 0; \mu_2)$. Let the component Γ_2 near to $(0, 0; \mu_2)$ be denoted by

$$\Gamma_2 = \{(u(s), v(s); \mu_2(s)) : s \in (0, \delta)\}. \quad (3.18)$$

By integrating two equations of (2.2) over $\overline{\Omega}$ and $\overline{\Omega}_1$ respectively and by computing simply,

$$c\lambda \int_{\Omega} u(s) dx - c \int_{\Omega} [u(s)]^2 dx + b\mu_2(s) \int_{\Omega_1} v(s) dx - b \int_{\Omega_1} [v(s)]^2 dx = 0. \quad (3.19)$$

Let $u(s) = s\varphi_2(x) + s^2\varphi_3(x) + o_1(s^2)$ and $v(s) = s\psi_2(x) + s^2\psi_3(x) + o_2(s^2)$ express as the Taylor expansions with respect to s for $0 < s \ll 1$, then by (3.17) and (3.19), we have

$$\begin{aligned} c\lambda \int_{\Omega} [s^2\varphi_3(x) + o_1(s^2)] dx - c \int_{\Omega} [u(s)]^2 dx + b[\mu_2(s) - \mu_2(0)] \int_{\Omega_1} v(s) dx \\ + b\mu_2(0) \int_{\Omega_1} [s^2\psi_3(x) + o_2(s^2)] dx - b \int_{\Omega_1} [v(s)]^2 dx = 0. \end{aligned} \quad (3.20)$$

Let (3.20) be divided by s^2 and let $s \rightarrow 0^+$, then

$$c\lambda \int_{\Omega} \varphi_3 dx - c \int_{\Omega} \varphi_2^2 dx + b\mu_2'(0) \int_{\Omega_1} \psi_2 dx + b\mu_2(0) \int_{\Omega_1} \psi_3 dx - b \int_{\Omega_1} \psi_2^2 dx = 0. \quad (3.21)$$

Thus,

$$\mu'_2(0) = \frac{c \int_{\Omega} \varphi_2^2 dx + b \int_{\Omega_1} \psi_2^2 dx - c\lambda \int_{\Omega} \varphi_3 dx - b\mu_2(0) \int_{\Omega_1} \psi_3 dx}{b \int_{\Omega_1} \psi_2 dx}. \quad (3.22)$$

Combined the arguments above with the local bifurcation results in Subsection 3.1, we obtain the following theorem.

Theorem 3.2. Suppose that b, c and λ be fixed.

(i) If $\lambda > b$, or $\lambda < b$ and $\lambda > \lambda_1^N(b\delta(x), \Omega)$, then there exists a continuum Γ^+ of the positive solutions of (2.2) satisfying

$$\text{Proj}_{\mu} \Gamma^+ = (-c, +\infty), \quad (3.23)$$

which implies that (2.2) possesses at least a positive solution for any $\mu \in (-c, +\infty)$.

(ii) If $\lambda < \lambda_1^N(b\delta(x), \Omega)$, then there exists a continuum Γ^+ of the positive solutions of (2.2) satisfying

$$\text{Proj}_{\mu} \Gamma^+ = (\mu_1, \mu^*) \quad \text{or} \quad (\mu_1, \mu^*], \quad (3.24)$$

and Γ^+ contains Γ_1 and Γ_2 , where $\mu_2(0) \leq \mu^* < +\infty$, $\mu_1 = -c$ and $-c < \mu_2 = \mu_2(0) < 0$, and the components Γ_1 and Γ_2 emanate from $(\lambda, 0; \mu_1)$ and $(0, 0; \mu_2)$ respectively. Therefore, (2.2) possesses at least a positive solution for any $\mu \in (-c, \mu^*)$. Furthermore, the curve Γ^+ at $(\lambda, 0; \mu_1)$ turns right (i.e. $\mu'_1(0) > 0$), while the turning direction of the curve Γ^+ at $(0, 0; \mu_2)$ is determined by

$$\mu'_2(0) = \frac{c \int_{\Omega} \varphi_2^2 dx + b \int_{\Omega_1} \psi_2^2 dx - c\lambda \int_{\Omega} \varphi_3 dx - b\mu_2(0) \int_{\Omega_1} \psi_3 dx}{b \int_{\Omega_1} \psi_2 dx}, \quad (3.25)$$

where $\varphi_2(x), \varphi_3(x), \psi_2(x)$ and $\psi_3(x)$ are determined by $u(s) = s\varphi_2(x) + s^2\varphi_3(x) + o_1(s^2)$ and $v(s) = s\psi_2(x) + s^2\psi_3(x) + o_2(s^2)$ for $0 < s \ll 1$.

4. Uniqueness or multiplicity of positive solutions of (2.2)

Theorem 3.2 only obtain a rough profile of the curve of the positive solutions of (2.2), and μ_2 and (φ_2, ψ_2) and $\mu'_2(0)$ are not exactly expressed. In this section, by using the Lyapunov–Schmidt reduction method, we will obtain a fine profile of the curve of the positive solutions of (2.2), and the more exact expressions of μ_2 and (φ_2, ψ_2) and $\mu'_2(0)$. Meantime, we may obtain the uniqueness or multiplicity of the positive solutions of (2.2) for any sufficient small $\varepsilon > 0$. One can see [16,26,43,44] and references therein where discuss some reaction-diffusion systems by the Lyapunov–Schmidt reduction method.

4.1. Lyapunov–Schmidt reduction

Let $u = \varepsilon w, v = \varepsilon z, \lambda = \varepsilon\alpha, \mu = \varepsilon\beta, b = \varepsilon b_1$ and $c = \varepsilon c_1$. Then (2.2) becomes

$$\begin{cases} \Delta w + \varepsilon(\alpha w - w^2 - \frac{b_1\delta(x)wz}{w+z}) = 0, & x \in \Omega, \\ \Delta z + \varepsilon(\beta z - z^2 + \frac{c_1 wz}{w+z}) = 0, & x \in \Omega_1, \\ \partial_{\nu} w = 0, & x \in \partial\Omega, \\ \partial_{\nu} z = 0, & x \in \partial\Omega_1. \end{cases} \quad (4.1)$$

It is obvious that (4.1) possesses two semitrivial solutions $(\alpha, 0)$ and $(0, \beta)$ provided $\beta > 0$, and the trivial solution $(0, 0)$.

Let $X := (X_1, X_2)$ and $Y := (Y_1, Y_2)$ be two Banach spaces, where X_1, X_2, Y_1 and Y_2 are defined in Subsection 3.1 and $((u, v); \mu) := (u, v; \mu)$, and let

$$\begin{cases} F(w, z) := \alpha w - w^2 - \frac{b_1 \delta(x) wz}{w+z}, \\ G((w, z); \beta) := \beta z - z^2 + \frac{c_1 wz}{w+z}, \end{cases}$$

and define a linear operator $H : X \rightarrow Y$, and a nonlinear operator $B : X \times R \rightarrow Y$ by

$$H(w, z) = (\Delta w, \Delta z), \quad B((w, z); \beta) = (F(w, z), G((w, z); \beta)).$$

Thus, (4.1) becomes

$$H(w, z) + \varepsilon B((w, z); \beta) = 0. \quad (4.2)$$

It is obvious that $\text{Ker} H = \text{span}\{(1, 0), (0, 1)\} = \mathbf{R}^2$. Thus, X and Y may be decomposed as

$$X = \mathbf{R}^2 \oplus X_3, \quad Y = \mathbf{R}^2 \oplus Y_3,$$

where X_3 and Y_3 denote the L^2 -orthogonal spaces of \mathbf{R}^2 in X and Y respectively. Let $P : X \rightarrow X_3$ and $Q : Y \rightarrow Y_3$ be the orthogonal projections, then any solution (w, z) of (4.2) (or (4.1)) can be uniquely decomposed as

$$(w, z) = (r, s) + \mathbf{u}, \quad \text{where } (r, s) \in \mathbf{R}^2, \quad \mathbf{u} = P(w, z) \in X_3.$$

It is easily checked that $H(r, s) = 0$ and $(I - Q)H(X_3) = 0$. Thus (4.2) may be reduced to

$$QH(\mathbf{u}) + \varepsilon QB((r, s) + \mathbf{u}; \beta) = 0, \quad (4.3)$$

$$(I - Q)B((r, s) + \mathbf{u}; \beta) = 0. \quad (4.4)$$

It is easily checked that $QH(\mathbf{u}) = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$. Thus, QH is reversible in X_3 . Furthermore, by the implicit function theorem and the Lyapunov–Schmidt reduction, we can obtain the following lemma.

Lemma 4.1. *For any fixed $C > 0$, there exist a small $\varepsilon_0 > 0$ and a neighborhood N of $\{((w, z); \beta, \varepsilon) = ((r, s); \beta, 0) \in \mathbf{R}^2 \times \mathbf{R}^2 : |r|, |s|, |\beta| \leq C\}$, such that the curve of the positive solutions of (4.2) contained in N can be parameterized as*

$$K = \{((r, s) + \varepsilon \mathbf{U}((r, s); \beta, \varepsilon); \beta, \varepsilon) : |r|, |s|, |\beta| \leq C + \varepsilon_0, |\varepsilon| \leq \varepsilon_0\}, \quad (4.5)$$

where $\mathbf{U}((r, s); \beta, \varepsilon)$ is an X_3 -valued smooth function vector. Therefore,

$$((w, z); \beta, \varepsilon) = ((r, s) + \varepsilon \mathbf{U}((r, s); \beta, \varepsilon); \beta, \varepsilon) \in K \quad (4.6)$$

is a solution of (4.2) if and only if

$$\Phi^\varepsilon((r, s); \beta) := (I - Q)B((r, s) + \varepsilon \mathbf{U}((r, s); \beta, \varepsilon); \beta) = 0. \quad (4.7)$$

4.2. Structure of the solutions of $\Phi^0((r, s); \beta) = 0$

By Lemma 4.1, the positive solution of (4.2) (or (4.1)) in K must satisfy (4.7). Thus, by investigating the structure of the solutions $((r, s); \beta)$ of $\Phi^0((r, s); \beta) = 0$, we will obtain some important information of the solutions of $\Phi^\varepsilon((r, s); \beta) = 0$ for $\varepsilon > 0$ small enough.

Since $(I - Q)(w, z) = (\frac{1}{|\Omega|} \int_{\Omega} w dx, \frac{1}{|\Omega_1|} \int_{\Omega_1} z dx)$, we have

$$\Phi^0((r, s); \beta) = \begin{pmatrix} \frac{1}{|\Omega|} \int_{\Omega} r(\alpha - r - \frac{b_1 \delta(x)s}{r+s}) dx \\ \frac{1}{|\Omega_1|} \int_{\Omega_1} s(\beta - s + \frac{c_1 r}{r+s}) dx \end{pmatrix}. \quad (4.8)$$

Thus, $\text{Ker} \Phi^0((r, s); \beta) = K_0 \cup K_w \cup K_z \cup K_p$, where

$$K_0 = \{((0, 0); \beta) : \beta \in \mathbf{R}\}, \quad K_w = \{((\alpha, 0); \beta) : \beta \in \mathbf{R}\}, \\ K_z = \{((0, \beta); \beta) : \beta \in \mathbf{R}^+\}, \quad K_p = \{((r, f(r)); g(r)) : r \in \mathbf{R}^+\},$$

$\alpha_0 := b_1 |\Omega_1| / |\Omega|$, and

$$\begin{cases} f(r) = \frac{(\alpha-r)r}{\alpha_0 - \alpha + r}, \\ g(r) = \frac{(\alpha-r)r}{\alpha_0 - \alpha + r} - \frac{c_1(\alpha_0 - \alpha)}{\alpha_0} - \frac{c_1 r}{\alpha_0}. \end{cases} \quad (4.9)$$

Since K_p contains the limiting set of the positive solutions of (4.2), it is very important to investigate the profiles of $f(r)$ and $g(r)$. On the other hand, since $0 \leq u \leq \lambda$, we may assume that $0 \leq r \leq \alpha$. Hence, we obtain the following lemma.

Lemma 4.2. *Let $f(r), g(r)$ and α_0 be defined in (4.9), then the following results hold:*

- (i) *For $f(r)$, (a) if $\alpha < \alpha_0$, then $f(r) > 0$ for $r \in (0, \alpha)$, and $\lim_{r \rightarrow 0^+} \frac{f(r)}{r} = \frac{\alpha}{\alpha_0 - \alpha}$;*
(b) if $\alpha > \alpha_0$, then $f(r) < 0$ for $r \in (0, \alpha - \alpha_0)$, $f(r) > 0$ for $r \in (\alpha - \alpha_0, \alpha)$, and $\lim_{r \rightarrow (\alpha - \alpha_0)^\pm} f(r) = \pm \infty$.
- (ii) *For $g(r)$, it is obvious that $g(0) = -\frac{c_1(\alpha_0 - \alpha)}{\alpha_0}$ and $g(\alpha) = -c_1$. Furthermore,*
(a) if $\alpha > \alpha_0$, then $g'(r) < 0$ for $r \in [0, \alpha - \alpha_0) \cup (\alpha - \alpha_0, \alpha]$, $\lim_{r \rightarrow (\alpha - \alpha_0)^\pm} g(r) = \pm \infty$;
(b) if $0 < \alpha \leq \frac{\alpha_0 c_1}{\alpha_0 + c_1}$, then $g'(r) < 0$ for $r \in (0, \alpha]$; while if $\frac{\alpha_0 c_1}{\alpha_0 + c_1} < \alpha < \alpha_0$, then there is a positive number $r_1 := -(\alpha_0 - \alpha) + \alpha_0 \sqrt{\frac{\alpha_0 - \alpha}{\alpha_0 + c_1}}$ such that $g'(r_1) = 0$, $g'(r) > 0$ for $r \in [0, r_1)$ and $g'(r) < 0$ for $r \in (r_1, \alpha]$.

Proof. From (4.9), Part (i) is obvious. In the following, we show Part (ii).

For $g(r)$, from (4.9), it is easily checked that

$$g(0) = -\frac{c_1(\alpha_0 - \alpha)}{\alpha_0}, \quad g(\alpha) = -c_1, \quad (4.10)$$

$$g'(r) = -\frac{\alpha_0 + c_1}{\alpha_0} + \frac{\alpha_0(\alpha_0 - \alpha)}{(\alpha_0 - \alpha + r)^2}, \quad (4.11)$$

$$g'(0) = \frac{\alpha_0 \alpha - c_1(\alpha_0 - \alpha)}{\alpha_0(\alpha_0 - \alpha)}, \quad g'(\alpha) = -\frac{c_1 + \alpha}{\alpha_0} < 0. \quad (4.12)$$

Thus, if $\alpha > \alpha_0$, then $g'(r) < 0$ for $r \in [0, \alpha - \alpha_0) \cup (\alpha - \alpha_0, \alpha]$, and $\lim_{r \rightarrow (\alpha - \alpha_0)^-} g(r) = -\infty$ and $\lim_{r \rightarrow (\alpha - \alpha_0)^+} g(r) = +\infty$.

If $\alpha < \alpha_0$, then $r = \alpha - \alpha_0 \notin [0, \alpha]$, and the solutions of $g'(r) = 0$ are

$$r_{1,2} = -(\alpha_0 - \alpha) \pm \alpha_0 \sqrt{\frac{\alpha_0 - \alpha}{\alpha_0 + c_1}}, \quad (4.13)$$

$$g(r_1) = 2\alpha_0 - \alpha - 2\sqrt{(\alpha_0 - \alpha)(\alpha_0 + c_1)}. \quad (4.14)$$

It is obvious that $r_2 < 0$. On the other hand, $r_1 > 0$ if and only if

$$\alpha > \frac{\alpha_0 c_1}{\alpha_0 + c_1} := \alpha_1. \quad (4.15)$$

Therefore, by (4.11), if $\frac{\alpha_0 c_1}{\alpha_0 + c_1} < \alpha < \alpha_0$, then $g'(r) > 0$ for $r \in [0, r_1)$ and $g'(r) < 0$ for $r \in (r_1, \alpha]$; while if $0 < \alpha \leq \frac{\alpha_0 c_1}{\alpha_0 + c_1}$, then $g'(r) < 0$ for $r \in (0, \alpha]$. \square

4.3. Construction of the positive solutions of (4.1)

In this subsection, by perturbing the curve of the solutions set $\{(r, f(r)); g(r)\}$ of $\Phi^0((r, s); \beta) = 0$, we will construct the positive solutions of (4.1) for $\varepsilon > 0$ small enough, which yields a fine profile of the curve of the positive solutions of (2.2). In the following, we discuss these problems according to the cases $\alpha > \alpha_0$ and $\alpha < \alpha_0$ respectively.

Case $\alpha > \alpha_0$

For $\alpha > \alpha_0$, from Lemma 4.2, we know that $f(r) > 0$ only when $r \in (\alpha - \alpha_0, \alpha)$, $f'(r) = -1 + \frac{\alpha_0(\alpha_0 - \alpha)}{(\alpha_0 - \alpha + r)^2} < 0$ and $g'(r) < 0$ when $(\alpha - \alpha_0, \alpha]$. Thus, for any positive number δ satisfying $0 < \delta < \alpha_0$, let $C_1 := \alpha - \alpha_0 + \delta$, then

$$A := \max_{r \in [C_1, \alpha]} g(r) = g(C_1). \quad (4.16)$$

On the other hand, since $\alpha > \alpha_0 = \frac{b_1 |\Omega_1|}{|\Omega|}$, we have

$$\lambda > \frac{b |\Omega_1|}{|\Omega|} \geq \inf_{u \in H^1(\Omega), u \neq 0} \frac{\int_{\Omega} [|\nabla u|^2 + b\delta(x)u^2] dx}{\int_{\Omega} u^2 dx} = \lambda_1^N(b\delta(x)). \quad (4.17)$$

By Theorem 3.2, since (2.2) has no positive solution emanating from any point on Γ_v , (4.1) has no positive solution emanating from $((0, \beta); \beta)$ for any $\beta > 0$. Furthermore, by perturbing the set $\{(r, f(r)); g(r) : C_1 \leq r \leq \alpha\}$, we may obtain the following proposition.

Proposition 4.3. Assume that $\alpha > \alpha_0$. Let C_1 and A be defined in (4.16), then there exist a positive number ε_0 and a family of bounded smooth curves

$$S(\xi, \varepsilon) := ((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon)) \in \mathbf{R}^3, \quad (\xi, \varepsilon) \in [C(\varepsilon), \alpha] \times [0, \varepsilon_0],$$

such that for $\varepsilon \in (0, \varepsilon_0]$, the component of the positive solutions of (4.1) is expressed by

$$\begin{aligned} \Gamma^\varepsilon &:= \{((w(\xi, \varepsilon), z(\xi, \varepsilon)); \beta(\xi, \varepsilon)) = ((r, s) + \varepsilon \mathbf{U}(r, s); \beta, \varepsilon); \beta) : \\ &((r, s); \beta) = (r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon)), \xi \in [C(\varepsilon), \alpha]\} \end{aligned} \quad (4.18)$$

with $S(\xi, 0) = ((\xi, f(\xi)); g(\xi))$ and $S(\alpha, \varepsilon) = ((\alpha, 0); \beta_*(\varepsilon))$, where $\beta_*(\varepsilon) = \mu_1/\varepsilon = -c_1$, \mathbf{U} is an X_3 -valued smooth function vector, $C(\varepsilon)$ is smooth with respect to ε and $\alpha - \alpha_0 < C(\varepsilon) < \alpha$, $C(0) = C_1$ and $\beta(C(\varepsilon), \varepsilon) = A$. Furthermore, Γ^ε can be extended to the range of $\beta \in [A, +\infty)$ as a curve of the positive solutions of (4.1).

By the following three lemmas, we will complete the proof of the Proposition 4.3. We first construct the local branch of the positive solutions of (4.1) near to $((\alpha, 0); \beta_*(\varepsilon))$.

Lemma 4.4. If $\alpha > \alpha_0$, then there exist two small positive numbers ε_1 and δ_0 , and a neighborhood U_0 of $((\alpha, 0); -c_1)$, such that for $\varepsilon \in (0, \varepsilon_1]$, it follows that

$$\text{Ker}\Phi^\varepsilon \cap U_0 \cap ((\mathbf{R}^+)^2 \times \mathbf{R}) = \{((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon)) : \xi \in (\alpha - \delta_0, \alpha)\} \cup \{((\alpha, 0); -c_1)\}, \quad (4.19)$$

where $((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon))$ is a smooth function vector satisfying

$$\begin{aligned} ((r(\xi, 0), s(\xi, 0)); \beta(\xi, 0)) &= ((\xi, f(\xi)); g(\xi)), \\ ((r(\alpha, \varepsilon), s(\alpha, \varepsilon)); \beta(\alpha, \varepsilon)) &= ((\alpha, 0); -c_1). \end{aligned}$$

Proof. By the arguments of the local bifurcation analysis of (2.2) at $((\lambda, 0); -c)$ in Subsection 3.1, for any small given $\varepsilon > 0$, let $s = \varepsilon \frac{(\alpha - \xi)\xi}{\alpha_0 - \alpha + \xi}$ of (3.10), then there exist a small positive number δ_0 and a neighborhood V_ε of $((\alpha, 0); -c_1)$, such that all positive solutions of (4.1) in V_ε are given by

$$\begin{aligned} ((w, z); \beta) &= ((w(\xi, \varepsilon), z(\xi, \varepsilon)); \beta(\xi, \varepsilon)) \\ &= ((\alpha - \frac{(\alpha - \xi)\xi}{\alpha_0 - \alpha + \xi}(\varphi_1 + W(\xi, \varepsilon)), \frac{(\alpha - \xi)\xi}{\alpha_0 - \alpha + \xi}(1 + Z(\xi, \varepsilon)); \beta(\xi, \varepsilon)) \end{aligned}$$

for $\xi \in (\alpha - \delta_0, \alpha)$, where φ_1 is obtained in (3.10), and $((W(\xi, \varepsilon), Z(\xi, \varepsilon)); \beta(\xi, \varepsilon))$ satisfies that $((W(\alpha, \varepsilon), Z(\alpha, \varepsilon)); \beta(\alpha, \varepsilon)) = ((0, 0); -c_1)$, $\int_{\Omega_1} Z(\xi, \varepsilon) dx = 0$. On the other hand, from the first equation of (4.1), it is easily checked that $\frac{1}{|\Omega|} \int_{\Omega} (\varphi_1 + W(\xi, 0)) dx = \frac{\alpha_0 - \alpha + \xi}{\xi}$.

We define an open set U_ε in \mathbf{R}^3 by

$$U_\varepsilon := \{((r, s); \beta) : r = \frac{1}{|\Omega|} \int_{\Omega} w dx, s = \frac{1}{|\Omega_1|} \int_{\Omega_1} z dx, ((w, z); \beta) \in V_\varepsilon\},$$

and let

$$r(\xi, \varepsilon) = \frac{1}{|\Omega|} \int_{\Omega} w(\xi, \varepsilon) dx, \quad s(\xi, \varepsilon) = \frac{1}{|\Omega_1|} \int_{\Omega_1} z(\xi, \varepsilon) dx.$$

Then by Lemma 4.1, we obtain that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\text{Ker}\Phi^\varepsilon \cap U_\varepsilon \cap ((\mathbf{R}^+)^2 \times \mathbf{R}) = \{((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon)) : \xi \in (\alpha - \delta_0, \alpha)\} \cup \{((\alpha, 0); -c_1)\}.$$

Since $((\alpha, 0); -c_1)$ is a bifurcation point of (4.1) for any $\varepsilon \in (0, \varepsilon_1]$, it is easily checked that U_ε contains a neighborhood U_0 of $((\alpha, 0); -c_1)$. Proof of Lemma 4.4 is completed. \square

Lemma 4.5. Assume that $\alpha > \alpha_0$. Let C_1 and A be defined in (4.16), then there exist a small positive number ε_2 and a neighborhood U of $\{((r, f(r)); g(r)) : C_1 \leq r \leq \alpha\}$ such that for $\varepsilon \in (0, \varepsilon_2]$, the curve of the positive solutions of (4.1) contained in $U \cap (X \times (-c_1, A])$ is expressed as (4.18).

Proof. Define $K_p([C_1, \alpha - \frac{\delta_0}{2}]) = \{((r, f(r)); g(r)) : r \in [C_1, \alpha - \frac{\delta_0}{2}]\}$, where δ_0 is defined in Lemma 4.4. From (4.8) and (4.9), we have

$$\Phi_{(r,s)}^0((r, s); \beta) = \begin{pmatrix} r(-1 + \alpha_0 \frac{s}{(r+s)^2}) & -\alpha_0 \frac{r^2}{(r+s)^2} \\ \frac{c_1 s^2}{(r+s)^2} & s(-1 - \frac{c_1 r}{(r+s)^2}) \end{pmatrix}, \quad (4.20)$$

$$\det \Phi_{(r,s)}^0((r, f(r)); g(r)) = rs(1 + \frac{c_1 r - \alpha_0 s}{(r+s)^2}) = rf(r)h(r), \quad (4.21)$$

where $h(r) := 2 - \frac{(\alpha + g(r))(\alpha_0 - \alpha + r)}{\alpha_0 r}$.

Let $((r_0, f(r_0)); g(r_0)) \in K_p([C_1, \alpha - \frac{\delta_0}{2}])$, then $f(r_0) > 0$ since $\alpha - \alpha_0 < r_0 < \alpha$. By a similar argument as that in [7,34], if $h(r_0) \neq 0$, then $\det \Phi_{(r,s)}^0(r_0, f(r_0); g(r_0)) \neq 0$ and $\Phi_{(r,s)}^0((r_0, f(r_0)); g(r_0))$ is invertible. Thus, by using the implicit function theorem, there exist two positive numbers δ_{r_0} and ε_{r_0} , and a neighborhood W_{r_0} of $(r_0, f(r_0))$, such that for any $\varepsilon \in (0, \varepsilon_{r_0}]$,

$$\text{Ker} \Phi^\varepsilon \cap U_{r_0} = \{((r(\beta, \varepsilon), s(\beta, \varepsilon)); \beta) : \beta \in (g(r_0) - \delta_{r_0}, g(r_0) + \delta_{r_0})\}, \quad (4.22)$$

where $U_{r_0} = W_{r_0} \times (g(r_0) - \delta_{r_0}, g(r_0) + \delta_{r_0})$, and $(r(\beta, \varepsilon), s(\beta, \varepsilon))$ is a smooth function vector satisfying $(r(g(r_0), 0), s(g(r_0), 0)) = (r_0, f(r_0))$. By Lemma 4.2, since $g'(\xi) < 0$ for $\xi \in (\alpha - \alpha_0, \alpha]$, $r(\beta(\xi, \varepsilon), \varepsilon)$ can be defined as $r(\xi, \varepsilon)$ for $\xi \in (r_0 - \delta'_{r_0}, r_0 + \delta'_{r_0})$ such that $r(\xi, 0) = \xi$, where δ'_{r_0} is a positive number depending on δ_{r_0} .

If $h(r_0) = 0$, then $\text{Range} \Phi_{(r,s)}^0((r_0, f(r_0)); g(r_0)) = \text{span}\left\{\begin{pmatrix} C(r_0) \\ K(r_0)C(r_0) \end{pmatrix}\right\}$, where $C(r_0)$ and $K(r_0)$ are two nonzero constants depending on r_0 . Furthermore,

$$\dim \text{Ker} \Phi_{(r,s)}^0((r_0, f(r_0)); g(r_0)) = \text{codim} \text{Range} \Phi_{(r,s)}^0((r_0, f(r_0)); g(r_0)) = 1, \quad (4.23)$$

$$\Phi_{\beta}^0((r_0, f(r_0)); g(r_0)) = \begin{pmatrix} 0 \\ f(r_0) \end{pmatrix} \notin \text{Range} \Phi_{(r,s)}^0((r_0, f(r_0)); g(r_0)). \quad (4.24)$$

Thus, by the spontaneous bifurcation theory of Crandall and Rabinowitz [9], there exist two positive numbers which are still denoted by δ_{r_0} and ε_{r_0} , and a neighborhood U_{r_0} of $((r_0, f(r_0)); g(r_0))$, such that for any $\varepsilon \in (0, \varepsilon_{r_0}]$,

$$\text{Ker} \Phi^\varepsilon \cap U_{r_0} = \{((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon)) : \xi \in (r_0 - \delta_{r_0}, r_0 + \delta_{r_0})\}, \quad (4.25)$$

where $((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon))$ is a smooth function vector in $(r_0 - \delta_{r_0}, r_0 + \delta_{r_0}) \times [0, \varepsilon_{r_0}]$ and satisfies that

$$((r(r_0, 0), s(r_0, 0)); \beta(r_0, 0)) = ((r_0, f(r_0)); g(r_0)).$$

Furthermore, by the covering theorem, for all U_{r_0} satisfying (4.22) or (4.25), we have

$$K_p([C_1, \alpha - \frac{\delta_0}{2}]) \subset \cup \{U_{r_0} : r_0 \in [C_1, \alpha - \frac{\delta_0}{2}]\}.$$

Since $K_p([C_1, \alpha - \frac{\delta_0}{2}])$ is compact, there are finite points r_i , ($i = 1, \dots, N$), such that

$$((r_i, f(r_i)); g(r_i)) \in K_p([C_1, \alpha - \frac{\delta_0}{2}]) \quad \text{for } 1 \leq i \leq N,$$

$$K_p([C_1, \alpha - \frac{\delta_0}{2}]) \subset \cup_{i=1}^N U_{r_i}$$

We may assume that $U_{r_i} \cap U_{r_{i+1}}$ is not the empty set for any $0 \leq i \leq N-1$, where U_0 is a neighborhood of $((\alpha, 0); -c_1)$.

Let $\delta_i = \delta_{r_i}$ and $\varepsilon_i = \varepsilon_{r_i}$, then for any $\varepsilon \in (0, \varepsilon_i]$ ($1 \leq i \leq N$),

$$\text{Ker} \Phi^\varepsilon \cap U_{r_i} = \{((r^i(\xi, \varepsilon), s^i(\xi, \varepsilon)); \beta^i(\xi, \varepsilon)) : \xi \in (r_i - \delta_i, r_i + \delta_i)\} := J_i^\varepsilon,$$

and $((r^i(r_i, 0), s^i(r_i, 0)); \beta^i(r_i, 0)) = ((r_i, f(r_i)); g(r_i))$. In addition, by Lemma 4.4, let

$$J_0^\varepsilon := \{((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon)) : \xi \in [\alpha - \delta_0, \alpha]\}$$

and $U = \cup_{i=0}^N U_{r_i}$. Then

$$\text{Ker}\Phi^\varepsilon \cap U \cap ((\mathbf{R}^+)^2 \times \mathbf{R}) = \cup_{i=0}^N J_i^\varepsilon \text{ for any } \varepsilon \in (0, \min_{0 \leq i \leq N} \varepsilon_i]. \quad (4.26)$$

Thus, $\text{Ker}\Phi^\varepsilon \cap U \cap ((\mathbf{R}^+)^2 \times \mathbf{R})$ forms a one-dimensional sub-manifold. It is possible to construct a smooth curve $S(\xi, \varepsilon) = ((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon))$, such that

$$\begin{aligned} \cup_{i=0}^N J_i^\varepsilon &= S([C(\varepsilon), \alpha], \varepsilon), \\ ((r(\xi, 0), s(\xi, 0)); \beta(\xi, 0)) &= ((\xi, f(\xi)); g(\xi)), \\ ((r(\alpha, \varepsilon), s(\alpha, \varepsilon)); \beta(\alpha, \varepsilon)) &= ((\alpha, 0); -c_1) \end{aligned}$$

for any $\varepsilon > 0$ small enough and $\xi \in [C(\varepsilon), \alpha]$, where $C(\varepsilon)$ is a constant depending on ε , and $S([C(\varepsilon), \alpha], \varepsilon) := \{S(\xi, \varepsilon) : \xi \in [C(\varepsilon), \alpha]\}$. \square

The following Lemma show that for any $\beta \in (-c_1, A]$ and any $\varepsilon > 0$ small enough, (4.1) has no positive solution outside of U .

Lemma 4.6. Assume that $\alpha > \alpha_0$. Let C_1 and A be defined in (4.16), and V be any neighborhood of $\{((r, f(r)); g(r)) : C_1 \leq r \leq \alpha\}$, then there exists $\varepsilon_3 > 0$, such that for $\varepsilon \in (0, \varepsilon_3]$, any positive solution of (4.1) with $\beta \in (-c_1, A]$ can be given by

$$(w, z) = (r, s) + \varepsilon \mathbf{U}((r, s); \beta, \varepsilon) \quad \text{for some } ((r, s); \beta) \in V.$$

Proof. Assume on the contrary that there exists a sequence $\{(\beta_n, \varepsilon_n)\}$ satisfying $\beta_n \in (-c_1, A]$ and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, such that (4.1) with $(\beta, \varepsilon) = (\beta_n, \varepsilon_n)$ has a positive solution $((w_n, z_n); \beta_n) \notin V$ for all $n \in \mathbf{N}$.

First, we give a priori bounds for $\{w_n\}$ and $\{z_n\}$. By using the maximum principle to (4.1), we have

$$w_n \leq \max_{x \in \bar{\Omega}} w_n \leq \alpha, \quad z_n \leq \max_{x \in \bar{\Omega}_1} z_n \leq \beta + c_1 \leq A + c_1$$

for all $n \in \mathbf{N}$. Thus, by using the estimates of the elliptic equations, $\{w_n\}$ and $\{z_n\}$ are bounded sequences in $C(\bar{\Omega})$ and $C(\bar{\Omega}_1)$ respectively.

Let $\bar{w}_n = \frac{w_n}{\max_{x \in \bar{\Omega}} w_n}$ and $\bar{z}_n = \frac{z_n}{\max_{x \in \bar{\Omega}_1} z_n}$, then (4.1) becomes

$$\begin{cases} \Delta \bar{w}_n + \varepsilon_n \bar{w}_n (\alpha - w_n - \frac{b_1 \delta(x) z_n}{w_n + z_n}) = 0, & x \in \Omega, \\ \Delta \bar{z}_n + \varepsilon_n \bar{z}_n (\beta_n - z_n + \frac{c_1 w_n}{w_n + z_n}) = 0, & x \in \Omega_1, \\ \partial_\nu \bar{w}_n = 0, & x \in \partial\Omega, \\ \partial_\nu \bar{z}_n = 0, & x \in \partial\Omega_1. \end{cases} \quad (4.27)$$

Since $\{((w_n, z_n); \beta_n)\}$ is uniformly bounded in $C(\bar{\Omega}) \times C(\bar{\Omega}_1) \times \mathbf{R}$, we have

$$\|\alpha - w_n - \frac{b_1 \delta(x) z_n}{w_n + z_n}\|_\infty \leq 2\alpha + b_1, \quad \|\beta - z_n + \frac{c_1 w_n}{w_n + z_n}\|_\infty \leq 2(A + c_1).$$

Thus, there exists a subsequence, which is still denoted by $\{((\bar{w}_n, \bar{z}_n); \beta_n)\}$, such that

$$\lim_{n \rightarrow \infty} ((\bar{w}_n, \bar{z}_n); \beta_n) = ((\bar{w}, \bar{z}); \beta_\infty) \quad \text{in } C(\bar{\Omega}) \times C(\bar{\Omega}_1) \times \mathbf{R},$$

and $((\bar{w}, \bar{z}); \beta_\infty)$ satisfies that

$$\Delta \bar{w} = 0, x \in \Omega, \partial_\nu \bar{w} = 0, x \in \partial\Omega; \quad \Delta \bar{z} = 0, x \in \Omega_1, \partial_\nu \bar{z} = 0, x \in \partial\Omega_1, \quad (4.28)$$

which implies that both \bar{w} and \bar{z} are constants. Combined these with $\|\bar{w}_n\|_\infty = \|\bar{z}_n\|_\infty = 1$, we obtain that $\bar{w} \equiv 1$ in $\bar{\Omega}$ and $\bar{z} \equiv 1$ in $\bar{\Omega}_1$. Furthermore, by the estimates of the elliptic equations, $\{w_n\}$ and $\{z_n\}$ are bounded sequences in $C^2(\bar{\Omega})$ and $C^2(\bar{\Omega}_1)$ respectively. Hence,

$$\lim_{n \rightarrow \infty} (w_n, z_n) = (r, s) \quad \text{in } C^1(\bar{\Omega}) \times C^1(\bar{\Omega}_1) \quad (4.29)$$

for some $r \geq 0$ and $s \geq 0$. Combined (4.29) with (4.6), for any n large enough, (w_n, z_n) may be given by

$$(w_n, z_n) = (r_n, s_n) + \varepsilon_n \mathbf{U}((r_n, s_n); \beta_n, \varepsilon_n)$$

with some sequence (r_n, s_n) such that $\lim_{n \rightarrow \infty} (r_n, s_n) = (r, s)$.

Next, we show that $s = f(r)$ and $\beta_\infty = g(r)$. By integrating the first equation of (4.27) over Ω , we have

$$\int_{\Omega} \bar{w}_n (\alpha - w_n - \frac{b_1 \delta(x) z_n}{w_n + z_n}) dx = 0.$$

Let $n \rightarrow \infty$, then $\alpha - r - \frac{|\Omega_1|}{|\Omega|} \frac{b_1 s}{r+s} = 0$, which implies that $s = \frac{(\alpha-r)r}{\alpha_0-\alpha+r} = f(r)$.

By integrating the second equation of (4.27) over Ω_1 , we have

$$\int_{\Omega_1} \bar{z}_n (\beta_n - z_n + \frac{c_1 w_n}{w_n + z_n}) dx = 0.$$

Let $n \rightarrow \infty$, then $\beta_\infty - s + \frac{c_1 r}{r+s} = 0$. Combined this with $s = f(r)$, we obtain that $\beta_\infty = g(r)$, which contradict our assumption. Therefore, the conclusion of Lemma 4.6 holds. \square

Proof of Proposition 4.3. Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, then by Lemmas 4.4–4.6, Proposition 4.3 holds for $\beta \in (-c_1, A]$ and $\varepsilon \in (0, \varepsilon_0]$. Now, we show Γ^ε can be extended to the range $\beta \in [A, \infty)$ as a curve of the positive solutions of (4.1). Let Γ_M^ε be a maximum extension of Γ^ε in the direction $\beta \geq A$ as a curve of the solutions of (4.1). By Theorem 3.2 (i), since $\text{proj}_\mu \Gamma^+ = (-c, +\infty)$, we have $\text{proj}_\beta \Gamma_M^\varepsilon = (-c_1, +\infty)$. \square

Combined Proposition 4.3 with Lemma 4.2, we obtain the following corollary.

Corollary 4.7. *If $\alpha > \alpha_0$, then there exists a small positive number ε_4 , such that for $\varepsilon \in (0, \varepsilon_4]$, (4.1) possesses a unique positive solution (w, z) for any $\beta \in (-c_1, +\infty)$. Furthermore, $\lim_{\beta \rightarrow +\infty} (w, z) = (\alpha - \alpha_0 + h(\varepsilon), +\infty)$, where $h(\varepsilon)$ is a function of ε .*

Proof. For $\alpha > \alpha_0$, by Lemma 4.2, it follows that

$$\begin{aligned} f(r) &> 0 \quad \text{only when } r \in (\alpha - \alpha_0, \alpha), \quad g(\alpha) = -c_1, \quad \lim_{r \rightarrow (\alpha - \alpha_0) + 0} g(r) = +\infty, \\ f'(r) &= -1 + \frac{\alpha_0(\alpha_0 - \alpha)}{(\alpha_0 - \alpha + r)^2} < 0, \quad g'(r) = -\frac{\alpha_0 + c_1}{\alpha_0} + \frac{\alpha_0(\alpha_0 - \alpha)}{(\alpha_0 - \alpha + r)^2} < 0. \end{aligned}$$

Therefore, (r, s) is unique for any $\beta \in (-c_1, +\infty)$. Since a small perturbation of the curve $\{(r, f(r)); g(r)\} : r \in (\alpha - \alpha_0, \alpha)\}$ doesn't change the main shape of this curve, there exists a small positive number ε_4 , such that when $\varepsilon \in (0, \varepsilon_4]$, the positive solution (w, z) of (4.1) is unique for $\beta \in (-c_1, +\infty)$.

Let $(r, s) = (r(\beta), s(\beta))$, then by the monotonicity, it follows that $\lim_{\beta \rightarrow +\infty} (r(\beta), s(\beta)) = (\alpha - \alpha_0, +\infty)$. Thus, $\lim_{\beta \rightarrow +\infty} (w, z) = (\alpha - \alpha_0 + h(\varepsilon), +\infty)$. \square

Case $\alpha < \alpha_0$

For $\alpha < \alpha_0$, from Lemma 4.2, there exists a large number A_1 , such that

$$\max_{r \in [0, \alpha]} g(r) < A_1. \quad (4.30)$$

By perturbing the set $\{(r, f(r)); g(r) : 0 \leq r \leq \alpha\}$, we obtain the following proposition.

Proposition 4.8. Assume that $\alpha < \alpha_0$. Let A_1 be defined in (4.30), then there exist a small positive number ε'_0 and a family of bounded smooth curves

$$S_1(\xi, \varepsilon) = ((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon)) \in \mathbf{R}^3, \quad (\xi, \varepsilon) \in [C(\varepsilon), \alpha] \times [0, \varepsilon'_0],$$

such that for $\varepsilon \in (0, \varepsilon'_0]$, the curve of the positive solutions of (4.1) can be expressed by

$$\begin{aligned} \Gamma_1^\varepsilon &= \{((w(\xi, \varepsilon), z(\xi, \varepsilon)); \beta(\xi, \varepsilon)) = ((r, s) + \varepsilon \mathbf{U}((r, s); \beta, \varepsilon); \beta) : \\ &((r, s); \beta) = ((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon)), \xi \in (C(\varepsilon), \alpha)\} \end{aligned} \quad (4.31)$$

with $S_1(\xi, 0) = ((\xi, f(\xi)); g(\xi))$, $S_1(\alpha, \varepsilon) = ((\alpha, 0); \beta_*(\varepsilon))$, and $S_1(C(\varepsilon), \varepsilon) = ((0, 0); \beta^*(\varepsilon))$, where, $\beta_*(\varepsilon) = \mu_1/\varepsilon = -c_1$ and $\beta^*(\varepsilon) = \mu_2/\varepsilon = -\frac{c_1 \alpha \int_{\Omega} \varphi_2(\varepsilon) dx}{b_1 \int_{\Omega_1} \psi_2(\varepsilon) dx}$, \mathbf{U} is an X_3 -valued smooth function, $C(\varepsilon)$ is a constant depending on ε , and $0 \leq C(\varepsilon) < \alpha$, $C(0) = 0$.

For $\alpha < \alpha_0$, Lemma 4.4 still holds when the component of the positive solution curve of (4.1) is near to $((\alpha, 0); -c_1)$. Next, we consider the expressions of the positive solution curve of (4.1) near to $((0, 0); \beta^*(\varepsilon))$.

Lemma 4.9. If $\alpha < \alpha_0$, then there exist two small positive numbers ε'_2 and δ_1 , and a neighborhood U'_1 of $((0, 0); -\frac{c_1(\alpha_0 - \alpha)}{\alpha_0})$, such that for any $\varepsilon \in (0, \varepsilon'_2]$, it follows that

$$\text{Ker} \Phi^\varepsilon \cap U'_1 \cap ((\mathbf{R}^+)^2 \times \mathbf{R}) = \{((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon)) : \xi \in (0, \delta_1]\} \cup \{((0, 0); \beta^*(\varepsilon))\}, \quad (4.32)$$

where $((r(\xi, \varepsilon), s(\xi, \varepsilon)); \beta(\xi, \varepsilon))$ is a smooth function vector satisfying

$$\begin{aligned} ((r(\xi, 0), s(\xi, 0)); \beta(\xi, 0)) &= ((\xi, f(\xi)); g(\xi)), \\ ((r(0, \varepsilon), s(0, \varepsilon)); \beta(0, \varepsilon)) &= ((0, 0); \beta^*(\varepsilon)), \end{aligned}$$

$$\beta^*(\varepsilon) = \mu_2/\varepsilon = -\frac{c_1 \alpha \int_{\Omega} \varphi_2(\varepsilon) dx}{b_1 \int_{\Omega_1} \psi_2(\varepsilon) dx} \text{ and } \beta^*(0) = -\frac{c_1(\alpha_0 - \alpha)}{\alpha_0}.$$

Proof. Let $\lambda = \varepsilon\alpha$, $\mu = \varepsilon\beta$, $b = \varepsilon b_1$ and $c = \varepsilon c_1$, then (3.16) becomes

$$\begin{cases} \Delta \varphi_2 + \varepsilon(\alpha \varphi_2 - \frac{b_1 \delta(x) \varphi_2 \psi_2}{\varphi_2 + \psi_2}) = 0, & x \in \Omega; \\ \Delta \psi_2 + \varepsilon(\beta \psi_2 + \frac{c_1 \varphi_2 \psi_2}{\varphi_2 + \psi_2}) = 0, & x \in \Omega_1; \\ \partial_\nu \varphi_2 = 0, & x \in \partial\Omega; \\ \partial_\nu \psi_2 = 0, & x \in \partial\Omega_1. \end{cases} \quad (4.33)$$

By using the Lyapunov–Schmidt reduction method, we obtain easily that $\lim_{\varepsilon \rightarrow +0}(\varphi_2, \psi_2) = (r_2, s_2)$, where r_2 and s_2 are two constants satisfying $s_2 = \frac{\alpha}{\alpha_0 - \alpha} r_2$, $\beta^*(0) = -\frac{c_1(\alpha_0 - \alpha)}{\alpha_0}$. Thus, there exists a small positive number ε'_2 , such that

$$((\varphi_2(\varepsilon), \psi_2(\varepsilon)); \beta^*(\varepsilon)) = ((r_2(\varepsilon), s_2(\varepsilon)) + \varepsilon \mathbf{U}((r_2(\varepsilon), s_2(\varepsilon)); \beta^*(\varepsilon), \varepsilon); \beta^*(\varepsilon)), \quad 0 < \varepsilon \leq \varepsilon'_2,$$

where, $\beta^*(\varepsilon) = \mu_2/\varepsilon = -\frac{c_1 \alpha \int_{\Omega} \varphi_2(\varepsilon) dx}{b_1 \int_{\Omega_1} \psi_2(\varepsilon) dx}$, $(r_2(\varepsilon), s_2(\varepsilon)) \in (\mathbf{R}^+)^2$, $(r_2(0), s_2(0)) = (r_2, s_2)$ and $\mathbf{U}((r_2(\varepsilon), s_2(\varepsilon)); \beta^*(\varepsilon), \varepsilon) \in X_3$.

Let $(r_2, s_2) = (1, \frac{\alpha}{\alpha_0 - \alpha})$. Since $(\varphi_2(\varepsilon), \psi_2(\varepsilon))$ is the tangential vector of the positive solution curve of (4.1) at $((0, 0); \beta^*(\varepsilon))$, there exists a small positive number δ_1 , such that

$$\begin{aligned} ((w, z); \beta) &= ((\xi r_2(\varepsilon), \xi s_2(\varepsilon) + o_1(\xi, \varepsilon)) \\ &+ \varepsilon \xi \mathbf{U}((\xi r_2(\varepsilon), \xi s_2(\varepsilon) + o_1(\xi, \varepsilon)); \beta^*(\varepsilon), \varepsilon) + o_2(\xi, \varepsilon); \beta^*(\varepsilon) + h_1(\xi, \varepsilon)) \end{aligned}$$

for any $\xi \in [0, \delta_1]$, where $\xi s_2(0) + o_1(\xi, 0) = f(\xi)$, $\beta^*(0) + h_1(\xi, 0) = g(\xi)$, $o_1(0, \varepsilon) = h_1(0, \varepsilon) = 0$, and $o_2(\xi, 0) = o_2(0, \varepsilon) = \mathbf{0}$. Thus, Lemma 4.9 holds. \square

Lemma 4.10. Assume that $\alpha < \alpha_0$. Let A_1 be defined in (4.30), then there exist a small positive number ε'_3 and a neighborhood U' of $\{((r, f(r)); g(r)) : 0 \leq r \leq \alpha\}$, such that for any $\varepsilon \in (0, \varepsilon'_3]$, the curve of the positive solutions of (4.1) contained in $U' \cap (X \times (-c_1, A_1])$ can be expressed by (4.31).

Proof. Let δ_0 and δ_1 be the positive constants obtained in Lemmas 4.4 and 4.9 respectively, and define

$$K_p([\frac{\delta_1}{2}, \alpha - \frac{\delta_0}{2}]) := \{((r, f(r)); g(r)) : r \in [\frac{\delta_1}{2}, \alpha - \frac{\delta_0}{2}]\},$$

where $K_p([\frac{\delta_1}{2}, \alpha - \frac{\delta_0}{2}])$ is a compact set, and both $K_p([\frac{\delta_1}{2}, \alpha - \frac{\delta_0}{2}]) \cap U_0$ and $K_p([\frac{\delta_1}{2}, \alpha - \frac{\delta_0}{2}]) \cap U'_1$ are not the empty sets, and U_0 and U'_1 be the open sets obtained in Lemmas 4.4 and 4.9 respectively. Furthermore, by a similar argument as those of Lemma 4.5, we may construct the positive solution curve of (4.1) in a neighborhood U' of $\{((r, f(r)); g(r)) : \frac{\delta_1}{2} \leq r \leq \alpha - \frac{\delta_0}{2}\}$. Since both $U' \cap U_0$ and $U' \cap U'_1$ are not the empty sets, combined with Lemmas 4.4 and 4.9, we obtain the assertion of Lemma 4.10. \square

Lemma 4.11. Assume that $\alpha < \alpha_0$. Let A_1 be defined in (4.30), and V' be any given neighborhood of $\{((r, f(r)); g(r)) : 0 \leq r \leq \alpha\}$, then there exists a small $\varepsilon'_4 > 0$, such that for $\varepsilon \in (0, \varepsilon'_4]$, every positive solution of (4.1) with $\beta \in (-c_1, A_1]$ can be given by

$$(w, z) = (r, s) + \varepsilon \mathbf{U}((r, s); \beta, \varepsilon) \quad \text{for some } ((r, s); \beta) \in V'.$$

Proof. Proof of Lemma 4.10 is completely similar to that of Lemma 4.6. Thus, we omit the procedures of its proof. \square

Proposition 4.8 follows from Lemmas 4.10 and 4.11. Furthermore, combined Proposition 4.8 with Lemma 4.2, we obtain the following corollary.

Corollary 4.12. Assume that $\alpha < \alpha_0$. Then there exists a small positive number ε'_5 , such that for $\varepsilon \in (0, \varepsilon'_5]$, the following results hold:

(i) When $0 < \alpha \leq \frac{\alpha_0 c_1}{\alpha_0 + c_1}$, (4.1) possesses a unique positive solution (w, z) for $\beta \in (-c_1, \beta^*(\varepsilon))$. Furthermore, $\lim_{\beta \rightarrow -c_1+0}(w, z) = (\alpha, 0)$ and $\lim_{\beta \rightarrow \beta^*(\varepsilon)-0}(w, z) = (0, 0)$.

(ii) If $\frac{\alpha_0 c_1}{\alpha_0 + c_1} < \alpha < \alpha_0$, then there exists a positive number $r_1(\varepsilon)$, such that (4.1) possesses a unique positive solution (w, z) for $\beta \in (-c_1, \beta^*(\varepsilon))$ or $\beta = g(r_1(\varepsilon))$; while (4.1) possesses two positive solutions for $\beta \in (\beta^*(\varepsilon), g(r_1(\varepsilon)))$, where $r_1(0) = r_1$, r_1 is defined in (4.13). Furthermore, $\lim_{\beta \rightarrow -c_1+0} (w, z) = (\alpha, 0)$; while the minimal positive solution of (4.1) satisfies that $\lim_{\beta \rightarrow \beta^*(\varepsilon)+0} (w, z) = (0, 0)$. Therefore, the curve of the positive solutions of (4.1) is a type \supset when $\frac{\alpha_0 c_1}{\alpha_0 + c_1} < \alpha < \alpha_0$.

4.4. Conclusion

By the relationships between (4.1) and (2.2), it is easily checked that $((w, z); \beta)$ is a positive solution of (4.1) if and only if $((u, v); \mu) = \varepsilon((w, z); \beta)$ is a positive solution of (2.2). Thus, by Corollaries 4.7 and 4.12 and Theorem 3.2, we obtain the following conclusions.

Theorem 4.13. Assume that $\lambda_0 = \varepsilon \alpha_0 = \frac{b|\Omega_1|}{|\Omega|}$ and $\lambda_1 = \frac{\lambda_0 c}{\lambda_0 + c}$. Then there exists a small positive number ε_0 , such that for any $\varepsilon \in (0, \varepsilon_0]$,

(i) if $\lambda > \lambda_1^N(b\delta(x), \Omega)$, then (2.2) possesses at least a positive solution (u, v) for $\mu \in (-c, +\infty)$ and $\lim_{\mu \rightarrow +\infty} (u, \frac{v}{\mu}) = (\lambda - \lambda_0 + \varepsilon h(\varepsilon), 1)$. Furthermore, for any $\varepsilon > 0$ small enough, the positive solution of (2.2) is unique for $\mu \in (-c, +\infty)$.

(ii) For $\lambda < \lambda_1^N(b\delta(x), \Omega)$, (a) if $0 < \lambda \leq \lambda_1$, then (2.2) possesses at least a positive solution (u, v) for $\mu \in (-c, \mu_2)$, and $\lim_{\mu \rightarrow -c+0} (u, v) = (\lambda, 0)$ and $\lim_{\mu \rightarrow \mu_2-0} (u, v) = (0, 0)$; while when $\mu \geq \mu_2$, (2.2) has no positive solution. Furthermore, for any $\varepsilon > 0$ small enough, the positive solution of (2.2) is unique for $\mu \in (-c, \mu_2)$.

(b) If $\lambda_1 < \lambda < \lambda_1^N(b\delta(x), \Omega)$, then there exists a constant $\mu^* > \mu_2$, such that (2.2) possesses at least a positive solution for $\mu \in (-c, \mu_2)$ or $\mu = \mu^*$, (2.2) possesses at least two positive solutions for $\mu \in (\mu_2, \mu^*)$, and (2.2) has no positive solution for $\mu > \mu^*$, where $\lim_{\varepsilon \rightarrow +0} \frac{\mu^*}{\varepsilon} = g(r_1)$ and r_1 is defined in (4.13); $\lim_{\mu \rightarrow -c+0} (u, v) = (\lambda, 0)$, while the minimal positive solution of (2.2) satisfies that $\lim_{\mu \rightarrow \mu_2+0} (u, v) = (0, 0)$. Furthermore, for any $\varepsilon > 0$ small enough, the number of the positive solutions of (2.2) is really one for $\mu \in (-c, \mu_2)$ or $\mu = \mu^*$, two for $\mu \in (\mu_2, \mu^*)$, and zero for $\mu > \mu^*$. Therefore, the shape of the positive solution curve of (2.2) is a type \supset when $\lambda_1 < \lambda < \lambda_1^N(b\delta(x), \Omega)$.

Remark 4.14. From Theorem 3.2 and Theorem 4.13, we notice that: (i) by using the bifurcation theories, the parameters of (2.2) may be any range and (2.2) can present the complicated bifurcation diagrams, but these bifurcation diagrams are not elaborate;

(ii) by using the Lyapunov–Schmidt reduction, we may obtain a fine profile of the positive solution curve of (2.2), but the ranges of the parameters of (2.2) are restricted, namely, the parameters of (2.2) are small.

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