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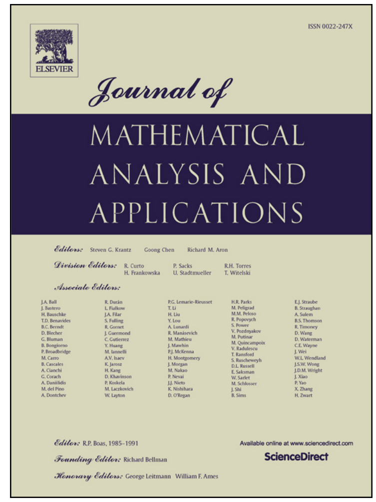

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On external fields created by fixed charges

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Abstract

In this paper equilibrium measures in the presence of external fields created by fixed charges are analyzed. These external fields are a particular case of the so-called rational external fields (in the sense that their derivatives are rational functions). Along with some general results, a thorough analysis of the particular case of two fixed negative charges ("attractors") is presented; indeed, the main result of the paper deals with this particular case. As for the main tools used, this paper is a natural continuation of [33], where polynomial external fields were thoroughly studied, and [39], where rational external fields with a polynomial part were considered. However, the absence of the polynomial part in the external fields analyzed in the current paper adds a considerable difficulty to solve the problem and justifies its separated treatment; moreover, it is noteworthy to point out the simplicity and beauty of the results obtained.

Keywords and phrases: Equilibrium measures, External fields, Phase transitions.

1. Introduction

This paper is devoted to the study of equilibrium measures in the real axis in the presence of rational external fields created by fixed charges. These are external fields of the form:

$$\varphi(x) = \sum_{j=1}^{q} \gamma_j \log |x - z_j| \,, \ \gamma_j \in \mathbb{R} \,, \ z_j \in \mathbb{C} \,, \tag{1}$$

where for $\gamma_k > 0$, z_k must lie on $\mathbb{C} \setminus \mathbb{R}$, and it is assumed that $\sum_{j=1}^{q} \gamma_j = T > 0$. These conditions ensure that given any $t \in (0, T)$, there exists a measure $\lambda_t = \lambda_{t,\varphi}$, such that $\lambda_t(\mathbb{R}) = t$, with compact support $S_t \subset \mathbb{R}$, uniquely determined by the equilibrium condition (see e.g. [41])

$$V^{\lambda_t}(x) + \varphi(x) \begin{cases} = c_t \,, & x \in S_t \,, \\ \geq c_t \,, & x \in \mathbb{R} \,, \end{cases} \tag{2}$$

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where c_t is called the equilibrium constant and for a measure σ, $V^{\sigma}(x) = -\int \log|x-s|\, d\sigma(s)$. The measure λ_t is called the equilibrium measure in the presence of φ and minimizes the weighted energy

$$I_{\varphi}(\sigma) = -\iint \log|x-z|\, d\sigma(x)\, d\sigma(z) + 2\int \varphi(x) d\sigma(x)$$

among all measures σ supported in the real axis and such that $\sigma(\mathbb{R}) = t$.

It is easy to see that

$$\lambda_t = t\, \lambda_{1,\varphi/t}\,, \tag{3}$$

where $\lambda_{1,\varphi/t}$ denotes the unit equilibrium measure in the external field $\frac{1}{t}\varphi$. External fields (1) are called rational since their derivatives are rational functions:

$$\varphi'(x) = \sum_{j=1}^{q} \gamma_j \frac{x - \operatorname{Re} z_j}{(x-z_j)(x-\overline{z}_j)}\,,\ x \in \mathbb{R}.$$

In this sense, this paper completes the analysis started in [39], where rational external fields of the form

$$\varphi(x) = P(x) + \sum_{j=1}^{q} \gamma_j \log|x - z_j|\,,\ \gamma_j \in \mathbb{R}\,,\ z_j \in \mathbb{C}\,, \tag{4}$$

P being a polynomial of even degree $2p$, with $p \geq 1$, were considered. There, a particular case was treated in detail: a generalized Gauss-Penner model for which $p = q = 2$. When $p \geq 1$, the polynomial part makes the external field strong enough to be admissible for any $t \in (0, +\infty)$. On the contrary, when the polynomial part is absent, the external field is weaker and it is admissible just for $t \in (0, T)$. This important difference is one of the reasons for studying these weaker rational external field in a separated paper.

Of course, it is also possible to deal with rational external fields (1) with some $z_j \in \mathbb{R}$ and the corresponding $\gamma_j > 0$; but in that case, the conductor where the equilibrium problem is posed cannot be the real axis. This is the situation, for instance, when the asymptotics of Jacobi polynomials with varying non–classical parameters are handled (see e.g. [23], [29] and [32]). In this situation, the support of the equilibrium measure consists of a finite union of arcs and curves in the complex plane; these arcs/curves satisfy a kind of symmetry with respect to the external field (the so-called "S-symmetry" introduced by H. Stahl during the eighties, see [43]). In a similar fashion, when asymptotics of Laguerre polynomials with varying non–classical parameters are studied, a rational external field with a polynomial part (4) takes place (see [3], [12], [25], [26], [30] and, from the viewpoint of the Gauss-Penner Random Matrix models, [2]). Other setting where it is feasible having $z_j \in \mathbb{R}$ and $\gamma_j > 0$ is when the conductor is a proper subset of the real axis not containing points z_j; in that case, hard edges at the endpoints of the conductor arise (see e.g. [39], where the conductor $[0, \infty)$ is considered).

Rational external fields appear in a number of applications in approximation theory, for instance when dealing with the asymptotic distribution of zeros of orthogonal or Heine-Stieltjes polynomials; in particular, the application of "purely rational" external fields of the form (1) to the asymptotics of Heine-Stieltjes polynomials will be recalled below with more detail. But there are also important applications in random matrix theory, for example in the study of Gauss-Penner type models. The rest of this section will be devoted to describing briefly these applications.

In the second section, equilibrium problems in the presence of external fields (1) are handled in general, studying some properties such as the asymptotic behavior of the equilibrium measure when t (the size of the equilibrium measure or, equivalently in other contexts, the "time" or the "temperature", [5]-[7], [24] and [33], among others) tends to T, as well as the evolution of this equilibrium measure when other parameters (the "heights", $\operatorname{Im} z_j$, or the "masses", γ_j) of the external field vary.

The main contribution of this paper is in Section 3, where the particular case of a rational external field created by two fixed charges is treated in detail. In this case, the support of the equilibrium measure may consist of one or two intervals ("one-cut" or "two-cut", respectively), and we are mainly interested in the evolution of this support when t travels through the interval $(0, T)$. Our main result is Theorem 3.1 below, though other results necessary for its proof, presented in Sections 2 (Theorem 2.2) and 3 (Theorems 3.2–3.4), are also of interest themselves; their proofs are collected in Section 4 in order to make the paper more readable. Finally, the geometrical aspects of the solution of our main problem presented in Theorem 3.1 below are illustrated in the final appendix.

It is convenient to recall again that, regarding the methodology used, this paper, as well as the previous [39], is a natural continuation of [33], where this "dynamical" approach was thoroughly carried out for the case of polynomial external fields.

1.1. Generalized Lamé equations and Heine-Stieltjes Polynomials

In a series of seminal papers (see [44]-[47]), T. J. Stieltjes (1856-1894) provided an elegant model for the electrostatic interpretation of the zeros of classical families of orthogonal polynomials (Jacobi, Laguerre and Hermite) and polynomial solutions of certain linear differential equations (the so-called Heine-Stieltjes polynomials, see also [48, §6.8] for a more detailed study). Regarding the latter case, in [38] the authors considered the following equilibrium problem in the real axis (for a counterpart of this problem in the Unit Circle, see [17] and [31]).

Let $m, n \in \mathbb{N}$ and consider m prescribed negative charges at points $z_k \in \mathbb{C} \setminus \mathbb{R}$, $k = 1, \ldots, m$. Then, suppose there are n positive unit charges that can move freely through the real axis, and denote by $x_k \in \mathbb{R}$, $k = 1, \ldots, n$, their positions. Observe that in the classical Heine-Stieltjes setting, the fixed charges are located in the real axis and are positive, that is, they are "repellents", while now they are negative ("attractors"). Then, the free charges will be located in such a way that the (logarithmic) energy of the system

$$E(x_1, \ldots, x_n) = -\sum_{1 \le j < k \le n} \log |x_k - x_j| + \sum_{j=1}^{n} \sum_{k=1}^{m} \omega_k \log |z_k - x_j| \tag{5}$$

is minimized. Hereafter, let us denote by $s = \sum_{k=1}^{m} \omega_k$, the total mass of the prescribed charges. Then, it was shown (see [38, Theorem 1]) that if $s > n-1$, the energy functional (5) has a global minimum in $\mathbb{R}^n$. This minimum is attained at some point $(x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$, where $-\infty < x_1^* < \ldots < x_n^* < +\infty$. However, in this case, unlike what happens in the classical setting, the global minimum does not need to be unique (see [38, Section 2.2]). In addition, it was shown that each generalized Heine-Stieltjes polynomial, $y(x) = \prod_{j=1}^{n} (x - x_j^*)$, is solution of a generalized Lamé differential equation of the form $A(x)y'' + B(x)y' + C(x)y = 0$, where

$$A(x) = \prod_{k=1}^{m} (x - z_k)(x - \overline{z}_k), \; h(x) = \prod_{k=1}^{m} \left((x - z_k)(x - \overline{z}_k) \right)^{\omega_k}, \; B(x) = -A(x) \frac{h'(x)}{h(x)},$$

for some Van Vleck polynomial $C \in \mathbb{P}_{2m-2}$. (For more information about the Lamé equation, and the Van Vleck and Heine-Stieltjes polynomials in the classical setting, see e.g. [28] and [48]).

In [38] the asymptotics of Heine-Stieltjes polynomials was also considered when both n and $s = s(n)$ tend to ∞, in such a way that $\lim_{n\to\infty} \frac{s}{n} = \theta > 1$, extending the asymptotic analysis carried out by A. Martínez Finkelshtein and E. B. Saff in the classical scenario when the number of free charges increases to infinity [35]. To describe the situation considered in [38] , for each n denote $\nu_n = \frac{1}{n}\sum_{k=1}^{m(n)} \omega_{nk}\delta_{z_{nk}}$, which is an atomic measure such that $\nu_n(\mathbb{R}) = \frac{s(n)}{n} > \frac{n-1}{n}$, and suppose that

$$\nu_n \xrightarrow{*} \nu\,,\; n \in \Lambda \subset \mathbb{N} \text{ and } n \to \infty\,,$$

in the weak-* topology, for some measure ν of size $\theta = \lim_{n\to\infty} \frac{s}{n} > 1$, with compact support in $\mathbb{C}\setminus\mathbb{R}$ and some infinite subsequence $\Lambda \subset \mathbb{N}$. Now, suppose that, for each $n \in \mathbb{N}$, $\{x^*_{nj} : j = 1, \ldots, n\}$ is an equilibrium configuration (that is, a global minimum) for the discrete equilibrium problem (5). Then, denoting by $\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{x^*_{nj}}$, in [38, Theorem 3] it was proved (following the same approach as in [35, Th. 2] and [31, Th. 3.2]) that $\mu_n \xrightarrow{*} \mu\,,\; n \in \Lambda \subset \mathbb{N}$ and $n \to \infty$, where μ is the equilibirum measure of $\mathbb{R}$ in the external field $\varphi = -V(\nu, \cdot)$. That is, the unit counting measures of zeros of Heine-Stieltjes polynomials converge, in the weak-* star topology, to the equilibrium measure in the external field due to the potential of the negative charge ν .

In the general case not much more can be said, but the situation is different if the limit measure ν is atomic. This situation was considered in [38], where the case $\nu = \gamma_1\delta_{z_1} + \gamma_2\delta_{z_2}$, with $z_1, z_2 \in \mathbb{C}\setminus\mathbb{R}$ and $\gamma_1 + \gamma_2 > 1$ was analyzed, studying in detail the so-called "totally symmetric" case, i.e. when $z_2 = -\bar{z}_1$ and $\gamma_2 = \gamma_1$. One of the main goals of this paper is to deal with the general situation where the "heights", $\operatorname{Im} z_1, \operatorname{Im} z_2$, and the "charges", γ_1, γ_2 , are arbitrary positive real numbers.

In adition, let us point out that it is also possible to consider sequences of critical configurations (relative extrema or saddle points), not necessarily global minima, of the discrete energy (5) for $n \to \infty$ and $s/n \to \lambda > 1$. The limit measures of such sequences will be the so-called (continuous) critical measures, a class of measures to which the equilibrium measure belongs. In Section 2, a little bit more will be said about these critical measures (see [34] for an extensive study of them).

Finally, it is noteworthy to mention that the zeros of Heine-Stieltjes polynomials are actually a particular case of the so-called *weighted Fekete points* (see e.g. [41]).

1.2. Applications of rational external fields to random matrix models

It is well known that another circle of applications of equilibrium problems in the presence of external fields comes from the Random Matrix models (see e.g. [33] and the exhaustive bibliography therein). This is an important theory within mathematical physics and, more precisely, statistical mechanics. For the sake of completeness, let us briefly recall the connection between this theory and our problem in potential theory.

To do it, consider that the set of $N \times N$ Hermitian matrices

$$\left\{ M = (M_{jk})_{j,k=1}^{N} :\; M_{kj} = \overline{M_{jk}} \right\}$$

is equipped with the joint probability distribution

$$d\nu_N(M) = \frac{1}{\widetilde{Z}_N} \exp\left(-\operatorname{Tr} V(M)\right) dM,$$

with

$$dM = \prod_{j=1}^{N} dM_{jj} \prod_{j\neq k}^{N} d\operatorname{Re} M_{jk}\, d\operatorname{Im} M_{jk},$$

where $V : \mathbb{R} \to \mathbb{R}$ is a function such that the integral

$$\widetilde{Z}_N = \int \exp\left(-\operatorname{Tr} V(M)\right) dM.$$

is convergent. Then, it is well-known (see e.g. [36]) that ν_N induces a joint probability distribution μ_N on the eigenvalues $\lambda_1 < \cdots < \lambda_N$ of these matrices, with the density

$$\mu_N'(\boldsymbol{\lambda}) = \frac{1}{Z_N} \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left(-\sum_{i=1}^{N} V(\lambda_i)\right), \tag{6}$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$, and with the corresponding *partition function*

$$Z_N = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left(-\sum_{i=1}^{N} V(\lambda_i)\right) d\lambda_1 \ldots d\lambda_N.$$

In the framework of Random Matrix models, the *free energy* of this model is defined as

$$F_N = -\frac{1}{N^2} \log Z_N$$

and it is very important to study the limit

$$F_\infty = \lim_{N\to\infty} F_N .$$

The existence of this limit has been established under very general conditions on V, see e.g. [21].

Regarding the connection with the equilibrium problems in the presence of external fields, the subject of the current paper, take into account that the density (6) above may be expressed in the form

$$\mu_N'(\boldsymbol{\lambda}) = \frac{1}{Z_N} e^{-N^2 I_\varphi(\sigma_N)} ,$$

where $\sigma_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j}$ is the unit counting measure of the eigenvalues, and $\varphi = \dfrac{V}{2}$. This means that the value of F_∞ is related to the solution of a minimization problem for the weighted logarithmic energy. Therefore, the corresponding minimizer is the equilibrium measure associated to the external field $\varphi = \frac{V}{2}$.

The case of polynomial potentials V, and, in particular, the situation when V is a quartic polynomial has been extensively studied in the literature (see the monograph by Wang [49] or the papers [1], [5], [6], [9], [24] and [33], among many others), paying special attention to the so-called phase transitions. In [39], general rational external fields of type (4) are handled in connection with the generalized Gauss-Penner model considered in [22]: a 1-matrix model whose action is given by $V(M) = tr\ \left(M^4 - \log(v + M^2)\right)$, in order to get a computable toy-model for the gluon correlations in a baryon background. On the other hand, the purely rational external fields considered in the current paper are connected with the so-called "multi-Penner" matrix model, with action given by $W(M) = \sum_{j=1}^{N} \mu_j \log(M - q_i)$, which is of interest in Gauge Theory, as well as in Toda systems (see e.g. [15] and [13]).

2. Rational external fields

The subject of the present paper is the study of equilibrium measures in the presence of external fields of the form (1), with $\gamma_j > 0$ and $z_j \in \mathbb{C} \setminus \mathbb{R}$ for $j = 1, \ldots, q$, where

$$\sum_{j=1}^{q} \gamma_j \ = T \ > \ 0\,.$$

In this sense, and also regarding the methodology used, it is a continuation of [39], where general external fields containing a polynomial part of the form (4) were considered. However, the absence of the polynomial part in the present external field represents a significant increase in the difficulty to solve the problem (as will be seen later).

In [33] and [39] it was shown how a combined use of two main ingredients provide a full description of the evolution of the support of the equilibrium measure when the size of the measure, t, grows from 0 to ∞ (recall that in both previous papers, the external fields were strong enough to allow $t \in (0, +\infty)$). These main tools are an algebraic equation for the Cauchy transform of the equilibrium measure and a dynamical system for the zeros of the density function of this measure, based on the Buyarov-Rakhmanov seminal result in [10].

Indeed, suppose that the rational external field is of the form (1). Then, with respect to the first ingredient, as it was seen in [39, Theorem 2.1] within a more general context, denote by Φ the function, analytic in a neighborhood of $\mathbb{R}$, such that $\varphi(x) = \Phi(x) = \operatorname{Re}\Phi(x)$, $x \in \mathbb{R}$ (this is feasible, since the branch points z_j and $\overline{z}_j$ lie on $\mathbb{C} \setminus \mathbb{R}$). Namely, we have

$$\Phi(z) = \sum_{j=1}^{q} \frac{\gamma_j}{2} \left(\log(z - z_j) + \log(z - \overline{z_j})\right)\,.$$

Then, we have that the Cauchy transform of the equilibrium measure, $\widehat{\lambda}_t(x) = \displaystyle\int \frac{d\lambda_t(s)}{x-s}$, satisfies the relation:

$$(-\widehat{\lambda}_t + \Phi')\,(z) \ = \ (T-t)\,\sqrt{R(z)} \ = \ (T-t)\,\frac{B(z)\,\sqrt{A(z)}}{D(z)}\,,\ z \in \mathbb{C} \setminus S_t\,, \tag{7}$$

for some monic polynomials $A(z) = \prod_{j=1}^{2k} (z - a_j)$ and $B(z) = \prod_{j=1}^{2q-k-1} (z - b_j)$, with $a_1 < \ldots < a_{2k} \in \mathbb{R}$, (and thus, R is a rational function). We take the branch of $\sqrt{R(z)}$ which is positive for $z \in (a_{2k}, +\infty)$.

Here and on the sequel, we denote $D(z) = \prod_{j=1}^{q} (z - z_j)(z - \overline{z}_j)$, which is a polynomial of degree $2q$. In fact, (7), as well as the previous results in [33, Theorem 2.2] and [39, Theorem 1.1], holds in the more general context of critical measures.

In addition, since the support of λ_t consists of the union of the cuts $[a_{2j-1}, a_{2j}]$, , $j = 1, \ldots, k$, (7) and the Sokhotski—Plemelj inversion formula provides an expression for the density of the equilibrium measure:

$$\lambda_t'(x) = \frac{T-t}{\pi i} \sqrt{R(x)^+} = \frac{T-t}{\pi} \sqrt{|R(x)|}\,,\ x \in \cup_{j=1}^{k} [a_{2j-1}, a_{2j}]\,, \tag{8}$$

where $\sqrt{R(x)^+}$ stands for the limit of $\sqrt{R(x)}$ as z approaches x from the upper half-plane.

Therefore, the zeros of R are the main parameters of the equilibrium problem; the ones with odd multiplicity are the endpoints of the intervals comprising the support S_t. Equating the residues of both members in (7) at z_j , $j = 1, \ldots, q$, we obtain (taking real and imaginary parts) $2q$ nonlinear equations for the $2q+k-1$ zeros of R. However, only when the one-cut case occurs, that is, $k = 1$, this system completely determines the unknowns. Indeed, when $k > 1$, this system has solutions where the so-called chemical potential takes a constant value on each connected component of the support (see the first equation in (2)), but in general, these constants are different, providing what is often referred to as critical measures. Thus, when $k > 1$, in order to fulfill condition (2), the extra conditions

$$\int_{a_{2j}}^{a_{2j+1}} \sqrt{R(x)}\, dx = 0\,,\ j = 1, \ldots, k-1 \tag{9}$$

must hold, which means that B must have an odd number of zeros (counting their multiplicities) on each gap (a_{2j}, a_{2j+1}) , $j = 1, \ldots, k-1$, of the support. Observe that, in addition, it provides a bound for k, the number of intervals comprising the support S_t: namely, we have that $k \leq q$.

Recall that for rational external fields of the form (4) with $p \geq 1$, the computation of the coefficients in z^m , with $m \geq -1$, of the expansions at infinity in both members of equality (7) always provides some simple equations helping us to find the value of the endpoints and the other zeros of the density function in a rather easy way, or other useful properties of them (such as the value of their arithmetic mean or others, see [39]); unfortunately, this does not work in our purely rational situation ($P \equiv 0$) since a trivial identity is obtained. Thus, in the current setting even the one-cut case is difficult to be explicitly solved, as well as finding the values of parameters where phase transitions occur. However, despite the complicated calculations needed for solving explicitly the problem of determining the support, it is worth to point out the simplicity and beauty of the results (see Theorem 3.1 below).

The second ingredient is based on a "dynamical" description of the support of the equilibrium measure λ_t in the real axis in the presence of an external field, which was proposed by Buyarov and Rakhmanov in [10]. This seminal result basically asserts that at a certain "instant" t_0, the derivative of the equilibrium measure λ_t with respect to t is given by the Robin measure (that is, the equilibrium measure in the absence of an external field) for the support S_{t_0}. Hereafter, by the derivative of a compactly supported measure σ with respect to a certain parameter τ at $\tau = \tau_0$, we mean the limit measure (provided it is unique; the existence is guaranteed by a weak compactness argument)

$$\lim_{h \to 0} \frac{1}{h} \left(\sigma(\tau_0 + h) - \sigma(\tau_0)\right) . \tag{10}$$

In the current rational case, taking into account these results and the well-known expression for the Robin measure of a finite union of compact intervals, we have that except for an at most denumerable set of values of t,

$$\frac{\partial}{\partial t}\left((T-t)\,\frac{B(z)\,\sqrt{A(z)}}{D(z)}\right) \;=\; -\,\frac{F(z)}{\sqrt{A(z)}}\,, \tag{11}$$

where F is a monic polynomial of degree $k-1$ such that $\displaystyle\int_{a_{2j}}^{a_{2j+1}} \frac{F(x)}{\sqrt{A(x)}}\,dx \;=\; 0\,,\; j=1,\ldots,k-1\,,$ which means that F has a simple zero on each gap (a_{2j}, a_{2j+1}) of the support (see [39]). Making use of the abbreviate physical notation for the derivative with respect to the "time" t: $\dot{f} = \dfrac{\partial f}{\partial t}$, and evaluating (11) at points a_i and b_i, it immediately yields (see [39, Theorem 1.2]):

Theorem 2.1. *Except for a denumerable set of values of t, it holds*

$$\begin{aligned}
\dot{a_i} &= \frac{1}{T-t}\,\frac{2D(a_i)F(a_i)}{B(a_i)\,\prod_{j\neq i}(a_i-a_j)}\,,\; i=1,\ldots,2k\,,\\
\dot{b_i} &= \frac{1}{T-t}\,\frac{D(b_i)F(b_i)}{A(b_i)\,\prod_{j\neq i}(b_i-b_j)}\,,\; i=1,\ldots,2q-k-1\,,
\end{aligned} \tag{12}$$

with $1\leq k\leq q$.

From (12), it is clear that always $\dot{a_1}<0$ and $\dot{a_{2k}}>0$. Moreover, taking into account that on each gap there is an even number of zeros of the product $B(x)F(x)$, it is also possible to assert that the a_i when i is odd are decreasing, while for even values of i are increasing; this is coherent with the well-known fact that the support S_t is increasing with t (see [10]).

Indeed, (12) is a dynamical system for the positions of all the important points determining the equilibrium measure and its support. Previously, in [33] a similar result was extensively used to study the dynamics of the equilibrium measure and its support for the case of polynomial external fields; in particular, the so-called "quartic" case was analyzed in detail. Similarly, in [39] the case of a rational external field consisting of a polynomial part plus a logarithmic term (a generalized Gauss-Penner model) was studied.

Remark 2.1. Bearing in mind the results in [33], for polynomial external fields, and those in [39] and (12), for the rational case, it is easy to find a general structure of these dynamical systems. Indeed, for the zeros of the density (8) of the equilibrium measure of the interval $[c,d]\subset\mathbb{R}$ (bounded or not) in the presence of the external field (4), the following system of differential equations holds (except for a finite number of bifurcations/collisions),

$$\dot{\xi_j} = h_j(t)\,\frac{D(\xi_j)F(\xi_j)}{(AB)'(\xi_j)}\,, \tag{13}$$

where $h_j(t)$ is a positive function of t which reduces to a constant if the external field (4) has a polynomial part, D is a real polynomial of even degree whose zeros are located at the point masses and their conjugates, and A,B play the same role as in (7)-(8). At first sight, there seems to be an important difference between the purely rational case handled in the current paper and the rational cases with a polynomial part: now, the system is not an autonomous one, since function h_j depends on the variable t; in the other rational cases, the presence of the polynomial part in the external field makes $h_j\equiv\kappa_j$, i.e., a constant independent of t. However, this is only an apparent difference: after a simple change of variable, the system easily becomes autonomous. Indeed, it is easy to check that the change $u = -\log(T-t) + \log T$, with the new "time" u lying on $(0, +\infty)$, transforms (12) in an autonomous system.

The shape of these dynamical systems (13) resembles in a certain sense to a system of ODEs studied by Dubrovin in [14] for the dynamics of the Korteweg-de Vries equation in the class of finite-zone or finite-band potentials, as it was pointed out in [33].

As it was said above, under mild conditions on φ, the equilibrium measure depends analytically on t except for a (possible) small set of values, which are called the critical points or the singularities of the problem. At these critical values of t, the so-called *phase transitions* occur; in most of them, it implies a change in the number of cuts (connected components of the support S_t), but not always. The study of these phase transitions is one of the main issues of this problem. Let us recall, briefly, the basic type of singularities we can find (using the classification in [11], also used then in [24] and [33]). In this case, we prefer recalling the version of these definitions used in [33], namely:

- **Singularity of type I:** at a time $t = \tau$ a real zero b of B is such that $(V^{\lambda_\tau} + \varphi)(b) = c_\tau$, $b \notin S_\tau$ (see (2)), in such a way that for $t = \tau$, b is a simple zero of B. Therefore, at this time $t = \tau$ a real zero b of B (a double zero of R_τ) splits into two simple zeros $a_- < a_+$ (of A), and the interval $[a_-, a_+]$ becomes part of S_t for $t > \tau$ (*birth of a cut*).

- **Singularity of type II:** at a time $t = \tau$, a real zero b of B (of even multiplicity) belongs to the interior of the support S_τ; according to (7)-(8), the density of λ_τ vanishes in the interior of its support, in such a way that for $t = \tau$, b is a double zero of B. Thus, at this time $t = \tau$ two simple zeros a_{2s} and a_{2s+1} of A (simple zeros of R_t, i.e., endpoints of the support) and a simple zero of B have collided (*fusion of two cuts*). After that collision, they become a pair of imaginary zeros of B.

- **Singularity of type III:** at a time $t = \tau$, polynomials A and B have a common real zero a; the only additional assumption is that a is a double zero of B, so that $\lambda'_\tau(x) = \mathcal{O}(|x-a|^{5/2})$ as $x \to a$. Then, at this time $t = \tau$ a pair of complex conjugate zeros b and $\overline{b}$ of B (double zeros of R_t) have collided with a simple zero a of A (endpoint), so that $\lambda'_\tau(x) = \mathcal{O}(|x-a|^{5/2})$ as $x \to a$. Observe that in this case no topological change takes place: the number of cuts does not vary.

There is another special situation which plays a similar role as the previous ones, but is not properly a singularity, and was also considered in [33] and [39]. It takes place when polynomial B has two conjugate imaginary roots, b and $\overline{b}$, which collide at a certain time, producing a double real root of B (quadruple real root of R) in the real axis, which immediately splits into a pair of simple real roots, b_1 and b_2, which tend to move away each other. In fact, what we have in this case is the birth of two new local extrema of the total (or "chemical") potential (2). From this point of view, a type III singularity may be seen as a limit case of these situations, when the pair of imaginary zeros of B collide simultaneously with a zero of A (endpoint).

In the case where the number of cuts is bounded by 2 (precisely, the case we will deal in Section 3 of this paper), these are just all the possible types of singularities, while if it can be bigger than 2 more intriguing phenomena may occur when two or more of these singularities take place simultaneously.

Now, we are dealing with what may be called, in a colloquial style, "the beginning and the end of the movie", that is, the situation when $t \searrow 0$ and $t \nearrow T$. The answer to the first question is clear: the critical points of Φ are the initial conditions of the dynamical system (12), as we can see taking $t = 0$ in (7). Then, since the family of supports $\{S_t\}_{t\in(0,T)}$ is increasing with t, as $t \searrow 0$ the support tends to reduce to some subset of these critical points. But (2) implies that this limit support must be included in the set of points where the global minimum of Φ is attained. On the other hand, for $x \notin S_t$, the function $V^{\mu_t}(x) + \Phi(x) - c$ is strictly increasing with respect to t (see for details the previous proof of Theorem 2.2) and, so, applying (2) again, there can not be a global minimum of Φ not belonging to S_t. Hence, we have

$$\bigcap_{t\in(0,T)} S_t = \{y \in \mathbb{R} \,:\, \Phi(y) = \min_{x\in\mathbb{R}} \Phi(x)\}\,, \tag{14}$$

and since Φ' is a rational function whose numerator has degree $2q-1$, we easily conclude that the cardinality of the set (14) belongs to the set $\{1,\ldots,q\} \subset \mathbb{N}$. Regarding the second question, we have

Theorem 2.2. *Denote by* λ_t *the equilibrium measure in the external field* (1)*. Then,*

i) There exists the limit of the equilibrium measure when $t \to T$*:*

$$\lim_{t\to T} \lambda_t = \lambda_T\,,$$

in the sense that

$$\lim_{t\to T} \lambda_t(I) = \lambda_T(I)\,,$$

for any Borel set $I \subset \mathbb{R}$.

ii) For t *sufficiently close to* T*, the support* S_t *consists of a single interval. In particular, using the AB-representation* (7)*, the zeros of polynomial* A *(endpoints) diverge:*

$$\lim_{t\to T} a_1 = -\infty\,, \qquad \lim_{t\to T} a_2 = +\infty\,,$$

and the $2q-2$ *zeros of polynomial* B *converge to points in* $\mathbb{C}\setminus\mathbb{R}$*; in particular, they converge to the zeros of the rational function:*

$$\sum_{j=1}^{q} \left(\frac{\gamma_j}{2(z-z_j)} - \frac{\gamma_j}{2(z-\overline{z}_j)} \right)\,.$$

iii) The density of the limit measure λ_T *is given by*

$$\frac{d\lambda_T}{dx} = \frac{1}{T\,\pi} \sum_{j=1}^{q} \frac{\gamma_j \operatorname{Im} z_j}{((x-\operatorname{Re} z_j)^2 + \operatorname{Im} z_j^2)}$$

The proof of Theorem 2.2 will be displayed in Section 4, along with the proofs of the other important results in this paper.

Remark 2.2. The asymptotic result in Theorem 2.2 above deals with the so-called weakly admissible external fields (see [8], [19] and[42], among others). In this sense, given some $t > 0$ (the size of the equilibrium measure) an external field ϕ is said to be weakly admissible if

$$\liminf_{|x|\to\infty} (\phi(x) - t\,\log|x|) = M > -\infty\,, \tag{15}$$

in place of the usual requirement that

$$\lim_{|x|\to\infty} (\phi(x) - t \log|x|) = +\infty \,.$$

For these weakly admissible external fields satisfying condition (15), the equilibrium measure also exists and it is unique, but the support is possibly unbounded. In this framework, the asymptotic measure λ_T, found as a limit measure in Theorem 2.2, is the equilibrium measure λ_t, for $t = T$, in the weakly admissible external field (1).

To end this section, let us consider the variation of the equilibrium measure when some of the parameters of the external field (1) vary; that is, we mean the evolution of the equilibrium measure when one of the "masses" γ_j or "heights" $\operatorname{Im} z_j$ is varying.

In order to do it, we present now a simplified version of [33, Theorem 5], where the authors extended the seminal Buyarov-Rakhmanov result for the variation of the equilibrium measure with respect to other parameters in the external field (recall the definition of the derivative of a measure with respect to some parameter (10)), which is sufficient for our purposes (see also [39, Theorem 1.3]).

Theorem 2.3. *Let $t > 0$ be fixed and suppose that the function $\varphi(x;\tau)$ is a real-analytic function for $x \in \mathbb{R}$ and $\tau \in (c,d)$, where (c,d) is a real interval. Let $\lambda = \lambda_{t,\tau}$ denote the equilibrium measure in the external field $\varphi(x;\tau)$, for $\tau \in (c,d)$, with support $S_{t,\tau}$. Then, for any $\tau_0 \in (c,d)$,*

$$\frac{\partial \lambda}{\partial \tau}\,|_{\tau=\tau_0} = \omega \,,$$

where the measure ω is uniquely determined by the conditions

$$\operatorname{supp}\omega = S_{t,\tau_0}\,,\ \omega(S_{t,\tau_0}) = 0\,,\ V^{\omega} + \frac{\partial\varphi(x;\tau)}{\partial\tau}\,|_{\tau=\tau_0} = \frac{\partial c_t}{\partial \tau}\,|_{\tau=\tau_0} = const \ \text{ on } \ S_{t,\tau_0}\,, \tag{16}$$

where c_t is the equilibrium constant given in (2).

Observe that the second formula in (16) means that ω is a type of signed measure which is often called a *neutral* measure; it is a natural consequence of the fact that t, the total mass of λ, does not depend on parameter τ.

We are concerned, first, with the situation when one of the "masses" γ_j, with $j \in \{1,\ldots,q\}$, varies in $(0,+\infty)$. In this case, taking into account that $\dfrac{\partial\varphi(z)}{\partial\gamma_j} = \log|z - z_j|$ and that the measure ω in Theorem 2.3 above is supported in $\cup_{j=1}^{k}[a_{2j-1}, a_{2j}]$, then its Cauchy transform $\widehat{\omega}$ is such that

$$-\widehat{\omega}(z) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi(z)}{\partial\gamma_j}\right) = \frac{H(z)}{(z-z_j)(z-\overline{z_j})\sqrt{A(z)}}\,,$$

where H is a monic polynomial of degree $(k+1)$ which has a zero of odd order on each of the $(k-1)$ gaps of the support of the equilibrium measure, and such that at least one of the other two zeros lies inside the convex hull of the support. The coefficients of H can be obtained from the residues at z_j and $\overline{z_j}$ together with the corresponding equations of the same nature as previous conditions (9) (in order to have the same equilibrium constant in each connected component of the support). Hence, Theorem 2.3 asserts

$$\frac{\partial}{\partial\gamma_j}\left((T-t)\,\frac{B(z)\sqrt{A(z)}}{D(z)}\right) = \frac{H(z)}{(z-z_j)(z-\overline{z_j})\sqrt{A(z)}}\,, \tag{17}$$

In particular, when $k = 1$, that is, the one-cut case, taking derivatives at the left-hand side of (17) and evaluating at the endpoints a_1 and a_2 of the single interval comprising the support, the following dynamical system holds:

$$\frac{\partial a_1}{\partial \gamma_j} = -\,\frac{2}{T-t}\,\frac{(a_1-h_1)(a_1-h_2)\,\widetilde{D_j}(a_1)}{(a_1-a_2)\,B(a_1)}\,,$$
$$\frac{\partial a_2}{\partial \gamma_j} = -\,\frac{2}{T-t}\,\frac{(a_2-h_1)(a_2-h_2)\,\widetilde{D_j}(a_2)}{(a_2-a_1)\,B(a_2)}\,,$$

where B, given in (7), is, in this case, a monic polynomial of degree $2q-2$ being positive in the interval (a_1, a_2), h_1, h_2, the roots of polynomial H in (17), is a pair or real numbers, such that at least one of them belongs to (a_1, a_2), and

$$\widetilde{D_j}(z) = \frac{D(z)}{(z-z_j)(z-\overline{z_j})} = \prod_{l\neq j}(z-z_l)(z-\overline{z_l}) = \prod_{l\neq j}\left((z-\operatorname{Re} z_l)^2 + (\operatorname{Im} z_l)^2\right) > 0\,.$$

Therefore, the increase or decrease of the endpoints when γ_j grows depends on the position of the points h_1, h_2, which, in turn, is determined by the relative position of the charge z_j in the set $\{z_1, \ldots, z_q\}$. In a similar way, the dynamical system for the other zeros of the density of the equilibrium measure ($2q-2$ zeros of polynomial B) may be displayed.

In a similar fashion, the evolution of the support when one of the heights $\beta_j = \operatorname{Im} z_j$ varies may be handled. In this case, we have that $\dfrac{\partial \varphi(z)}{\partial \beta_j} = \dfrac{\gamma_j\,\beta_j}{(z-z_j)(z-\overline{z_j})}$, and thus, Theorem 2.3 implies that

$$\frac{\partial}{\partial \beta_j}\left((T-t)\,\frac{B(z)\sqrt{A(z)}}{D(z)}\right) = \frac{K(z)}{(z-z_j)^2\,(z-\overline{z_j})^2\,\sqrt{A(z)}}\,, \tag{18}$$

where now the polynomial K, not necessarily monic, has degree $\leq (k+2)$ and a zero of odd multiplicity on each of the $(k-1)$ gaps of the support, and such that at least one of the other zeros lies inside the convex hull of the support. As above, for $k = 1$, that is, when the one-cut case takes place, (18) yields the following dynamical system for the endpoints of the single interval comprising the support:

$$\frac{\partial a_1}{\partial \beta_j} = \frac{2}{T-t}\,\frac{K(a_1)D(a_1)}{(a_1-a_2)\,B(a_1)\,(a_1-z_j)^2\,(a_1-\overline{z}_j)^2}\,,$$
$$\frac{\partial a_2}{\partial \beta_j} = \frac{2}{T-t}\,\frac{K(a_2)D(a_2)}{(a_2-a_1)\,B(a_2)\,(a_2-z_j)^2\,(a_2-\overline{z}_j)^2}\,,$$

Of course, it is possible to combine two or more of these dynamical systems to obtain a full description of the evolution of the support of the equilibrium measure. Indeed, a suitably combined use of the dynamical system with respect to the total mass t and the corresponding with respect to one of the prescribed charges γ_j will play a key role in proving some results stated in the next section (see the proofs in Section 4 below).

The next section is devoted to the simplest (but quite difficult) non-trivial case, where the external field is created by two prescribed charges, that is, $q = 2$. The full description of the dynamics of the equilibrium measure may be done for this situation.

3. An external field created by two prescribed charges

Throughout this section, we restrict to the case of the equilibrium problem in the presence of a couple of (attractive) prescribed charges. In particular, and without lack of generality (see Remark 3.3 below), we consider external fields of the form:

$$\varphi(x) = \log|x - z_1| + \gamma \log|x - z_2|\,, \ \gamma > 0\,, \ z_1, z_2 \in \mathbb{C}\setminus\mathbb{R}\,. \tag{19}$$

That is, we are concerned with the case where $q = 2$ in (1) and, thus, we know that the number of intervals ("cuts") comprising the support S_t is given by 1 or 2. We can assume that $\operatorname{Re} z_1 = -\operatorname{Re} z_2 = -1$, as well as $\operatorname{Im} z_1 = \beta_1 > 0, \operatorname{Im} z_2 = \beta_2 > 0$, also without loss of generality. Now, the evolution of the equilibrium measure λ_t and, in particular, of its support S_t, in the presence of the external field (19), depending on three parameters, $\beta_1, \beta_2 > 0$ and $\gamma > 0$, for $t \in (0, T)$, with $T = 1 + \gamma$, is investigated.

In the particular case handled in this section, where the external field is due to a couple of prescribed charges (19), we have that (7) holds, with $degA \in \{2, 4\}$ and $degB = 3 - \frac{degA}{2}$. Thus, taking residues at $z = z_1$ and $z = z_2$ in (7) yields

$$\begin{cases} (T-t)\,B(z_1)\,\sqrt{A(z_1)} \,-\, i\,\operatorname{Im} z_1\,(z_1 - z_2)(z_1 - \overline{z}_2) & = 0\,, \\ (T-t)\,B(z_2)\,\sqrt{A(z_2)} \,-\, i\,\gamma\,\operatorname{Im} z_2\,(z_2 - z_1)(z_2 - \overline{z}_1) & = 0\,, \end{cases} \tag{20}$$

and, after taking real and imaginary parts, we finally arrive to a nonlinear system of four equations. Thus, system (20) determines uniquely polynomials A and B in the one–cut case; but if the support consists of two disjoint intervals, then an additional condition (9) is also necessary.

Now, combining the two ingredients above, that is, formulas (7) and (12), we have the following possible settings for the support S_t of the equilibrium measure and its density (for non-singular values of $t \in (0, T)$). On the sequel, $D(x) = (x - z_1)(x - \overline{z}_1)(x - z_2)(x - \overline{z}_2)$.

(one-cut) $S_t = [a_1, a_2]\,,\ a_1 = a_1(t) < a_2 = a_2(t)$ and

$$\lambda_t'(x) = \frac{T-t}{\pi}\,\frac{(x-b_1)(x-b_2)\,\sqrt{(x-a_1)(a_2-x)}}{D(x)}\,,$$

with $b_2 < b_1 < a_1 < a_2$ and

$$\int_{b_2}^{a_1} \frac{(x-b_1)(x-b_2)\,\sqrt{(x-a_1)(x-a_2)}}{D(x)}\,dx \,>\, 0\,,$$

or $a_1 < a_2 < b_1 < b_2$ and

$$\int_{a_1}^{b_2} \frac{(x-b_1)(x-b_2)\,\sqrt{(x-a_1)(x-a_2)}}{D(x)}\,dx \,>\, 0\,,$$

or, finally, $b_2 = \overline{b}_1 \in \mathbb{C}\setminus\mathbb{R}$. In this scenario, we also have, for $i, j = 1, 2$ and $j \neq i$,

$$\dot{a_i} = \frac{1}{T-t}\,\frac{2D(a_i)}{(a_i-a_j)(a_i-b_1)(a_i-b_2)}\,,\ \dot{b_i} = \frac{1}{T-t}\,\frac{D(b_i)}{(b_i-b_j)(b_i-a_1)(b_i-a_2)}\,. \tag{21}$$

(**two-cut**) $S_t = [a_1, a_2] \cup [a_3, a_4]$, $a_1 = a_1(t) < a_2 = a_2(t) < a_3 = a_3(t) < a_4 = a_4(t)$ and

$$\lambda_t'(x) = \frac{T-t}{\pi} \frac{|x-b_1| \sqrt{(x-a_1)(x-a_2)(x-a_3)(a_4-x)}}{D(x)},$$

with $a_2 < b_1 < a_3$ and

$$\int_{a_2}^{a_3} \frac{(x-b_1)\sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}}{D(x)}\, dx = 0.$$

In this case, the following dynamical system holds

$$\dot{a}_i = \frac{1}{T-t} \frac{2D(a_i)\, F(a_i)}{A'(a_i)(a_i - b_1)}, \quad \dot{b}_1 = \frac{1}{T-t} \frac{D(b_1)\, F(b_1)}{A(b_1)}, \quad i = 1, 2, 3, 4,$$

where $F(x) = x - \zeta$, with ζ uniquely determined by the condition

$$\int_{a_2}^{a_3} \frac{(x-\zeta)}{\sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}}\, dx = 0.$$

3.1. Main Result

Now, we are going to state our main result. In order to do it, consider the bivariate polynomial

$$f(x,y) = 27xy(x-y)^2 - 4(x^3+y^3) + 204xy(x+y) - 48(x^2 - 7xy + y^2 + 4x + 4y) - 256. \qquad (22)$$

For $x, y > 0$, it is a symmetric function with respect to its arguments, and the graph of the curve

$$\mathcal{C} = \left\{ (\beta_1, \beta_2) \in (0, +\infty) \times (0, +\infty) \,:\, f(\beta_1^2, \beta_2^2) = 0 \right\} \qquad (23)$$

is decreasing and splits the open first quadrant of the (β_1, β_2)-plane into two domains: Ω_0, with the origin belonging to its closure, and Ω_∞ (see Figure 1). The curve $\mathcal{C}$ has two asymptotes at $\beta_1 = \frac{2}{3\sqrt{3}}$ and $\beta_2 = \frac{2}{3\sqrt{3}}$.

Theorem 3.1. *Let S_t be the support of λ_t, the equilibrium measure in the external field* (19)*, with $z_1 = -1 + i\beta_1, z_2 = 1 + i\beta_2$. Then, we have*

- *If $(\beta_1, \beta_2) \in \Omega_\infty \cup \mathcal{C}$ (that is, $f(\beta_1^2, \beta_2^2) \geq 0$), then S_t consists of a single interval ("one-cut") for any $\gamma > 0$ and $t \in (0, T)$.*
- *If $(\beta_1, \beta_2) \in \Omega_0$ ($f(\beta_1^2, \beta_2^2) < 0$), then there exist two values $0 < \Gamma_1 = \Gamma_1(\beta_1, \beta_2) < \Gamma_2 = \Gamma_2(\beta_1, \beta_2)$ such that for $\gamma \in (\Gamma_1, \Gamma_2)$ there are two critical values $0 \leq T_1 = T_1(\beta_1, \beta_2, \gamma) < T_2 = T_2(\beta_1, \beta_2, \gamma) < T$, in such a way that S_t consists of two disjoint intervals for $t \in (T_1, T_2)$ ("two-cut"). Otherwise, S_t consists of a single interval.*

Remark 3.1. The expression of the "boundary-curve" $\mathcal{C}$ may be easily obtained by imposing that the derivative $\varphi'(x)$ of the external field (19) has a triple real root. The recipe to determine the values of Γ_1, Γ_2 and T_1, T_2 will be shown within the proofs in Section 4 below. In the final Appendix, a more detailed description, along with illustrative figures, of the region where the two-cut phase is feasible will be given.

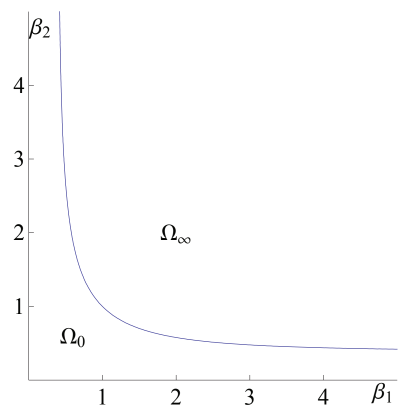

Figure 1: Regions Ω_0 and Ω_∞ and the curve $\mathcal{C}$

Remark 3.2. The result in Theorem 3.1 means that the relationship between the distances of the two attractive charges to the real axis ("heights") determines the possible existence of a two-cut phase. It seems natural, but a curious phenomenon also takes place: if one of the charges is close enough to the real axis, say $\beta_1 < \frac{2\sqrt{3}}{9} \sim 0.385$, then for any value of the other height β_2, there always exists an interval of values of the "mass" γ for which a two–cut phase takes place. Roughly speaking, it seems to tell us that if one of the charges is sufficiently close to the real axis, then it is always possible to distinguish both charges from there (in the sense that they are able to split the support of the equilibrium measure), whatever the distance of the other one is (provided a suitable fit between the masses, of course).

Obviously, this "positive" result has a "negative" counterpart: if the couple of attractive charges are sufficiently far from the real axis (i.e., $(\beta_1, \beta_2) \in \Omega_\infty \cup C$), they are indistinguishable, in the sense mentioned above, from there whatever the masses are.

Remark 3.3. For the sake of simplicity, Theorem 3.1 was stated for the external field (19). But obviously the result may be immediately translated for external fields of the more general shape:

$$\phi(x) = \gamma_1 \log|x - z_1| + \gamma_2 \log|x - z_2|\,,\ \gamma_1, \gamma_2 > 0\,,\ z_j = \alpha_j + i\,\beta_j\,,\ j = 1, 2$$

In this case, it is easy to see that $\beta_j\,,\ j = 1, 2$, in the expression of the critical curve (23) must be replaced by the respective ratios $\dfrac{2\,\beta_j}{|\alpha_2 - \alpha_1|}$. In addition, (3) provides the recipe to translate the results for general $\gamma_1, \gamma_2 > 0$.

Remark 3.4. The result in previous Theorem 2.2 may be easily illustrated in the case of two attractors considered in Theorem 3.1. Indeed, we have that the density of the limit measure (as $t \to T = 1 + \gamma$) is given by

$$\frac{d\mu_T}{dx} = \frac{1}{T\pi}\left(\frac{\beta_1}{(x+1)^2 + \beta_1^2} + \frac{\gamma\,\beta_2}{(x-1)^2 + \beta_2^2}\right),\ x \in \mathbb{R}\,,$$

whose zeros are imaginary for any $\beta_1, \beta_2, \gamma > 0$.

Remark 3.5. As it was said above, some particular situations were considered in [38]; in particular, the so-called "totally symmetric" case, that is, where heights and masses are equal ($\beta_2 = \beta_1 = \beta$ and

$\gamma = 1$ in our current notation) and a "partially symmetric" case, where just the heights are supposed to be equal. With respect to the latter one, it is easy to see that the intersection between the curve $\mathcal{C}$ and the bisector $\beta_2 = \beta_1$ consists of the point $\beta_1 = \beta_2 = 1$. Thus, we conclude that a two-cut phase is feasible in this partially symmetric situation if and only if the common height $\beta < 1$ or, what is the same, if the two charges are close enough to the real axis to be able to split the support.

The special situation of the totally symmetric case will be revisited in detail after Theorem 3.4 below, where the evolution of the support S_t is described.

Remark 3.6. It is well-known that the convexity of the external field ensures that the support of the equilibrium measure is an interval (see e.g. [41]). In [4] a weaker sufficient condition is given, namely, the convexity of the function $\exp(\varphi)$. We can check whether this condition is fulfilled when $(\beta_1, \beta_2) \in \Omega_\infty \cup \mathcal{C}$ and, thus, whether in this sense the first part of Theorem 3.1 is a consequence of that previous result. However, it is possible to find examples with $(\beta_1, \beta_2) \in \Omega_\infty \cup \mathcal{C}$ and $\gamma > 0$ such that $\exp\varphi$ is not convex. Indeed, it is easy to check that $(\beta_1, \beta_2) \in \Omega_\infty$ for $\beta_1 = 0.5$ and $\beta_2 = 2.7$, but $\exp\varphi$ is non-convex for these values when we take, for instance, $\gamma = 5.6$.

Remark 3.7. In a similar fashion as in the applications to the asymptotics of Heine-Stieltjes polynomials or to Random Matrix models considered in Sections 1.1-1.2, results in Theorem 3.1 are also related to the asymptotic behavior of certain families of varying orthogonal polynomials (i.e., when the weight depends on the degree of the polynomial); this connection was developed in some seminal papers during the eighties, see e.g. [16], [27], [37] and also [41]. In the current case, the results in Theorem 3.1 allow to describe the support of the limit zero distribution of polynomials P_n, with $deg\, P_n = n$, satisfying varying orthogonality relations of the form

$$\int x^k\, P_n(x)\, \omega_n(x)\, dx \;=\; 0\,,\;\; k = 0, \ldots, n-1\,,$$

where the varying weight ω_n is a generalized Jacobi–type weight given by

$$\omega_n(x) = \frac{1}{|x - z_1|^{\alpha_n}\, |x - z_2|^{\beta_n}}\,,\;\; x \in \mathbb{R}\,,$$

with $\alpha_n + \beta_n > 2n\,,\; n \in \mathbb{N}$, in such a way that $\lim\limits_{n\to\infty} \dfrac{\alpha_n}{n} \;=\; A > 0$ and $\lim\limits_{n\to\infty} \dfrac{\beta_n}{n} \;=\; B$, and $A+B > 2$. Indeed, it is enough to set $\gamma = \dfrac{B}{A}$ and $t = \dfrac{2}{A}$.

3.2. Auxiliary Results

For the proof of Theorem 3.1 we need a number of results which are also of interest themselves, in such a way that all together describe the different scenarios in the evolution of the equilibrium measure when t grows from 0 to $T = 1 + \gamma$. Indeed, our main result, Theorem 3.1, is a synthesis of such a full description.

First, in [33] and [39], it was shown that the knowledge about the set of minima of the external field plays a key role in describing the evolution of the equilibrium measure λ_t when t varies. In those cases, the existence of two relative minima of the external field was shown as a sufficient condition for the existence of a two-cut phase (that is, a range of values of t for which the support comprises two disjoint intervals). It will be also true in the current case (see Theorem 3.4 below; in fact, the existence of a two-cut phase when the external field has at least two minima is true for a more general class of external fields).

First, taking into account that the relative minima of the external field φ are roots of the polynomial

$$P(x) = (x^2-1)((x-1)+\gamma(x+1)) + \gamma\beta_1^2(x-1) + \beta_2^2(x+1)\,, \tag{24}$$

it is easy to see that the real relative minima lie on the interval $(-1,1)$. Moreover, we have,

Theorem 3.2. *Consider the external field* (19)*, with $z_1 = -1+\beta_1 i$, $z_1 = 1+\beta_2 i$ and $\gamma > 0$. Then,*

- *If $f(\beta_1^2, \beta_2^2) < 0$, with f given by* (22)*, or, what is the same, $(\beta_1, \beta_2) \in \Omega_0$, there exist two values $0 < \widetilde{\Gamma}_1 = \widetilde{\Gamma}_1(\beta_1,\beta_2) < \widetilde{\Gamma}_2 = \widetilde{\Gamma}_2(\beta_1,\beta_2)$, such that* (19) *has two real local minima for $\gamma \in (\widetilde{\Gamma}_1, \widetilde{\Gamma}_2)$. If $\gamma \notin (\widetilde{\Gamma}_1, \widetilde{\Gamma}_2)$, then* (19) *has a single real local (and, in fact, absolute) minimum.*
- *If $f(\beta_1^2, \beta_2^2) \geq 0$, that is, $(\beta_1, \beta_2) \in \Omega_\infty \cup \mathcal{C}$, then* (19) *has a single real local minimum for any $\gamma > 0$.*

Remark 3.8. In a similar fashion as in the previous Theorem 3.1, a simple recipe to compute the critical values $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ is feasible. In fact, it is enough to compute the values of γ for which the external field has a double (real) critical point.

Although the fact that the external field has two real minima is a sufficient condition for the existence of a two-cut phase, it is not a necessary one. Indeed, roughly speaking, if the external field is "sufficiently non-convex" a two-cut phase is still feasible. In order to get it, it is necessary the appearance of a new local extrema of (2), that is, a double real root of polynomial B outside the support, as it was said in Section 2. The border external fields in this sense are those for which a type III singularity (that is, the confluence of a couple of imaginary zeros of B with a zero of A in the AB-representation given by (7)), takes place for some critical value of t. Indeed, we have

Theorem 3.3. *For the equilibrium measure λ_t in the external field* (19)*, it holds:*

- *If $(\beta_1, \beta_2) \in \Omega_0$, there exist two values $\Gamma_1 = \Gamma_1(\beta_1,\beta_2), \Gamma_2 = \Gamma_2(\beta_1,\beta_2)$, with $0 < \Gamma_1 < \widetilde{\Gamma}_1 < \widetilde{\Gamma}_2 < \Gamma_2$, such that for $\gamma = \Gamma_i$, $i = 1,2$, a type III singularity occurs at certain critical values of $t \in (0,T)$.*
- *If $(\beta_1, \beta_2) \in \Omega_\infty \cup \mathcal{C}$, no type III singularity takes place.*

Remark 3.9. In the situations discussed in Theorem 3.3, if (19) has a single minimum then for t sufficiently small we have a one-cut phase and the polynomial B has a couple of imaginary roots $b_1 = b$ and $b_2 = \overline{b}$. The sign of $\dfrac{\partial \operatorname{Im} b}{\partial t}$ plays a central role in the description of the dynamics of the support. In this sense, it is possible that $\operatorname{Im} b$ is always increasing and, thus, b and $\overline{b}$ are always going away from the real axis; but also $\operatorname{Im} b$ could be initially increasing but becomes decreasing at a certain moment, and so on. Indeed, for each value of t there is a critical curve such that if $b = b(t)$ belongs to this curve, then $\dfrac{\partial \operatorname{Im} b}{\partial t} = 0$. This curve determines the regions where the zeros of B approach (or not) the real axis. In previous [33] and [39], this curve was shown to take the form of a hyperbola and a circle, respectively. In the present case, its shape is much more involved. Indeed, we have by (21),

$$\frac{\partial \operatorname{Im} b}{\partial t} < 0 \;\Leftrightarrow\; \operatorname{Re}\left(\frac{D(b)}{A(b)}\right) > 0 \;\Leftrightarrow\; \operatorname{Re} D(b) \operatorname{Re} A(b) \,+\, \operatorname{Im} D(b) \operatorname{Im} A(b) \,>\, 0\,,$$

where $A(b) = (b - a_1)(b - a_2)$ and $D(b) = (b^2 - z_1^2)(b^2 - z_2^2)$. Thus, in this case the critical curve is given in terms of a bivariate polynomial of degree 6 in $x = \operatorname{Re} b$ and $y = \operatorname{Im} b$.

Therefore, now the geometry of the problem is much more complicated. Furthermore, it is easy to check that in this situation, while the support plays the role of a "repellent" for the couple of conjugate roots b and $\overline{b}$, the rest of the real line acts as an "attractor". Indeed, for a general rational external field, the support always repels the couple of imaginary roots, while each gap is split in an odd number of subintervals by the roots of F and B in such a way that the first one acts as an "attractor", the second one as a "repellent", and so on (of course, it is also necessary to take into account the multiplicity of each zero).

Now, we have all the ingredients for describing the evolution of the equilibrium measure and, especially, its support S_t when t grows from 0 to T. Our main result, Theorem 3.1, is a simplified version of the following result.

Theorem 3.4. *Let $\beta_1, \beta_2 > 0$, and $0 < \Gamma_1 < \widetilde{\Gamma}_1 < \widetilde{\Gamma}_2 < \Gamma_2$, as given in Theorems 3.2-3.3.*

(a) If $(\beta_1, \beta_2) \in \Omega_0$ and $\gamma \in (\widetilde{\Gamma}_1, \widetilde{\Gamma}_2)$, consider the threshold values $T_1 = T_1(\beta_1, \beta_2, \gamma)$ and $T_2 = T_2(\beta_1, \beta_2, \gamma)$, with $0 < T_1 < T_2 < T$, whose existence is established in Theorem 3.1. Then, we have the following phase diagram for the support of the equilibrium measure, S_t:

***one-cut**, for $t \in (0, T_1) \longrightarrow$ **two-cut**, for $t \in (T_1, T_2) \longrightarrow$ **one-cut**, for $t \in (T_2, T)$.*

At $t = T_1$ ($t = T_2$), a type I (respect., type II) singularity occurs. If the external field φ takes the same value in its two relative minima, then $T_1 = 0$ in the phase diagram above and the initial one-cut phase is absent.

(b) If $(\beta_1, \beta_2) \in \Omega_0$ and $\gamma \in (\Gamma_1, \widetilde{\Gamma}_1] \cup [\widetilde{\Gamma}_2, \Gamma_2)$, the phase diagram for S_t is the same as in (a), but the appearance of a pair of new local extrema of (2) occurs at a certain $t = T_0$, with $T_0 < T_1$.

*(c) If $(\beta_1, \beta_2) \in \Omega_0$ and $\gamma \in (0, \Gamma_1] \cup [\Gamma_2, \infty)$, **one-cut** phase holds for any $t \in (0, T)$.*

*(d) If $(\beta_1, \beta_2) \in \Omega_\infty \cup \mathcal{C}$, we have **one-cut** phase for any $\gamma > 0$ and $t \in (0, T)$.*

In Figures 2-4 below, the phase diagram corresponding to the scenario (a) in Theorem 3.4 is illustrated. We assume that the absolute minimum is the one on the left. Then, during the first phase the support S_t consists of the interval $[a_1, a_2]$, which enlarges as t increases; b_1 (local maximum of the total potential) moves to the left while b_2 (local minimum of the total potential) moves to the right. After the split of b_2 in two new points, the support becomes comprised by two intervals, $S_t = [a_1, a_2] \cup [a_3, a_4]$, and both of them enlarge as t increases; b_1 (the local maximum of the total potential) can move to the left or to the right depending on its relative position with respect to the unique root of F, which is also in the gap. Finally, after the collision of the endpoints a_2 and a_3 (and hence, also of b_1), during the final phase the support is comprised again by a single interval, which enlarges as t increases, tending to fill the whole real axis, while the two roots of B are imaginary and evolve according to (21).

Remark 3.10. The so-called totally symmetric case studied in [38], that is, when $z_2 = -\overline{z}_1$ and $\gamma = 1$ (equal heights and masses), may be now revisited in the light of the results in Theorem 3.4. In this case, the symmetry of the external field is inherited by the support, what means that $a_1 = -a_2 = -a$ and $b = 0$ when type II transition (fusion of the two cuts) occurs. Hence, (20) yields

$$(2 - T_2)^2\, z_1^4\, (z_1^2 - a^2) \,+\, 16\, (\operatorname{Im} z_1)^2\, (\operatorname{Re} z_1)^2\, z_1 \,=\, 0\,,$$

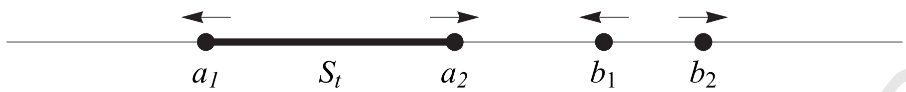

Figure 2: $(\beta_1, \beta_2) \in \Omega_0$ and $\gamma \in (\widetilde{\Gamma}_1, \widetilde{\Gamma}_2)$. Phase 1 ($t \in (0, T_1)$): one-cut.

Figure 3: $(\beta_1, \beta_2) \in \Omega_0$ and $\gamma \in (\widetilde{\Gamma}_1, \widetilde{\Gamma}_2)$. Phase 2 ($t \in (T_1, T_2)$): two-cut.

and thus, the following system of nonlinear equations arises, with T_2 and a as unknowns (of course, we are looking for solutions for which $T_2 < T = 2$),

$$\begin{cases} K\ \left((1-\beta^2)(1-\beta^2-a^2) - 4\beta^2\right) + 16\beta^2 = 0\,, \\ 2(1-\beta^2) - a^2 = 0\,, \end{cases} \tag{25}$$

where $K = (2 - T_2)^2 > 0$. From the second identity in (25), it is clear that necessarily $\beta < 1$. Under this condition, it is easy to check that the fusion of cuts takes place for

$$T_2 = 2\,\frac{(1-\beta)^2}{1+\beta^2}\ <\ 2\,.$$

Finally, for $t \in (T_2, 2)$, the one–cut phase takes place and Theorem 2.2 implies, for the density λ_t' of the equilibrium measure, that

$$\lim_{t\to 2} \lambda_t'(x) = \frac{1}{\pi}\,\frac{x^2+\beta^2+1}{D(x)^2}\,,$$

with $D(x) = ((x+1)^2 + \beta^2)\,((x-1)^2 + \beta^2)$.

Thus, in the totally symmetric case, when $\beta < 1$, we always have the phase diagram:

two-cut $(0 < t < T_2)$ $\longrightarrow$ **one-cut** $(T_2 \le t < T = 2)$

On the other hand, when $\beta \ge 1$, it is easy to check that φ only has a real critical point, at $x = 0$, where it attains its absolute minimum. Therefore, the support S_t starts being of the form

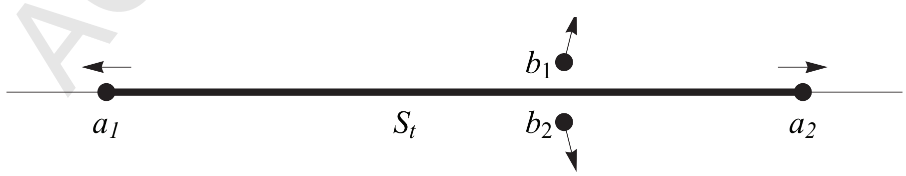

Figure 4: $(\beta_1, \beta_2) \in \Omega_0$ and $\gamma \in (\widetilde{\Gamma}_1, \widetilde{\Gamma}_2)$. Phase 3 ($t \in (T_2, T)$): one-cut.

$S_t = [-a(t), a(t)]$, with $a(t)$ an increasing function as above. No phase transition occurs, since it would imply by symmetry a three-cut situation, which is not possible.

The reader can check that the conclusions above agree with the results in [38].

4. Proofs

Throughout this section the proofs of Theorems 2.2 and 3.2–3.4 above will be displayed. As it was said, they all together render the proof of the main result Theorem 3.1 and enrich it with auxiliary results which are of interest themselves.

4.1. Proof of Theorem 2.2

Along this proof we will use the set $S^T = \{x \in \mathbb{R} : V^{\lambda_t}(x) + \varphi(x) = c_t\}$, where c_t is the equilibrium constant given by (2). Clearly, from (2), $S_T \subseteq S^T$.

As it was shown in [10, Theorem 2 (3)], we have that λ_t is increasing and continuous in the weak topology of the set of measures with compact support in $\mathbb{R}$. In addition, for any Borel set $I \subset \mathbb{R}$, it holds $\lambda_t(I) \leq \lambda_t(\mathbb{R}) = t \in (0, T)$ and, thus, there exists $\lim_{t\to T} \lambda_t = \lambda_T$ in the sense mentioned above.

Now, let us show that $\lim_{t\to T} S_t = \mathbb{R}$. For this, consider the function

$$\phi_t(x) = V^{\lambda_t}(x) + \varphi(x) - c_t\,,\ t \in (0, T)\,,\ x \in \mathbb{R}\,.$$

It is clear that $\phi_t(x) \geq 0\,, x \in \mathbb{R}$ and $t \in (0, T)$. Let $\tau \in (0, T)$ fixed. If $x \in S_\tau$, then $\phi_\tau(x) = 0$ and since the family of supports $\{S_t\}$ is increasing ([10, Theorem 2, (1)]), we have that $\phi_t(x) = 0, t \geq \tau$. On the other hand, if $x \in \mathbb{R} \setminus S_\tau$, formulas (1.8) and (1.13) in [10] yield,

$$\frac{\partial \phi_t(x)}{\partial t}|_{t=\tau^-} = -g_\tau(x) < 0\,,\quad \frac{\partial \phi_t(x)}{\partial t}|_{t=\tau^+} = -g^\tau(x) < 0\,,$$

where g_τ (g^τ) denotes the Green function of $\mathbb{R} \setminus S_\tau$ (respect., $\mathbb{R} \setminus S^\tau$) with a pole at infinity, and $S^\tau = \{x \in \mathbb{R} : \phi_\tau(x) = 0\} \supseteq S_\tau$. Hence, we have that $\phi_t(x) \geq 0$, for any $x \in \mathbb{R}$ and $t \in (0, +\infty)$, and that $\phi_t(x)$ is a decreasing function of t for any fixed $x \in \mathbb{R}$. This shows that there exists

$$\lim_{t\to T} \phi_t(x) = \phi_T(x)\,,\ x \in \mathbb{R}\,.$$

Now, let us see that $\phi_T \equiv 0$. To do it, recall that for $x \in S_t$, we have that $V^{\lambda_t}(x) + \varphi(x) = c_t$ and, thus,

$$V^{\lambda_t}(x) = -\frac{t}{T}\,\varphi(x) - \frac{T-t}{T}\,\varphi(x) + c_t = V^{\nu_t}(x) - \frac{T-t}{T}\,\varphi(x) + c_t\,, \tag{26}$$

where $\nu_t = \displaystyle\sum_{j=1}^{q} \frac{t\gamma_j}{T}\delta_{z_j}$, with $\nu_t(\mathbb{C}) = t$. Since φ has an absolute minimum on the real axis, $m = \min_{x\in\mathbb{R}} \varphi(x) > \displaystyle\sum_{j=1}^{q} \gamma_j \operatorname{Im} z_j > -\infty$, then (26) implies that

$$V^{\lambda_t}(x) \leq V^{\nu_t}(x) - \frac{T-t}{T}\,m + c_t\,,\ x \in S_t\,.$$

Thus, the Domination Principle [41, Theorem II.3.2] asserts that this inequality holds for any real x (and, in fact, for any complex x). Thus,

$$\phi_t(x) \leq \frac{T-t}{T}(\varphi(x)-m)\,,\; x\in\mathbb{R}\,,$$

which shows that

$$\lim_{t\to T}\phi_t(x)=0\,,\; x\in\mathbb{R}$$

and, then, that $S^T=\mathbb{R}$. The fact that for rational external fields the difference $S^t\setminus S_t$ consists of, at most, a finite set of points, for each $t\in(0,T)$ (see [33] or [39]), implies that $S_T=\mathbb{R}$.

Now, we are going to prove the rest of the results. First, we are dealing with the function $\widehat{\lambda}_T$, that is, the Cauchy transform of the limit function λ_T. Taking into account (7) and the previous analysis, we have that $\widehat{\lambda}_T$ must be analytic on $\mathbb{C}\setminus(\mathbb{R}\cup\bigcup_{j=1}^{q}\{z_j,\overline{z}_j\})$, in such a way that

- $\left(-\widehat{\lambda}_T+\Phi'\right)(x^+)=-\left(-\widehat{\lambda}_T+\Phi'\right)(x^-)$ for $x\in\mathbb{R}$,
- $\left(-\widehat{\lambda}_T+\Phi'\right)(x^+)\in\mathbb{R}^+ i$, taking into account the positivity of the measure,
- For $z\to z_j$,
$$\left(-\widehat{\lambda}_T+\Phi'\right)(z)=\frac{\gamma_j}{2(z-z_j)}+O(1)$$
- For $z\to \overline{z}_j$,
$$\left(-\widehat{\lambda}_T+\Phi'\right)(z)=\frac{\gamma_j}{2(z-\overline{z}_j)}+O(1)\,.$$

Therefore, having in mind the Liouville Theorem and some immediate consequences, we have that

$$\left(-\widehat{\lambda}_T+\Phi'\right)(z)=\begin{cases}\sum_{j=1}^{q}\left(\frac{\gamma_j}{2(z-z_j)}-\frac{\gamma_j}{2(z-\overline{z}_j)}\right), \operatorname{Im} z>0\,,\\ -\sum_{j=1}^{q}\left(\frac{\gamma_j}{2(z-z_j)}-\frac{\gamma_j}{2(z-\overline{z}_j)}\right), \operatorname{Im} z<0\,.\end{cases}$$

The fact that the limit function $\left(-\widehat{\lambda}_T+\Phi'\right)(z)$ has exactly $2q-2$ imaginary roots implies that, for t close to T, the support is necessarily comprised by a single interval. Then, for the expression of the density λ_T' it is enough to apply the Cauchy Theorem integrating along curves surrounding the components of the support, but not the points z_j and $\overline{z_j}$; finally, by shrinking the curves to fit the components of the support, the conclusion easily follows (alternatively, the application of the Sokhotski-Plemelj inversion formula may be considered).

4.2. Proof of Theorem 3.2

Polynomial P in (24) may be rewritten in the form

$$P(x)=P(x,\gamma)=(x+1)\left((x-1)^2+\beta_2^2\right)+\gamma\,(x-1)\left((x+1)^2+\beta_1^2\right)=u(x)+\gamma\,v(x) \tag{27}$$

Let us study the zeros of (27) when γ increases. For $\gamma=0$, P has a single real zero at $x=-1$ and a couple of conjugate imaginary zeros at z_2 and $\overline{z}_2$. On the other hand, since $\gamma>0$, then

$P(x) = \gamma \left(u(x) + \dfrac{v(x)}{\gamma} \right)$ and, thus, the zeros of $P(x)$ are the same as the zeros of $u(x) + \dfrac{v(x)}{\gamma}$, yielding that when γ tends to infinity, the real root of P approaches $x = 1$ and the couple of imaginary zeros tend to z_1 and $\bar{z}_1$. Let us start with $\gamma > 0$ small enough. Since

$$\frac{\partial P}{\partial \gamma} = v(x) < 0 \,, \quad \text{for } \; x < 1 \,, \tag{28}$$

the real zero, say ζ_1, move to the right as γ increases. On the other hand, writing (27) in powers of x yields that the arithmetic mean of the zeros of P equals $\dfrac{1}{3}\dfrac{1-\gamma}{1+\gamma}$ and, thus, this mean decreases as γ increases, which means that the real parts of the couple of imaginary roots $\xi, \bar{\xi}$, move to the left. It implies there are just two possible scenarios for the evolution of the critical points of φ as γ travels across $(0, +\infty)$.

- The pair of imaginary roots never reaches the real axis. In such a case, φ has a single minimum for any $\gamma > 0$.
- There exists a real number $\widetilde{\Gamma}_1 > 0$, such that for $\gamma = \widetilde{\Gamma}_1$ the pair of imaginary roots reach the real axis, giving birth to a double real root ξ located to the right of the simple real root ζ_1 (this is due to the fact that $P(x) < 0$ to the left of this simple real root with decreasing values of $P(x)$ as γ increases). Immediately after the collision, a pair of new simple real roots arises, say ζ_2, ζ_3, in such a way that $-1 < \zeta_1 < \zeta_2 < \xi < \zeta_3 < 1$ and with ζ_3 moving to the right and ζ_2 to the left (because of (28)). This situation holds until ζ_1 and ζ_2 collide, creating a double real root before immediately going to $\mathbb{C} \setminus \mathbb{R}$.

It is easy to see that the boundary between these possible evolutions is the case where the pair of imaginary roots $\xi, \bar{\xi}$ collide with the real one, ζ_1, giving birth to a triple real root for a certain value of γ: that is, when $(\beta_1, \beta_2) \in \mathcal{C}$, with $\mathcal{C}$ given by (23). It is also easy to check that the region where two minima are feasible is Ω_0: to check it, consider, for instance, the case with $\beta_2 = \beta_1$ taking small positive values.

4.3. Proof of Theorem 3.3

This is, in fact, the most important theorem in order to prove our main result (Theorem 3.1). Indeed, it is a direct consequence of Proposition 4.1 below, to whose proof it is devoted the most part of this subsection.

From the results in the previous [33] and [39] we know that when the external field has two local minima, then a two-cut phase may occur; but this is not the unique way to reach that phase. Indeed, when the external field has a single minimum, the necessary and sufficient condition for the existence of such a phase is the birth of a new local minimum of the total potential (2) in a previous "instant" t. Proposition 4.1 below analyzes the possible birth of this minimum of (2) as γ varies.

The setting for the result below is as follows. Let $(\beta_1, \beta_2) \in (\mathbb{R}^+)^2$ be fixed, and suppose that for a certain $\gamma = \gamma_0$ and $t = t_0$ the polynomial B in the AB–representation (7) has a multiple real root (i.e., double or triple), not belonging to the interior of the support (it means, as an immediate consequence, that the support consists of a single interval). Now, let $I = I(\beta_1, \beta_2)$ be the largest interval containing γ_0 such that B has a multiple real root (outside the interior of the support, too) for some $t = t(\gamma)$. Though it is not obvious, in principle, that the mass t must vary with γ in order to maintain the multiple root of B, this is indeed the case (it will be shown in (30) and (40) below). In this setting, let us denote, as usual, by a_1 and a_2 the endpoints of the support and by b the multiple root of B. Then, we have,

Proposition 4.1. *The interval I is compact and the functions $a_1 = a_1(\gamma)$, $a_2 = a_2(\gamma)$, $b = b(\gamma)$ (see (7)) and $t = t(\gamma)$ are analytic functions in the interior of I and continuous in I, with $a_1(t)$, $a_2(t)$ and $b(t)$ being monotonic. In particular, it holds,*

- *If $a_1 \le a_2 \le b$, then $\dfrac{\partial a_1}{\partial \gamma} > 0$, $\dfrac{\partial a_2}{\partial \gamma} < 0$, $\dfrac{\partial b}{\partial \gamma} > 0$ and there exists $\Gamma_1 > 0$ such that $I = [\Gamma_1, \tilde{\Gamma}_1]$, with $\tilde{\Gamma}_1$ given in Theorem 3.2, in such a way that a type III singularity takes place for $\gamma = \Gamma_1$ and a certain value of t.*
- *If $b \le a_1 \le a_2$, then $\dfrac{\partial a_1}{\partial \gamma} < 0$, $\dfrac{\partial a_2}{\partial \gamma} > 0$, $\dfrac{\partial b}{\partial \gamma} > 0$, and there exists $\Gamma_2 > 0$ such that $I = [\tilde{\Gamma}_2, \Gamma_2]$, with $\tilde{\Gamma}_2$ given in Theorem 3.2, in such a way that a type III singularity occurs for $\gamma = \Gamma_2$ and a certain value of t.*

The following result, which in turn yields Theorem 3.3, is a direct consequence of Proposition 4.1 and Theorem 3.2.

Corollary 4.1. *Let $(\beta_1, \beta_2) \in (\mathbb{R}^+)^2$ be fixed. Then, the following statements are equivalent:*

i) $(\beta_1, \beta_2) \in \Omega_0$

ii) *There exists $\gamma > 0$ such that Φ' has a double root.*

iii) *There exist $\gamma > 0$ and $t \ge 0$ for which polynomial B in (7) has a multiple real root.*

iv) *There exist $\gamma > 0$ such that a type III singularity takes place for some $t > 0$.*

Furthermore, if some of these statements holds, there exist exactly two values of γ satisfying it.

Proof. The proof is immediate. It is enough to take into account that Theorem 3.2 implies the equivalence between (i)-(ii), while implications (iii) $\Longrightarrow$ (iv) and (iii) $\Longrightarrow$ (ii) comes from Proposition 4.1. Finally, (iv) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iii) are straightforward.

$\square$

Remark 4.1. Therefore, the two-cut phase is feasible only when $(\beta_1, \beta_2) \in \Omega_0$, and the number of type III singularities is 2 at most. Indeed, a double root of B outside S_t can only appear for $(\beta_1, \beta_2) \in \Omega_0$ and $\gamma \in [\Gamma_1, \widetilde{\Gamma_1}] \cup [\widetilde{\Gamma_2}, \Gamma_2]$.

Now, let us proceed with the proof of Proposition 4.1. Since it deals with the case where polynomial B in (7) has a double (at least) root b, let us start pointing out that in this case (20) implies that the following system of equations holds

$$\begin{cases} (T-t)\,(z_1-b)^2\,\sqrt{(z_1-a_1)(z_1-a_2)} \;-\; i\,(z_1-z_2)(z_1-\overline{z_2})\,\operatorname{Im} z_1 & = 0\,, \\ (T-t)\,(z_2-b)^2\,\sqrt{(z_2-a_1)(z_2-a_2)} \;-\; i\,\gamma\,(z_2-z_1)(z_2-\overline{z_1})\,\operatorname{Im} z_2 & = 0\,. \end{cases} \tag{29}$$

Rewriting (29) in the form

$$\begin{cases} \frac{(z_1-b)^2\sqrt{(z_1-a_1)(z_1-a_2)}}{i\,(z_1-z_2)(z_1-\overline{z_2})\ \mathrm{Im}\, z_1} & = \frac{1}{T-t}\,, \\ \frac{(z_2-b)^2\sqrt{(z_2-a_1)(z_2-a_2)}}{i\,(z_2-z_1)(z_2-\overline{z_1})\ \mathrm{Im}\, z_2} & = \frac{\gamma}{T-t}\,, \end{cases} \tag{30}$$

the geometry of our problem becomes clear: once a_1, a_2 and b are obtained in such a way that both left-hand sides in (30) are real, the values of t and γ are uniquely determined. Therefore, t and γ must vary together in order to keep the double root b of the polynomial B.

Now, we first need the following technical results. On the sequel, Arg$z$ denotes the branch of the argument of the complex number z belonging to $(-\pi, \pi]$.

Lemma 4.1. *Let $c, d \in \mathbb{R}$. Then,*

$$\mathrm{Arg}(z_1 - c) + \mathrm{Arg}(z_1 - d) < \pi \ \textit{ if and only if } \ c + d < -2\,, \tag{31}$$

$$\mathrm{Arg}(z_2 - c) + \mathrm{Arg}(z_2 - d) < \pi \ \textit{ if and only if } \ c + d < 2\,, \tag{32}$$

Proof. We know that

$$\mathrm{Arg}(z_j - c) + \mathrm{Arg}(z_j - d) \in (0, 2\pi)\,.$$

Thus, making use of well-known trigonometric identities, we have,

$$\begin{aligned} &\sin\left(\mathrm{Arg}(z_j - c) + \mathrm{Arg}(z_j - d)\right) \\ =&\frac{\beta_j(2\,\mathrm{Re}\, z_j - c - d)}{\sqrt{(\mathrm{Re}\, z_j - c)^2 + \beta_j^2}\sqrt{(\mathrm{Re}\, z_j - d)^2 + \beta_j^2}}\,, \end{aligned}$$

which shows that

$$\mathrm{Arg}(z_j - c) + \mathrm{Arg}(z_j - d) < \pi \ \text{ iff } \ 2\,\mathrm{Re}\, z_j - c - d > 0 \ \text{ iff } \ c + d < 2\,\mathrm{Re}\, z_j\,,$$

and it settles the proof. □

Now, it is convenient to introduce the point

$$x_0 = \frac{-\beta_1^2 + \beta_2^2}{4}\,,$$

that is, the intersection between the mediatrix of the segment joining $[z_1, z_2]$ and the real axis, and the points $x_1 < x_2$, where the circumference with center at x_0 and passing through z_1 and z_2 meets the real axis (see Figure 2). It is also worth to point out that

$$\mathrm{Arg}(z_j - x_1) = \frac{1}{2}\,\mathrm{Arg}(z_j - x_0)\,, \ \ \mathrm{Arg}(z_j - x_2) = \frac{\pi}{2} + \frac{1}{2}\,\mathrm{Arg}(z_j - x_0)\,, \ j = 1, 2\,. \tag{33}$$

Lemma 4.2. *Suppose that for some fixed $(\beta_1, \beta_2, \gamma) \in (\mathbb{R}^+)^3$ the polynomial B has a double root b. Then, it holds*

$$\frac{1}{2}\mathrm{Arg}(z_1 - a_1) + \frac{1}{2}\mathrm{Arg}(z_1 - a_2) + 2\mathrm{Arg}(z_1 - b) - \mathrm{Arg}(z_1 - x_0) - \frac{3\pi}{2} = 0\,, \tag{34}$$

$$\frac{1}{2}\mathrm{Arg}(z_2 - a_1) + \frac{1}{2}\mathrm{Arg}(z_2 - a_2) + 2\mathrm{Arg}(z_2 - b) - \mathrm{Arg}(z_2 - x_0) - \frac{\pi}{2} = 0\,, \tag{35}$$

where a_1, a_2 denote the endpoints of the support S_t.

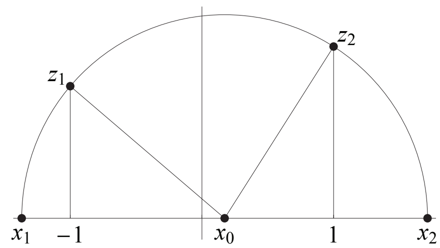

Figure 5: Location of points x_0, x_1, x_2.

Proof. From (29), the following system must hold:

$$\frac{1}{2}\mathrm{Arg}(z_1 - a_1) + \frac{1}{2}\mathrm{Arg}(z_1 - a_2) + 2\mathrm{Arg}(z_1 - b) - \mathrm{Arg}(z_1 - x_0) - \frac{3\pi}{2} = 2k_1\pi\,, \tag{36}$$

$$\frac{1}{2}\mathrm{Arg}(z_2 - a_1) + \frac{1}{2}\mathrm{Arg}(z_2 - a_2) + 2\mathrm{Arg}(z_2 - b) - \mathrm{Arg}(z_2 - x_0) - \frac{\pi}{2} = 2k_2\pi\,, \tag{37}$$

with $k_j \in \mathbb{Z}$. Now, it will be shown that $k_1 = k_2 = 0$. First, let us see that these k_j just can take some particular values.

We initially deal with (36). First, since $\mathrm{Arg}(z_1 - a_1) \in (0, \pi)$, $\mathrm{Arg}(z_1 - a_2) \in (\pi/2, \pi)$ (because $a_2(0) > -1$ and $\partial a_2/\partial t > 0$), $\mathrm{Arg}(z_1 - b) \in (0, \pi)$ and $\mathrm{Arg}(z_1 - x_0) \in (0, \pi)$, it yields

$$\begin{aligned}&\frac{1}{2}\mathrm{Arg}(z_1 - a_1) + \frac{1}{2}\mathrm{Arg}(z_1 - a_2) + 2\mathrm{Arg}(z_1 - b) - \mathrm{Arg}(z_1 - x_0) - \frac{3\pi}{2}\\ &\in \left(\frac{-9\pi}{4}, \frac{3\pi}{2}\right)\,,\end{aligned}$$

and we conclude that $k_1 = -1$ or $k_1 = 0$. In a similar way, it is easy to check that $k_2 \in \{0, 1\}$ in (37).

Now, let us show that $k_1 = -1$ cannot occur. Let us see, first, that if $k_1 = -1$, then we would necessarily have that $a_1 + a_2 < -2$, $b < x_1$. Indeed,

- We have $a_1 + a_2 < -2$, since otherwise, (31) would yield

  $$\frac{1}{2}\mathrm{Arg}(z_1 - a_1) + \frac{1}{2}\mathrm{Arg}(z_1 - a_2) \geq \frac{\pi}{2}\,,$$

  and thus,

  $$\begin{aligned}&\frac{1}{2}\mathrm{Arg}(z_1 - a_1) + \frac{1}{2}\mathrm{Arg}(z_1 - a_2) + 2\mathrm{Arg}(z_1 - b) - \mathrm{Arg}(z_1 - x_0) - \frac{3\pi}{2}\\ \geq&\frac{\pi}{2} + 2\mathrm{Arg}(z_1 - b) - \mathrm{Arg}(z_1 - x_0) - \frac{3\pi}{2} = 2\mathrm{Arg}(z_1 - b) - \mathrm{Arg}(z_1 - x_0) - \pi\\ >&0 - \pi - \pi = -2\pi\,,\end{aligned}$$

  which would imply that $k_1 \neq -1$.

- $b < x_1$, since

$$\begin{aligned}
&-2\pi = \frac{1}{2}\mathrm{Arg}(z_1-a_1)+\frac{1}{2}\mathrm{Arg}(z_1-a_2)+2\mathrm{Arg}(z_1-b)-\mathrm{Arg}(z_1-x_0)-\frac{3\pi}{2}\\
\Longrightarrow & 2\mathrm{Arg}(z_1-b) = -\frac{1}{2}\mathrm{Arg}(z_1-a_1)-\frac{1}{2}\mathrm{Arg}(z_1-a_2)+\mathrm{Arg}(z_1-x_0)-\frac{\pi}{2}\\
\Longrightarrow & \mathrm{Arg}(z_1-b) < \frac{1}{2}\mathrm{Arg}(z_1-x_0) = \mathrm{Arg}(z_1-x_1) \Longrightarrow b < x_1 .
\end{aligned}$$

Thus, taking into account (37), one has,

$$\begin{aligned}
2k_2\pi &= \frac{1}{2}\mathrm{Arg}(z_2-a_1)+\frac{1}{2}\mathrm{Arg}(z_2-a_2)+2\mathrm{Arg}(z_2-b)-\mathrm{Arg}(z_2-x_0)-\frac{\pi}{2}\\
&< \frac{1}{2}\mathrm{Arg}(z_2-a_1)+\frac{1}{2}\mathrm{Arg}(z_2-a_2)+2\mathrm{Arg}(z_2-x_1)-\mathrm{Arg}(z_2-x_0)-\frac{\pi}{2}\\
&= \frac{1}{2}\mathrm{Arg}(z_2-a_1)+\frac{1}{2}\mathrm{Arg}(z_2-a_2)-\frac{\pi}{2} < 0 ,
\end{aligned}$$

where for the last inequality we have used (32). But this inequality would imply that $k_2 < 0$, while it is known that $k_2 = 0$ or $k_2 = 1$. Hence, we conclude that $k_1 = 0$ and (34) is established.

In a similar fashion, (35) is established.

□

The following result, concerning the location of the point b and the arithmetic mean of the endpoints of S_t, is also useful.

Lemma 4.3.

i) The point $\dfrac{a_1+a_2}{2} \in (-1,1)$ *or, equivalently,*

$$\begin{aligned}
\mathrm{Arg}(z_1-a_1)+\mathrm{Arg}(z_1-a_2) > \pi ,\\
\mathrm{Arg}(z_2-a_1)+\mathrm{Arg}(z_2-a_2) < \pi .
\end{aligned}$$

ii) $b \in (-1,1)$.

Proof. Let us show, first, that $b \in (x_1, x_2)$. Indeed, (34) yields

$$\begin{aligned}
2\mathrm{Arg}(z_1-b) &= -\frac{1}{2}\mathrm{Arg}(z_1-a_1)-\frac{1}{2}\mathrm{Arg}(z_1-a_2)+\mathrm{Arg}(z_1-x_0)+\frac{3\pi}{2}\\
&> -\frac{\pi}{2}-\frac{\pi}{2}+\mathrm{Arg}(z_1-x_0)+\frac{3\pi}{2} = \mathrm{Arg}(z_1-x_0)+\frac{\pi}{2} > \mathrm{Arg}(z_1-x_0) ,
\end{aligned}$$

which implies that

$$\mathrm{Arg}(z_1-b) > \frac{1}{2}\mathrm{Arg}(z_1-x_0) = \mathrm{Arg}(z_1-x_1)$$

and, hence, it holds $b > x_1$. Analogously, from (35) it is easy to get that $b < x_2$.

Although this first bound for b is rough, it allows to get the bounds for the arithmetic mean. In fact, from (34) and using (33) and the bounds obtained for b, one has,

$$\begin{aligned}\frac{1}{2}\mathrm{Arg}(z_1-a_1)+\frac{1}{2}\mathrm{Arg}(z_1-a_2) &= -2\mathrm{Arg}(z_1-b)+\mathrm{Arg}(z_1-x_0)+\frac{3\pi}{2}\\ &> -2\mathrm{Arg}(z_1-x_2)+\mathrm{Arg}(z_1-x_0)+\frac{3\pi}{2}\\ &= -2\left(\frac{\pi}{2}+\frac{1}{2}\mathrm{Arg}(z_1-x_0)\right)+\mathrm{Arg}(z_1-x_0)+\frac{3\pi}{2}=\frac{\pi}{2}\,,\end{aligned}$$

which, by (31), yields $(a_1+a_2)/2 > -1$. In the same way, from (35) and (32), the inequality $(a_1+a_2)/2 < 1$ is easily obtained.

Finally, using these bounds for the mass center of the endpoints, it is possible to precise the location of b. Let us start showing that $b < 1$. Indeed, if $b \geq 1$, we have, using (35),

$$\begin{aligned}0 &\geq \frac{1}{2}\mathrm{Arg}(z_2-a_1)+\frac{1}{2}\mathrm{Arg}(z_2-a_2)+2\frac{\pi}{2}-\mathrm{Arg}(z_2-x_0)-\frac{\pi}{2}\\ &= \frac{1}{2}\mathrm{Arg}(z_2-a_1)+\frac{1}{2}\mathrm{Arg}(z_2-a_2)-\mathrm{Arg}(z_2-x_0)+\frac{\pi}{2}\end{aligned}$$

and, hence,

$$\frac{1}{2}\mathrm{Arg}(z_2-a_1)+\frac{1}{2}\mathrm{Arg}(z_2-a_2)\leq \mathrm{Arg}(z_2-x_0)-\frac{\pi}{2}\,. \tag{38}$$

Inequality (38) implies some consequences. First, it is easy to check that it would be possible as long as $\mathrm{Arg}(z_2-x_0) > \pi/2$, which means that $x_0 > 1$.

Moreover, let $a \in \mathbb{R}$ such that

$$\frac{1}{2}\mathrm{Arg}(z_2-a_1)+\frac{1}{2}\mathrm{Arg}(z_2-a_2)=\mathrm{Arg}(z_2-a)\,,$$

that is, $a \in (a_1, a_2)$ is the point where the bisector of the angle $\widehat{a_1, z_2, a_2}$ meets the real axis. On the other hand, let $\tilde{a} \in \mathbb{R}$ such that

$$\mathrm{Arg}(z_2-x_0)-\frac{\pi}{2}=\mathrm{Arg}(z_2-\tilde{a})\,.$$

This last point may be seen as the point where the tangent line to the circumference with center x_0 and passing through z_2 intersects the real axis. Since $x_0 > 1$, then $\tilde{a} < x_1$ (see Figure 6). Thus, inequality (38) yields $a \leq \tilde{a} < x_1 < 1$, but this is a contradiction with the fact that $a_1 + a_2 > 1$. Indeed, it is enough to make use of the following property from elementary geometry:

"Let ABC be a triangle and consider the bisector of the angle a, which splits the segment BC into two parts, BD and DC. Then, the length of BD is less than the length of DC if and only if the angle B is greater than the angle C".

This simple property applied to the triangle $a_1a_2z_2$, and taking the bisector joining z_2 with a, means that $a_2 - a < a - a_1$ but, then,

$$\frac{a_1+a_2}{2}=a-\frac{(a-a_1)-(a_2-a)}{2}<a\leq\tilde{a}<x_1<-1,$$

which contradicts the result in Lemma 4.1 above.

Proceeding in an analogous way, we can prove the lower bound, i.e. $b > -1$, using now (34). $\square$

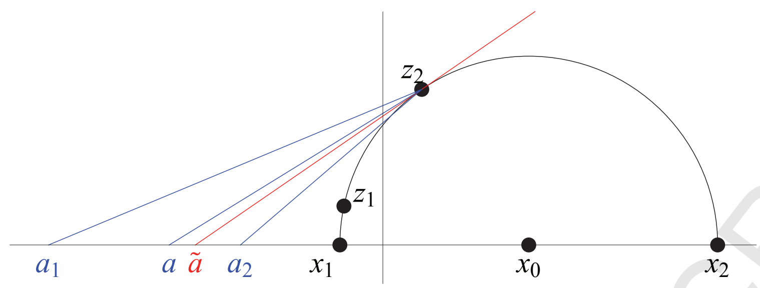

Figure 6: location of points a and $\widetilde{a}$ if $b \geq 1$.

Proof of Proposition 4.1

Let us take (β_1, β_2) fixed and γ and t varying in such a way that B has a double root: $B(z) = (z-b)^2$, with $b \notin (a_1, a_2)$. It implicitly means a dependency relationship between t and γ, which will indeed be shown below.

Let us consider first t and γ as being two independent variables, then, from (17), we have

$$\frac{\partial}{\partial \gamma} \frac{(T-t)\, B(z)\, \sqrt{A(z)}}{D(z)} = \frac{H(z)}{\sqrt{A(z)}\, D_2(z)}$$

with $D_2(z) = (z - z_2)(z - \overline{z_2})$ and H being a monic polynomial of degree 2 and having at least a root in (a_1, a_2). Furthermore, taking into account that z_2 and $\overline{z_2}$ do not depend on γ, the residue at both points in the left hand side of the previous expression is equal to $1/2$ and thus, the same must happen for the right hand side, which yields

$$H(z_2) = \frac{\sqrt{A(z_2)}(z_2 - \overline{z_2})}{2}, \qquad H(\overline{z_2}) = \frac{\sqrt{A(\overline{z_2})}(\overline{z_2} - z_2)}{2}$$

and, hence, H can be written as $H(z) = D_2(z) + P(z)$, where

$$P(z) = \frac{\sqrt{A(z_2)}}{2}(z - \overline{z_2}) + \frac{\sqrt{A(\overline{z_2})}}{2}(z - z_2)\,.$$

On the other hand, from (11) we also have

$$\frac{\partial}{\partial t} \frac{(T-t)\, B(z)\, \sqrt{A(z)}}{D(z)} = -\frac{1}{\sqrt{A(z)}}\,.$$

Therefore, considering now $t = t(\gamma)$ in such a way that the double root of B takes place, and applying the chain rule, one has, on the one hand,

$$\frac{\partial}{\partial \gamma} \frac{(T-t)\sqrt{A(z)}(z-b)^2}{D(z)} = \frac{H(z)}{\sqrt{A(z)} D_2(z)} + \frac{-1}{\sqrt{A(z)}} \frac{\partial t}{\partial \gamma} = \frac{H(z) - \frac{\partial t}{\partial \gamma} D_2(z)}{\sqrt{A(z)} D_2(z)} =: \frac{L(z)}{\sqrt{A(z)}\, D_2(z)}$$

and, on the other,

$$\frac{\partial}{\partial\gamma}\frac{(T-t)\sqrt{A(z)}(z-b)^2}{D(z)} =$$

$$\frac{1}{D(z)}\left(-\frac{\partial t}{\partial\gamma}\sqrt{A(z)}\,(z-b)^2 + (T-t)\,\frac{\frac{\partial A(z)}{\partial\gamma}}{2\sqrt{A(z)}}\,(z-b)^2 + (T-t)\,\sqrt{A(z)}\,2(z-b)(-\frac{\partial b}{\partial\gamma})\right).$$

Then, it yields,

$$-2\,\frac{\partial t}{\partial\gamma}\,A(z)(z-b)^2 + (T-t)\,\frac{\partial A(z)}{\partial\gamma}\,(z-b)^2 - 4(T-t)A(z)(z-b)\,\frac{\partial b}{\partial\gamma} = 2L(z)D_1(z) \tag{39}$$

where $D_1(z) = (z-z_1)(z-\overline{z_1}) = (D/D_2)(z)$. The left and right hand sides of (39) are polynomials of degree 4, with the left-hand one vanishing for $z = b$. This implies that $L(b) = 0$ and thus,

$$\frac{\partial t}{\partial\gamma} = \frac{H(b)}{D_2(b)} = \frac{D_2(b) + P(b)}{D_2(b)}. \tag{40}$$

In particular, (40) indeed shows the real dependence of t on γ in the current setting, in such a way that the derivative in (40) is well–defined and bounded. Therefore, computing the derivatives in (39) and evaluating them at a_1, a_2 and b, the following differential equations hold

$$\begin{aligned}
\frac{\partial a_1}{\partial\gamma} &= \frac{-2D_1(a_1)L(a_1)}{(T-t)(a_1-a_2)(a_1-b)^2},\\
\frac{\partial a_2}{\partial\gamma} &= \frac{-2D_1(a_2)L(a_2)}{(T-t)(a_2-a_1)(a_2-b)^2},\\
\frac{\partial b}{\partial\gamma} &= \frac{-D_1(b)L'(b)}{2(T-t)(b-a_1)(b-a_2)}.
\end{aligned}$$

Now, since polynomial L is very important for our analysis, we are concerned with its expression. Indeed, we have for $x \in \mathbb{R}$,

$$\begin{aligned}
L(x) = H(x) - \frac{\partial t}{\partial\gamma}\,D_2(x) &= P(x) - \frac{P(b)}{D_2(b)}D_2(x)\\
&= \mathrm{Re}\left(\sqrt{A(z_2)}(x-\overline{z_2})\left(1 - \frac{x-z_2}{b-z_2}\right)\right)\\
&= (b-x)\,\mathrm{Re}\left(\sqrt{A(z_2)}\frac{x-\overline{z_2}}{b-z_2}\right).
\end{aligned}$$

Thus, if ℓ denotes the other real root of L, then

$$L(x) = \mathrm{Re}\left(\frac{\sqrt{A(z_2)}}{z_2-b}\right)(x-b)(x-\ell),$$

and ℓ is such that

$$\mathrm{Re}\left(\sqrt{A(z_2)}\frac{\ell-\overline{z_2}}{b-z_2}\right) = 0.$$

Consequently, taking into account Lemma 4.3, the following relation for the arguments holds:

$$\frac{1}{2}\,\mathrm{Arg}(z_2 - a_1) \,+\, \frac{1}{2}\,\mathrm{Arg}(z_2 - a_2) \,-\, \mathrm{Arg}(z_2 - \ell) \,-\, \mathrm{Arg}(z_2 - b) \,=\, -\frac{\pi}{2}\,. \tag{41}$$

Now, we are in a position to study what happens in each of the scenarios in the statement of Proposition 4.1.

- If $a_1 \le a_2 \le b$ holds, then recalling $b < 1$ (see Lemma 4.3), we obtain

$$\mathrm{Arg}\left(\frac{\sqrt{A(z_2)}}{z_2 - b}\right) = \frac{1}{2}\mathrm{Arg}(z_2 - a_1) + \frac{1}{2}\mathrm{Arg}(z_2 - a_2) - \mathrm{Arg}(z_2 - b) \in \Big(-\frac{\pi}{2}, 0\Big)\,,$$

and thus, the leading coefficient of L is given by

$$\mathrm{Re}\,\frac{\sqrt{A(z_2)}}{z_2 - b} > 0\,. \tag{42}$$

Now, let us show that $\ell > b$. Indeed, if this were not the case, by (41) we would have,

$$\frac{1}{2}\mathrm{Arg}(z_2 - a_1) + \frac{1}{2}\mathrm{Arg}(z_2 - a_2) + \frac{\pi}{2} \le 2\mathrm{Arg}(z_2 - b)$$

but, using (35),

$$2\mathrm{Arg}(z_2 - b) \,=\, -\frac{1}{2}\mathrm{Arg}(z_2 - a_1) \,-\, \frac{1}{2}\mathrm{Arg}(z_2 - a_2) \,+\, \mathrm{Arg}(z_2 - x_0) \,+\, \frac{\pi}{2}$$

and, hence,

$$\begin{aligned}
&\frac{1}{2}\mathrm{Arg}(z_2 - a_1) + \frac{1}{2}\mathrm{Arg}(z_2 - a_2) + \frac{\pi}{2} \le \frac{-1}{2}\mathrm{Arg}(z_2 - a_1) + \frac{-1}{2}\mathrm{Arg}(z_2 - a_2) + \mathrm{Arg}(z_2 - x_0) + \frac{\pi}{2} \\
\Longrightarrow &\frac{1}{2}\mathrm{Arg}(z_2 - a_1) + \frac{1}{2}\mathrm{Arg}(z_2 - a_2) \le \frac{1}{2}\mathrm{Arg}(z_2 - x_0) = \mathrm{Arg}(z_2 - x_1)\,,
\end{aligned}$$

where the last inequality is due to (33). Thus, taking a such that $\dfrac{\mathrm{Arg}(z_2 - a_1) + \mathrm{Arg}(z_2 - a_2)}{2} =$ $\mathrm{Arg}(z_2 - a)$, we would get $a_2 - a < a - a_1$ and, hence,

$$\frac{a_1 + a_2}{2} = a + \frac{(a_2 - a) - (a - a_1)}{2} < a < x_1 < -1\,,$$

which is not possible by Lemma 4.3. Therefore, the inequality $\ell > b$ has been established, that is, we have that $a_1 < a_2 < \ell$ and taking into account the positivity of the leading coefficient of L (by (42)), we have

$$\begin{aligned}
\frac{\partial a_1}{\partial \gamma} &= \frac{-2D_1(a_1)L(a_1)}{(T-t)(a_1 - a_2)(a_1 - b)^2} > 0\,, \\
\frac{\partial a_2}{\partial \gamma} &= \frac{-2D_1(a_2)L(a_2)}{(T-t)(a_2 - a_1)(a_2 - b)^2} < 0\,, \\
\frac{\partial b}{\partial \gamma} &= \frac{-D_1(b)L'(b)}{2(T-t)(b - a_1)(b - a_2)} > 0\,,
\end{aligned} \tag{43}$$

Moreover, having in mind (35), the fact that $\frac{\partial b}{\partial \gamma} > 0$ necessarily implies that $\frac{\partial a}{\partial \gamma} < 0$, what means, by (41), that $\mathrm{Arg}(z_2 - \ell)$ is a decreasing function of γ and, hence, $\frac{\partial \ell}{\partial \gamma} < 0$.

Thus, considering the setting where (β_1, β_2) are fixed and γ is such that there exists $t > 0$ with $a_1 < a_2 < b$, if γ is allowed to increase as far as possible, we see that the endpoints a_1 and a_2 tend to collide, as well as points b and ℓ on the interval $(a_2, +\infty)$; but we know this last collision cannot take place since the inequality $b < \ell$ is strict. Therefore, the unique feasible final setting consists in the collision $a_1 = a_2$, which obviously means that the support disappears or, equivalently, that $t = 0$: we reach the situation where φ has two minima (for $\gamma = \widetilde{\Gamma}_1$ in Theorem 3.2).

- Finally, the reciprocal case $b \leq a_1 \leq a_2$ may be easily reduced to the previous one by means of the transformation $x \to -x$, with $\gamma \to \frac{1}{\gamma}$, which yields

$$a_1 \to -a_2\,,\ a_2 \to -a_1\,,\ b \to -b\,,\ t \to \frac{t}{\gamma}\,.$$

4.4. Proof of Theorem 3.4

The full description of the dynamics of the equilibrium measure runs parallelling to the proof of [33, Theorems 15–16] and [39, Theorem 2.1] (which are strongly based in the results for the dynamics as t increases). Therefore, we restrict here to outline the proof, omitting certain details.

When $(\beta_1, \beta_2) \in \Omega_0$ and $\gamma \in (\widetilde{\Gamma}_1, \widetilde{\Gamma}_2)$, Theorem 3.2 shows that (19) has two relative minima $-1 < \zeta_1 < \zeta_3 < 1$. First, assume that $\varphi(\zeta_1) < \varphi(\zeta_3)$ and, thus, that the leftmost relative minimum is the absolute one; in addition, φ has a relative maximum ζ_2 such that $-1 < \zeta_1 < \zeta_2 < \zeta_3 < 1$.

Therefore, by (7) we have for the endpoints of the support (zeros of A) and the zeros of B that $-1 < a_1(0) = a_2(0) = \zeta_1 < b_1(0) = \zeta_2 < b_2(0) = \zeta_3 < 1$, in such a way that (21) yields: $\dot{a_1} < 0, \dot{a_2} > 0, \dot{b_1} < 0$ and $\dot{b_2} > 0$ (see Fig. 2 above). Thus, points a_2 and b_1 tend to collide, and this collision would take place in a time $T^* < T$ by Theorem 2.2; but this would contradict equilibrium condition (2). Hence, there must exist a critical value $T_1 < T^*$ where the initial configuration changes. The unique change which takes care of condition (2) is the birth of a couple of real zeros a_3, a_4 from the rightmost double zero b_2. That is, at this critical value T_1 a type I singularity occurs and immediately a new cut arises.

Then, for $t > T_1$, combining again Theorem 2.1 and (2), we have that the central endpoints a_2 and a_3 tend to collide (and, of course, also collide with b_1, see Fig. 3), producing a double root of B, that is, a type II singularity. Finally, this double root necessarily splits into a couple of conjugate imaginary roots which tend to some prescribed points in $\mathbb{C} \setminus \mathbb{R}$, while $a_1 \to -\infty$ and $a_4 \to +\infty$, as established in Theorem 2.2 (see Fig. 4 above).

If $\varphi(\zeta_1) = \varphi(\zeta_3)$, which is possible for any $(\beta_1, \beta_2) \in \Omega_0$ and a suitable value of γ, then the evolution is the same as above, but now $T_1 = 0$ and, thus, the initial one–cut phase does not take place. Obviously, when $\varphi(\zeta_1) > \varphi(\zeta_3)$, the evolution is also the same as above but starting at ζ_3 in place of ζ_1.

The description of the dynamics in scenario (b) is similar, with the unique difference that we start with a couple of conjugate imaginary roots b_1 and $b_2 = \overline{b}_1$ which at a certain time $T_0 < T^*$ collide at the real axis, becoming a double real root of B, in such a way that this double root immediately produces a pair of simple real roots $b_1 < b_2$, as at the beginning of the dynamics described above. The fact that the double root immediately splits into two roots follows from (40), since the existence of such a double root for an interval of t of positive length would imply that $\partial\gamma/\partial t = 0$ or, equivalently, an unbounded $\partial t/\partial\gamma$, which is not possible.

Conversely, in scenario (c), the initial couple of conjugate imaginary roots b_1 and $b_2 = \bar{b}_1$ never attain the real axis and, consequently, the support always consists of a single interval.

Observe that the boundary between scenarios (b) and (c) occurs precisely when the pair of roots of B collide with the root of A, producing a type III singularity, which has been studied in previous Theorem 3.3

There is still an open question: can the roots of B fall back towards the real axis after a type II singularity? Or, in other terms, can the two–cut phase repeat several times through the dynamics with respect to t? In virtue of Proposition 4.1 (see Remark 4.1), this could happen only for $\gamma \in (\Gamma_1, \widetilde{\Gamma_1})$ or $\gamma \in (\widetilde{\Gamma_2}, \Gamma_2)$, but let us see that even under this assumption, a second two–cut phase can not happen. Suppose, on the contrary, that it does happen, and consider for instance that $\gamma \in (\Gamma_1, \widetilde{\Gamma_1})$, so there are two values of t for which B has a double root outside the support: the previously described T_0 and a new T_3; additionally, there is another value T_2 where B has a double root inside the support (type II singularity for the merger of two cuts). Let us denote $a_1^j = a_1(T_j)$, $a_2^j = a_2(T_j)$ and $b^j = b(T_j)$ for $j = 1, 2, 3$. In particular, it holds

$$a_1^3 < a_1^2 < b^2 < a_2^2 < a_2^3 < b^3 . \tag{44}$$

Now, using Proposition 4.1, we can vary γ (and therefore the values T_j) in such a way that the same setting remains valid, which provides $T_1(\gamma) < T_2(\gamma) < T_3(\gamma)$, with its corresponding $a_1^j(\gamma)$, $a_2^j(\gamma)$ and $b^j(\gamma)$, and property (44) holds for any γ. But increasing γ up to $\widetilde{\Gamma_1}$ and applying Proposition 4.1, we see that every a_k^j collide at the global minimum of the external field and, by (44), b^2 does the same too. But, on the other hand, b^3 is increasing in γ (see the dynamical system (43)); hence, we have

$$\lim_{\gamma\nearrow\widetilde{\Gamma_1}} b^2 < \lim_{\gamma\nearrow\widetilde{\Gamma_1}} b^3 .$$

This last property, along with the continuity of all the functions involved, imply that for $\gamma = \widetilde{\Gamma_1}$, the derivative of the external field, Φ', has two double roots, which is an absurd. The contradiction comes from the assumption that, after the merger of the two cuts, the pair of complex roots of polynomial B can fall to the real axis again.

Appendix: The “two–cut” body

Throughout this appendix, along with the three–dimension body

$$\widetilde{\Delta} = \{(\beta_1, \beta_2, \gamma) \in (\mathbb{R}^+)^3 \,:\, (\beta_1, \beta_2) \in \Omega_0 \,,\, \gamma \in (\widetilde{\Gamma}_1(\beta_1, \beta_2), \widetilde{\Gamma}_2(\beta_1, \beta_2))\} , \tag{45}$$

where the external field φ has two local minima, we consider the larger body, strictly containing the former one, given by

$$\Delta = \{(\beta_1, \beta_2, \gamma) \in (\mathbb{R}^+)^3 \,:\, (\beta_1, \beta_2) \in \Omega_0 \,,\, \gamma \in (\Gamma_1(\beta_1, \beta_2), \Gamma_2(\beta_1, \beta_2))\} , \tag{46}$$

for which range of parameters a two–cut phase takes place.

In Theorems 3.2-3.3 above, the geometry of the problem has been mainly depicted in terms of the admissible values of the heights (β_1, β_2) in order to guarantee the existence of a two-cut phase for a

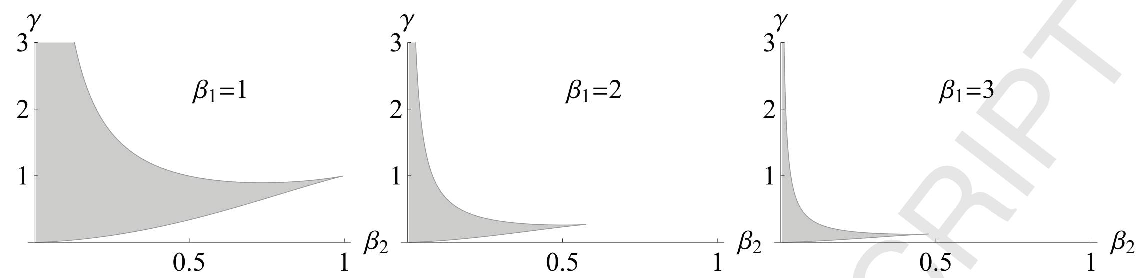

Figure 7: Vertical sections of Δ for $\beta_1 = 1, 2, 3$.

certain range of the other parameter, the charge γ. However, next a description of the three-dimension "two–cut" body Δ will be provided to get a better knowledge of the solution of the problem. As shown in Theorem 3.3, it is a body in the first (positive) octant of the $(\beta_1, \beta_2, \gamma)$–space whose projection in the (β_1, β_2)–plane is given by the Ω_0 region in Figure 2. Theorems 3.2 also shows that the projection of both Δ and $\widetilde{\Delta}$ on the (β_1, β_2)–plane is given by the same region Ω_0. On the largest body, the existence of a two-cut phase is provided; the unique difference between the phase diagram corresponding to $\widetilde{\Delta}$ and $\Delta \setminus \widetilde{\Delta}$ lies on the fact that for the range of parameters belonging to $\widetilde{\Delta}$, the "life" of the two-cut phase is longer.

In this section we restrict ourselves to show the main characteristics of these admissible bodies, especially of Δ, defined in (46). This is a three-dimension body bounded by two surfaces ("top and lower covers").

In this sense, some results will be presented (without proof) and some graphics will be displayed. First, the following properties holds for the three–dimension body Δ

- The intersection of both surfaces for $(\beta_1, \beta_2) \in \mathcal{C}$ is given by the curve

$$\gamma = -\frac{3\beta_1^2 + 3\beta_2^2 - 4}{2(3\beta_1^2 - 4)} + \frac{1}{2}\sqrt{\left(\frac{3\beta_1^2 + 3\beta_2^2 - 4}{(3\beta_1^2 - 4)}\right)^2 - 4\frac{3\beta_2^2 - 4}{3\beta_1^2 - 4}}$$

- The intersection with the plane $\beta_2 = 0$ is the whole quadrant $(\mathbb{R}^+)^2$. The same occurs with respect to the plane $\beta_1 = 0$.

- The following limits hold:

$$\lim_{\beta_1 \searrow 0} \Gamma_1 = 0\,, \qquad \lim_{\beta_1 \searrow 0} \Gamma_2 = +\infty\,,$$
$$\lim_{\beta_2 \searrow 0} \Gamma_1 = 0\,, \qquad \lim_{\beta_2 \searrow 0} \Gamma_2 = +\infty\,.$$

In Figures 7-9 different sections of Δ are shown.

Finally, with respect to the body $\widetilde{\Delta}$, in (45), the following are its main features:

$$\lim_{\beta_2 \searrow 0} \widetilde{\Gamma}_1 = 0\,, \qquad \lim_{\beta_2 \searrow 0} \widetilde{\Gamma}_2 = \frac{-1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + \frac{1}{\beta_1^2}}\,.$$

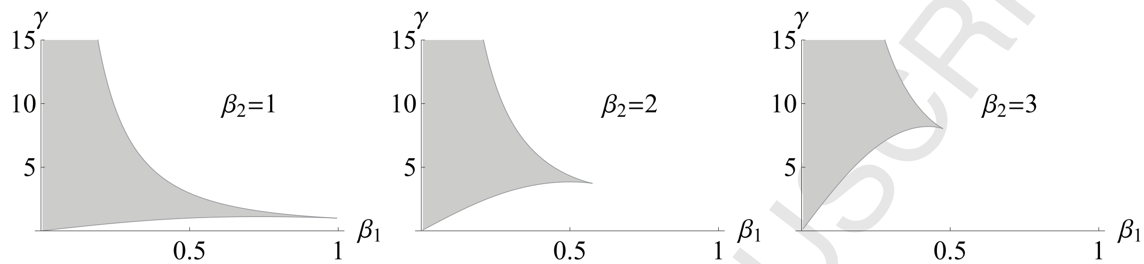

Figure 8: Vertical sections of Δ for $\beta_2 = 1, 2, 3$.

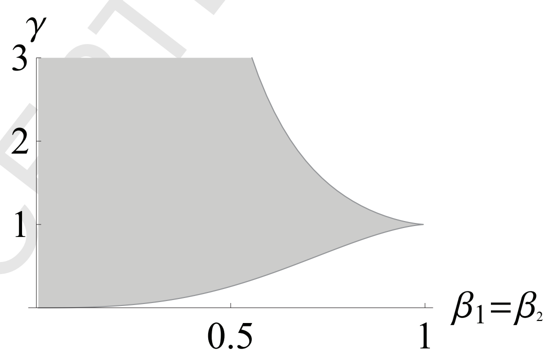

Figure 9: Section of Δ for $\beta_1 = \beta_2$.

$$\lim_{\beta_1 \searrow 0} \widetilde{\Gamma}_1 = \frac{\beta_2^2}{2} + \frac{\beta_2 \sqrt{4+\beta_2^2}}{2}, \qquad \lim_{\beta_1 \searrow 0} \widetilde{\Gamma}_2 = +\infty .$$

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