



# A NOTE ON HAMILTON SEQUENCES OF EXTREMAL BELTRAMI DIFFERENTIALS

ZHONG LI AND YEZHOU LI

ABSTRACT. Let  $X$  be a hyperbolic Riemann surface and let  $\mu$  be an extremal Beltrami differential on  $X$  with  $\|\mu\|_\infty \in (0, 1)$ . It is proved that, if  $\{\phi_n\}$  is a Hamilton sequence of  $\mu$ , then  $\{\phi_n\}$  must be a Hamilton sequence of any extremal Beltrami differential  $\nu$  contained in  $[\mu]$ . This result proved a conjecture of the first author of this paper in 1996. This result is also a generalization of two known results.

## §1. INTRODUCTION

Let  $X$  be a given hyperbolic Riemann surface and let  $Belt(X)$  be the Banach space of bounded measurable Beltrami differentials  $\mu = \mu(z)\overline{dz}/dz$  on  $X$  with  $L_\infty$ -norm.

By  $\mathcal{M}(X)$  we denote the open unit ball of  $Belt(X)$ , namely

$$\mathcal{M}(X) := \{\mu \in Belt(X) : \|\mu\| < 1\}.$$

For any given element  $\mu$  of  $\mathcal{M}(X)$ , there is a quasiconformal mapping  $f^\mu$  of  $X$  onto  $X$  with the Beltrami coefficient  $\mu$ , which is normalized by

$$f^\mu(\pm 1) = \pm 1 \quad \text{and} \quad f^\mu(i) = i.$$

Such a quasiconformal mapping  $f^\mu : X \rightarrow X$  is uniquely determined by  $\mu$ .

Two elements  $\mu$  and  $\nu$  in  $\mathcal{M}(X)$  are said to be Teichmüller equivalent to each other, denoted by  $\mu \sim \nu$ , if there is a conformal mapping  $\Phi$  of  $X$  onto  $X$  such that

$$(f^\nu)^{-1} \circ \Phi \circ f^\mu \text{ is homotopic to the identity map of } X \text{ (Mod } \partial X).$$

The Teichmüller space  $T(X)$  of  $X$  is defined as the quotient space  $\mathcal{M}(X)/\sim$ :

$$T(X) := \{[\mu] : \mu \in \mathcal{M}(X)\},$$

where  $[\mu]$  is the Teichmüller equivalence class of  $\mu$ .

An element  $\mu$  in  $\mathcal{M}(X)$  is said to be an extremal, if

$$\|\mu\|_\infty \leq \|\mu'\|_\infty, \quad \forall \mu' \in [\mu].$$

Let  $\mathcal{Q}(X)$  be the Banach space of integrable holomorphic quadratic differentials  $\phi = \phi(z)dz^2$  on  $X$  with  $L_1$ -norm.

2000 *Mathematics Subject Classification.* Primary 37F30, Secondary 30F70.

*Key words and phrases.* quasiconformal mappings, extremal quasiconformal mappings, Hamilton sequences.

The research is partially supported by the National Natural Science Foundation of China (Grant number No. 11571049 and No. 10571009).

As is well known, the following theorem plays an important role in the study of extremal problems of quasiconformal mappings:

**Theorem A** (Hamilton-Krushkal-Reich-Strebel) *Let  $\mu$  be any element of  $\mathcal{M}(X)$ . Then  $\mu$  is extremal, if and only if, there exists a sequence  $\{\phi_n\}$  in  $\mathcal{Q}(X)$  with  $\|\phi_n\| = 1$  ( $n = 1, 2, \dots$ ) such that*

$$\lim_{n \rightarrow \infty} \operatorname{Re} \int_X \mu \phi_n = \|\mu\|_\infty. \quad (*)$$

A sequence  $\{\phi_n\}$  in  $\mathcal{Q}(X)$  with  $\|\phi_n\| = 1$  ( $n = 1, 2, \dots$ ) which satisfies  $(*)$  is called a Hamilton sequence of  $\mu$ .

For the proof of Theorem A, refer to [7] by Hamilton, [8] by Krushkal, and [15] by Reich and Strebel, or see the book [5] by Gardiner and Lakic.

For a given element  $\mu \in \mathcal{M}(X)$ , we define

$$K(\mu) := \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$$

and called it the maximal dilatation of  $\mu$ . The quantity

$$K_0([\mu]) := \inf_{\mu'} \{K(\mu') : \mu' \sim \mu\}$$

is called the extremal maximal dilatation of the point  $[\mu]$ . The boundary dilatation of  $[\mu]$  is defined as the following:

$$H_0([\mu]) := \inf_E \inf_{\mu'} \{K(\mu'|_{X \setminus E}) : \mu' \sim \mu\},$$

where  $E$  ranges compact subsets of  $X$ .

Obviously, we have  $H_0([\mu]) \leq K_0([\mu])$  for any point  $[\mu]$  in  $T(X)$ .

Following [4] by Earle and Li, a point  $[\mu] \neq [0]$  is called a Strebel point, if

$$H_0([\mu]) < K_0([\mu]).$$

Similarly, a point  $[\mu] \neq [0]$  is called a non-Strebel point, if

$$H_0([\mu]) = K_0([\mu]).$$

The following theorem is a known result (see [4]):

**Theorem B (I)** *If  $[\mu] \in T(X) \setminus \{[0]\}$  is a Strebel point, then there is a  $\tilde{\mu} \in \mathcal{M}(X)$  with  $\mu' \sim \mu$  and  $\phi_0 \in \mathcal{Q}(X)$  with  $\|\phi_0\| = 1$  such that*

$$\tilde{\mu} = k_0([\mu]) \frac{\phi_0}{|\phi_0|},$$

where  $k_0([\mu]) = (K_0([\mu]) - 1)/(K_0([\mu]) + 1)$ .

**(II)** *If  $\mu \in \mathcal{M}(X)$  is extremal and  $[\mu]$  is a non-Strebel point, then its Hamilton sequence  $\{\phi_n\}$  must be degenerating sequence.*

Now we introduce a conjecture on Hamilton sequences by the second author of this paper.

In the 80s of last century, he found the following fact: For **some** points  $[\mu] \in T(X)$ , we can construct an extremal Beltrami differential  $\tilde{\mu}$  in  $[\mu]$ , such that any Hamilton sequence of  $\tilde{\mu}$  is a common Hamilton sequence of all extremal Beltrami differentials  $\nu$  in  $[\mu]$ . It is very natural to ask the question: whether or not **every** point  $[\mu]$  in  $T(X)$  possesses the same property (see [10]):

**Problem:** Let  $X$  be a hyperbolic Riemann surface. Suppose that  $\{\phi_n\}$  is a Hamilton sequence of  $\mu \in \mathcal{M}(X)$ . Whether  $\{\phi_n\}$  must be a common Hamilton sequence for any extremal Beltrami differential  $\nu$  contained in  $[\mu]$ ?

Actually, in 1996 Z. Li conjectured that the answer to this question is yes, by oral communications at the seminars of the Department of Mathematics of Peking University.

The first progress about this conjecture is made by S. Wu (see [18]). He proved this conjecture is true for the case that  $X$  is the unit disk. Later, G. Yao proved that this conjecture is true for doubly-connected domains (see [17]).

Unfortunately, this conjecture has so far failed to confirm.

In this paper, we are going to prove the conjecture is true for any hyperbolic Riemann surface  $X$ . We proved the following theorem:

**Theorem 1.1** *Let  $X$  be any hyperbolic Riemann surface and let  $\mu$  be an extremal Beltrami differential of  $\mathcal{M}(X)$  with  $\|\mu\|_\infty \in (0, 1)$ . If  $\{\phi_n\}$  is any given Hamilton sequence of  $\mu$ , then it must be a common Hamilton sequence for all extremal Beltrami differentials  $\nu$  contained in  $[\mu]$ .*

When  $[\mu]$  is not a Strebel point nor a non-Strebel point, what can we say about  $[\mu]$  in such a case? By using a result on  $T_0$  problem (cf.[9]), we have the following result, as a supplement of Theorem 1.1:

**Remark 1.1** *Suppose  $\mu$  is a Beltrami differential on  $X$  with  $\|\mu\|_\infty \neq 0$  and  $[\mu]$  is not a Strebel point nor a non-Strebel point. Then we have*

$$\pi([\mu]) = [[0]],$$

where  $\pi : [\sigma] \rightarrow [[\sigma]]$  is the natural projection of  $T(X)$  onto  $AT(X)$ .<sup>1</sup>

The proof of Remark 1.1 is very easy. In fact, if  $\mu \in \mathcal{M}(X)$  satisfies the conditions in Remark 1.1, then we have

$$T(X) = T_s(X) \cup T_n(X) \cup \{[0]\},$$

where  $T_s(X)$  is the set of all Strebel points in  $T(X)$  and  $T_n(X)$  is the set of all non-Strebel points in  $T(X)$ .

Because  $[\mu]$  is not a Strebel point nor a non-Strebel point, we have

$$[\mu] \in T(X) \setminus \{T_s(X) \cup T_n(X)\} = \{[0]\}.$$

Recalling Lemma 2.4 of this paper and making use of Lemma 3.1 and Lemma 3.2 of [9], we have

$$\pi([\mu]) = \pi([0]) = [[0]].$$

Remark 1.1 is proved.  $\square$

**Remark 1.2** The Referee pointed out that the whole discussing of this paper is essentially related to the conception of essential boundary points. To understand his observation, we refer E. Reich's paper [14] and two papers [12] and [13] of M. Mathelević.

The structure of this paper is as follows: §2 is devoted to introduce some conceptions and results we need. In §3 we will prove Theorem 1.1.

<sup>1</sup> In the next section, we will introduce the asymptotic Teichmüller space  $AT(X)$  and the main result of the  $T_0$  problem in [9].

## §2. PRELIMINARY

In this section, we will introduce the conception of asymptotic Teichmüller spaces  $AT(X)$ , the Delta inequality, the  $T_0$  problem and its solution.

The asymptotic Teichmüller space  $AT(X)$  of a hyperbolic Riemann surface  $X$  was introduced by Gardiner and Sullivan for the upper half plane (cf. [7]) and by Earle, Gardiner and Lakic for arbitrary hyperbolic Riemann surfaces (cf. [2,3,6]).

Let  $g : \Delta \rightarrow \Delta$  be a quasiconformal mapping. We say  $g$  is asymptotically conformal, if its Beltrami coefficient  $\mu_g$  possesses the following properties: for any  $\varepsilon > 0$ , there is a compact subset  $E_\varepsilon$  of  $\Delta$  such that

$$\|\mu_g|_{\Delta \setminus E_\varepsilon}\|_\infty < \varepsilon.$$

Let  $\mu$  and  $\nu$  be two elements of  $\mathcal{M}(X)$  and let  $g_{\mu,\nu} := f^\mu \circ (f^\nu)^{-1}$ . We say  $\mu$  and  $\nu$  are asymptotically Teichmüller equivalent to each other, denoted by  $\mu \approx \nu$ , if  $g_{\mu,\nu}$  is asymptotically conformal.

The asymptotic Teichmüller space of a hyperbolic Riemann surface  $X$  is defined as the quotient space of  $\mathcal{M}(X)/\approx$ :

$$AT(X) := \{[\mu] : \mu \in \mathcal{M}(X)\},$$

where  $[\mu]$  is the asymptotical Teichmüller equivalence of  $\mu$ .

It is proved by Earle, Gardiner and Lakic (cf. [2,3,6]) that  $AT(X)$  has a natural Teichmüller metric  $d_{AT}([\mu], [\nu])$ , which is a pseudo metric.

The following lemma is a known result (cf. [2] and [3]):

**Lemma 2.1** *Let  $\mu$  and  $\nu$  be two elements of  $\mathcal{M}(X)$ . If  $\mu \sim \nu$ , then we have  $\mu \approx \nu$ .*

The following lemma is known as the Delta inequality, see [1] by V. Bož, N. Lakic, V. Markovi and M. Mateljević:

**Lemma 2.2 ( the Delta inequality)** *Let  $\mu$  and  $\nu$  be two Beltrami differentials on  $X$  with*

$$\|\nu\|_\infty \leq k_0 := \|\mu\|_\infty < 1.$$

*Then there is a constant  $C = C(k_0)$ , which is determined only by  $k_0$ , such that*

$$\int_X \left| \frac{\tilde{\mu} - \tilde{\nu}}{1 - \tilde{\mu}\tilde{\nu}} \right|^2 \circ f^\mu |\phi| \leq C \left( k_0 \|\phi\| - \operatorname{Re} \int_X \mu \phi \right),$$

*where  $\tilde{\mu}$  and  $\tilde{\nu}$  are the Beltrami coefficients of  $(f^\mu)^{-1}$  and  $(f^\nu)^{-1}$ , respectively, and  $\phi$  is any element of  $\mathcal{Q}(X)$ .*

Now we introduce  $T_0$  problem and its solution (cf.[9]).

Let  $T(\mathbb{D})$  be the universal Teichmüller space and let  $T_0(\mathbb{D})$  be the subspace of  $T(\mathbb{D})$  consisting of points  $[\mu]$  in  $T(\mathbb{D})$  with  $H_0([\mu]) = 1$ . Suppose  $[\mu]$  is a point of  $T_0(\mathbb{D})$ . The  $T_0$  problem is asking whether the geodesic in  $T(\mathbb{D})$  between  $[0]$  and  $[\mu]$  belongs to  $T_0(\mathbb{D})$ .

It is proved in [9] that the answer to  $T_0$  problem is yes:

**Lemma 2.3** *If  $\mu$  is an extremal Beltrami differential on  $\mathbb{D}$  with  $\|\mu\|_\infty = k < 1$  and  $[\mu] \in T_0(\mathbb{D})$ , then we have*

$$[t\mu] \in T_0(\mathbb{D}) \quad \text{for all } t \in [0, 1/k].$$

*In particular, the geodesic from  $[0]$  to  $[\mu]$  belongs to  $T_0(\mathbb{D})$ .*

The following result in [9] is very useful:

**Lemma 2.4** *Let  $\pi$  be the nature projection of  $T(\mathbb{D})$  onto  $AT(\mathbb{D})$ :*

$$\pi : [\mu] \rightarrow [[\mu]].$$

*Then we have*

$$T_0(\mathbb{D}) = \pi^{-1}([0]).$$

One can find the proof of Lemma 2.4 at the page 1463 in [9].

### §3 PROOF OF THEOREM 1.1

*Proof of Theorem 1.1 :*

Suppose  $\mu \in \mathcal{M}(X)$  is extremal with  $\|\mu\|_\infty = k_0 \in (0, 1)$  and suppose  $\{\phi_n\}$  is a Hamilton sequence of  $\mu$ . Let  $\nu$  be any given extremal Beltrami differential in  $[\mu]$ . We want to show that  $\{\phi_n\}$  is a Hamilton sequence of  $\nu$ .

If  $[\mu]$  is a Strebel point, from Theorem B (see §2) we see

$$\mu = k_0 \frac{\phi_0}{|\phi_0|},$$

where  $\phi_0$  is an element in  $\mathcal{Q}(X) \setminus \{0\}$ . In this case,  $[\mu]$  contains only one extremal Beltrami differential. So Theorem 1.1 is automatically established. We do not need to show any thing.

Now we assume  $[\mu]$  is a non-Strebel point. From Theorem B we see that  $\mu$ 's Hamilton sequence  $\{\phi_n\}$  must be a degenerating sequence.

To show Theorem 1.1, it is sufficient to prove that the sequence  $\{\phi_n\}$  is also a Hamilton sequence of  $\nu$ .<sup>2</sup>

Our proof is a bit long. Now we point out that the key step of the proof is to show (3.12).

By making use of Theorem D, we have a constant  $C = C(k_0)$ , which is only determined by  $k_0$ , such that for any  $\phi \in \mathcal{Q}(X)$ , we have

$$\int_X \left| \frac{\tilde{\mu} - \tilde{\nu}}{1 - \tilde{\mu}\tilde{\nu}} \right|^2 \circ f^\mu |\phi| \leq C \left( k_0 \|\phi\| - \operatorname{Re} \int_X \mu \phi \right), \quad (3.1)$$

where  $\tilde{\mu}$  and  $\tilde{\nu}$  are the Beltrami coefficients of  $(f^\mu)^{-1}$  and  $(f^\nu)^{-1}$ , respectively.

Now we replace  $\phi$  in (3.1) by  $\phi_n$ . Then we get

$$\int_X \left| \frac{\tilde{\mu} - \tilde{\nu}}{1 - \tilde{\mu}\tilde{\nu}} \right|^2 \circ f^\mu |\phi_n| \leq C \left( k_0 - \operatorname{Re} \int_X \mu \phi_n \right). \quad (3.2)$$

Here we have used the fact that  $\|\phi_n\| = 1$  (as  $n = 1, 2, \dots$ ).

Since  $\{\phi_n\}$  is a Hamilton sequence of  $\mu$ , from (3.2) we get

$$\lim_{n \rightarrow \infty} \int_X \left| \frac{\tilde{\mu} - \tilde{\nu}}{1 - \tilde{\mu}\tilde{\nu}} \right|^2 \circ f^\mu |\phi_n| = 0. \quad (3.3)$$

Suppose  $F \geq 0$  and  $G \geq 0$  are two real functions on  $X$  which are integrable on  $X$ . The Cauchy-Schwartz inequality tells us:

$$\left( \int_X F \times G \right)^2 \leq \int_X F^2 \int_X G^2.$$

<sup>2</sup> In this case,  $[\mu]$  contains infinitely many extremal Beltrami differentials  $\nu$ . This can be proved by using the result of K. Strebel (cf. [16]).

Now we let

$$F = \left( \left| \frac{\tilde{\mu} - \tilde{\nu}}{1 - \tilde{\mu}\tilde{\nu}} \right|^2 \circ f^\mu |\phi_n| \right)^{1/2} |\phi_n|^{1/2}$$

and let  $G = |\phi_n|$ . Applying the Cauchy-Schwartz inequality, we get

$$\left( \int_X \left| \frac{\tilde{\mu} - \tilde{\nu}}{1 - \tilde{\mu}\tilde{\nu}} \right|^2 \circ f^\mu |\phi_n| \right)^2 \leq \int_X \left| \frac{\tilde{\mu} - \tilde{\nu}}{1 - \tilde{\mu}\tilde{\nu}} \right|^2 \circ f^\mu |\phi_n| \|\phi_n\|.$$

Hence, from (3.3) we have

$$\lim_{n \rightarrow \infty} \int_X \left| \frac{\tilde{\mu} - \tilde{\nu}}{1 - \tilde{\mu}\tilde{\nu}} \right|^2 \circ f^\mu |\phi_n| = 0. \quad (3.4)$$

Since  $\mu \sim \nu$ , we have  $\|\mu\|_\infty = \|\nu\|_\infty$ , and hence,  $\|\tilde{\mu}\|_\infty = \|\tilde{\nu}\|_\infty$ . Thus it follows from (3.4) that

$$\frac{1}{1 - k_0^2} \int_X |\tilde{\mu} - \tilde{\nu}| \circ f^\mu |\phi_n| \leq \int_X \left| \frac{\tilde{\mu} - \tilde{\nu}}{1 - \tilde{\mu}\tilde{\nu}} \right|^2 \circ f^\mu |\phi_n|. \quad (3.5)$$

Then it follows from (3.3)-(3.5) that

$$\lim_{n \rightarrow \infty} \int_X |\tilde{\mu} - \tilde{\nu}| \circ f^\mu |\phi_n| = 0, \quad (3.6)$$

On the other hand, we have

$$\operatorname{Re} \int_X (\tilde{\mu} - \tilde{\nu}) \circ f^\mu \phi_n \leq \int_X |\tilde{\mu} - \tilde{\nu}| \circ f^\mu |\phi_n|.$$

Thus it follows from (3.6) that

$$\limsup_{n \rightarrow \infty} \operatorname{Re} \int_X (\tilde{\mu} - \tilde{\nu}) \circ f^\mu \phi_n = 0. \quad (3.7)$$

A simple computation shows

$$\tilde{\mu} \circ f^\mu = -\mu \cdot \beta_{f^\mu} \quad (3.8)$$

where  $\beta_{f^\mu} = (\overline{\frac{\partial_z}{\partial_{\bar{z}}} f^\mu}) / (\frac{\partial_z}{\partial_{\bar{z}}} f^\mu)$ . Similarly, we have

$$\tilde{\nu} \circ f^\mu = -\nu \cdot \beta_{f^\nu} \circ f^\mu. \quad (3.9)$$

Then it follows from (3.7)-(3.9) that

$$\limsup_{n \rightarrow \infty} \operatorname{Re} \int_X (\nu \cdot \beta_{f^\nu} \circ f^\mu - \mu \cdot \beta_{f^\mu}) \phi_n = 0. \quad (3.10)$$

It is easy to see that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \operatorname{Re} \int_X (\nu \cdot \beta_{f^\nu} \circ f^\mu - \mu \cdot \beta_{f^\mu}) \phi_n \\ & \geq \liminf_{n \rightarrow \infty} \operatorname{Re} \int_X \nu \cdot \beta_{f^\nu} \circ f^\mu \phi_n - \|\mu\|_\infty. \end{aligned} \quad (3.11)$$

Now we claim that

$$\liminf_{n \rightarrow \infty} \operatorname{Re} \int_X \nu \cdot \beta_{f^\nu} \circ f^\mu \phi_n \geq \|\mu\|_\infty. \quad (3.12)$$

We are going to show (3.12) in Reductio ad absurdum.

Now we suppose (3.12) is not true. Then there exists a number  $\delta > 0$ , such that

$$\liminf_{n \rightarrow \infty} \operatorname{Re} \int_X \nu \cdot \beta_{f\nu} \circ f^\mu \phi_n > \|\mu\|_\infty + \delta. \quad (3.13)$$

Now from (3.13) and (3.11), we have

$$\begin{aligned} \delta &< \liminf_{n \rightarrow \infty} \operatorname{Re} \int_X \nu \cdot \beta_{f\nu} \circ f^\mu \phi_n - \|\mu\|_\infty \\ &\leq \limsup_{n \rightarrow \infty} \operatorname{Re} \int_X (\nu \cdot \beta_{f\nu} \circ f^\mu - \mu \beta_{f\mu}) \phi_n. \end{aligned} \quad (3.14)$$

However, it follows from (3.10) that

$$\limsup_{n \rightarrow \infty} \operatorname{Re} \int_X (\nu \cdot \beta_{f\nu} \circ f^\mu - \mu \beta_{f\mu}) \phi_n = 0. \quad (3.15)$$

Then it follows from (3.14) and (3.15) that  $\delta < 0$ .

We have the assumed that  $\delta > 0$ . So we get a contradiction with our assumption before.

So we have proved our claim (3.12).

However, noting the facts that  $\nu \in [\mu]$  and  $\nu$  is extremal, we see  $\|\mu\|_\infty = \|\nu\|_\infty$ . From (3.12) we get

$$\lim_{n \rightarrow \infty} \operatorname{Re} \int_X \nu \beta_{f\nu} \circ f^\mu \phi_n = \|\nu\|_\infty. \quad (3.16)$$

Since  $\beta_{f\nu} \circ f^\mu$  is independent of  $n$  and  $|\beta_{f\nu} \circ f^\mu| = 1$ , it is easy to check from (3.16) that  $\{\phi_n\}$  is a Hamilton sequence of  $\nu$ .

The proof of Theorem 1.1 is completed.  $\square$

## REFERENCES

- [1] V. Božin, N. Lakic, V. Marković and M. Mateljević, *Unique extremality*, Journal D'Analysis Math. **75**(1998), 299-337.
- [2] C. J. Earle, F. P. Gardiner and N. Lakic, *Asymptotic Teichmüller spaces, Part I, The complex structure*. Contem. Math. **256** (2000), 17-38.
- [3] C. J. Earle, F. P. Gardiner and N. Lakic, *Asymptotic Teichmüller spaces, Part II, The metric structure*. Contem. Math. **355** (2004), 187-210.
- [4] C. J. Earle and Z. Li, *Isometrically embedded polydisks in infinite-dimensional Teichmüller spaces*. *Journal of Geometric Analysis*, **9**(1998), 51-71.
- [5] F. P. Gardiner and N. Lakic, *Quasi-conformal Teichmüller theorem*. American Mathematical Society, New York, 2000.
- [6] F. P. Gardiner and D. P. Sullivan, *Symmetric structures on a closed curve*, *American Journal of Math.* **114**(1992) 683-736.
- [7] R. S. Hamilton, *Extremal quasiconformal mappings with prescribed boundary values*. Trans. Amer. Math. Soc. **138**(1969), 399-406.
- [8] S. L. Krushkaí, *Extremal quasiconformal mappings*, *Silbisk. Mat. Zh.* **10**(1969), 411-418.
- [9] Z. Li, *A problem on  $T_0$  Teichmüller space*, *J. Math. Anal. Appl.* **465**(2017), 1457-1469.



- [10] Z. Li, A note on extremal quasiconformal mappings, *Sci. China Ser. A.* **53**(2010), 63-70.
- [11] Z. Li, Strebel differentials and Hamilton sequences, *Sci. China Ser. A.* **44**(2001), 696-979.
- [12] M. Mateljevic, Unique extremality of quasiconformal mappings, *Anal Math.* **107**(2009), 39-63.
- [13] M. Mateljevic, Quasiconformal maps and Teichmüller theory, extremal mappings, Overview, Sulletin T. CXLV de l'Academie Serbe des sciences et des arts **38**(2013), 129-172.
- [14] E. Reich, Extremal Quasiconformal Mappings of the Disk, Chapter 3, in Geometric function theory, **1**(2002), Edited by *R. Kötter - hnau*, Elsevier Science B. V. , 104-106.
- [15] E. Reich and K. Strebel, Extremal quasiconformal mappings with given boundary values, in the book: Contributions to Analysis (L. Ahlfors et al, eds), Academic Press, New York, pp. 375-392.
- [16] K. Strebel, The point-shift differentials and extremal quasiconformal mappings. *Ann. Acad. Sci. Fenn. Math.*, **23** (1998), 475-494.
- [17] G. Yao, Hamilton sequence for extremal quasiconformal mappings of doubly-connected domains, *Bull. Aust. Math. Soc.*, **88**(2013), 376-379.
- [18] S. Wu, Hamilton sequence for extremal quasiconformal mappings of the unit disk, *Sci. China, Ser. A.* **42**(1991), 1033-1042.

Z. LI: SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING 100875, CHINA.

Y. LI: SCIENCE OF SCHOOL, BEIJING UNIVERSITY OF POSTS AND TELECOMMUNICATIONS, BEIJING 100876, CHINA

*E-mail address:* lizhong@math.pku.edu.cn

*E-mail address:* yezhouli@bupt.edu.cn