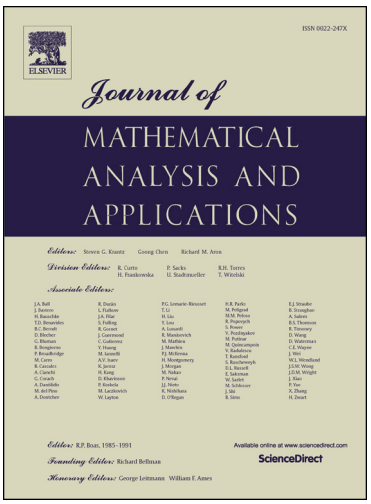
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NON-UNIQUENESS FOR THE FOKAS-OLVER-ROSENAU-QIAO EQUATION

A. ALEXANDROU HIMONAS & CURTIS HOLLIMAN

Abstract. For the Fokas-Olver-Rosenau-Qiao equation (FORQ), on both the line and the circle, it is proved that there exist initial data in the Sobolev spaces H^s, with $s < 3/2$, which have nonunique solutions. This phenomenon is demonstrated by constructing a 2-peakon solution with an asymmetric peakon-antipeakon initial profile. The peakon and antipeakon eventually collide, and the solution at the collision time coincides with a second solitary peakon solution. Using the time-reversibility of the equation, this scenario proves nonuniqueness.

1. Introduction

We consider the following initial value problem (IVP) for the Fokas-Olver-Rosenau-Qiao equation (FORQ)

$$\begin{cases} u_t + u^2 u_x - \frac{1}{3}u_x^3 + D^{-2}\partial_x\left(\frac{2}{3}u^3 + uu_x^2\right) + D^{-2}\left(\frac{1}{3}u_x^3\right) = 0, \\ u(x,0) = u_0(x) \in H^s, \quad x \in \mathbb{R} \text{ or } \mathbb{T}, \end{cases} \tag{1.1}$$

where the operator D is given by $\widehat{D} = (1+\xi^2)^{1/2}$, and we construct specific 2-peakon solutions $u(t)$ consisting of a peakon and antipeakon that collide at a finite time T. At the collision time, these solutions converge to single antipeakons $u(T)$, which can be thought of as the superposition of both peakons. Using the time-reversibility of the equation, this scenario shows that given the initial data of a single peakon, there are two solutions to the corresponding FORQ Cauchy problem: the peakon traveling wave solution and the time-reversed peakon-antipeakon collision. This ill-posedness result is summarized in the following theorem.

Theorem 1. *If $s < 3/2$, then the Cauchy problem for FORQ equation (1.1) is not well-posed in Sobolev spaces H^s, in the sense that uniqueness of solutions fails.*

We recall that FORQ is well-posed in the sense of Hadamard (see [14]) in Sobolev spaces H^s with exponents $s > 5/2$ (see [20]). More precisely, if u_0 belongs to the space H^s on the circle or the line, $s > 5/2$, then there exists $T_s = T_s(||u_0||_{H^s}) > 0$ and a unique solution $u \in C([0,T_s];H^s)$ of the Cauchy problem for FORQ (1.1) satisfying the estimate

$$\|u(t)\|_{H^s} \le 2\|u_0\|_{H^s}, \quad \text{for} \quad 0 \le t \le T_s, \quad \text{with} \quad T_s = \frac{1}{4c_s\|u_0\|_{H^s}^2}, \tag{1.2}$$

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where $c_s > 0$ is a constant depending on s. Furthermore, the data-to-solution map $u_0 \mapsto u$ is continuous but not uniformly continuous. As FORQ is well-posed for $s > 5/2$ and ill-posed for $s < 3/2$, we refer to exponents $3/2 \le s \le 5/2$ as lying in the critical gap.

The FORQ equation was derived separately by Fokas [12], Olver and Rosenau [35], Fuchssteiner [13] and Qiao [38]. FORQ also appeared as an integrable member of a family of Camassa-Holm-type equations with cubic nonlinearities in [34]. The entire integrable hierarchy related to the FORQ equation was proposed in [39]. Its bi-Hamiltonian structure was first derived in [35] and then in [38], and it is given by

$$q_t = -[q(u^2 - u_x^2)]_x = -J\frac{\delta H_0}{\delta q} = -K\frac{\delta H_1}{\delta q},$$

where

$$J = \partial q \partial^{-1} q \partial, \qquad K = \partial - \partial^3,$$
$$H_0(u) = 2\int qu\,dx, \qquad H_1(u) = \frac{1}{4}\int [u^4 + 2u^2(\partial u)^2]\,dx.$$

The Lax pair for FORQ has been derived in [38] as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U(q,\lambda)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = V(q,u,\lambda)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where

$$U(q,\lambda) = \frac{1}{2}\begin{pmatrix} -1 & \lambda q \\ -\lambda q & 1 \end{pmatrix} \quad \text{and} \quad V(q,u,\lambda) = -\frac{1}{2}\begin{pmatrix} \frac{1}{\lambda^2} + \frac{1}{2}vw & -\frac{w}{\lambda} - \frac{1}{2}\lambda vwq \\ \frac{v}{\lambda} + \frac{1}{2}\lambda qvw & -\frac{1}{\lambda^2} - \frac{1}{2}vw \end{pmatrix}$$

and $v = u + u_x$ and $w = u - u_x$. For more discussion about Lax integrability and peakon solutions of FORQ we refer to [6], where this equation is also referred as the modified Camassa-Holm equation.

FORQ admits peakon traveling wave solutions and multipeakon solutions. The peakon traveling wave solutions on the line are given by

$$u_c(x,t) = \sqrt{\tfrac{3}{2}c}\; e^{-|x-ct|}. \tag{1.3}$$

On the circle, they are given by

$$u_c(x,t) = \sqrt{c}\Big(1 + \frac{2}{3}\sinh^2\pi\Big)^{-1/2}\cosh\Big([x-ct]_p - \pi\Big), \tag{1.4}$$

where c is a positive constant and $[\cdot]_p$ makes a quantity 2π-periodic via

$$[x]_p = x - 2\pi\left\lfloor \frac{x}{2\pi} \right\rfloor. \tag{1.5}$$

Equations having quadratic nonlinearities that admit multipeakon solutions include the much studied Camassa-Holm (CH) equation

$$u_t + uu_x + \partial_x D^{-2}\Big(u^2 + \frac{u_x^2}{2}\Big) = 0,$$

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and Degasperis-Procesi (DP) equation

$$u_t + uu_x + \frac{3}{2}D^{-2}u^2 = 0.$$

These equations are members of the so called b-family of equations that all posses multipeakon solutions. In the case of cubic nonlinearities, [21] studies the multipeakon solutions for the ab-family of nonlocal evolution equations given by

$$u_t + u^2u_x - au_x^3 + \partial_x D^{-2}\left[\frac{b}{3}u^3 + \frac{6-6a-b}{2}uu_x^2\right] + D^{-2}\left[\frac{2a+b-2}{2}u_x^3\right] = 0, \tag{1.6}$$

where $a, b \in \mathbb{R}$. FORQ is obtained from this ab-equation by setting $a = 1/3$ and $b = 2$.

In addition to FORQ, the ab-family contains another integrable equation that has been the focus of much study, Novikov's equation (NE). Setting $a = 0$ and $b = 3$ gives NE

$$u_t + u^2u_x + \partial_x\left(1-\partial_x^2\right)^{-1}\left(u^3 + \tfrac{3}{2}uu_x^2\right) + \left(1-\partial_x^2\right)^{-1}\left(\tfrac{1}{2}u_x^3\right) = 0.$$

The local form of the ab-family (1.6), which reads

$$\left(1-\partial_x^2\right)u_t + (b+1)u^2u_x - 3au_x^3 - (6a+b)uu_xu_{xx} + 6au_xu_{xx}^2 - u^2u_{xxx} + 3au_x^2u_{xxx} = 0,$$

is convenient for showing that under the condition

$$3a + b = 3, \tag{1.7}$$

the ab-family conserves the Sobolev H^1 norm,

$$\|u(t)\|_{H^1}^2 \doteq \int_{\mathbb{R} \text{ or } \mathbb{T}} \left[u(x,t)^2 + u_x(x,t)^2\right] dx.$$

Indeed, using integration by parts and (1.7) we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^1}^2 = \int_{\mathbb{R} \text{ or } \mathbb{T}} \left[\left(uu_{xt} - \tfrac{b+1}{4}u^4 + \tfrac{3a}{2}u^2u_x^2 - 3auu_x^2u_{xx} + u^3u_{xx} + \tfrac{3a}{4}u_x^4\right)_x\right] dx,$$

which vanishes. Condition (1.7) holds for FORQ and NE, so these equations conserve the H^1 norm. Moreover, the H^1 norm is also conserved by CH but not by DP, which instead conserves a twisted L^2 norm.

It is worth comparing the ill-posedness of FORQ to equations with similar properties such as NE which also has cubic nonlinearities and CH and DP, which have quadratic nonlinearities. NE, CH and DP are well-posed in H^s for $s > 3/2$. The nonquasilinearity of FORQ currently obstructs establishing well-posedness for $3/2 < s \leq 5/2$ with known techniques. Unlike these other three equations, FORQ has nonunique solutions for all $s < 3/2$. For each of these other equations, there is an interval of s just below $3/2$ for which the equations exhibit the phenomenon of *norm-inflation*. For a discussion of norm-inflation and its relationship to the well-posedness of CH, DP and NE, we refer the reader to [15], [16], [18].

The method used here for proving the ill-posedness of FORQ is similar to that used by many authors for other nonlinear evolution equations. Bourgain and Pavlovic proved ill-posedness for the 3D Navier-Stokes equations in Besov spaces in the sense of norm-inflation [3]. Similar methods for establishing ill-posedness for dispersive equations have been used by Kenig, Ponce and Vega [27] and Christ, Colliander and Tao [7]. The ill-posedness for the generalized KdV and nonlinear Schrödinger equations in Sobolev spaces has been handled in [1]. Linares, Pilod

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and Saut tackle the ill-posedness of the Burgers equation in $H^{3/2}$ [30]. For more results on traveling wave solutions, well-posedness, ill-posedness and other analytic and geometric properties of nonlinear evolution equations, we refer the reader to the following works and the references therein [2], [5], [8], [11], [14], [19], [22], [26], [29], [33], [37], [36].

This paper is organized as follows. In Section 2, we prove Theorem 1 on the line. We do this by first constructing specific 2-peakon solutions and calculating the collision time. We then prove that the collision profile is indeed a single peakon and use this to establish the nonuniqueness. Section 3 adapts the above results to the circle. See figure 2 for a visualization of our construction on the line and figure 4 for the circle.

2. Construction of the 2-peakon solution

Our proof revolves around utilizing a 2-peakon solution of the form

$$u(x,t) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|}. \tag{2.1}$$

The 2-peakon system for FORQ was first derived in [40]; however, we will approach this problem by starting with the ab-equation. From [21], this 2-peakon u solves the ab-equation iff the positions q_1, q_2 and the momenta p_1, p_2 satisfy the 4×4 system of ODEs

$$\begin{aligned}
q_1' &= (1-a)p_1^2 + 2p_1p_2e^{-|q_1-q_2|} + (1-3a)p_2^2e^{-2|q_1-q_2|},\\
q_2' &= (1-a)p_2^2 + 2p_1p_2e^{-|q_1-q_2|} + (1-3a)p_1^2e^{-2|q_1-q_2|},\\
p_1' &= (2-b)\mathrm{sgn}(q_2-q_1)p_1p_2e^{-|q_1-q_2|}(p_1 + p_2e^{-|q_1-q_2|}),\\
p_2' &= (2-b)\mathrm{sgn}(q_1-q_2)p_1p_2e^{-|q_1-q_2|}(p_1e^{-|q_1-q_2|} + p_2).
\end{aligned}$$

Setting $a = 1/3$ and $b = 2$ we get the ODE system for FORQ as

$$\begin{aligned}
q_1' &= \frac{2}{3}p_1^2 + 2p_1p_2e^{-|q_1-q_2|}, \qquad p_1' = 0,\\
q_2' &= \frac{2}{3}p_2^2 + 2p_1p_2e^{-|q_1-q_2|}, \qquad p_2' = 0.
\end{aligned}$$

From the last two equations, we immediately see that

$$p_1(t) = p_1(0) \quad \text{and} \quad p_2(t) = p_2(0).$$

Additionally, we note that if we take the initial data to be a symmetric peakon-antipeakon, $p_1(0) = -p_2(0)$, we will have $q_1' = q_2'$ which implies both peaks will move at the same velocity, preventing any collision from happening. Thus we will take the initial data for the momenta to be asymetric. Finally, we note that the ODEs for q_1 and q_2 are in fact solvable, but our argument will allow us to bypass the need for exact solutions.

Initial Data. We take

$$\begin{aligned}
q_1(0) &= 0, \quad p_1(0) = b + \delta,\\
q_2(0) &= a, \quad p_2(0) = -b.
\end{aligned}$$

These initial data choices are summarized in the picture below.

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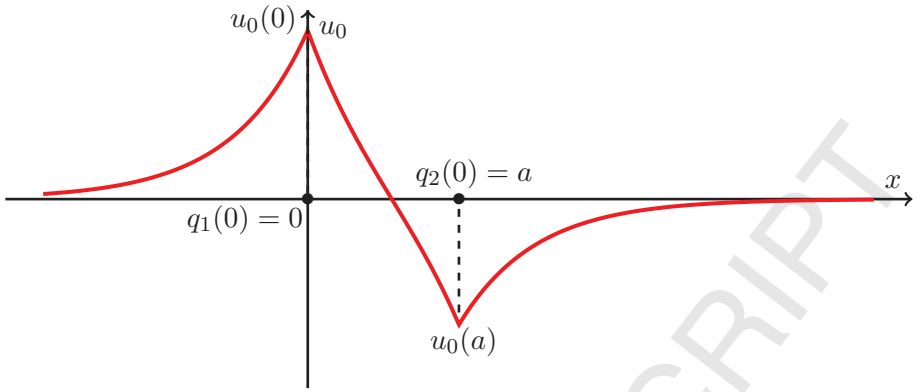


Figure 1: Initial profile $u_0(x)$

Collision Time and Function. There is a positive time T at which the peakon and antipeakon collide. Defining q to be the difference between the positions of the antipeakon and peakon, we demonstrate that q has a zero. We see that,

$$q' = q_2' - q_1' = \frac{2}{3}p_2^2 - \frac{2}{3}p_1^2 = \frac{2}{3}\Big(b^2 - (b+\delta)^2\Big).$$

Integrating from 0 to t gives us,

$$q(t) = \frac{2}{3}\Big(b^2 - (b+\delta)^2\Big)t + a = -\frac{2}{3}\Big(2b\delta + \delta^2\Big)t + a.$$

We see that we get a collision when $q(t) = 0$. Defining this collision time to be T, we get

$$T \doteq \frac{3}{2} \cdot \frac{a}{2b\delta + \delta^2}.$$

We define q_T to be the location of the collision, or

$$q_T \doteq \lim_{t \to T^-} q_1(t) = \lim_{t \to T^-} q_2(t),$$

and we define the collision function v to be

$$v(x) \doteq \delta e^{-|x - q_T|}.$$

Proposition 1 (Real-Line Case)**.** *The H^s limit of u as t approaches T is v, or*

$$\lim_{t \to T^-} \|u(t) - v\|_{H^s} = 0. \tag{2.2}$$

Proof. Taking the Fourier transform of u, we have

$$\widehat{u}(\xi, t) = \frac{2p_1 e^{-i\xi q_1}}{1+\xi^2} + \frac{2p_2 e^{-i\xi q_2}}{1+\xi^2} = \frac{2(b+\delta)e^{-i\xi q_1}}{1+\xi^2} + \frac{2(-b)e^{-i\xi q_2}}{1+\xi^2}.$$

Similarly, we compute the Fourier transform of v as

$$\widehat{v}(\xi) = \frac{2\delta e^{-i\xi q_T}}{1+\xi^2}.$$

Calculating the H^s norm of $u(t) - v$ gives us

$$\lim_{t \to T^-} \|u(t) - v\|_{H^s}^2 = \lim_{t \to T^-} 4 \int_{\mathbb{R}} (1+\xi^2)^{s-2} \Big| (b+\delta)e^{-i\xi q_1} - be^{-i\xi q_2} - \delta e^{-i\xi q_T} \Big|^2 d\xi.$$

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We can bound the quantity inside the absolute value by

$$\left|(b+\delta)e^{-i\xi q_1} - be^{-i\xi q_2} - \delta e^{-i\xi q_T}\right| \leq (b+\delta)|e^{-i\xi q_1}| + b|e^{-i\xi q_T}| + \delta|e^{-i\xi q_T}| = 2(b+\delta).$$

Setting

$$g(\xi) \doteq (1+\xi^2)^{s-2} \cdot 4(b+\delta)^2,$$

we see that g dominates our original integrand and g is itself integrable when $s < 3/2$. Therefore, we may apply the Dominated Convergence Theorem and bring the limit inside the integral. We thus get

$$\begin{aligned}\lim_{t\to T^-} 4\int_{\mathbb{R}} (1+\xi^2)^{s-2}&\left|(b+\delta)e^{-i\xi q_1} - be^{-i\xi q_2} - \delta e^{-i\xi q_T}\right|^2 d\xi \\ &= 4\int_{\mathbb{R}} (1+\xi^2)^{s-2}\left|(b+\delta)e^{-i\xi q_1(T)} - be^{-i\xi q_2(T)} - \delta e^{-i\xi q_T}\right|^2 d\xi = 0.\end{aligned}$$

□

Proof of Theorem 1 (Real-Line Case). Given the two-peakon solution we have constructed, the crux of our proof will be to construct a traveling wave solution $w(x,t)$ with $w(x,T) = u(x,T) = v(x)$. Starting with the 4×4 system, we will take $p_1 = \delta$, $p_2 = 0$, $q_1(0) = \alpha$ and $q_2(0) = 0$. This reduced system is easily solved as

$$p_1(t) = \delta, \quad q_1(t) = \frac{2}{3}\delta^2 t + \alpha.$$

Thus, we will choose α such that $q_1(T)$ for w equals $q_1(T)$ for u. To distinguish these q_1's we will introduce q_1^u for the q_1 corresponding to u and q_1^w for the q_1 corresponding to w. Thus, having these solutions coincide at time T precisely means $q_1^w(T) = q_1^u(T)$. This equation allows us to solve for α as

$$\alpha = q_1^u(T) - \frac{2}{3}\delta^2 T.$$

This completes the construction of our traveling wave w. To demonstrate non-uniqueness, we define new solutions

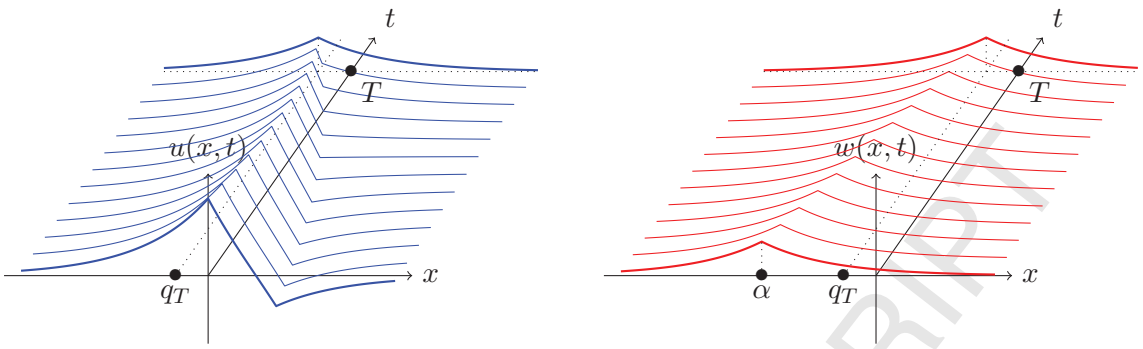
$$\tilde{u}(x,t) = u(-x,-t+T) \quad \text{and} \quad \tilde{w}(x,t) = w(-x,-t+T).$$

We see that since $u(x,t)$ solves FORQ iff $u(-x,-t)$ solves FORQ, that $\tilde{u}$ and $\tilde{w}$ are both solutions to FORQ. Furthermore, we see that the initial conditions of these solutions are equal by Proposition 1 we have in H^s

$$\lim_{t\to 0^+} \tilde{u}(x,0) = \lim_{t\to T^-} u(-x,t) = v(-x) = \lim_{t\to T^-} w(-x,t) = \lim_{t\to 0^+} \tilde{w}(x,t).$$

A visualization of u and w is shown below.

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(a) Peakon-Antipeakon Solution (b) Traveling Wave Solution

Figure 2: Construction of u and w.

Finally, we note that the initial data for these nonunique solutions can be made arbitrarily small. As we have our new initial data being the collision function $v = \delta e^{-|x-q_T|}$, we compute

$$\|v\|_{H^s}^2 = 4c_s\delta^2 \quad \text{where} \quad c_s = \int_{\mathbb{R}} (1+\xi^2)^{s-2} d\xi.$$

Thus, we can control the size of the initial data via the choice of δ. In particular, given $\varepsilon > 0$, we will take

$$\delta < \frac{\varepsilon}{2\sqrt{c_s}} \quad \text{which gives} \quad \|v\|_{H^s} < \varepsilon.$$

□

3. Periodic Case

We search for a 2-peakon solution of the form

$$u(x,t) = p_1(t)\cosh([x-q_1(t)]_p - \pi) + p_2\cosh([x-q_2(t)]_p - \pi) \tag{3.1}$$

where $[\cdot]_p$ is given in (1.5). From [20], the 2-peakon (3.1) solves the ab-equation if and if the positions q_1, q_2, and the momenta p_1, p_2, satisfy the 4×4 ODE system

$$\begin{aligned}
q_1' &= p_1^2[1 + (1-a)\sinh^2\pi + 2p_1p_2\cosh\pi\cosh([q_1-q_2]_p - \pi) \\
&\quad + p_2^2[1 + (1-3a)\sinh^2([q_1-q_2]_p - \pi), \\
q_2' &= p_2^2[1 + (1-a)\sinh^2\pi + 2p_1p_2\cosh\pi\cosh([q_2-q_1]_p - \pi) \\
&\quad + p_1^2[1 + (1-3a)\sinh^2([q_2-q_1]_p - \pi), \\
p_1' &= (2-b)p_1p_2\sinh([q_1-q_2]_p - \pi) \\
&\quad \times [p_1\cosh\pi + p_2\cosh([q_1-q_2]_p - \pi)], \\
p_2' &= (2-b)p_1p_2\sinh([q_2-q_1]_p - \pi) \\
&\quad \times [p_2\cosh\pi + p_1\cosh([q_2-q_1]_p - \pi)].
\end{aligned}$$

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Noting again that FORQ corresponds to $a = 1/3$ and $b = 2$, this system reduces to

$$q_1' = p_1^2[1 + \frac{2}{3}\sinh^2\pi] + 2p_1p_2\cosh\pi\cosh([q_1 - q_2]_p - \pi) + p_2^2, \qquad p_1' = 0,$$
$$q_2' = p_2^2[1 + \frac{2}{3}\sinh^2\pi] + 2p_1p_2\cosh\pi\cosh([q_2 - q_1]_p - \pi) + p_1^2, \qquad p_2' = 0.$$

From here, we again immediately obtain

$$p_1 = p_1(0) \quad \text{and} \quad p_2 = p_2(0).$$

As in the real-line case, we must take asymmetric initial data and can can bypass solving the q_1 and q_2 equations.

Initial Data. We follow the strategy from the real-line case and take

$$p_1(0) = b + \delta, \quad q_1(0) = 0,$$
$$p_2(0) = -b, \qquad q_2(0) = a.$$

These initial data choices are summarized in the picture below.

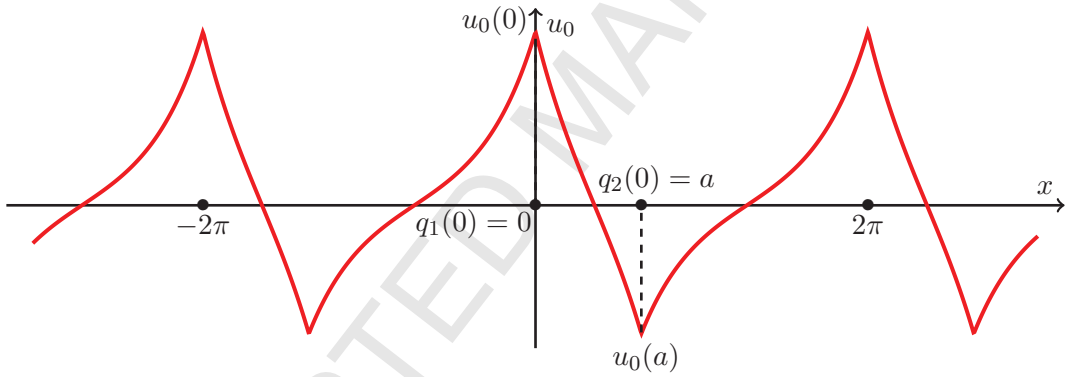


Figure 3: Initial profile $u_0(x)$

These choices immediately imply

$$p_1(t) = b + \delta \quad \text{and} \quad p_2(t) = -b.$$

The fact that $\cosh([x]_p - \pi)$ is an *even* function allows for many cancellations when we simplify the formula for the difference $q = q_2 - q_1$. The ODE for q is thus given by

$$q' = -\frac{2}{3}\sinh^2\pi[(b + \delta)^2 - b^2].$$

Integrating from 0 to t gives us

$$q(t) = a - \frac{2}{3}\sinh^2\pi[(b + \delta)^2 - b^2] \cdot t.$$

Setting this equation equal to zero, we can easily calculate the collision time T as

$$T \doteq \frac{a}{\frac{2}{3}\sinh^2\pi[(b + \delta)^2 - b^2]}.$$

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We again define q_T to be the location of the collision, or

$$q_T \doteq \lim_{t\to T^-} q_1(t) = \lim_{t\to T^-} q_2(t),$$

and we similarly define the periodic collision function v to be

$$v(x) \doteq \delta \cosh([x - q_1(t)]_p - \pi).$$

Proposition 2 (Circle Case)**.** *The limit of u as t approaches T is v in H^s, or*

$$\lim_{t\to T^-} \|u(t) - v\|_{H^s} = 0. \tag{3.2}$$

Proof. Taking the Fourier transform of u, we have

$$\widehat{u}(n,t) = \sinh\pi\Big(\frac{2p_1e^{-inq_1}}{1+n^2} + \frac{2p_2e^{-inq_2}}{1+n^2}\Big) = \sinh\pi\Big(\frac{2(b+\delta)e^{-inq_1}}{1+n^2} + \frac{2(-b)e^{-inq_2}}{1+n^2}\Big).$$

Similarly,

$$\widehat{v}(n) = \sinh\pi\cdot\frac{2\delta e^{-inq_T}}{1+n^2}.$$

Calculating the norm of $u(t) - v$ and taking the limit $t \to T^-$ gives us

$$\lim_{t\to T^-} \|u(t) - v\|_{H^s}^2 = \lim_{t\to T^-} 4\sinh^2\pi \sum_{n\in\mathbb{Z}} (1+n^2)^{s-2}\Big|(b+\delta)e^{-inq_1} - be^{-inq_2} - \delta e^{-inq_T}\Big|^2.$$

We can bound the quantity inside the absolute value by

$$\Big|(b+\delta)e^{-inq_1} - be^{-inq_2} - \delta e^{-inq_T}\Big| \le (b+\delta)|e^{-inq_1}| + b|e^{-inq_T}| + \delta|e^{-inq_T}| = 2(b+\delta). \tag{3.3}$$

Setting

$$g(n) \doteq (1+n^2)^{s-2}\cdot 4(b+\delta)^2,$$

we see that g dominates our original summand and g is itself summable. Therefore, we may apply the Dominated Convergence Theorem and bring the limit inside the summation. We thus get

$$\begin{aligned}\lim_{t\to T^-} 4\sinh^2\pi \sum_{n\in\mathbb{Z}} (1+n^2)^{s-2}&\Big|(b+\delta)e^{-inq_1} - be^{-inq_2} - \delta e^{-inq_T}\Big|^2 \\ &= 4\sinh^2\pi \sum_{n\in\mathbb{Z}} (1+n^2)^{s-2}\Big|(b+\delta)e^{-inq_1(T)} - be^{-inq_2(T)} - \delta e^{-inq_T}\Big|^2 = 0.\end{aligned}$$

□

Proof of Theorem 1 (Circle Case). Given the two-peakon solution we have constructed, the crux of our proof will be to construct a traveling wave solution $w(x,t)$ with $w(x,T) = u(x,T) = v(x)$. Starting with the periodic 4×4 system for FORQ, we will take $p_1 = \delta$, $p_2 = 0$, $q_1(0) = \alpha$, $q_2(0) = 0$ to construct a solitary traveling peakon. This reduced system is solved as

$$p_1(t) = \delta \qquad \text{and} \qquad q_1(t) = \frac{2}{3}\delta^2[1 + \frac{2}{3}\sinh^2\pi]t + \alpha.$$

Thus, we will choose $q_1(0) = \alpha$ such that $q_1(T)$ for w equals $q_1(T)$ for u. To distinguish these q_1's, we will write q_1^u for the q_1 corresponding to u and q_1^w for the q_1 corresponding to w.

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Having these solutions coincide at the collision time T precisely means $q_1^w(T) = q_1^u(T)$. We can now use this equation to solve for α as

$$\alpha = q_1^u(T) - \frac{2}{3}\delta^2[1 + \frac{2}{3}\sinh^2\pi]T.$$

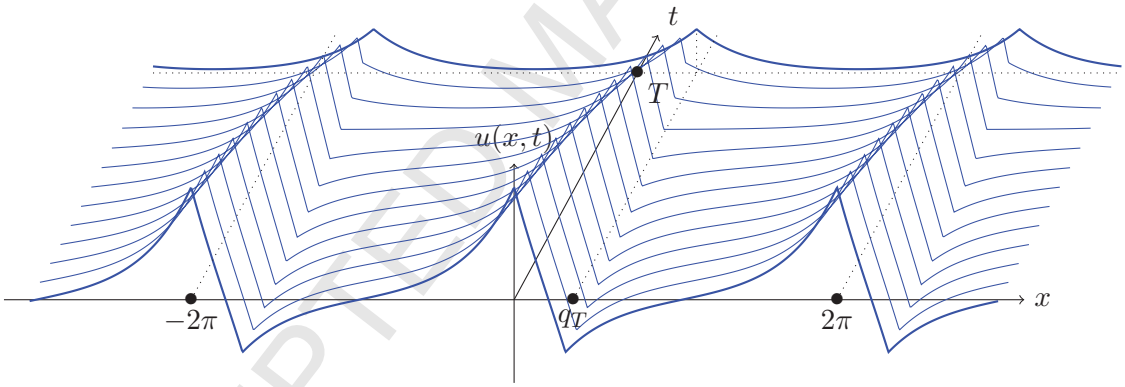
This completes the construction of our traveling wave w. To demonstrate non-uniqueness, we define new solutions

$$\tilde{u}(x,t) = u(-x,-t+T) \quad \text{and} \quad \tilde{w}(x,t) = w(-x,-t+T).$$

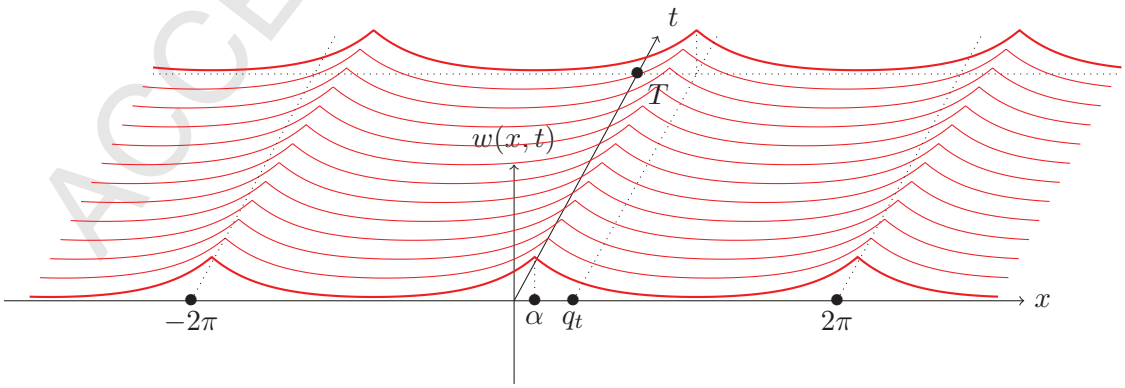
We see that since $u(x,t)$ solves FORQ iff $u(-x,-t)$ solves FORQ, that $\tilde{u}$ and $\tilde{w}$ are both solutions to FORQ. Furthermore, analogously to the real-line case, we see that the initial conditions of these solutions are equal by Proposition 2 we have in H^s

$$\lim_{t\to 0^+} \tilde{u}(x,0) = \lim_{t\to T^-} u(-x,t) = v(-x) = \lim_{t\to T^-} w(-x,t) = \lim_{t\to 0^+} \tilde{w}(x,t).$$

A visualization of the periodic u and w is shown below.



(a) Peakon-Antipeakon Solution



(b) Traveling Wave Solution

Figure 2: Construction of periodic u and w.

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We again note that the initial data which has nonunique solutions can be made arbitrarily small. Taking the H^s norm of the new initial data which is the collision function v, we have

$$\|v\|_{H^s}^2 = 4c_s \sinh^2 \pi\delta^2 \quad \text{where} \quad c_s = \sum_{n\in\mathbb{Z}} (1+n^2)^{s-2}.$$

Thus, we can control the size of the initial data via the choice of δ. In particular, given $\varepsilon > 0$, we will take

$$\delta < \frac{\varepsilon}{2\sinh \pi\sqrt{c_s}} \quad \text{which gives} \quad \|v\|_{H^s} < \varepsilon.$$

□

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